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Research article

Existence and multiplicity of solutions for a Schrödinger type equations involving the fractional $p(x)$-Laplacian

Shuhai Zhu*

College of Basic Science, Ningbo University of Finance & Economics, Ningbo 315175, China

* **Correspondence:** Email: zhushuhai@nbufe.edu.cn.

Abstract: We are concerned with the following Schrödinger type equation with variable exponents

$$(-\Delta_{p(x)})^s u + V(x)|u|^{p(x)-2}u = f(x,u) \quad \text{in} \quad \mathbb{R}^N,$$

where $(-\Delta_{p(x)})^s$ is the fractional $p(x)$-Laplace operator, $s \in (0,1)$, $V : \mathbb{R}^N \to (0,+\infty)$ is a continuous potential function, and $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition. We study the nonlinearity of this equation which is superlinear but does not satisfy the Ambrosetti-Rabinowitz type condition. By using variational techniques and the fountain theorem, we obtain the existence and multiplicity of nontrivial solutions. Furthermore, we show that the problem has a sequence of solutions with high energies.

Keywords: fractional $p(x)$-Laplacian; fractional Sobolev space with variable exponent; variational method; fountain theorem
Mathematics Subject Classification: 35B08, 35J60, 35A15

1. Introduction

In this paper, we are concerned with the existence and multiplicity of nontrivial solutions for the following nonlinear Schrödinger type equation involving fractional $p(x)$-Laplacian:

$$(-\Delta_{p(x)})^s u + V(x)|u|^{p(x)-2}u = f(x,u) \quad \text{in} \quad \mathbb{R}^N, \tag{1.1}$$

where $(-\Delta_{p(x)})^s$ is the fractional $p(x)$-Laplacian operator, $s \in (0,1)$ and potential function $V(x)$ satisfies the following conditions:

(V_1) $V(x) \in C(\mathbb{R}^N,\mathbb{R})$, $\inf_{x\in\mathbb{R}^N} V(x) \geq V_0 > 0$;
(V_2) For every constant $M > 0$, the Lebesgue measure of the set $\{x \in \mathbb{R}^N : V(x) \leq M\}$ is finite.

The nonlocal operator $(-\Delta_{p(x)})^s$ is defined as

$$(-\Delta_{p(x)})^s u(x) = \text{P.V.} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p(x,y)-2}(u(x) - u(y))}{|x - y|^{N+sp(x,y)}} dy, \quad \forall x \in \mathbb{R}^N,$$

where P.V. denotes the Cauchy principle value and for brevity. Notice that the operator $(-\Delta_{p(x)})^s$ is the fractional version of the well known $p(x)$-Laplacian operator $\Delta_{p(x)} u = \mathrm{div}(|\nabla u|^{p(x)-2} u)$, which was first introduced by Kaufmann, Rossi and Vidal in [24]. Some properties of fractional Sobolev space with variable exponent and the existence and multiple results of elliptic equations with fractional $p(x)$-Laplace operator are studied in [1, 3, 5, 8, 14, 24, 38].

In recent years, problems involving nonlocal operators have gained a lot of attention due to their occurrence in real world applications, such as the thin obstacle problem, optimization, finance, phase transitions and also in pure mathematical research, such as minimal surfaces, conservation laws etc. The celebrated work of Nezza et al. [33] provides the necessary functional set-up to study these nonlocal operator problems using variational methods. We refer [30, 34] and references therein for more details on problems involving fractional Laplace operator. In (1.1), when $p(\cdot) = p$ (constant), $(-\Delta_{p(x)})^s$ reduce to the usual fractional p-Laplace operator. In [9, 19, 28, 31, 32], the authors studied various aspects, viz., existence, multiplicity and regularity of the solutions of the nonlinear elliptic type problems involving fractional p-Laplace operator.

When $s \equiv 1$, problem (1.1) becomes the following $p(x)$-Laplacian equation:

$$-\Delta_{p(x)} u + V(x)|u|^{p(x)-2} u = f(x, u) \quad \text{in } \mathbb{R}^N. \tag{1.2}$$

These equations involving the $p(x)$-Laplacian arise in the modeling of electrorheological fluids and image restorations among other problems in physics and engineering, see [12, 13, 25, 29, 36]. Different from the Laplacian Δ and the p-Laplacian Δ_p, the $p(x)$-Laplacian is nonlinear and nonhomogeneous. It is worth pointing out that Eq (1.2) received much attention after Kovacik and Rakosnik [25] set up the variable exponent Soboev space. For example, in [16], Fan considered a constrained minimization problem involving $p(x)$-Laplacian in $\mathbb{R}^N$, and in [17] considered $p(x)$-Laplacian equations in $\mathbb{R}^N$ with periodic data and non-periodic perturbations. Moreover, some other nonlinear problems with variable exponent can be found in [2, 4, 10, 15, 18, 20–23, 26, 27, 36] and references therein.

A natural question is what results can be recovered when the $p(x)$-Laplacian operator is replaced by the fractional $p(x)$-Laplacian of the form $(-\Delta_{p(x)})^s$. To our best knowledge, Kaufmann et al. [24] and Del Pezzo et al. [14] first introduced some results on fractional Sobolev spaces with variable exponent $W^{s,q(x),p(x,y)}(\Omega)$ and the fractional $p(x)$-Laplacian. Then, the authors established compact embedding theorems of these spaces into variable exponent Lebesgue spaces. As an application, they also prove an existence result for nonlocal problems involving the fractional $p(x)$-Laplacian. In [8], Bahrouni et al. obtained some further qualitative properties of the variable exponent fractional Sobolev space $W^{s,q(x),p(x,y)}(\Omega)$ and the fractional $p(x)$-Laplacian operator $(-\Delta_{p(x)})^s$. After that, some studies on such problems are performed by using different approaches, see [1, 3, 5, 11] and references therein.

Motivated by the results on the $p(x)$-Laplacian problem and some results on the theory of fractional Sobolev spaces with variable exponent in [1, 7, 8, 14, 24], we study the existence and multiplicity of weak solutions for the problem (1.1) via variational techniques and fountain theorem. Moreover, we show that the equation has a sequence of solutions with high energies. To the best of the author's knowledge, the present paper seems to be the first to study the infinitely solutions to the Schrödinger type problem with fractional $p(x)$-Laplacian operator. In order to state the main results, we introduce some basic definitions of fractional Sobolev space with variable exponent.

Throughout this paper, we assume that the continuous function $p : \mathbb{R}^N \times \mathbb{R}^N \to (1, \infty)$ satisfies

$$p(x) := p(x,x) \le q(x) < p_s^*(x) := \frac{Np(x,x)}{N - sp(x,x)}, \quad sp(x,y) < N, \ \ \forall x, y \in \mathbb{R}^N,$$

where $p_s^*(x)$ is the so-called critical exponent in fractional Sobolev space with variable exponent. Moreover, we make the following assumptions:

(P_1) $1 < p^- := \inf_{\mathbb{R}^N \times \mathbb{R}^N} p(x,y) \le p(x,y) \le p^+ := \sup_{\mathbb{R}^N \times \mathbb{R}^N} p(x,y) < +\infty$;

(P_2) p is symmetric, i.e., $p(x,y) = p(y,x)$ for all $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$.

Let $F(x,t) = \int_0^t f(x,\tau)d\tau$. Assume that

(f_1) $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ satisfies $f(x,t)t \ge 0$ for all $(x,t) \in \mathbb{R}^N \times (0, +\infty)$ and

$$|f(x,t)| \le C(|t|^{p(x)-1} + |t|^{q(x)-1}), \ \ \forall (x,t) \in \mathbb{R}^N \times \mathbb{R},$$

with $p(x) \le q(x) \ll p^*(x)$ for all $x \in \mathbb{R}^N$.

(f_2) There exist $C_0 > 0$ and $\mu > p^+$ such that

$$\liminf_{|t|\to\infty} \frac{f(x,t)t}{|t|^\mu} \ge C_0 \quad \text{uniformly for } \ x \in \mathbb{R}^N.$$

(f_3) $\limsup_{|t|\to 0} \frac{f(x,t)t}{|t|^{p^+}} = 0$ uniformly for $x \in \mathbb{R}^N$.

(f_4) There exist two constants $C_1, C_2 > 0$ such that

$$G(x,u) \le C_1 G(x,v) \le C_2 H(x,v), \ \text{for} \ \ 0 \le u \le v,$$

where $G(x,t) := tf(x,t) - p^- F(x,t)$ and $H(x,t) := tf(x,t) - p^+ F(x,t)$.

(f_5) $f(x,-t) = -f(x,t)$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$.

The main result of this paper is as follows.

Theorem 1.1. *Assume that* (V_1)*,* (V_2)*,* (P_1)*,* (P_2) *and* (f_1)*–*(f_5) *hold. Then the problem* (1.1) *has infinitely many solution* $\{u_k\}$ *satisfying*

$$\Phi(u_k) = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_k(x) - u_k(u)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}} dxdy + \int_{\mathbb{R}^N} \frac{V(x)|u_k|^{p(x)}}{p(x)} dx - \int_{\mathbb{R}^N} F(x,u_k)dx \to \infty$$

as $n \to \infty$*, where* $\Phi : W^{s,p(x,y)}(\mathbb{R}^N) \to \mathbb{R}$ *is the energy functional corresponding to problem* (1.1).

Remark 1.2. *Let us consider*

$$f(x,t) = |t|^{q(x)-2}t, \quad \forall t \in \mathbb{R},$$

where $q(x) \in C_+(\mathbb{R}^N)$ *satisfying* $q(x) \ll p_s^*(x)$ *and* $p^+ < q^-$*. It is easy to check that* (f_1)*–*(f_3) *and* (f_5) *hold. For* (f_4)*, since* $F(x,t) = \frac{|t|^{q(x)}}{q(x)}$*,* $f(x,t)t = |t|^{q(x)}$ *and* $G(x,t) = (1 - \frac{p^-}{q(x)})|t|^{q(x)}$*,* $H(x,t) = (1 - \frac{p^+}{q(x)})|t|^{q(x)}$*, we get that* $G(x,t)$ *is nondecreasing in* $t \ge 0$*. Moreover, in view of* $G, H \ge 0$*, we know that*

$$\frac{G(x,t)}{H(x,t)} = \frac{q(x) - p^-}{q(x) - p^+} \le \frac{q^+ - p^-}{q^- - p^+}.$$

Choosing $C_1 = \frac{q^+ - p^-}{q^- - p^+}$*, we obtain* $G(x,t) \le C_1 H(x,t)$*, that is,* (f_4) *holds.*

Remark 1.3. *Condition (f_1) means that $f(x,t)$ is subcritical in the variable sense. Different from things in constant case (i.e. $p^+ = p^-$), we need $q(x) \ll p^*(x)$. Condition (f_5) assures the energy functional Φ is an even functional. So this condition is necessary for us to take advantage of the fountain geometry. Furthermore, It's known that (f_4) is much weaker than the Ambrosetti-Rabinowitz type condition in the constant exponent case ($p^+ = p^-$).*

This paper is organized as follows. In Section 2, certain basic results on fractional Sobolev spaces with variable exponent are stated, and abstract critical point theory is presented based on fountain theorem. Moreover, under condition (V_1) and (V_2), we could get some compact embedding theorems. In Section 3, under various conditions on the nonlinear growth term f, the compactness condition for the energy functional Φ is obtained. The existence and multiplicity of nontrivial solutions for the problem (1.1) are established by the fountain theorem without the (AR)-condition.

Notation. For two functions $a(x)$, $b(x) \in C(\mathbb{R}^N)$, $a(x) \ll b(x)$ means that $\inf_{x\in\mathbb{R}^N}(b(x)-a(x)) > 0$; "$\rightharpoonup$", "$\to$" denoted the weak convergence and strong convergence in a Banach space respectively; "$\hookrightarrow$", "$\hookrightarrow\hookrightarrow$" will be used to denote continuous embedding and compact embedding between spaces respectively. Moreover, we use C, C_i ($i = 1, 2, \cdots$) to denote some generic positive constants.

2. Fractional Sobolev spaces with variable exponent

In this section, we recall some definitions and basic properties of the variable exponent Lebesgue space $L^{p(\cdot)}(\mathbb{R}^N)$ and the fractional Sobolev space with variable exponent $W^{s,q(\cdot),p(\cdot,\cdot)}(\mathbb{R}^N)$, which will be treated in the next section.

Set $C_+(\mathbb{R}^N) := \{q(x) \in C(\mathbb{R}^N) : \inf_{x\in\mathbb{R}^N} q(x) > 1\}$. For any $q \in C_+(\mathbb{R}^N)$, we define

$$q^- := \inf_{x\in\mathbb{R}^N} q(x) \quad \text{and} \quad q^+ := \sup_{x\in\mathbb{R}^N} q(x).$$

For any $q(x) \in C_+(\mathbb{R}^N)$, we introduce the variable exponent Lebesgue space

$$L^{q(\cdot)}(\mathbb{R}^N) = \left\{ u : u \text{ is a measurable function}, \int_{\mathbb{R}^N} |u(x)|^{q(x)} dx < \infty \right\},$$

endowed with the Luxemburg norm

$$\|u\|_{L^{q(\cdot)}(\mathbb{R}^N)} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^N} \left| \frac{u(x)}{\lambda} \right|^{q(x)} dx \le 1 \right\}.$$

Lemma 2.1 ([20, 21]). *The space $L^{q(\cdot)}(\mathbb{R}^N)$ is a separable, reflexive and uniformly convex Banach space, and its conjugate space is $L^{\widehat{q}(\cdot)}(\mathbb{R}^N)$, where $\frac{1}{q(x)} + \frac{1}{\widehat{q}(x)} = 1$. For any $u \in L^{q(\cdot)}(\mathbb{R}^N)$, $v \in L^{\widehat{q}(\cdot)}(\mathbb{R}^N)$, we have the following Hölder type inequality*

$$\left| \int_{\mathbb{R}^N} uv dx \right| \le \left(\frac{1}{q^-} + \frac{1}{\hat{q}^-}\right) \|u\|_{L^{q(\cdot)}(\mathbb{R}^N)} \|v\|_{L^{\widehat{q}(\cdot)}(\mathbb{R}^N)} \le 2 \|u\|_{L^{q(\cdot)}(\mathbb{R}^N)} \|v\|_{L^{\widehat{q}(\cdot)}(\mathbb{R}^N)}.$$

Lemma 2.2. *If $\frac{1}{q(x)} + \frac{1}{r(x)} + \frac{1}{t(x)} = 1$, then for any $u \in L^{q(\cdot)}(\mathbb{R}^N)$, $v \in L^{r(\cdot)}(\mathbb{R}^N)$ and $w \in L^{t(\cdot)}(\mathbb{R}^N)$, we have*

$$\begin{aligned} \left| \int_{\mathbb{R}^N} uvw dx \right| &\le \left(\frac{1}{q^-} + \frac{1}{r^-} + \frac{1}{t^-} \right) \|u\|_{L^{q(\cdot)}(\mathbb{R}^N)} \|v\|_{L^{r(\cdot)}(\mathbb{R}^N)} \|v\|_{L^{t(\cdot)}(\mathbb{R}^N)} \\ &\le 3 \|u\|_{L^{q(\cdot)}(\mathbb{R}^N)} \|v\|_{L^{r(\cdot)}(\mathbb{R}^N)} \|w\|_{L^{t(\cdot)}(\mathbb{R}^N)}. \end{aligned}$$

The modular of the space $L^{q(\cdot)}(\mathbb{R}^N)$, which is the mapping $\rho_{q(\cdot)}: L^{q(\cdot)}(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$\rho_{q(\cdot)}(u) = \int_{\mathbb{R}^N} |u(x)|^{q(x)} dx, \qquad \forall u \in L^{q(\cdot)}(\mathbb{R}^N).$$

Then, we have the following well-known results.

Lemma 2.3 ([20, 21]). *If u, $u_n \in L^{q(\cdot)}(\mathbb{R}^N)$, then*

(1) $\|u\|_{L^{q(\cdot)}(\mathbb{R}^N)} > 1 (= 1, < 1)$ *if and only if* $\rho_{q(\cdot)}(u) > 1 (= 1, < 1\ resp.)$;
(2) if $\|u\|_{L^{q(\cdot)}(\mathbb{R}^N)} < 1 \Longrightarrow \|u\|^{q^+}_{L^{q(\cdot)}(\mathbb{R}^N)} \le \rho_{q(\cdot)}(u) \le \|u\|^{q^-}_{L^{q(\cdot)}(\mathbb{R}^N)}$;
(3) if $\|u\|_{L^{q(\cdot)}(\mathbb{R}^N)} > 1 \Longrightarrow \|u\|^{q^-}_{L^{q(\cdot)}(\mathbb{R}^N)} \le \rho_{q(\cdot)}(u) \le \|u\|^{q^+}_{L^{q(\cdot)}(\mathbb{R}^N)}$;
(4) for any $u_n \in L^{q(\cdot)}(\mathbb{R}^N)$, $\rho_{q(\cdot)}(u_n) \to 0 \Longleftrightarrow \|u\|_{L^{q(\cdot)}(\mathbb{R}^N)} \to 0$ *as* $n \to \infty$;
(5) for any $u_n \in L^{q(\cdot)}(\mathbb{R}^N)$, $\rho_{q(\cdot)}(u_n) \to \infty \Longleftrightarrow \|u\|_{L^{q(\cdot)}(\mathbb{R}^N)} \to \infty$ *as* $n \to \infty$.

Remark 2.4. *As a consequence of (2) and (3), for all $u \in L^{q(\cdot)}(\mathbb{R}^N)$, we have*

$$\|u\|_{L^{q(\cdot)}(\mathbb{R}^N)} \le \left(\int_{\mathbb{R}^N} |u(x)|^{q(x)} dx \right)^{\frac{1}{q^-}} + \left(\int_{\mathbb{R}^N} |u(x)|^{q(x)} dx \right)^{\frac{1}{q^+}}. \tag{2.1}$$

Let $0 < s < 1$ and assume that $p \in C(\mathbb{R}^N \times \mathbb{R}^N, (1, +\infty))$ satisfy (P_1) and (P_2). For $q \in C_+(\mathbb{R}^N)$, the fractional Sobolev space with variable exponent $X := W^{s,q(\cdot),p(\cdot,\cdot)}(\mathbb{R}^N)$ is defined as follows

$$X = \left\{ u \in L^{q(\cdot)}(\mathbb{R}^N) : \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p(x,y)}}{|x - y|^{N+sp(x,y)}} dxdy < +\infty \right\}.$$

Let

$$[u]_{s,p(\cdot,\cdot)} := \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p(x,y)}}{\lambda^{p(x,y)} |x - y|^{N+sp(x,y)}} dxdy < 1 \right\}$$

be the variable exponent Gagliardo seminorm and define

$$\|u\|_X := \|u\|_{L^{q(\cdot)}} + [u]_{s,p(\cdot,\cdot)}.$$

On X, we shall sometimes work with the norm

$$\|u\|_{\rho,X} := \inf \left\{ \lambda > 0 : \rho\Big(\frac{u}{\lambda}\Big) < 1 \right\},$$

where

$$\rho(u) := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p(x,y)}}{|x - y|^{N+sp(x,y)}} dxdy + \int_{\mathbb{R}^N} |u(x)|^{q(x)} dx.$$

It is not difficult to see that $\|\cdot\|_{\rho,X}$ is an equivalent norm of $\|\cdot\|_X$ with the relation

$$\frac{1}{2} \|u\|_X \le \|u\|_{\rho,X} \le 2\|u\|_X.$$

The following relations between the norm $\|\cdot\|_{\rho,X}$ and the modular $\rho(\cdot)$ can be easily obtained from their definitions.

Proposition 2.5. *On X it holds that*

(i) *for $u \in X \backslash \{0\}$, $\lambda = \|u\|_{\rho,X}$ if and only if $\rho(\frac{u}{\lambda}) = 1$;*
(ii) $\rho(u) > 1$ $(= 1; < 1)$ *if and only if* $\|u\|_{\rho,X} > 1$ $(= 1; < 1)$, *respectively;*
(iii) *if $\|u\|_{\rho,X} \geq 1$, then $\|u\|^{p^-}_{\rho,X} \leq \rho(u) \leq \|u\|^{p^+}_{\rho,X}$;*
(iv) *if $\|u\|_{\rho,X} < 1$, then $\|u\|^{p^+}_{\rho,X} \leq \rho(u) \leq \|u\|^{p^-}_{\rho,X}$.*

Proof. The proof is similar to [21, Theorem 3.1] and the details are omitted. □

For the bounded domain $\Omega \subset \mathbb{R}^N$, the following main embedding result was obtained in [24, Theorem 1.1].

Theorem 2.6. *Let Ω be a bounded domain in $\mathbb{R}^N$ and assume that p, q, s be as above such that $q(x) > p(x,x)$ for all $x \in \overline{\Omega}$. Then, it holds that*

$$W^{s,q(\cdot),p(\cdot,\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^{\beta(\cdot)}(\Omega)$$

for any $\beta \in C_+(\overline{\Omega})$ with $\beta(x) < p^*_s(x)$ for all $x \in \overline{\Omega}$.

Remark 2.7. *(i) It is worth pointing out that in existing articles [5, 8, 24] working on $W^{s,q(\cdot),p(\cdot,\cdot)}(\Omega)$, the function q is actually assumed that $q(x) > p(x,x)$ for all $x \in \overline{\Omega}$ due to some technical reason. Such spaces are actually not a generalization of the fractional Sobolev space $W^{s,p}(\Omega)$.*
(ii) We could like to mention that the Theorem 2.6 is holds if Ω is bounded and $q(x) \geq p(x,x)$ for all $x \in \overline{\Omega}$, and $\beta \in C_+(\overline{\Omega})$ with $\beta(x) < p^*_s(x)$ for all $x \in \overline{\Omega}$. Some detail see [38].

In what follows, for brevity, in some places we write $p(x)$ instead of $p(x,x)$ and in this sense, $p \in C_+(\mathbb{R}^N)$. If $q(x) = p(x) = p(x,x)$, we denote $W^{s,q(\cdot),p(\cdot,\cdot)}(\mathbb{R}^N)$ by $W^{s,p(\cdot,\cdot)}(\mathbb{R}^N)$. Moreover, we have the following embeddings.

Theorem 2.8. *Let $s \in (0,1)$. Assume that $p \in C_+(\mathbb{R}^N \times \mathbb{R}^N)$ be a uniformly continuous and satisfying the conditions (P_1) and (P_2). Then*

(i) $W^{s,p(\cdot,\cdot)}(\mathbb{R}^N) \hookrightarrow L^{r(\cdot)}(\mathbb{R}^N)$ for any uniform continuous function $r \in C_+(\mathbb{R}^N)$ satisfying $p(x) \leq r(x) \ll p^*_s(x)$ for all $x \in \mathbb{R}^N$;
(ii) $W^{s,p(\cdot,\cdot)}(\mathbb{R}^N) \hookrightarrow\hookrightarrow L^{r(\cdot)}_{Loc}(\mathbb{R}^N)$ for any uniform continuous function $r \in C_+(\mathbb{R}^N)$ with $r(x) < p^*_s(x)$ for all $x \in \mathbb{R}^N$.

Proof. (i) It suffices to prove for the case $p(x) \ll r(x)$. Decompose $\mathbb{R}^N$ by cubes B_i, $(i = 0,\ 1,\ 2,\ \cdots)$ with sides of length $\varepsilon > 0$ and parallel to the coordinate axes, where B_0 is the cube centered at the origin.

By the uniform continuity of p and r, we can choose ε sufficiently small and $t \in (0,s)$ such that

$$p^-_i \leq r^-_i \leq r^+_i \leq (p^-_i)^*_t, \quad \forall\, i \in \mathbb{N},$$

where $p^-_i := \inf\limits_{(x,y)\in B_i\times B_i} p(x,y)$, $r^-_i := \inf\limits_{x\in B_i} r(x)$, $r^+_i := \sup\limits_{x\in B_i} r(x)$ and $(p^-_i)^*_t = \frac{Np^-_i}{N-tp^-_i}$.

Let $u \in W^{s,p(\cdot,\cdot)}(\mathbb{R}^N) \setminus \{0\}$. Set $v := \frac{u}{\|u\|_{\rho,X}}$. Then, by Proposition 2.5, we get

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v(x)-v(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy + \int_{\mathbb{R}^N} |v(x)|^{p(x)}dx = 1. \tag{2.2}$$

So, for each $i \in \mathbb{N}$, we have that

$$\int_{B_i}\int_{B_i} \frac{|v(x)-v(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy + \int_{B_i} |v(x)|^{p(x)}dx \le 1. \tag{2.3}$$

Now, we claim that there exists a constant $C = C(p^+, p^-, s, t, \varepsilon, B_0) > 0$ such that

$$C\|v\|_{L^{r(\cdot)}(B_i)} \le \|v\|_{s,p(\cdot,\cdot),B_i}, \quad \forall i \in \mathbb{N} \tag{2.4}$$

where

$$\|v\|_{s,p(\cdot,\cdot),B_i} = \inf\left\{\lambda > 0 : \rho_{B_i}(\frac{u}{\lambda}) < 1\right\}$$

with

$$\rho_{B_i}(u) = \int_{B_i}\int_{B_i} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy + \int_{\mathbb{R}^N} |u(x)|^{p(x)}dx.$$

Indeed, for any $r \in C_+(B_i)$, from Corollary 3.3.4 of [12], for each $i \in \mathbb{N}$, we get

$$\|v\|_{L^{r(x)}(B_i)} \le 2(1+|B_i|)\|v\|_{L^{r_i^+}(B_i)}, \tag{2.5}$$

and

$$\begin{aligned}
&\left(\int_{B_i}\int_{B_i} \frac{|v(x)-v(y)|^{p_i^-}}{|x-y|^{N+tp_i^-}}dxdy\right)^{\frac{1}{p_i^-}} \\
&= \left(\int_{B_i}\int_{B_i} \frac{|v(x)-v(y)|^{p_i^-}}{|x-y|^{sp_i^-+N+(t-s)p_i^-}}dxdy\right)^{\frac{1}{p_i^-}} \\
&= \left(\int_{B_i}\int_{B_i} \left(\frac{|v(x)-v(y)|}{|x-y|^{s}}\right)^{p_i^-} \frac{1}{|x-y|^{N+(t-s)p_i^-}}dxdy\right)^{\frac{1}{p_i^-}} \\
&= \left\|\frac{|v(x)-v(y)|}{|x-y|^s}\right\|_{L^{p_i^-}(B_i\times B_i, |x-y|^{-(N+(t-s)p_i^-)}dxdy)} \\
&\le 2\Big(1+|B_i\times B_i|\Big)\left\|\frac{|v(x)-v(y)|}{|x-y|^s}\right\|_{L^{p(\cdot,\cdot)}(B_i\times B_i, |x-y|^{-(N+(t-s)p_i^-)}dxdy)}.
\end{aligned} \tag{2.6}$$

Let $\lambda > 0$ be such that

$$\int_{B_i}\int_{B_i} \frac{|v(x)-v(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x-y|^{N+sp(x,y)}}dxdy < 1.$$

Then, taking $K = \sup\limits_{B_i\times B_i}\{1, |x-y|^{s-t}\} \in [1, +\infty)$, for $\widehat{\lambda} = K\lambda$, we have

$$\begin{aligned}
&\int_{B_i}\int_{B_i} \Big(\frac{|v(x)-v(y)|}{\widehat{\lambda}|x-y|^s}\Big)^{p(x,y)} \frac{1}{|x-y|^{N+(t-s)p_i^-}}dxdy \\
&= \int_{B_i}\int_{B_i} \frac{|v(x)-v(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x-y|^{N+sp(x,y)}} \frac{1}{K^{p(x,y)}|x-y|^{(t-s)p_i^-}}dxdy \\
&\le \int_{B_i}\int_{B_i} \frac{|v(x)-v(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x-y|^{N+sp(x,y)}} \frac{1}{K^{p_i^-}|x-y|^{(t-s)p_i^-}}dxdy \\
&\le \int_{B_i}\int_{B_i} \frac{|u(x)-u(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x-y|^{N+sp(x,y)}}dxdy \\
&< 1,
\end{aligned} \tag{2.7}$$

which implies that

$$\left\| \frac{|v(x) - v(y)|}{|x - y|^s} \right\|_{L^{p(\cdot,\cdot)}(B_i \times B_i, |x-y|^{-(N+(t-s)p_i^-)} dxdy)} \le \widehat{\lambda} = K\lambda. \tag{2.8}$$

Taking infimum over all λ, (2.7) and (2.8) imply that

$$\left\| \frac{|v(x) - v(y)|}{|x - y|^s} \right\|_{L^{p(\cdot,\cdot)}(B_i \times B_i, |x-y|^{-(N+(t-s)p_i^-)} dxdy)} \le K[v]_{s,p(\cdot,\cdot),B_i}. \tag{2.9}$$

Together (2.6) with (2.9), there exists a constant $C > 0$ such that

$$\left(\int_{B_i} \int_{B_i} \frac{|v(x) - v(y)|^{p_i^-}}{|x - y|^{N+tp_i^-}} dxdy \right)^{\frac{1}{p_i^-}} \le C[v]_{s,p(\cdot,\cdot),B_i}. \tag{2.10}$$

Arguing as in that obtained (2.10), we get

$$\|v\|_{t,p_i^-,B_i} \le C\|v\|_{s,p(\cdot,\cdot),B_i}, \quad \forall i \in \mathbb{N}, \tag{2.11}$$

where

$$\|u\|_{t,p_i^-,B_i} = \left(\int_{B_i} \int_{B_i} \frac{|v(x) - v(y)|^{p_i^-}}{|x - y|^{N+tp_i^-}} dxdy \right)^{\frac{1}{p_i^-}} + \left(\int_{B_i} |u|^{p_i^-} dx \right)^{\frac{1}{p_i^-}}.$$

Using the same methods of Theorems 5.4 and 6.5 in [33], there exists an extension function $\widetilde{v} \in W^{t,p_i^-}(\mathbb{R}^N)$ of v such that $\widetilde{v} = v$ on B_i,

$$\|\widetilde{v}\|_{t,p_i^-,\mathbb{R}^N} \le C\|v\|_{t,p_i^-,B_i} \tag{2.12}$$

and

$$\|\widetilde{v}\|^{p_i^-}_{L^{(p_i^-)^*_t}(\mathbb{R}^N)} \le tp_i^- \, \omega_N^{\frac{N+tp_i^-}{N}} 2^{p_i^- + (p_i^-)^*_t} \|\widetilde{v}\|_{t,p_i^-,\mathbb{R}^N}, \tag{2.13}$$

where ω_N is the volume of the unit ball in $\mathbb{R}^N$. So, by (2.13), the space $W^{t,p_i^-}(\mathbb{R}^N)$ is continuously embedded in $L^q(\mathbb{R}^N)$ for any $q \in [\, p_i^-\,,\, (p_i^-)^*_t]$, i.e., there exists a constant $C = C(N, p_i^-, t) > 0$ such that

$$\|v\|_{L^{r_i^+}(B_i)} = \|\widetilde{v}\|_{L^{r_i^+}(B_i)} \le \|\widetilde{v}\|_{L^{r_i^+}(\mathbb{R}^N)} \le C\|\widetilde{v}\|_{t,p_i^-,\mathbb{R}^N}. \tag{2.14}$$

Thus, combining (2.14) with (2.12), (2.11) and (2.5) that there exists $C > 0$ such that (2.4) holds.

If $\|v\|_{L^{r(\cdot)}(B_i)} \ge 1$. Invoking Lemma 2.3 and Proposition 2.5 with taking (2.4) into account, we have

$$\begin{aligned} \int_{B_i} |v|^{r(x)} dx &\le \|v\|^{r_i^+}_{L^{r(\cdot)}(B_i)} \\ &\le C^{r_i^+} \left(\int_{B_i} \int_{B_i} \frac{|v(x) - v(y)|^{p(x,y)}}{|x - y|^{N+sp(x,y)}} dxdy + \int_{B_i} |v|^{p(x)} dx \right)^{\frac{r_i^+}{p_i^+}} \\ &\le C^{r_i^+} \left(\int_{B_i} \int_{B_i} \frac{|v(x) - v(y)|^{p(x,y)}}{|x - y|^{N+sp(x,y)}} dxdy + \int_{B_i} |v|^{p(x)} dx \right). \end{aligned} \tag{2.15}$$

If $\|v\|_{L^{r(\cdot)}(Q_i)} < 1$. Invoking Lemma 2.3 and Proposition 2.5 with taking (2.4) into account again, we have

$$\begin{aligned}\int_{B_i} |v|^{r(x)}dx &\le \|v\|_{L^{r(\cdot)}(B_i)}^{r_i^-} \\ &\le C^{r_i^-}\left(\int_{B_i}\int_{B_i} \frac{|v(x)-v(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy + \int_{B_i}|v|^{p(x)}dx\right)^{\frac{r_i^-}{p_i^+}} \\ &\le C^{r_i^-}\left(\int_{B_i}\int_{B_i} \frac{|v(x)-v(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy + \int_{B_i}|v|^{p(x)}dx\right).\end{aligned} \tag{2.16}$$

Then, taking (2.15) with (2.16), for any $i \in \mathbb{N}$, we have

$$\int_{B_i} |v|^{r(x)}dx \le \left(C^{r_i^-} + C^{r_i^+}\right)\left(\int_{B_i}|v|^{p(x)}dx + \int_{B_i}\int_{B_i} \frac{|v(x)-v(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy\right).$$

Summing up the last inequality over all $i \in \mathbb{N}$, combining with (2.2), there exists a constant $C > 0$ such that

$$\int_{\mathbb{R}^N} |v|^{r(x)}dx \le C.$$

Thus, $W^{s,p(\cdot,\cdot)}(\mathbb{R}^N) \subset L^{r(\cdot)}(\mathbb{R}^N)$ and hence, $W^{s,p(\cdot,\cdot)}(\mathbb{R}^N) \hookrightarrow L^{r(\cdot)}(\mathbb{R}^N)$ due to the closed graph theorem. The proof of assertion (i) is complete.

(ii) Let B be any ball in $\mathbb{R}^N$. Let $u_n \rightharpoonup 0$ in $W^{s,p(\cdot,\cdot)}(\mathbb{R}^N)$ and thus $u_n \rightharpoonup 0$ in $W^{s,p(\cdot,\cdot)}(B)$. Invoking Theorem 2.6, we have $u_n \to 0$ in $L^{r(\cdot)}(B)$. This completed the proof of Theorem 2.8. □

Now, define the following linear subspace

$$E := \left\{u \in W^{s,p(\cdot,\cdot)}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)|u|^{p(x)}dx < \infty\right\}$$

with the norm

$$\|u\|_E = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x-y|^{N+sp(x,y)}}dxdy + \int_{\mathbb{R}^N} V(x)\left|\frac{u}{\lambda}\right|^{p(x)}dx \le 1\right\}.$$

Under the conditions (V_1) and (V_2), E is continuously embedded in $W^{s,p(\cdot,\cdot)}(\mathbb{R}^N)$ as a closed subspace, and $E \hookrightarrow L^{p(\cdot)}(\mathbb{R}^N)$, $E \hookrightarrow L^{q(x)}(\mathbb{R}^N)$ if $q(x) \in C_+(\mathbb{R}^N)$ satisfies $p(x) \le q(x) \ll p_s^*(x)$ for all $x \in \mathbb{R}^N$. Therefore, E is also a separable reflexive Banach space. It is easy to see that with the norm $\|\cdot\|_E$, the following proposition remains valid.

Lemma 2.9. *The functional $\psi : E \to \mathbb{R}$ defined by*

$$\psi(u) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy + \int_{\mathbb{R}^N} V(x)\,|u|^{p(x)}\,dx$$

has the following properties:

(i) *If* $\|u\|_E \ge 1$, *then* $\|u\|_E^{p^-} \le \psi(u) \le \|u\|_E^{p^+}$.
(ii) *If* $\|u\|_E \le 1$, *then* $\|u\|_E^{p^+} \le \psi(u) \le \|u\|_E^{p^-}$.

Proof. We first prove the pair of inequalities. Indeed, for any $\lambda \in (0, 1)$ it is easy to see that

$$\lambda^{p_+}\,\psi(u) \le \psi(\lambda u) \le \lambda^{p_-}\,\psi(u). \tag{2.17}$$

Now, if $\|u\|_E > 1$, we have $0 < \frac{1}{\|u\|_E} < 1$ and $\psi(\frac{1}{\|u\|_E}u) = 1$. Taking $\lambda = \frac{1}{\|u\|_E}$ in (2.17), we get

$$\frac{\psi(u)}{\|u\|_E^{p_+}} \le 1 \le \frac{\psi(u)}{\|u\|_E^{p_-}}.$$

This completes the proof of Lemma 2.9 (i). The proof of the second is essentially the same. □

Lemma 2.10. *Assume that V satisfies (V_1) and (V_2). Then*

(i) $E \hookrightarrow\hookrightarrow L^{p(x)}(\mathbb{R}^N)$;
(ii) $E \hookrightarrow\hookrightarrow L^{\beta(x)}(\mathbb{R}^N)$ *for any* $\beta(x) \in C_+(\mathbb{R}^N)$ *with* $p(x) < \beta(x) \ll p_s^*(x)$.

Proof. (i) Assume $u_n \rightharpoonup 0$ in E. We will show that $u_n \to 0$ in $L^{p(x)}(\mathbb{R}^N)$, that is, $\int_{\mathbb{R}^N} |u_n(x)|^{p(x)}dx \to 0$ as $n \to \infty$. For any given $R \in (0, +\infty)$, we write

$$\begin{aligned}\int_{\mathbb{R}^N} |u_n|^{p(x)}dx &= \int_{B_R(0)} |u_n|^{p(x)}dx + \int_{\mathbb{R}^N\backslash B_R(0)} |u_n|^{p(x)}dx\\ &:= I_1(u_n) + I_2(u_n).\end{aligned}$$

Since $E \hookrightarrow W^{s,p(\cdot,\cdot)}(\mathbb{R}^N)$ and $W^{s,p(\cdot,\cdot)}(B_R(0)) \hookrightarrow\hookrightarrow L^{p(x)}(B_R(0))$, we have $E \hookrightarrow\hookrightarrow L^{p(x)}(B_R(0))$, which implies $I_1(u_n) \to 0$ as $n \to \infty$.

Next, we need to show that for any $\varepsilon > 0$, there exists $R > 0$ such that

$$I_2(u_n) := \int_{\mathbb{R}^N\backslash B_R(0)} |u_n(x)|^{p(x)}dx \le \varepsilon. \tag{2.18}$$

Note that $\|u_n\|_E < +\infty$. Given $\varepsilon > 0$, set $M = \frac{2}{\varepsilon}\sup_n\left(\|u_n\|_E^{p^+} + \|u_n\|_E^{p^-}\right)$. Denote

$$\mathcal{A} = \{x \in \mathbb{R}^N\backslash B_R(0) : \; V(x) \ge M\}$$

and

$$\mathcal{B} = \{x \in \mathbb{R}^N\backslash B_R(0) : \; V(x) < M\}.$$

Then, using Lemma 2.9, we have

$$\begin{aligned}\int_{\mathcal{A}} |u_n|^{p(x)}dx &\le \int_{\mathcal{A}} \frac{V(x)}{M}|u_n|^{p(x)}dx\\ &\le \frac{1}{M}\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy + \int_{\mathbb{R}^N} V(x)\,|u|^{p(x)}\,dx\right)\\ &\le \frac{1}{M}(\|u_n\|_E^{p^+} + \|u_n\|_E^{p^-}) \le \frac{\varepsilon}{2}.\end{aligned} \tag{2.19}$$

On the other hand, by Hölder inequality and Theorem 2.6, we get

$$\begin{aligned}\int_{\mathcal{B}}|u_n|^{p(x)}dx &\le \left(\int_{\mathcal{B}}|u_n|^{\alpha p(x)}dx\right)^{\frac{1}{\alpha}}\left(\int_{\mathcal{B}}1^{\frac{\alpha}{\alpha-1}}dx\right)^{\frac{\alpha-1}{\alpha}}\\ &= \left(\int_{\mathcal{B}}|u_n|^{\alpha p(x)}dx\right)^{\frac{1}{\alpha}}(\mathrm{meas}(\mathcal{B}))^{\frac{\alpha-1}{\alpha}}\\ &\le C\Big(\|u_n\|_E^{\frac{p^-}{\alpha}}+\|u_n\|_E^{\frac{p^+}{\alpha}}\Big)(\mathrm{meas}(\mathcal{B}))^{\frac{\alpha-1}{\alpha}},\end{aligned}\tag{2.20}$$

where the number $\alpha\in(1,+\infty)$ such that $p(x)<\alpha p(x)<p_s^*(x)$ for all $x\in\mathbb{R}^N$.

From (V_2), we can choose R large enough such that

$$\mathrm{meas}(\mathcal{B})\le\left(\frac{\varepsilon}{2C(\|u_n\|_E^{\frac{p^-}{\alpha}}+\|u_n\|_E^{\frac{p^+}{\alpha}})}\right)^{\frac{\alpha}{\alpha-1}}.\tag{2.21}$$

Then, (2.20) and (2.21) imply that

$$\int_{\mathcal{B}}|u_n(x)|^{p(x)}dx\le\frac{\varepsilon}{2}.\tag{2.22}$$

It follows from (2.19) and (2.22) that (2.18) holds and completes the proof of (i).

(ii) Let $u_n\rightharpoonup 0$ in E. We need to show $u_n\to 0$ in $L^{\beta(x)}(\mathbb{R}^N)$. That is

$$\int_{\mathbb{R}^N}|u_n|^{\beta(x)}dx\to 0\quad\text{as }\ n\to\infty.\tag{2.23}$$

Since $E\hookrightarrow L^{p_s^*(x)}(\mathbb{R}^N)$ is continuous and $\{u_n\}$ is bounded in E, we have

$$\sup_n\int_{\mathbb{R}^N}|u_n|^{p_s^*(x)}dx<+\infty.\tag{2.24}$$

Since $q(x)<\beta(x)\ll p_s^*(x)$, there exists a function $\gamma\in C(\mathbb{R}^N,(0,1))$ such that

$$\frac{1}{\beta(x)}=\frac{\gamma(x)}{p(x)}+\frac{1-\gamma(x)}{p_s^*(x)}\ \text{ a.e. in }\mathbb{R}^N.$$

Then, by Lemma 2.1 and Remark 2.4, we get

$$\begin{aligned}&\int_{\mathbb{R}^N}|u_n|^{\beta(x)}dx\\ &=\int_{\mathbb{R}^N}|u_n|^{p(x)\frac{\beta(x)\gamma(x)}{p(x)}}|u_n|^{p_s^*(x)\frac{\beta(x)(1-\gamma(x))}{p_s^*(x)}}dx\\ &\le 2\Big(\int_{\mathbb{R}^N}|u_n|^{p(x)}dx\Big)^{\frac{\beta(x)\gamma(x)}{p(x)}}\Big(\int_{\mathbb{R}^N}|u_n|^{p_s^*(x)}dx\Big)^{\frac{\beta(x)(1-\gamma(x))}{p_s^*(x)}}\\ &\le 2\left[\Big(\int_{\mathbb{R}^N}|u_n|^{p(x)}dx\Big)^{\left(\frac{\beta(x)\gamma(x)}{p(x)}\right)^+}+\Big(\int_{\mathbb{R}^N}|u_n|^{p(x)}dx\Big)^{\left(\frac{\beta(x)\gamma(x)}{p(x)}\right)^-}\right]\\ &\quad\times\left[\Big(\int_{\mathbb{R}^N}|u_n|^{p_s^*(x)}dx\Big)^{\left(\frac{\beta(x)(1-\gamma(x))}{p_s^*(x)}\right)^+}+\Big(\int_{\mathbb{R}^N}|u_n|^{p_s^*(x)}dx\Big)^{\left(\frac{\beta(x)(1-\gamma(x))}{p_s^*(x)}\right)^-}\right].\end{aligned}\tag{2.25}$$

Therefore, from (2.24), (2.25) and (i), we get that $u_n \to 0$ in $L^{\beta(x)}(\mathbb{R}^N)$, and the proof of (ii) is completed. □

Remark 2.11. *From Lemma 2.10, we know that the conditions* (V_1) *and* (V_2) *play an important role in enables E to be compactly embedded into* $L^{\beta(x)}(\mathbb{R}^N)$ *type spaces.*

3. Existence of weak solutions

In this section, the proof of the existence and multiplicity of nontrivial solutions for (1.1) by applying the fountain theorem under some assumptions on f.

Equation (1.1) has a variational structure and its associated energy functional $\Phi : E \to R$ is defined by

$$\Phi(u) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x) - u(u)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}}dxdy + \int_{\mathbb{R}^N} \frac{V(x)|u|^{p(x)}}{p(x)}dx - \int_{\mathbb{R}^N} F(x,u)dx.$$

Under the assumptions (f_1)–(f_3), Φ is of class $C^1(E,\mathbb{R})$. We say that $u \in E$ is a weak solution of (1.1), if

$$\langle u, v\rangle_{s,p(\cdot,\cdot)} + \int_{\mathbb{R}^N} V(x)|u|^{p(x)-2}uvdx = \int_{\mathbb{R}^N} f(x,u)vdx$$

for all $v \in E$, where

$$\langle u, v\rangle_{s,p(\cdot,\cdot)} = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x) - u(u)|^{p(x,y)-2}(u(x) - u(y))(v(x) - v(y))}{|x-y|^{N+sp(x,y)}}dxdy.$$

Clearly, the critical points of Φ are exactly the weak solutions of problem (1.1).

Define the functional $\Psi : E \to \mathbb{R}$ by

$$\Psi(u) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}}dxdy + \int_{\mathbb{R}^N} \frac{V(x)|u|^{p(x)}}{p(x)}dx.$$

Then, $\Psi \in C^1(E,\mathbb{R})$ and its Fréchet derivative is

$$\langle \Psi'(u), v\rangle = \langle u, v\rangle_{s,p(\cdot,\cdot)} + \int_{\mathbb{R}^N} V(x)|u|^{p(x)-2}uvdx, \quad \forall u, v \in E.$$

According to the analogous arguments in [8, Lemma 4.2], the following lemma is easily checked, so we omit the proof.

Lemma 3.1. *Assume that* (V_1) *and* (P_1)*,* (P_2) *hold. Then, the functional* $\Psi : E \to \mathbb{R}$ *is convex and weakly lower semicontinuous on E. Moreover, the operator* Ψ' *is a mapping of* (S_+)*-type, that is,* $u_n \rightharpoonup u$ *and* $\limsup_{n\to\infty}\langle \Psi'(u_n) - \Psi'(u), u_n - u\rangle \le 0$ *implies* $u_n \to u$ *strongly in E as* $n \to \infty$*.*

Definition 3.2. *For* $c \in \mathbb{R}$*, we say that* Φ *satisfies the* $(C)_c$*-condition if for any sequence* $\{u_n\} \subset E$ *with*

$$\Phi(u_n) \to c, \qquad (1 + \|u_n\|_E)\|\Phi'(u_n)\| \to 0,$$

there is a subsequence $\{u_n\}$ *such that* $\{u_n\}$ *converges strongly in E.*

Let W be a reflexive and separable Banach space. It is well-known that there exist $\{e_i\}_{i=1}^{\infty} \subset W$ and $\{f_i^*\}_{i=1}^{\infty} \subset W^*$ such that

$$W = \overline{\operatorname{span}\{e_i : i = 1, 2, \cdots\}}, \qquad W^* = \overline{\operatorname{span}\{f_i^* : i = 1, 2, \cdots\}},$$

and

$$\langle f_i^*, e_j \rangle = \begin{cases} 1, & \text{if } \ i = j, \\ 0, & \text{if } \ i \neq j. \end{cases}$$

Let $W_i = \operatorname{span}\{e_i\}$, then $W = \overline{\bigoplus_{i=1}^{\infty} W_i}$. Now we define

$$Y_k = \bigoplus_{i=1}^{k} W_i, \qquad Z_k = \overline{\bigoplus_{i=k}^{\infty} W_i}. \tag{3.1}$$

Then, we will use the following fountain theorem [6] (see also [35]) to prove our result.

Theorem 3.3 (Fountain theorem)**.** *Let W be a real reflexive Banach space, $\mathcal{I} \in C^1(W, \mathbb{R})$ satisfies the (C)-condition, $\mathcal{I}(-u) = \mathcal{I}(u)$. If for each sufficiently large $k \in \mathbb{N}$, there exist $\rho_k > \delta_k > 0$ such that the following conditions hold:*

(i) $a_k := \inf\{\mathcal{I}(u) : u \in Z_k, \|u\|_W = \delta_k\} \to \infty$ *as* $k \to \infty$*;*
(ii) $b_k := \max\{\mathcal{I}(u) : u \in Y_k, \|u\|_W = \rho_k\} \leq 0$.

Then $\mathcal{I}$ has a sequence of critical points $\{u_k\} \subset W$ such that $\mathcal{I}(u_k) \to +\infty$ as $k \to +\infty$.

In the following, for the reflexive and separable Banach space E, define Y_k and Z_k as in (3.1), we will show that the energy functional Φ satisfies the geometric structure. We now give a useful lemma.

Lemma 3.4. *Let $q(x) \in C_+(\mathbb{R}^N)$ with $p(x) \leq q(x) \ll p^*(x)$ and denote*

$$\alpha_k = \sup\{\|u\|_{L^{q(\cdot)}(\mathbb{R}^N)} : \ \|u\|_E = 1, \ u \in Z_k\}.$$

Then $\alpha_k \to 0$ as $k \to \infty$.

Proof. Suppose to the contrary that there exist $\varepsilon_0, k_0 > 0$, and the sequence $\{u_k\} \subset Z_k$ such that

$$\|u_k\|_E = 1 \ \text{ and } \ \|u_k\|_{L^{q(\cdot)}(\mathbb{R}^N)} \geq \varepsilon_0 > 0$$

for all $k \geq k_0$. Since $\{u_k\}$ is bounded in E, there exists $u \in E$ such that $u_k \rightharpoonup u$ in E as $k \to \infty$ and

$$\langle f_j^*, u \rangle = \lim_{k \to \infty} \langle f_j^*, u_k \rangle = 0 \ \text{ for } \ j = 1, 2, \cdots.$$

Hence, we get $u = 0$. However, we obtain that

$$0 < \varepsilon_0 \leq \lim_{k \to \infty} \|u_k\|_{L^{q(\cdot)}(\mathbb{R}^N)} = \|u\|_{L^{q(\cdot)}(\mathbb{R}^N)} = 0,$$

which provides a contradiction. Thus, we have proved that $\alpha_k \to 0$ as $k \to \infty$. □

Lemma 3.5. *Under the assumptions of Theorem 1.1, the geometry conditions of the fountain theorem hold, that is,* (i) *and* (ii) *of Theorem 3.3 hold.*

Proof. (i) By (f_1) and (f_3), for any $\varepsilon > 0$, there exists a $C(\varepsilon) > 0$ such that

$$|F(x,u)| \leq \varepsilon|u|^{p^+} + C(\varepsilon)|u|^{q(x)}, \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}. \tag{3.2}$$

Let

$$\theta_k = \sup_{u \in Z_k,\, \|u\|_E = 1} \|u\|_{L^{p^+}(\mathbb{R}^N)}, \quad \eta_k = \sup_{u \in Z_k,\, \|u\|_E = 1} \|u\|_{L^{q(\cdot)}(\mathbb{R}^N)}. \tag{3.3}$$

Then, by Lemma 3.4, we obtain $\theta_k \to 0^+$ and $\eta_k \to 0^+$ as $k \to \infty$. So, for any $u \in Z_k$ with $\|u\|_E = \delta_k > 1$, from (3.2), (3.3), Remark 2.4 and Lemma 2.9, we get

$$\begin{aligned}
\Phi(u) &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}}dxdy + \int_{\mathbb{R}^N} \frac{V(x)|u|^{p(x)}}{p(x)}dx - \int_{\mathbb{R}^N} F(x,u)dx \\
&\geq \frac{1}{p^+}\|u\|_E^{p^-} - \varepsilon\|u\|_{L^{p^+}(\mathbb{R}^N)}^{p^+} - C\max\{\|u\|_{L^{q(\cdot)}(\mathbb{R}^N)}^{q^-},\ \|u\|_{L^{q(\cdot)}(\mathbb{R}^N)}^{q^+}\} \\
&\geq \frac{1}{p^+}\|u\|_E^{p^-} - \varepsilon\Big(\theta_k\|u\|_E\Big)^{p^+} - C\max\left\{\Big(\eta_k\|u\|_E\Big)^{q^-},\ \Big(\eta_k\|u\|_E\Big)^{q^+}\right\}.
\end{aligned}$$

Let $\varepsilon > 0$ small enough such that $\varepsilon(\theta_k\|u\|_E)^{p^+} \leq \frac{1}{2p^+}\|u\|_E^{p^-}$. If $\|u\|_E^{q^-} \geq \|u\|_E^{q^+}$, let $\delta_k = (\frac{1}{4p^+C\eta_k^{q^-}})^{\frac{1}{q^- - p^-}}$, for sufficiently large k,

$$\Phi(u) \geq \frac{1}{4p^+}\Big(\frac{1}{4p^+C\eta_k^{q^-}}\Big)^{\frac{p^-}{q^- - p^-}}.$$

Now $\eta_k \to 0$ and $q^- > p^+$ implies that

$$\inf_{u \in Z_k, \|u\|_E = \delta_k} \Phi(u) \to +\infty \ \text{ as } \ k \to \infty.$$

If $\|u\|_E^{q^-} < \|u\|_E^{q^+}$, we can similarly derive that $\inf_{u \in Z_k, \|u\|_E = \delta_k} \Phi(u) \to +\infty$ as $k \to \infty$, hence (i) is satisfied.

(ii) By (f_2) and (f_3), for any $\varepsilon > 0$, there exists a $C(\varepsilon) > 0$ such that

$$F(x,u) \geq C(\epsilon)|u|^{\mu} - \varepsilon|u|^{p^+}, \quad \forall (x,u) \in \mathbb{R}^N \times \mathbb{R}. \tag{3.4}$$

From (3.4) and Lemma 2.9, for some $v \in Y_k$ with $\|v\|_E = 1$ and $t > 1$, we have

$$\begin{aligned}
\Phi(tv) &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|tv(x)-tv(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}}dxdy + \int_{\mathbb{R}^N} \frac{V(x)|tv|^{p(x)}}{p(x)}dx - \int_{\mathbb{R}^N} F(x,tv)dx \\
&\leq \frac{t^{p^+}}{p^-}\|v\|_E^{p^+} + \varepsilon t^{p^+}\int_{\mathbb{R}^N}|v|^{p^+}dx - C(\varepsilon)t^{\mu}\int_{\mathbb{R}^N}|v|^{\mu}dx \\
&\to -\infty \ \text{ as } \ t \to +\infty,
\end{aligned}$$

due to $\mu > p^+$ and all norms on Y_k are equivalent. So there exists $\rho_k > \delta_k$ such that $t = \rho_k$ concludes $\Phi(tv) \leq 0$, and then

$$\max_{u \in Y_k, \|u\|_E = \rho_k} \Phi(u) \leq 0.$$

Hence (ii) is satisfied. □

Lemma 3.6. *Assume that (f_1), (f_2) and (f_4) hold. Then $(C)_c$-sequence of Φ is bounded.*

Proof. Suppose that $\{u_n\} \subset E$ is a $(C)_c$-sequence for Φ, that is,

$$\Phi(u_n) \to c \quad \text{and} \quad (1 + \|u_n\|_E)\|\Phi'(u_n)\| \to 0 \quad \text{as} \quad n \to \infty,$$

which show that

$$\Phi(u_n) = c + o_n(1), \quad \langle \Phi'(u_n), u_n \rangle = o_n(1), \tag{3.5}$$

where $o_n(1) \to 0$ as $n \to \infty$. We now prove that $\{u_n\}$ is bounded in E. We argue by contradiction. Suppose that the sequence $\{u_n\}$ is unbounded in E, let $w_n = \frac{u_n}{\|u_n\|_E}$, then $w_n \in E$ with $\|w_n\|_E = 1$. Hence, up to a subsequence, still denoted by itself, there exists a function $w \in E$ such that

$$\begin{cases} w_n \rightharpoonup w \text{ in } E, \\ w_n \to w \text{ in } L^{\beta(x)}(\mathbb{R}^N), \text{ for } p(x) \le \beta(x) \ll p_s^*(x), \\ w_n \to w \text{ a.e. in } \mathbb{R}^N. \end{cases} \tag{3.6}$$

If $w = 0$, we can define a sequence $\{t_n\} \subset \mathbb{R}$, as argued in [37], such that

$$\Phi(t_n u_n) = \max_{t \in [0,1]} \Phi(t u_n).$$

Then, for any $L > 1$, and n large enough, we have

$$\begin{aligned} \Phi(t_n u_n) \ge \Phi(L w_n) &\ge \frac{1}{p^+} L^{p^-} \|w_n\|_E^{p^-} - \int_{\mathbb{R}^N} F(x, L w_n) dx \\ &= \frac{1}{p^+} L^{p^-} - \int_{\mathbb{R}^N} F(x, L w_n) dx. \end{aligned} \tag{3.7}$$

Moreover, from (f_1) and (3.6), we get $\int_{\mathbb{R}^N} F(x, L w_n) dx \to 0$. Hence, this and (3.7) imply that $\Phi(t_n u_n) \to \infty$ as $n \to \infty$ by the fact L can be large arbitrarily.

Noting that $\Phi(0) = 0$ and $\Phi(u_n) \to c$, then $t_n \in (0, 1)$ when n is large enough. Hence, $\langle \Phi'(t_n u_n), t_n u_n \rangle \to 0$ and

$$\Phi(t_n u_n) - \frac{1}{p^-} \langle \Phi'(t_n u_n), t_n u_n \rangle \to \infty \quad \text{as} \quad n \to \infty. \tag{3.8}$$

Moreover,

$$\begin{aligned} &\Phi(t_n u_n) - \frac{1}{p^-} \langle \Phi'(t_n u_n), t_n u_n \rangle \\ &= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \Big(\frac{1}{p(x,y)} - \frac{1}{p^-} \Big) \frac{|t_n u_n(x) - t_n u_n(y)|^{p(x,y)}}{|x - y|^{N + sp(x,y)}} dx dy \\ &\quad + \int_{\mathbb{R}^N} \Big(\frac{1}{p(x)} - \frac{1}{p^-} \Big) V(x) |t_n u_n|^{p(x)} dx \\ &\quad + \frac{1}{p^-} \int_{\mathbb{R}^N} \Big(f(x, t_n u_n) t_n u_n - p^- F(x, t_n u_n) \Big) dx \\ &\le \frac{1}{p^-} \int_{\mathbb{R}^N} G(x, t_n u_n) dx. \end{aligned}$$

This and (3.8) deduce that

$$\int_{\mathbb{R}^N} G(x, t_n u_n)dx \to \infty \text{ as } n \to \infty. \tag{3.9}$$

On the other hand, in view of (f_4), there exist two constants $C_1, C_2 > 0$ such that

$$H(x, u_n) \geq C_1 G(x, u_n) \geq C_2 G(x, t_n u_n). \tag{3.10}$$

So, (3.9) and (3.10) imply that

$$\begin{aligned}
\infty > c + o_n(1) &= \Phi(u_n) - \frac{1}{p^+}\langle \Phi'(u_n), u_n\rangle \\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\Big(\frac{1}{p(x,y)} - \frac{1}{p^+}\Big)\frac{|u_n(x) - u_n(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy \\
&\quad + \int_{\mathbb{R}^N}\Big(\frac{1}{p(x)} - \frac{1}{p^+}\Big)V(x)|u_n|^{p(x)}dx + \frac{1}{p^+}\int_{\mathbb{R}^N} H(x, u_n)dx \\
&\geq \frac{1}{p^+}\int_{\mathbb{R}^N} H(x, u_n)dx \geq \frac{C_1}{p^+}\int_{\mathbb{R}^N} G(x, u_n)dx \\
&\geq \frac{C_2}{p^+}\int_{\mathbb{R}^N} G(x, t_n u_n)dx \to \infty
\end{aligned}$$

which is contradictory.

If $w \neq 0$. Assume $\|u_n\|_E > 1$, by $\langle \Phi'(u_n), u_n\rangle = o_n(1)$ and Lemma 2.9, we have

$$\begin{aligned}
1 + o_n(1) &= \int_{\mathbb{R}^N} \frac{f(x, u_n)u_n}{\psi(u_n)}dx \geq \int_{\mathbb{R}^N} \frac{f(x, u_n)u_n}{\|u_n\|_E^{p^+}}dx \\
&\geq \int_{\mathbb{R}^N} \frac{f(x, u_n)u_n}{\|u_n\|_E^{\mu}}dx = \int_{\mathbb{R}^N} \frac{f(x, u_n)u_n}{|u_n|^{\mu}}|w_n|^{\mu}dx.
\end{aligned} \tag{3.11}$$

Denote $\Omega_0 = \{x \in \mathbb{R}^N : w(x) = 0\}$. Then, for $x \in \mathbb{R}^N \backslash \Omega_0$, we have $|u_n| = |w_n|\|u_n\|_E \to +\infty$ as $n \to \infty$. Hence, by (f_2), we have

$$\int_{\mathbb{R}^N \backslash \Omega_0} \frac{f(x, u_n)u_n}{|u_n|^{\mu}}|w_n|^{\mu}dx \to +\infty \text{ as } n \to \infty. \tag{3.12}$$

Combining (3.11) and (3.12), we obtain a contradiction. Therefore, $\{u_n\}$ is bounded in E and the proof is complete. □

Lemma 3.7. *Assume that conditions (f_1)–(f_4) hold. Then Φ satisfies $(C)_c$-condition, that is, for all $c \in \mathbb{R}$, any $(C)_c$-sequence of Φ has a convergent subsequence.*

Proof. Let $\{u_n\} \subset E$ be a $(C)_c$-sequence of Φ. According to Lemma 3.6, we deduce that $\{u_n\}$ is bounded in E. Up to a subsequence, we may assume that $u_n \rightharpoonup u$ weakly in E, $u_n \to u$ strongly in $L^{\beta(x)}(\mathbb{R}^N)$ for $p(x) \leq \beta(x) \ll p_s^*(x)$, and $u_n(x) \to u(x)$ a.e. in $\mathbb{R}^N$.

Using (2.1) and Lemma 2.1, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} |u_n|^{p(x)-1}|u_n-u|dx \\
&\le 2\left\||u_n|^{p(x)-1}\right\|_{L^{\hat{p}(x)}(\mathbb{R}^N)}\|u_n-u\|_{L^{p(x)}(\mathbb{R}^N)} \\
&\le 2\|u_n-u\|_{L^{p(x)}(\mathbb{R}^N)}\left[\Big(\int_{\mathbb{R}^N}\left||u_n|^{p(x)-1}\right|^{\hat{p}(x)}dx\Big)^{\hat{p}^-}+\Big(\int_{\mathbb{R}^N}||u_n|^{p(x)-1}|^{\hat{p}(x)}dx\Big)^{\hat{p}^+}\right] \\
&= 2\|u_n-u\|_{L^{p(x)}(\mathbb{R}^N)}\left[\Big(\int_{\mathbb{R}^N}|u_n|^{p(x)}dx\Big)^{\hat{p}^-}+\Big(\int_{\mathbb{R}^N}|u_n|^{p(x)}dx\Big)^{\hat{p}^+}\right].
\end{aligned} \tag{3.13}$$

Hence, (f_1) and (3.13) give that

$$\begin{aligned}
&\int_{\mathbb{R}^N}(f(x,u_n)-f(x,u))(u_n-u)dx \\
&\le C\int_{\mathbb{R}^N}\left(|u_n|^{p(x)-1}+|u|^{p(x)-1}+|u_n|^{q(x)-1}+|u|^{q(x)-1}\right)|u_n-u|dx \\
&= C\int_{\mathbb{R}^N}|u_n|^{p(x)-1}|u_n-u|dx+C\int_{\mathbb{R}^N}|u|^{p(x)-1}|u_n-u|dx \\
&\quad +C\int_{\mathbb{R}^N}|u_n|^{q(x)-1}|u_n-u|dx+C\int_{\mathbb{R}^N}|u|^{q(x)-1}|u_n-u|dx \\
&\le C\|u_n-u\|_{L^{p(x)}(\mathbb{R}^N)}\left[\Big(\int_{\mathbb{R}^N}|u_n|^{p(x)}dx\Big)^{\hat{p}^-}+\Big(\int_{\mathbb{R}^N}|u_n|^{p(x)}dx\Big)^{\hat{p}^+}\right. \\
&\qquad\qquad \left.+\Big(\int_{\mathbb{R}^N}|u|^{p(x)}dx\Big)^{\hat{p}^-}+\Big(\int_{\mathbb{R}^N}|u|^{p(x)}dx\Big)^{\hat{p}^+}\right] \\
&\quad +C\|u_n-u\|_{L^{q(x)}(\mathbb{R}^N)}\left[\Big(\int_{\mathbb{R}^N}|u_n|^{q(x)}dx\Big)^{\hat{q}^-}+\Big(\int_{\mathbb{R}^N}|u_n|^{q(x)}dx\Big)^{\hat{q}^+}\right. \\
&\qquad\qquad \left.+\Big(\int_{\mathbb{R}^N}|u|^{q(x)}dx\Big)^{\hat{q}^-}+\Big(\int_{\mathbb{R}^N}|u|^{q(x)}dx\Big)^{\hat{q}^+}\right].
\end{aligned} \tag{3.14}$$

By the boundedness of $\{u_n\}$ in E, we have that

$$\sup_n\int_{\mathbb{R}^N}|u_n|^{p(x)}dx<+\infty, \qquad \sup_n\int_{\mathbb{R}^N}|u_n|^{q(x)}dx<+\infty \tag{3.15}$$

and

$$\int_{\mathbb{R}^N}|u|^{p(x)}dx<+\infty, \qquad \int_{\mathbb{R}^N}|u|^{q(x)}dx<+\infty. \tag{3.16}$$

Therefore, we can deduce from (3.14)–(3.16) and $u_n\to u$ in $L^{q(x)}(\mathbb{R}^N)$ that

$$\int_{\mathbb{R}^N}(f(x,u_n)-f(x,u))(u_n-u)dx\to 0 \text{ as } n\to\infty. \tag{3.17}$$

Since $\Phi'(u_n)\to 0$ in E^* and $u_n\rightharpoonup u$ in E, we have

$$\langle\Phi'(u_n)-\Phi'(u),u_n-u\rangle\to 0 \text{ as } n\to\infty.$$

That is,

$$\begin{aligned}\langle \Psi'(u_n) - \Psi'(u), u_n - u\rangle &= \langle \Phi'(u_n) - \Phi'(u), u_n - u\rangle \\ &\quad + \int_{\mathbb{R}^N} (f(x, u_n) - f(x, u))(u_n - u)dx \to 0.\end{aligned} \tag{3.18}$$

Then, (3.17), (3.18) and Ψ' is a (S_+)-type operator imply that $u_n \to u$ in E. This completes the proof. □

Proof of Theorem 1.1. According to Lemma 3.7 and (f_5), Φ is an even functional and satisfies $(C)_c$-condition for all $c \in \mathbb{R}$. Lemma 3.5 implies that the functional Φ has the fountain theorem geometry conditions. So, from Theorem 3.3 we deduce that Φ has a sequence of critical points $\{u_k\}$ with $\Phi(u_k) \to +\infty$ and Theorem 1.1 follows.

4. Conclusions

This paper considers a class of Schrödinger equations involving the fractional $p(x)$-Laplacian in the whole space $\mathbb{R}^N$. We use the variational method and the fountain theorem to prove that this nonlocal problem has infinitely many high-energy solutions.

Conflict of interest

The author declares there is no conflict of interest.

References

1. E. Azroul, A. Benkirane, M. Shimi, General fractional Sobolev space with variable exponent and applications to nonlocal problems, *Adv. Oper. Theory*, **5** (2020), 1512–1540. https://doi.org/10.1007/s43036-020-00062-w
2. C. O. Alves, S. B. Liu, On superlinear $p(x)$-Laplacian equations in $\mathbb{R}^N$, *Nonlinear Anal.: Theory, Methods Appl.*, **73** (2010), 2566–2579. https://doi.org/10.1016/j.na.2010.06.033
3. E. H. Azroul, M. Shimi, Nonlocal eigenvalue problems with variable exponent, *Moroccan J. Pure Appl. Anal.*, **4** (2018), 46–61. https://doi.org/10.1515/mjpaa-2018-0006
4. C. O. Alves, M. A. S. Souto, Existence of solutions for a class of problems in $\mathbb{R}^N$ involving the $p(x)$-Laplacian, In: *Contributions to nonlinear analysis*, Progress in Nonlinear Differential Equations and Their Applications, Vol. 66, Birkhauser, Basel, 2005. https://doi.org/10.1007/3-7643-7401-2_2
5. A. Bahrouni, Comparison and sub-super solution principles for the fractional $p(x)$-Laplacian, *J. Math. Anal. Appl.*, **458** (2018), 1363–1372. https://doi.org/10.1016/j.jmaa.2017.10.025
6. T. Bartsch, Infinitely many solutions of a symmetric Dirichlet problem, *Nonlinear Anal.: Theory, Methods Appl.*, **20** (1993), 1205–1216. https://doi.org/10.1016/0362-546X(93)90151-H
7. A. Baalal, M. Berghout, Traces and fractional Sobolev extension domains with variable exponent, *Int. J. Math. Anal.*, **12** (2018), 85–98.

8. A. Bahrouni, V. D. Rădulescu, On a new fractional Sobolev space and applications to nonlocal variational problems with variable exponent, *Discrete Contin. Dyn. Syst. Ser. S*, **11** (2017), 379–389.

9. B. Barrios, E. Colorado, R. Servadei, F. Soria, A critical fractional equation with concave-convex power nonlinearities, *Ann. Inst. H. Poincaré C, Anal. Non Linéaire*, **32** (2015), 875–900. https://doi.org/10.1016/j.anihpc.2014.04.003

10. J. Chabrowski, Y. Q. Fu, Existence of solutions for $p(x)$-Laplacian problems on bounded domains, *J. Math. Anal. Appl.*, **306** (2005), 604–618. https://doi.org/10.1016/j.jmaa.2004.10.028

11. N. T. Chung, Eigenvalue problems for fractional $p(x, y)$-Laplacian equations with indefinite weight, *Taiwaness J. Math.*, **23** (2019), 1153–1173. https://doi.org/10.11650/tjm/190404

12. L. Diening, P. Harjulehto, P. Hästö, M. Ružicka, *Lebesgue and Sobolev spaces with variable exponents*, Lecture Notes in Mathematics, Springer-Verlag, Heidelberg, 2011.

13. L. Diening, P. H*ästö*, A. Nekvinda, Open problems in variable exponent Lebesgue and Sobolev spaces, *FSDONA04 Proc.*, 2004, 38–58.

14. L. Del Pezzo, J. D. Rossi, Traces for fractional Sobolev spaces with variable exponents, *Adv. Oper. Theory*, **2** (2017), 435–446. https://doi.org/10.22034/aot.1704-1152

15. X. L. Fan, Global $C^{1,\alpha}$ regularity for variable exponent elliptic equations in divergence form, *J. Differ. Equ.*, **235** (2007), 397–417. https://doi.org/10.1016/j.jde.2007.01.008

16. X. L. Fan, A constrained minimization problem involving the $p(x)$-Laplacian in $\mathbb{R}^N$, *Nonlinear Anal.: Theory, Methods Appl.*, **69** (2008), 3661–3670. https://doi.org/10.1016/j.na.2007.10.002

17. X. L. Fan, $p(x)$-Laplacian equations in $\mathbb{R}^N$ with periodic data and nonperiodic perturbations, *J. Math. Anal. Appl.*, **341** (2008), 103–119. https://doi.org/10.1016/j.jmaa.2007.10.006

18. X. L. Fan, S. G. Deng, Remarks on Ricceri's variational principle and applications to the $p(x)$-Laplacian equations, *Nonlinear Anal.: Theory, Methods Appl.*, **67** (2007), 3064–3075. https://doi.org/10.1016/j.na.2006.09.060

19. G. Franzina, G. Palatucci, Fractional p-eigenvalues, *Riv. Math. Univ. Parma (N.S.)*, **2** (2014), 373–386.

20. X. L. Fan, Q. H. Zhang, Existence of solutions for $p(x)$-Laplacian Dirichlet problem, *Nonlinear Anal.: Theory, Methods Appl.*, **52** (2003), 1843–1852. https://doi.org/10.1016/S0362-546X(02)00150-5

21. X. L. Fan, D. Zhao, On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$, *J. Math. Anal. Appl.*, **263** (2001), 424–446. https://doi.org/10.1006/jmaa.2000.7617

22. J. Giacomoni, S. Tiwari, G. Warnault, Quasilinear parabolic problem with $p(x)$-Laplacian: existence, uniqueness of weak solutions and stabilization, *Nonlinear Differ. Equ. Appl.*, **23** (2016), 24.

23. J. M. Kim, Y. H. Kim, J. Lee, Existence and multiplicity of solutions for equations of $p(x)$-Laplace type in $\mathbb{R}^N$ without (AR)-condition, *Differ. Integral Equ.*, **31** (2018), 435–464. https://doi.org/10.57262/die/1516676437

24. U. Kaufmann, J. Rossi, R. Vidal, Fractional Sobolev spaces with variable exponents and fractional $p(x)$-Laplacians, *Electron. J. Qual. Theory Differ. Equ.*, **76** (2017), 1–10.

25. O. Kovacik, J. Rákosník, On spaces $L^{p(x)}(\Omega)$ and $W^{k.p(x)}(\Omega)$, *Czechoslovak Math. J.*, **41** (1991), 592–618.

26. J. Lee, J. M. Kim, Y. H. Kim, Existence and multiplicity of solutions for Kirchhoff-Schrödinger type equations involving $p(x)$-Laplacian on the entire space $\mathbb{R}^N$, *Nonlinear Anal.: Real World Appl.*, **45** (2019), 620–649. https://doi.org/10.1016/j.nonrwa.2018.07.016

27. M. Mihăilescu, V. Rădulescu, On a nonhomogeneous quasilinear eigenvalue problem in Sobolev spaces with variable exponent, *Proc. Amer. Math. Soc.*, **135** (2007), 2929–2937.

28. G. Molica Bisci, V. Rădulescu, Multiplicity results for elliptic fractional equations with subcritical term, *Nonlinear Differ. Equ. Appl.*, **22** (2015), 721–739. https://doi.org/10.1007/s00030-014-0302-1

29. M. Mihăilescu, V. Rădulescu, A multiplicity result for a nonlinear degenerate problem arising in the theory of electrorheological fluids, *Proc. R. Soc. A: Math. Phys. Eng. Sci.*, **462** (2006), 2625–2641.

30. G. Molica Bisci, V. D. Rădulescu, R. Servadei, *Variational methods for nonlocal fractional problems*, Vol. 162, Cambridge University Press, Cambridge, 2016.

31. S. Mosconi, M. Squassina, Nonlocal problems at nearly critical growth, *Nonlinear Anal.: Theory, Methods Appl.*, **136** (2016), 84–101. https://doi.org/10.1016/j.na.2016.02.012

32. S. Mosconi, M. Squassina, Recent progresses in the theory of nonlinear nonlocal problems, *Bruno Pini Math. Anal. Semin.*, **7** (2017), 147–164. https://doi.org/10.6092/issn.2240-2829/6696

33. E. D. Nezza, G. Palatucci, E. Valdinoci, Hitchhiker's guide to the fractional Sobolev spaces, *Bull. Sci. Math.*, **136** (2012), 521–573. https://doi.org/10.1016/j.bulsci.2011.12.004

34. R. Servadei, E. Valdinoci, Variational methods for non-local operators of elliptic type, *Discrete Contin. Dyn. Syst.*, **33** (2013), 2105–2137. https://doi.org/10.3934/dcds.2013.33.2105

35. M. Willem, *Minimax theorems*, Birkhauser, Basel, 1996.

36. J. H. Yao, X. Y. Wang, On an open problem involving the $p(x)$-Laplacian–A further study on the multiplicity of weak solutions to $p(x)$-Laplacian equations, *Nonlinear Anal.: Theory, Methods Appl.*, **69** (2008), 1445–1453. https://doi.org/10.1016/j.na.2007.06.044

37. A. B. Zang, $p(x)$-Laplacian equations satisfying Cerami condition, *J. Math. Anal. Appl.*, **337** (2008), 547–555. https://doi.org/10.1016/j.jmaa.2007.04.007

38. C. Zhang, X. Zhang, Renormalized solutions for the fractional $p(x)$-Laplacian equation with L^1 date, *Nonlinear Anal.*, **190** (2020), 111610. https://doi.org/10.1016/j.na.2019.111610

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