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Research article

An application of Pascal distribution involving Kamali type related to leaf like domain

K. Saritha and K. Thilagavathi*

Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology, Vellore 632014, Tamilnadu, India

* **Correspondence:** Email: kthilagavathi@vit.ac.in.

Abstract: This paper aims to study the Geometric properties of analytic function in the open unit disk. In the present investigation, we obtain some geometric properties of Pascal distribution involving Kamali type related to leaf like domain. In this paper, we find coefficient inequality, Radii Properties, convolution product, partial sum of the class $\Sigma(\delta, \Phi, \beta, s, t, m)$. Furthermore, we examine the distortion bounds belonging to the same class.

Keywords: univalent function; convolution; Pascal distribution; leaf like domain; Kamali type; subordination
Mathematics Subject Classification: 52A41, 32W50

1. Introduction

Let $\mathcal{G}$ be the holomorphic function in the open unit disc $\mathbb{U}$ which is defined by

$$\jmath(\zeta) = \zeta + \sum_{k=2}^{\infty} b_k \zeta^k, \quad \zeta \in \mathbb{U}, \tag{1.1}$$

and let $\Bbbk$ be the subclass of $\mathcal{G}$ consisting of functions of the form

$$\jmath(\zeta) = \zeta - \sum_{k=2}^{\infty} b_k \zeta^k, \quad \zeta \in \mathbb{U}, \tag{1.2}$$

which are univalent and normalized in $\mathbb{U}$. For $\jmath \in \mathcal{G}$ and of the form (1.1) and $\imath(\zeta) \in \mathcal{G}$ given by

$$\imath(\zeta) = \zeta + \sum_{k=2}^{\infty} c_k \zeta^k, \quad \zeta \in \mathbb{U}, \tag{1.3}$$

convolution of $\jmath(\zeta)$ and $\imath(\zeta)$ is given by

$$(\jmath \star \imath)(\zeta) = \zeta + \sum_{k=2}^{\infty} b_k c_k \zeta^k, \quad \zeta \in \mathbb{U}. \tag{1.4}$$

Let $\jmath(\zeta)$ and $\imath(\zeta)$ be holomorphic functions and b and c are coefficient of the function which is belonging to $\mathcal{G}$, so we state that $\jmath$ is subordinate to $\imath$ it is denoted by $\jmath \prec \imath$ (see [10]), then there is a Schwarz function $\wp$ that is holomorphic in $\mathbb{U}$ including $\wp(0) = 0$ and $|\wp(\zeta)| < 1$ for every $\zeta \in \mathbb{U}$, such that $\jmath(\zeta) = \imath(\wp(\zeta))$, for $\zeta \in \mathbb{U}$. Moreover, we have $\jmath$ is univalent in $\mathbb{U}$.

$$\jmath \prec \imath \quad \Leftrightarrow \quad \jmath(0) \ = \ \imath(0) \ \ and \ \ \jmath(\mathbb{U}) \subset \imath(\mathbb{U}).$$

A Variable Y is said to have the Pascal distribution if it takes the values 0,1,2,3... with the probabilities $(1-t)^m$, $\frac{tm(1-t)^m}{1!}$, $\frac{t^2 m(m+1)(1-t)^m}{2!}$, $\frac{t^3 m(m+1)(m+2)(1-t)^m}{3!}$,... respectively, where t, m are called the parameters and thus,

$$P(Y = k) = \binom{k+m-1}{m-1} t^k (1-t)^m, k \in \{0, 1, 2, ...\}. \tag{1.5}$$

Many essentially interesting proof techniques involving a power series,whose co-efficients are probabilities of the Pascal distribution series introduced by El-Deeb et al. [15] that is

$$Q_t^m(\zeta) = \zeta + \binom{k+m-1}{m-1} t^{(k-1)} (1-t)^m \zeta^k, k \in \mathbb{U}, (m \geq 1, 0 \leq t \leq 1). \tag{1.6}$$

The family of an holomorphic function as follows

$$\mathcal{E} = \{F : F(\zeta) = \jmath * Q_t^m(\zeta) = \zeta + \sum_{k=2}^{\infty} \binom{k+m-2}{m-1} t^{k-1} (1-t)^m b_k \zeta^k, \jmath \in \mathcal{G}\}. \tag{1.7}$$

Muhammet Kamali et al. [11] introduced the class of function and gave the following condition.

Let $\Phi : \mathbb{U} \to C$ be holomorphic and for $0 \leq \delta \leq 1$. We define the class $\Sigma(\delta, \Phi, \beta, s, t, m)$ as

$$\begin{aligned} \Sigma(\delta, \Phi, \beta, s, t, m) &= \left\{ \jmath \in \mathcal{G} : \frac{\delta\zeta^3 F'''(\zeta) + (1+2\delta)F''(\zeta)\zeta^2 + \zeta F'(\zeta)}{\delta\zeta^2 F''(\zeta) + \zeta F'(\zeta)} \right\} \\ &= \zeta + \sum_{k=2}^{\infty} \binom{k+m-2}{m-1} t^{k-1} (1-t)^m k(\delta k - \delta + 1) b_k \zeta^k. \end{aligned} \tag{1.8}$$

Atshan et al. [19] was stuided in the class of function and the following condition is given in Eq (1.8). Paprocki and Sokół [12] studied the class of analytic and univalent functions defined by $S^*(\alpha, b)$, where $\alpha \geq 1, b \geq \frac{1}{2}$. For the choice of $\alpha = 1$, the class of $S^*(\alpha, b)$ investigated by Janowski [9]. For the choice of $\alpha = 2, b = 1$, the class $S^*(2, 1)$ investigated by Sokół [18]. It is easy to see that $f \in S^*(\alpha, b)$ iff

$$\frac{\zeta f'(\zeta)}{f(\zeta)} \prec q_0(\zeta) = \left(\frac{1+\zeta}{1 + \left(\frac{1-b}{b}\right)\zeta} \right)^{\frac{1}{\alpha}}, \quad q_0(0) = 1,$$

which is a leaf like set.

Making use of this the class $C(\beta, s)$ and is a leaf like set. For the choice of $\beta = 2, s = 1$, the class $C(2, 1)$ are investigated by Paprocki and Sokół [12]. The concept of leaf like domain was investigated by Paprocki and Sokół [12]. For more details related to the leaf-like domain, one may refer to the recent papers (see [1, 17]).

For fixed parameter β, s we say that $F \in \mathcal{G}$ is in the class $\Sigma(\delta, \Phi, \beta, s, t, m)$ if it satisfies the following subordination condition.

$$1 + \frac{\zeta\left(\Sigma_{\beta,s}^{t,m}F(\zeta)\right)''}{\left(\Sigma_{\beta,s}^{t,m}F(\zeta)\right)'} \prec \left(\frac{1 + \omega(\zeta)}{1 + \left(\frac{1-s}{s}\right)\omega(\zeta)}\right)^{\frac{1}{\beta}}. \tag{1.9}$$

In view of the definition of subordination is equivalent to the following conditions:

$$|\omega(\zeta)| < 1.$$

$$\left|\frac{\left[\left(\Sigma_{\beta,s}^{t,m}\right)' + \zeta\left(\Sigma_{\beta,s}^{t,m}\right)''\right]^{\beta} - \left[\left(\Sigma_{\beta,s}^{t,m}\right)'\right]^{\beta}}{\left[\left(\Sigma_{\beta,s}^{t,m}\right)'\right]^{\beta} - \left(\frac{1-s}{s}\right)\left[\left(\Sigma_{\beta,s}^{t,m}\right)' + \zeta\left(\Sigma_{\beta,s}^{t,m}\right)''\right]^{\beta}}\right| < 1. \tag{1.10}$$

This is used to obtain geometric properties like coefficient inequality, Radius of starlikeness, convolution properties, partial sum of the class $\Sigma(\delta, \Phi, \beta, s, t, m)$ involving Pascal distribution series related to the leaf like domain. The following theorem gives a necessary and sufficient condition for a function F to be in the class $\Sigma(\delta, \Phi, \beta, s, t, m)$.

2. Main result

Theorem 2.1. *Let a function $F \in \mathcal{G}$ which is belonging to the class $\Sigma(\delta, \Phi, \beta, s, t, m)$ if only if*

$$\sum_{k=2}^{\infty}\binom{k+m-2}{m-1}t^{k-1}(1-t)^m k^2(\delta k - \delta + 1)\Big(\frac{\beta k(s+1) + s(1-2\beta)}{s}\Big)b_k \leq \frac{2s-1}{s}. \tag{2.1}$$

Proof. Let $F \in \Sigma(\delta, \Phi, \beta, s, t, m)$ then,

$$1 + \frac{\zeta\left(\Sigma_{\beta,s}^{t,m}F(\zeta)\right)''}{\left(\Sigma_{\beta,s}^{t,m}F(\zeta)\right)'} \prec \left(\frac{1 + \omega(\zeta)}{1 + \left(\frac{1-s}{s}\right)\omega(\zeta)}\right)^{\frac{1}{\beta}}. \tag{2.2}$$

Therfore, there exists an holomorphic function ω such that,

$$\omega(\zeta) = \frac{\left[\left(\Sigma_{\beta,s}^{t,m}\right)' + \zeta\left(\Sigma_{\beta,s}^{t,m}\right)''\right]^{\beta} - \left[\left(\Sigma_{\beta,s}^{t,m}\right)'\right]^{\beta}}{\left[\left(\Sigma_{\beta,s}^{t,m}\right)'\right]^{\beta} - \left(\frac{1-s}{s}\right)\left[\left(\Sigma_{\beta,s}^{t,m}\right)' + \zeta\left(\Sigma_{\beta,s}^{t,m}\right)''\right]^{\beta}}. \tag{2.3}$$

Hence,

$$|\omega(\zeta)| = \left|\frac{\left[\left(\Sigma_{\beta,s}^{t,m}\right)' + \zeta\left(\Sigma_{\beta,s}^{t,m}\right)''\right]^{\beta} - \left[\left(\Sigma_{\beta,s}^{t,m}\right)'\right]^{\beta}}{\left[\left(\Sigma_{\beta,s}^{t,m}\right)'\right]^{\beta} - \left(\frac{1-s}{s}\right)\left[\left(\Sigma_{\beta,s}^{t,m}\right)' + \zeta\left(\Sigma_{\beta,s}^{t,m}\right)''\right]^{\beta}}\right| < 1, \tag{2.4}$$

let

$$\frac{\left[\left(\Sigma_{\beta,s}^{t,m}\right)' + \zeta\left(\Sigma_{\beta,s}^{t,m}\right)''\right]^{\beta} - \left[\left(\Sigma_{\beta,s}^{t,m}\right)'\right]^{\beta}}{\left[\left(\Sigma_{\beta,s}^{t,m}\right)'\right]^{\beta} - \left(\frac{1-s}{s}\right)\left[\left(\Sigma_{\beta,s}^{t,m}\right)' + \zeta\left(\Sigma_{\beta,s}^{t,m}\right)''\right]^{\beta}} < 1.$$

$$\left[\left(\Sigma_{\beta,s}^{t,m}\right)' + \zeta\left(\Sigma_{\beta,s}^{t,m}\right)''\right]^{\beta} - \left[\left(\Sigma_{\beta,s}^{t,m}\right)'\right]^{\beta} < \left[\left(\Sigma_{\beta,s}^{t,m}\right)'\right]^{\beta} - \left(\frac{1-s}{s}\right)\left[\left(\Sigma_{\beta,s}^{t,m}\right)' + \zeta\left(\Sigma_{\beta,s}^{t,m}\right)''\right]^{\beta}.$$

To solve this we get,

$$\left[\left(\Sigma_{\beta,s}^{t,m}\right)' + \zeta\left(\Sigma_{\beta,s}^{t,m}\right)''\right]^{\beta} < 2\left[\left(\Sigma_{\beta,s}^{t,m}\right)'\right]^{\beta} - \left(\frac{1-s}{s}\right)\left[\left(\Sigma_{\beta,s}^{t,m}\right)' + \zeta\left(\Sigma_{\beta,s}^{t,m}\right)''\right]^{\beta}.$$

Here,

$$\left[\left(\Sigma_{\beta,s}^{t,m}\right)' + \zeta\left(\Sigma_{\beta,s}^{t,m}\right)''\right]^{\beta} = 1 + \beta\sum_{k=2}^{\infty}\binom{k+m-2}{m-1}t^{k-1}(1-t)^m k^2(\delta k - \delta + 1)b_k\zeta^{n-1}, \tag{2.5}$$

and

$$\begin{aligned}&2\left[\left(\Sigma_{\beta,s}^{t,m}\right)'\right]^{\beta} - \left(\frac{1-s}{s}\right)\left[\left(\Sigma_{\beta,s}^{t,m}\right)' + \zeta\left(\Sigma_{\beta,s}^{t,m}\right)''\right]^{\beta} \\ =&\left(\frac{3s-1}{s}\right) + \sum_{k=2}^{\infty}\binom{k+m-2}{m-1}t^{k-1}(1-t)^m \times k^2(\delta k - \delta + 1)\left(\frac{2\beta s - \beta k + s}{s}\right)b_k\zeta^{n-1}.\end{aligned} \tag{2.6}$$

Compare the Eqs (2.4) and (2.5) we get,

$$\sum_{k=2}^{\infty}\binom{k+m-2}{m-1}t^{k-1}(1-t)^m k^2(\delta k - \delta + 1)\left(\frac{\beta k(s+1) + s(1-2\beta)}{s}\right)b_k \leq \frac{2s-1}{s}.$$

Conversely, let

$$\sum_{k=2}^{\infty}\binom{k+m-2}{m-1}t^{k-1}(1-t)^m k^2(\delta k - \delta + 1)\left(\frac{\beta k(s+1) + s(1-2\beta)}{s}\right)b_k \leq \frac{2s-1}{s}.$$

Then from Eq (1.10), we have

$$\begin{aligned}&2\left[\left(\Sigma_{\beta,s}^{t,m}\right)'\right]^{\beta} - \left(\frac{1-s}{s}\right)\left[\left(\Sigma_{\beta,s}^{t,m}\right)' + \zeta\left(\Sigma_{\beta,s}^{t,m}\right)''\right]^{\beta} \\ =&\left(\frac{3s-1}{s}\right) + \sum_{k=2}^{\infty}\binom{k+m-2}{m-1}t^{k-1}(1-t)^m \times k^2(\delta k - \delta + 1)\left(\frac{2\beta s - \beta k + s}{s}\right)b_k\zeta^{n-1}.\end{aligned} \tag{2.7}$$

$$\left[\left(\Sigma_{\beta,s}^{t,m}\right)' + \zeta\left(\Sigma_{\beta,s}^{t,m}\right)''\right]^{\beta} < 2\left[\left(\Sigma_{\beta,s}^{t,m}\right)'\right]^{\beta} - \left(\frac{1-s}{s}\right)\left[\left(\Sigma_{\beta,s}^{t,m}\right)' + \zeta\left(\Sigma_{\beta,s}^{t,m}\right)''\right]^{\beta}.$$

$$|\omega(\zeta)| = \left|\frac{\left[\left(\Sigma_{\beta,s}^{t,m}\right)' + \zeta\left(\Sigma_{\beta,s}^{t,m}\right)''\right]^{\beta} - \left[\left(\Sigma_{\beta,s}^{t,m}\right)'\right]^{\beta}}{\left[\left(\Sigma_{\beta,s}^{t,m}\right)'\right]^{\beta} - \left(\frac{1-s}{s}\right)\left[\left(\Sigma_{\beta,s}^{t,m}\right)' + \zeta\left(\Sigma_{\beta,s}^{t,m}\right)''\right]^{\beta}}\right| < 1. \tag{2.8}$$

Thus,

$$\omega(\zeta)=\frac{\left[\left(\Sigma_{\beta,s}^{t,m}\right)'+\zeta\left(\Sigma_{\beta,s}^{t,m}\right)''\right]^{\beta}-\left[\left(\Sigma_{\beta,s}^{t,m}\right)'\right]^{\beta}}{\left[\left(\Sigma_{\beta,s}^{t,m}\right)'\right]^{\beta}-\left(\frac{1-s}{s}\right)\left[\left(\Sigma_{\beta,s}^{t,m}\right)'+\zeta\left(\Sigma_{\beta,s}^{t,m}\right)''\right]^{\beta}}, \tag{2.9}$$

this proves that,

$$1+\frac{\zeta\left(\Sigma_{\beta,s}^{t,m}F(\zeta)\right)''}{\left(\Sigma_{\beta,s}^{t,m}F(\zeta)\right)'}\prec\left(\frac{1+\omega(\zeta)}{1+\left(\frac{1-s}{s}\right)\omega(\zeta)}\right)^{\frac{1}{\beta}}, \tag{2.10}$$

and hence $F\in\Sigma(\delta,\Phi,\beta,s,t,m)$. □

The concept of neighborhoods was first introduced by Goodman [7] and then generalized by Ruscheweyh [13] and studied by some authors, Atshan [2] and Atshan and Kulkarni [3].

3. Neighborhoods for the class $F\in\Sigma(\delta,\Phi,\beta,s,t,m)$

Theorem 3.1. *Let a function $F\in\mathcal{G}$ is in the class $F\in\Sigma(\delta,\Phi,\beta,s,t,m)$ then,*

$$\Omega=\frac{2s-1}{\binom{k+m-2}{m-1}t^{k-1}(1-t)^m k(\delta k-\delta+1)\big(\beta k(s+1)+s(1-2\beta)\big)}.$$

Proof. It follows from Theorem 2.1 that if $F\in\Sigma_{\beta,s}^{t,m}F_1(\zeta)$ then we have,

$$\sum_{k=2}^{\infty}\binom{k+m-2}{m-1}t^{k-1}(1-t)^m k^2(\delta k-\delta+1)\Big(\frac{\beta k(s+1)+s(1-2\beta)}{s}\Big)b_k\le\frac{2s-1}{s}.$$

Hence

$$\sum kb_k=\Big(\frac{2s-1}{s}\Big)\frac{s}{\binom{k+m-2}{m-1}t^{k-1}(1-t)^m k(\delta k-\delta+1)\big(\beta k(s+1)+s(1-2\beta)\big)},$$

which implies that

$$\begin{aligned}\sum kb_k&=\frac{2s-1}{\binom{k+m-2}{m-1}t^{k-1}(1-t)^m k(\delta k-\delta+1)\big(\beta k(s+1)+s(1-2\beta)\big)}\\&=\Omega.\end{aligned}$$

□

4. Convolution properties

Theorem 4.1. *For functions $F_j(\zeta)$ $(j=1,2)$ defined by (1.2) let $F_1(\zeta)\in\Sigma_{\beta,s}^{t,m}$ and $F_2(\zeta)\in\Sigma_{\beta,s}^{t,m}$ then $F_1(\zeta)*F_2(\zeta)\in\Sigma_{\beta,s}^{t,m}$ where*

$$\S=\frac{\beta k(s+1)+s(1-2\beta)\big[1-\beta k(r+1)+r(1-2\beta)\big]}{(2s-1)(2r-1)\big[\beta(k-2)+1\big]-2\big[\beta k(s+1)+s(1-2\beta)\big]\big[\beta k(r+1)+r(1-2\beta)\big]\Psi(k)}.$$

Here,

$$\Psi(k)=\binom{k+m-2}{m-1}t^{k-1}(1-t)^m k^2(\delta k-\delta+1).$$

Proof. In the view of Theorem 2.1 it suffices to prove that

$$\sum_{k=2}^{\infty}\frac{\beta k(\S+1)+\S(1-2\beta)}{2\S-1}\Psi(k)b_{k,1}b_{k,2}\le 1,$$

where § is defined by Theorem 2.1 under the hypothesis it follows from Theorem 2.1 and the Cauchy-Schwarz inequality that

$$\sum_{k=2}^{\infty}\frac{\left[\beta k(s+1)+s(1-2\beta)\right]^{\frac{1}{2}}\left[\beta k(r+1)+r(1-2\beta)\right]^{\frac{1}{2}}}{\sqrt{(2s-1)(2r-1)}}\Psi(k)\sqrt{b_{k,1}b_{k,2}}\le 1. \tag{4.1}$$

To find largest § such that

$$\begin{aligned}&\sum_{n=2}^{\infty}\frac{\beta k(\S+1)+\S(1-2\beta)}{2\S-1}\Psi(k)b_{k,1}b_{k,2}\\ \le&\sum_{k=2}^{\infty}\frac{\left[\beta k(s+1)+s(1-2\beta)\right]^{\frac{1}{2}}\left[\beta k(r+1)+r(1-2\beta)\right]^{\frac{1}{2}}}{\sqrt{(2s-1)(2r-1)}}\Psi(k)\sqrt{b_{k,1}b_{k,2}}\\ \le&1,\end{aligned}$$

or equivalently that

$$\sqrt{b_{k,1}b_{k,2}}\le\frac{\left[\beta k(s+1)+s(1-2\beta)\right]^{\frac{1}{2}}\left[\beta k(r+1)+r(1-2\beta)\right]^{\frac{1}{2}}}{\sqrt{(2s-1)(2r-1)}}\times\frac{2\S-1}{\beta k(\S+1)+\S(1-2\beta)}.$$

From (2.10) we have,

$$\sqrt{b_{k,1}b_{k,2}}\le\frac{\sqrt{(2s-1)(2r-1)}}{\left[\beta k(s+1)+s(1-2\beta)\right]^{\frac{1}{2}}\left[\beta k(r+1)+r(1-2\beta)\right]^{\frac{1}{2}}\Psi(k)}.$$

It is sufficient to find the largest Ψ such that

$$\begin{aligned}&\frac{\sqrt{(2s-1)(2r-1)}}{\left[\beta k(s+1)+s(1-2\beta)\right]^{\frac{1}{2}}\left[\beta k(r+1)+r(1-2\beta)\right]^{\frac{1}{2}}\Psi(k)}\\ \le&\frac{\left[\beta k(s+1)+s(1-2\beta)\right]^{\frac{1}{2}}\left[\beta k(r+1)+r(1-2\beta)\right]^{\frac{1}{2}}}{\sqrt{(2s-1)(2r-1)}}\times\frac{2\S-1}{\beta k(\S+1)+\S(1-2\beta)},\end{aligned}$$

which implies to

$$\S=\frac{\beta k(s+1)+s(1-2\beta)\left[1-\beta k(r+1)+r(1-2\beta)\right]}{(2s-1)(2r-1)\left[\beta(k-2)+1\right]-2\left[\beta k(s+1)+s(1-2\beta)\right]\left[\beta k(r+1)+r(1-2\beta)\right]\Psi(k)}.$$

Here,

$$\Psi(k) = \binom{k+m-2}{m-1} t^{k-1}(1-t)^m k^2(\delta k - \delta + 1).$$

This completes the proof. □

5. Distortion properties

A distortion property for the functions in the class $\Sigma(\delta, \Phi, \beta, s, t, m)$ is given as follows.

Theorem 5.1. *If the function $F \in \Sigma_{\beta,s}^{t,m}$ then,*

$$R - \frac{(2s-1)}{2(2\beta+s)}R^2 \leq |\Sigma_{\beta,s}^{t,m}(\zeta)| \leq R + \frac{(2s-1)}{2(2\beta+s)}R^2$$

with equality for

$$\Sigma_{\beta,s}^{t,m}(\zeta) = \zeta - \frac{(2s-1)}{2(2\beta+s)}\zeta^2.$$

Proof. If $F \in \Sigma_{\beta,s}^{t,m}$, Theorem 2.1 yields the inequality,

$$\sum_{k=2}^{\infty} \binom{k+m-2}{m-1} t^{k-1}(1-t)^m k^2(\delta k - \delta + 1)\Big(\frac{\beta k(s+1) + s(1-2\beta)}{s}\Big) b_k \leq \frac{2s-1}{s}.$$

Therefore, we have

$$\sum_{k=2}^{\infty} b_k = \frac{2s-1}{\binom{m}{m-1} t(1-t)^m 4(\delta+1)\Big(2\beta(s+1) + s(1-2\beta)\Big)}.$$

Thus,

$$\begin{aligned}
|\Sigma_{\beta,s}^{t,m}| &\leq \zeta + \sum_{k=2}^{\infty} \binom{k+m-2}{m-1} t^{k-1}(1-t)^m k(\delta k - \delta + 1) b_k \zeta^k \\
&\leq R + R^2 \binom{m}{m-1} t(1-t)^m 2(\delta+1) \sum_{k=2}^{\infty} b_k \\
&\leq R + R^2 \binom{m}{m-1} t(1-t)^m 2(\delta+1) \times \frac{2s-1}{\binom{m}{m-1} t(1-t)^m 4(\delta+1)\Big(2\beta(s+1) + s(1-2\beta)\Big)} \\
&\leq R + \frac{(2s-1)}{2(2\beta+s)}R^2.
\end{aligned}$$

Similarly,

$$
\begin{aligned}
|\Sigma_{\beta,s}^{t,m}| &\geq \zeta - \sum_{k=2}^{\infty} \binom{k+m-2}{m-1} t^{k-1}(1-t)^m k(\delta k - \delta + 1) b_k \zeta^k \\
&\geq R - R^2 \binom{m}{m-1} t(1-t)^m 2(\delta+1) \sum_{k=2}^{\infty} b_k \\
&\geq R - R^2 \binom{m}{m-1} t(1-t)^m 2(\delta+1) \times \frac{2s-1}{\binom{m}{m-1} t(1-t)^m 4(\delta+1)\Big(2\beta(s+1) + s(1-2\beta)\Big)} \\
&\geq R - \frac{(2s-1)}{2(2\beta+s)} R^2.
\end{aligned}
$$

□

Theorem 5.2. *If the function $F \in \Sigma_{\beta,s}^{t,m}$ then,*

$$
1 - \frac{(2s-1)}{(2\beta+s)} R \leq |\Big(\Sigma_{\beta,s}^{t,m}(\zeta)\Big)'| \leq 1 + \frac{(2s-1)}{(2\beta+s)} R
$$

with equality for

$$
\Sigma_{\beta,s}^{t,m}(\zeta) = 1 - \frac{(2s-1)}{2(2\beta+s)} \zeta.
$$

Proof. If $F \in \Sigma_{\beta,s}^{t,m}$, Theorem 2.1 yields the inequality,

$$
\sum_{k=2}^{\infty} \binom{k+m-2}{m-1} t^{k-1}(1-t)^m k^2(\delta k - \delta + 1)\Big(\frac{\beta k(s+1) + s(1-2\beta)}{s}\Big) b_k \leq \frac{2s-1}{s}.
$$

Therefore, we have

$$
\sum_{k=2}^{\infty} k b_k = \frac{2s-1}{\binom{m}{m-1} t(1-t)^m 2(\delta+1)\Big(2\beta(s+1) + s(1-2\beta)\Big)}.
$$

Thus,

$$
\begin{aligned}
|\Big(\Sigma_{\beta,s}^{t,m}\Big)'| &\leq 1 + \sum_{k=2}^{\infty} \binom{k+m-2}{m-1} t^{k-1}(1-t)^m k(\delta k - \delta + 1) b_k k \zeta^{k-1} \\
&\leq 1 + R \binom{m}{m-1} t(1-t)^m 2(\delta+1) \sum_{k=2}^{\infty} k b_k \\
&\leq 1 + R \binom{m}{m-1} t(1-t)^m 2(\delta+1) \times \frac{2s-1}{\binom{m}{m-1} t(1-t)^m 2(\delta+1)\Big(2\beta(s+1) + s(1-2\beta)\Big)} \\
&\leq 1 + \frac{(2s-1)}{(2\beta+s)} R.
\end{aligned}
$$

Similarly,

$$\begin{aligned}
|\left(\Sigma_{\beta,s}^{t,m}\right)'| &\geq 1-\sum_{k=2}^{\infty}\binom{k+m-2}{m-1}t^{k-1}(1-t)^m k(\delta k-\delta+1)b_k k\zeta^{k-1}\\
&\geq 1-R\binom{m}{m-1}t(1-t)^m 2(\delta+1)\sum_{k=2}^{\infty}kb_k\\
&\geq 1-R\binom{m}{m-1}t(1-t)^m 2(\delta+1)\frac{2s-1}{\binom{m}{m-1}t(1-t)^m 2(\delta+1)\left(2\beta(s+1)+s(1-2\beta)\right)}\\
&\geq 1-\frac{(2s-1)}{(2\beta+s)}R.
\end{aligned}$$

□

Theorem 5.3. *The class $F\in\Sigma(\delta,\Phi,\beta,s,t,m)$ is closed under convex linear combinations.*

Proof. Suppose that the functions $\Sigma_{\beta,s}^{t,m}F_1(\zeta)$, $\Sigma_{\beta,s}^{t,m}F_2(\zeta)$ defined by

$$\Sigma(\delta,\Phi,\beta,s,t,m)F_i(\zeta)=\zeta+\sum_{k=2}^{\infty}\binom{k+m-2}{m-1}t^{k-1}(1-t)^m k(\delta k-\delta+1)b_k\zeta^k,(i=1,2..,F\in\mathcal{G})$$

by set

$$\Sigma_{\beta,s}^{t,m}F(\zeta)=\ell\Sigma_{\beta,s}^{t,m}F_1(\zeta)+(1-\ell)\Sigma_{\beta,s}^{t,m}F_2(\zeta).$$

we find from (2.7) that,

$$\Sigma_{\beta,s}^{t,m}F(\zeta)=\zeta+\sum_{k=2}^{\infty}\{\ell c_{k,1}+(1-\ell)c_{k,2}\}\zeta^k\quad(0\leq\ell\leq 1),\zeta\in\mathcal{G}.$$

In view of Theorem 2.1 we have,

$$\begin{aligned}
&\sum_{k=2}^{\infty}\binom{k+m-2}{m-1}t^{k-1}(1-t)^m k^2(\delta k-\delta+1)\Big(\frac{\beta k(s+1)+s(1-2\beta)}{s}\Big)\{\ell c_{k,1}+(1-\ell)c_{k,2}\}\\
=&\ell\sum_{k=2}^{\infty}\binom{k+m-2}{m-1}t^{k-1}(1-t)^m k^2(\delta k-\delta+1)\Big(\frac{\beta k(s+1)+s(1-2\beta)}{s}\Big)\{\ell c_{k,1}+(1-\ell)c_{k,2}\}\\
+&(1-\ell)\sum_{k=2}^{\infty}\binom{k+m-2}{m-1}t^{k-1}(1-t)^m k^2(\delta k-\delta+1)\Big(\frac{\beta k(s+1)+s(1-2\beta)}{s}\Big)\{\ell c_{k,1}+(1-\ell)c_{k,2}\}\\
\leq&\ell\Big(\frac{2s-1}{s}\Big)+\frac{2s-1}{s}-\ell\Big(\frac{2s-1}{s}\Big)\\
=&\frac{2s-1}{s}.
\end{aligned}$$

□

6. Radius of starlikeness

Theorem 6.1. *(i) If the function $F \in \mathcal{G}$ be defined by (1.1) is in the class $\Sigma(\delta, \Phi, \beta, s, t, m)$, then F is starlike of order ρ in the disk $|\zeta| < r_1$ (i.e) $\Re\left(\frac{\zeta\left(\Sigma_{\beta,s}^{t,m}F(\zeta)\right)'}{\Sigma_{\beta,s}^{t,m}F(\zeta)}\right) > \rho$ $(|\zeta| < r_1; 0 \leq \rho \leq 1)$ where,*

$$r_1 = \left(\frac{1-\rho}{\rho+k-2}\right)^{\left(\frac{1}{k-1}\right)} \left(\frac{k(\alpha k(s+1) + s(1-2\alpha))}{2s-1}\right)^{\left(\frac{1}{k-1}\right)}.$$

(ii) If the function $F \in \mathcal{G}$ be defined by (1.1) is in the class $\Sigma(\delta, \Phi, \beta, s, t, m)$, then F is convex of order ρ in the disk $|\zeta| < r_2$ (i.e) $\Re\left(\frac{\zeta\Sigma_{\beta,s}^{t,m}F(\zeta)''}{\Sigma_{\beta,s}^{t,m}F(\zeta)'}\right) > \rho$ $(|\zeta| < r_2; 0 \leq \rho \leq 1)$ where,

$$r_2 = \left(\frac{1-\rho}{\rho+k-2}\right)^{\left(\frac{1}{k-1}\right)} \left(\frac{(\delta k-\delta+1)(\alpha k(s+1) + s(1-2\alpha))}{2s-1}\right)^{\left(\frac{1}{k-1}\right)}.$$

Proof. Let $F \in \mathcal{G}$ is starlike of order ρ we have,

$$\left|\frac{\zeta\left(\Sigma_{\beta,s}^{t,m}F(\zeta)\right)'}{\Sigma_{\beta,s}^{t,m}F(\zeta} - 1\right| < 1-\rho. \tag{6.1}$$

Thus,

$$\left|\frac{\sum_{k=2}^{\infty}(k-1)\binom{k+m-2}{m-1}t^{k-1}(1-t)^m k(\delta k-\delta+1)b_k\zeta^k}{\zeta + \sum_{k=2}^{\infty}\binom{k+m-2}{m-1}t^{k-1}(1-t)^m k(\delta k-\delta+1)b_k\zeta^k}\right| \leq 1-\rho. \tag{6.2}$$

Hence, (2.9) holds true if

$$\begin{aligned}
&\sum_{k=2}^{\infty}(k-1)\binom{k+m-2}{m-1}t^{k-1}(1-t)^m k(\delta k-\delta+1)b_k\zeta^k \\
&\quad \leq (1-\rho)\Big(\zeta + \sum_{k=2}^{\infty}\binom{k+m-2}{m-1}t^{k-1}(1-t)^m k(\delta k-\delta+1)b_k\zeta^k\Big). \\
&\sum_{k=2}^{\infty}(\rho+k-2)\binom{k+m-2}{m-1}t^{k-1}(1-t)^m k(\delta k-\delta+1)b_k\zeta^{k-1} \leq 1-\rho. \\
&\sum_{k=2}^{\infty}\Big(\frac{\rho+k-2}{1-\rho}\Big)\binom{k+m-2}{m-1}t^{k-1}(1-t)^m k(\delta k-\delta+1)b_k\zeta^{k-1} \leq 1.
\end{aligned}$$

From Theorem 2.1, we have

$$\sum_{k=2}^{\infty}\binom{k+m-2}{m-1}t^{k-1}(1-t)^m k^2(\delta k-\delta+1)\Big(\frac{\beta k(s+1) + s(1-2\beta)}{s}\Big)b_k \leq \frac{2s-1}{s}.$$

$$\sum_{k=2}^{\infty}\binom{k+m-2}{m-1}t^{k-1}(1-t)^{m}k^{2}(\delta k-\delta+1)\Big(\frac{2s-1}{s}\Big)\Big(\frac{\beta k(s+1)+s(1-2\beta)}{s}\Big)b_{k}\leq 1.$$

We say that,

$$\begin{aligned}&\sum_{k=2}^{\infty}\Big(\frac{\rho+k-2}{1-\rho}\Big)\binom{k+m-2}{m-1}t^{k-1}(1-t)^{m}k(\delta k-\delta+1)b_{k}\zeta^{k-1}\\ &\leq\sum_{k=2}^{\infty}\binom{k+m-2}{m-1}t^{k-1}(1-t)^{m}k^{2}(\delta k-\delta+1)\Big(\frac{2s-1}{s}\Big)\Big(\frac{\beta k(s+1)+s(1-2\beta)}{s}\Big)b_{k}.\end{aligned}$$

Equivalently,

$$|\zeta|^{k-1}=\frac{k(\beta k(s+1)+s(1-2\beta))(1-\rho)}{(2s-1)(\rho+k-2)}.$$

Therefore,

$$|\zeta|=\left(\frac{1-\rho}{\rho+k-2}\right)^{\left(\frac{1}{k-1}\right)}\left(\frac{k(\alpha k(s+1)+s(1-2\alpha))}{2s-1}\right)^{\left(\frac{1}{k-1}\right)}.$$

Hence $F\in\mathcal{G}$ is starlike of order ρ.

(ii) Let $F\in\mathcal{G}$ is convex of order ρ we have,

$$\left|\frac{\zeta\Sigma_{\beta,s}^{t,m}F(\zeta)''}{\Sigma_{\beta,s}^{t,m}F(\zeta)'}\right|<1-\rho. \tag{6.3}$$

$$\left|\frac{\zeta\Big(\sum_{k=2}^{\infty}(k-1)\binom{k+m-2}{m-1}t^{k-1}(1-t)^{m}k^{2}(\delta k-\delta+1)b_{k}\zeta^{k-2}\Big)}{1+\sum_{k=2}^{\infty}\binom{k+m-2}{m-1}t^{k-1}(1-t)^{m}k^{2}(\delta k-\delta+1)b_{k}\zeta^{k-1}}\right|\leq 1-\rho. \tag{6.4}$$

Hence (2.9) holds true if

$$\begin{aligned}&\sum_{k=2}^{\infty}(k-1)\binom{k+m-2}{m-1}t^{k-1}(1-t)^{m}k^{2}(\delta k-\delta+1)b_{k}\zeta^{k-1}\\ &\leq(1-\rho)\Big(\zeta+\sum_{k=2}^{\infty}\binom{k+m-2}{m-1}t^{k-1}(1-t)^{m}k^{2}(\delta k-\delta+1)b_{k}\zeta^{k-1}\Big).\\ &\sum_{k=2}^{\infty}(\rho+k-2)\binom{k+m-2}{m-1}t^{k-1}(1-t)^{m}k^{2}(\delta k-\delta+1)b_{k}\zeta^{k-1}\leq 1-\rho.\\ &\sum_{k=2}^{\infty}\Big(\frac{\rho+k-2}{1-\rho}\Big)\binom{k+m-2}{m-1}t^{k-1}(1-t)^{m}k^{2}(\delta k-\delta+1)b_{k}\zeta^{k-1}\leq 1.\end{aligned}$$

From Theorem 2.1, we have

$$\sum_{k=2}^{\infty}\binom{k+m-2}{m-1}t^{k-1}(1-t)^{m}k^{2}(\delta k-\delta+1)\Big(\frac{\beta k(s+1)+s(1-2\beta)}{s}\Big)b_{k}\leq\frac{2s-1}{s}.$$

$$\sum_{k=2}^{\infty}\binom{k+m-2}{m-1}t^{k-1}(1-t)^m k^2(\delta k-\delta+1)\Big(\frac{2s-1}{s}\Big)\Big(\frac{\beta k(s+1)+s(1-2\beta)}{s}\Big)b_k \leq 1.$$

We say that,

$$\sum_{k=2}^{\infty}\Big(\frac{\rho+k-2}{1-\rho}\Big)\binom{k+m-2}{m-1}t^{k-1}(1-t)^m k^2(\delta k-\delta+1)b_k\zeta^{k-1}$$
$$\leq \sum_{k=2}^{\infty}\binom{k+m-2}{m-1}t^{k-1}(1-t)^m k^2(\delta k-\delta+1)\Big(\frac{2s-1}{s}\Big)\Big(\frac{\beta k(s+1)+s(1-2\beta)}{s}\Big)b_k.$$

Equivalently,

$$|\zeta|^{k-1}=\frac{k(\beta(s+1)+s(1-2\beta))(1-\rho)}{(2s-1)(\rho+k-2)}.$$

Therefore,

$$|\zeta|=\left(\frac{1-\rho}{\rho+k-2}\right)^{\left(\frac{1}{k-1}\right)}\left(\frac{(\alpha k(s+1)+s(1-2\alpha))}{2s-1}\right)^{\left(\frac{1}{k-1}\right)}.$$

Hence $F\in\mathcal{G}$ is convex of order ρ. □

7. Partial sum

Partial sum is defined by Silverman [16]

$$\Sigma_{\beta,s}^{t,m}F_1(\zeta)=\zeta,\Sigma_{\beta,s}^{t,m}F_\gamma(\zeta)=\zeta+\sum_{k=2}^{\gamma}\binom{k+m-2}{m-1}t^{k-1}(1-t)^m k(\delta k-\delta+1)b_k\zeta^k.$$

In this paragraph, in the class $\Sigma_{\beta,s}^{t,m}F(\zeta)$, partial function sums can be considered and sharp lower limits can be reached for the function. For other investigation involving partial sum, one refer to [4, 5, 8, 14].

Theorem 7.1. *Let $F\in\Sigma_{\beta,s}^{t,m}$ is defined by (1.7), then*

$$\Re\{\frac{\Sigma_{\beta,s}^{t,m}F(\zeta)}{\Sigma_{\beta,s}^{t,m}F_\gamma(\zeta)}\}>1-\frac{1}{h_{\gamma+1}},\quad \zeta\in\mathbb{U},$$

where

$$F_k=\Big(\frac{2s-1}{\beta k^2(s+1)+ks(1-2\beta)}\Big).$$

Proof.

$$F_{d+1}>F_d>1,\quad d=2,3,....$$

Thus by Theorem 2.1, we obtain

$$\begin{aligned}&\sum_{k=2}^{\infty}\left|\binom{k+m-2}{m-1}t^{k-1}(1-t)^{m}k(\delta k-\delta+1)b_k\right|\\&+F_{d+1}\sum_{k=2}^{\infty}\left|\binom{k+m-2}{m-1}t^{k-1}(1-t)^{m}k(\delta k-\delta+1)b_k\right|\\&\leq F_m\sum_{k=2}^{\infty}\left|\binom{k+m-2}{m-1}t^{k-1}(1-t)^{m}k(\delta k-\delta+1)b_k\right|\\&\leq 1.\end{aligned}\tag{7.1}$$

By set

$$\begin{aligned}U\Big(\Sigma_{\beta,s}^{t,m}F(\zeta)\Big)&=F_{d+1}\left\{\frac{\Sigma_{\beta,s}^{t,m}F(\zeta)}{\Sigma_{\beta,s}^{t,m}F_d(\zeta)}-\Big(1-\frac{1}{F_{d+1}}\Big)\right\}\\&=1+F_{d+1}\Big(\frac{\Sigma_{\beta,s}^{t,m}F(\zeta)}{\Sigma_{\beta,s}^{t,m}F_d(\zeta)}-1\Big)\\&=1+F_{d+1}\left(\frac{\sum_{k=2}^{\infty}\binom{k+m-2}{m-1}t^{k-1}(1-t)^{m}k(\delta k-\delta+1)b_k\zeta^{k}}{z-\sum_{k=2}^{d}\binom{k+m-2}{m-1}t^{k-1}(1-t)^{m}k(\delta k-\delta+1)b_k\zeta^{k}}\right)\\&=1+F_{d+1}\left(\frac{\sum_{k=2}^{\infty}\binom{k+m-2}{m-1}t^{k-1}(1-t)^{m}k(\delta k-\delta+1)b_k\zeta^{k-1}}{1-\sum_{k=2}^{d}\binom{k+m-2}{m-1}t^{k-1}(1-t)^{m}k(\delta k-\delta+1)b_k\zeta^{k-1}}\right).\end{aligned}$$

$$U\Big(\Sigma_{\beta,s}^{t,m}F(\zeta)\Big)-1=F_{d+1}\left(\frac{\sum_{k=2}^{\infty}\binom{k+m-2}{m-1}t^{k-1}(1-t)^{m}k(\delta k-\delta+1)b_k\zeta^{k-1}}{1-\sum_{k=2}^{d}\binom{k+m-2}{m-1}t^{k-1}(1-t)^{m}k(\delta k-\delta+1)b_k\zeta^{k-1}}\right).$$

$$U\Big(\Sigma_{\beta,s}^{t,m}F(\zeta)\Big)+1=2+F_{d+1}\left(\frac{\sum_{k=2}^{\infty}\binom{k+m-2}{m-1}t^{k-1}(1-t)^{m}k(\delta k-\delta+1)b_k\zeta^{k-1}}{1-\sum_{k=2}^{d}\binom{k+m-2}{m-1}t^{k-1}(1-t)^{m}k(\delta k-\delta+1)b_k\zeta^{k-1}}\right).$$

And it is enough to show $\Re(\Sigma_{\beta,s}^{t,m}F(\zeta))>0,\zeta\in\mathbb{U}$ applying (2.5) we find

$$\begin{aligned}&\left|\frac{U\Big(\Sigma_{\beta,s}^{t,m}F(\zeta)\Big)-1}{U\Big(\Sigma_{\beta,s}^{t,m}F(\zeta)\Big)+1}\right|=\left|\frac{F_{d+1}\left(\frac{\sum_{k=2}^{\infty}\binom{k+m-2}{m-1}t^{k-1}(1-t)^{m}k(\delta k-\delta+1)b_k\zeta^{k-1}}{1-\sum_{k=2}^{d}\binom{k+m-2}{m-1}t^{k-1}(1-t)^{m}k(\delta k-\delta+1)b_k\zeta^{k-1}}\right)}{2+F_{d+1}\left(\frac{\sum_{k=2}^{\infty}\binom{k+m-2}{m-1}t^{k-1}(1-t)^{m}k(\delta k-\delta+1)b_k\zeta^{k-1}}{1-\sum_{k=2}^{d}\binom{k+m-2}{m-1}t^{k-1}(1-t)^{m}k(\delta k-\delta+1)b_k\zeta^{k-1}}\right)}\right|\\&\leq\left(F_{d+1}\frac{\sum_{k=2}^{\infty}\binom{k+m-2}{m-1}t^{k-1}(1-t)^{m}k(\delta k-\delta+1)|b_k|}{2-2\sum_{k=2}^{d}\binom{k+m-2}{m-1}t^{k-1}(1-t)^{m}k(\delta k-\delta+1)|b_k|+F_{d+1}\sum_{k=2}^{\infty}\binom{k+m-2}{m-1}t^{k-1}(1-t)^{m}k(\delta k-\delta+1)|b_k|}\right)\\&\leq 1,\end{aligned}$$

which gives

$$\Re\{\frac{\Sigma_{\beta,s}^{t,m}F(\zeta)}{\Sigma_{\beta,s}^{t,m}F_\gamma(\zeta)}\}>1-\frac{1}{h_{\gamma+1}},\ \zeta\in\mathbb{U}.$$

□

Theorem 7.2. *Let $F \in \Sigma_{\beta,s}^{t,m}$ is defined by (1.7), then*

$$\Re\left\{\frac{\Sigma_{\beta,s}^{t,m}F_k(\zeta)}{\Sigma_{\beta,s}^{t,m}F(\zeta)}\right\} > \frac{F_{n+1}}{1+F_{n+1}}, \quad \zeta \in \mathbb{U},$$

where

$$F_k = \Big(\frac{2s-1}{\beta k^2(s+1)+ks(1-2\beta)}\Big).$$

Proof.

$$F_{d+1} > F_d > 1, \quad d = 2, 3, ...$$

Thus by Theorem 2.1, we obtain

$$\begin{aligned}
&\sum_{k=2}^{\infty}\left|\binom{k+m-2}{m-1}t^{k-1}(1-t)^m k(\delta k-\delta+1)b_k\right| \\
&+ F_{d+1}\sum_{k=2}^{\infty}\left|\binom{k+m-2}{m-1}t^{k-1}(1-t)^m k(\delta k-\delta+1)b_k\right| \\
&\le F_m\sum_{k=2}^{\infty}\left|\binom{k+m-2}{m-1}t^{k-1}(1-t)^m k(\delta k-\delta+1)b_k\right| \\
&\le 1.
\end{aligned} \tag{7.2}$$

By set

$$\begin{aligned}
V\Big(\Sigma_{\beta,s}^{t,m}F(\zeta)\Big) &= 1 + F_{d+1}\left\{\frac{\Sigma_{\beta,s}^{t,m}F_\gamma(\zeta)}{\Sigma_{\beta,s}^{t,m}F(\zeta)} - \Big(\frac{d_{\gamma+1}}{1+d_{\gamma+1}}\Big)\right\} \\
&= 1 + F_{d+1}\left(\frac{\sum_{k=2}^{\gamma} z - \binom{k+m-2}{m-1}t^{k-1}(1-t)^m k(\delta k-\delta+1)b_k\zeta^k}{z - \sum_{k=2}^{\infty}\binom{k+m-2}{m-1}t^{k-1}(1-t)^m k(\delta k-\delta+1)b_k\zeta^k}\right) \\
&= 1 + \left(\frac{(1+d_{\gamma+1})\sum_{k=\gamma+1}^{\infty}\binom{k+m-2}{m-1}t^{k-1}(1-t)^m k(\delta k-\delta+1)b_k\zeta^k}{z - \sum_{k=2}^{\infty}\binom{k+m-2}{m-1}t^{k-1}(1-t)^m k(\delta k-\delta+1)b_k\zeta^k}\right) \\
&= 1 + \left(\frac{(1+F_{d+1})\sum_{k=\gamma+1}^{\infty}\binom{k+m-2}{m-1}t^{k-1}(1-t)^m k(\delta k-\delta+1)b_k\zeta^{k-1}}{1 - \sum_{k=2}^{\infty}\binom{k+m-2}{m-1}t^{k-1}(1-t)^m k(\delta k-\delta+1)b_k\zeta^{k-1}}\right).
\end{aligned}$$

$$V\Big(\Sigma_{\beta,s}^{t,m}F(\zeta)\Big) - 1 = \left(\frac{(1+F_{d+1})\sum_{k=\gamma+1}^{\infty}\binom{k+m-2}{m-1}t^{k-1}(1-t)^m k(\delta k-\delta+1)b_k\zeta^{k-1}}{1 - \sum_{k=2}^{\infty}\binom{k+m-2}{m-1}t^{k-1}(1-t)^m k(\delta k-\delta+1)b_k\zeta^{k-1}}\right).$$

$$V\Big(\Sigma_{\beta,s}^{t,m}F(\zeta)\Big) + 1 = 2 + \left(\frac{(1+F_{d+1})\sum_{k=\gamma+1}^{\infty}\binom{k+m-2}{m-1}t^{k-1}(1-t)^m k(\delta k-\delta+1)b_k\zeta^{k-1}}{1 - \sum_{k=2}^{\infty}\binom{k+m-2}{m-1}t^{k-1}(1-t)^m k(\delta k-\delta+1)b_k\zeta^{k-1}}\right).$$

And it is enough to show $\Re(\Sigma_{\beta,s}^{t,m}F(\zeta)) > 0, \zeta \in \mathbb{U}$ applying (7.2) we find

$$\left|\frac{V\Big(\Sigma_{\beta,s}^{t,m}F(\zeta)\Big) - 1}{V\Big(\Sigma_{\beta,s}^{t,m}F(\zeta)\Big) + 1}\right| \le 1,$$

which gives

$$\Re\left\{\frac{\Sigma_{\beta,s}^{t,m}F_k(\zeta)}{\Sigma_{\beta,s}^{t,m}F(\zeta)}\right\} > \frac{F_{k+1}}{1+F_{n+1}}, \quad \zeta \in \mathbb{U}.$$

□

8. Application of Pascal distribution

Every aspect of human endeavours depends on probability and statistics which are particularly Pascal distribution may be helpful in building models for inverse scattering problems and play a role in inferring the shape and physical properties of obstacles. Works are related to their study may refer [6, 20–22].

9. Conclusions

This paper deals with the application of Pascal distribution. The purpose of this article is to investigate the geometric properties of leaf-like domain, including co-efficient inequality, radius of starlikeness, convolution properties and partial sums of the class $\Sigma(\delta, \Phi, \beta, s, t, m)$ that involve Pascal distribution series. In addition, several theorems are presented which provide necessary and sufficient conditions for a function $F \in \Sigma(\delta, \Phi, \beta, s, t, m)$. Many interesting particular cases of main theorems are emphazied in the form of geometric properties. Furthermore to illustrate the results of application in various classes of analytic function. We anticipate the Pascal distribution may be helpful in building models for inverse scattering problems and play a role in inferring the shape and physical properties of obstacles. Pascal distribution will be important in several fields related to Mathematics, science and technology.

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Conflict of interest

The authors declare no conflicts of interest.

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