Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(7): 16308–16319.
DOI: 10.3934/math.2023835
Received: 09 March 2023
Revised: 19 April 2023
Accepted: 23 April 2023
Published: 08 May 2023

Research article

Min-max method for some classes of Kirchhoff problems involving the ψ-Hilfer fractional derivative

Hadeel Zaki Mohammed Azumi[1,*], **Wafa Mohammed Ahmed Shammakh**[1] **and Abdeljabbar Ghanmi**[2]

[1] Department of Mathematics, Faculty of Science, University of Jeddah, Jeddah 21493, Saudi Arabia
[2] Faculté des Sciences de Tunis, LR10ES09 Modélisation mathématique, analyse harmonique et théorie du potentiel, Université de Tunis El Manar, Tunis 2092, Tunisie

* **Correspondence:** Email: hzalzumi@uj.edu.sa.

Abstract: In this work, we develop some variational settings related to some singular p-Kirchhoff problems involving the ψ-Hilfer fractional derivative. More precisely, we combine the variational method with the min-max method in order to prove the existence of nontrivial solutions for the given problem. Our main result generalizes previous ones in the literature.

Keywords: variational methods; min-max method; p-Laplacian operator; ψ-Hilfer operators
Mathematics Subject Classification: 31B30, 35J35, 35J60

1. Introduction

Fractional differential equations have been recently studied by mathematicians and scientists due to the importance of fractional-order differential equations as a powerful tool for providing a full description of the physical behavior of various systems and their memory and hereditary properties. Several generalized formulations of fractional-order operators have been proposed and introduced into various phenomena in natural science and engineering, chemistry, and physics (see [6, 7, 17, 22, 24]). Recently, too many researchers have concentrated on the development of new fractional derivatives and integrals which generalize previous ones in the literature. One of these new operators is the Hilfer derivative which is introduced in 1999 by Hilfer [18]. We note that the Caputo fractional derivative and the Riemann-Liouville deriva ve are particular cases for this new derivative.

The authors in [26] proved the existence of solutions for some stochastic differential problems involving the Hilfer fractional derivative which is driven by fractional Brownian motion. Also, there are several important papers dealing with the studies of different fractional operators, in this direction, we can refer the interested readers to the monographs [1, 2, 13–16, 20, 23] and references therein.

Torres in [27], considered the following problem

$$\begin{cases} -{}_sD_1^{\theta}\ {}_0D_s^{\theta}z(s) = f(s, z(s)),\ \ s \in (0, T), \\ z(0) = z(T) = 0. \end{cases} \tag{1.1}$$

Here, ${}_sD_1^{\theta}$ and ${}_0D_s^{\theta}$ denote respectively the right and the left Riemann Liouville fractional derivatives. Precisely, to prove that problem (1.1) has a nontrivial weak solution, the author used the well-known Mountain Pass Theorem. We note that the variational approach is used for the first study of such a problem by Jiao and Zhou [19]. Next, several published works were studied using different methods like variational method, fixed points theorem method, iterative method, etc. We refer for instance to [2, 16, 28–30]. Particularly, César [28] studied the following Dirichlet problem with mixed derivatives:

$$\begin{cases} -{}_sD_1^{\theta}\ (\rho_p\left({}_0D_s^{\theta}\xi(s))\right) = k(s, \xi(s)),\ \ s \in (0, T), \\ \xi(0) = \xi(T) = 0, \end{cases} \tag{1.2}$$

where $0 < \frac{1}{p} < \theta < 1$, ρ_p denotes the p-Laplacian operator which is defined by $\rho_p(s) = |s|^{p-2}s$, and $k : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a function with Carathéodory type.

Since fractional calculus appear in many applications we cite for example, electrical circuits, chemistry, viscoelasticity and electromagnetism, many researchers are concentrated on the development of these operators. Recently, the fractional derivative with respect to another function is extensively studied by several authors. Kilbas et al. [20, Chapter 2], introduced the ψ-Riemann Liouville operators. Later, Almeida [3] introduced the ψ-Caputo operators. In 2018, Vanterler et al. [10] introduced the ψ-Hilfer fractional derivative. We noted that the ψ-Hilfer fractional derivative is a more general operators in the sense that if $\psi(s) = s$, then we get the Riemann fractional derivative, and if $\psi(s) = \ln(s)$, then we get the Hadamar fractional derivative. Da Sousa et al. [8] extended and developed other properties about the ψ-fractional operators. Due to their importance and huge of applications, there are several papers investigated problems involving these type of operators. Da Sousa et al. [9], considered some ψ-Hilfer problem with p-Laplacian operator, by the use of the variational approach, the authors proved that problem (1.2) admits multiple solutions. Very recently, Da Sousa et al. [11] studied some problems involving the ψ-Hilfer fractional derivatives. Precisely, they used the variational method approach in order to prove some existence results.

As far as we know, there are few works which discuss singular problems involving fractional operators, especially those that include the ψ-Hilfer fractional derivative. In this direction, we will study the following p-fractional boundary value problem with the ψ-Hilfer fractional derivative

$$\begin{cases} K\Big(\|{}^H\mathcal{D}_{0^+}^{\mu,\theta,\psi}z\|^p_{L^p(0,T)}\Big)\,{}^H\mathcal{D}_T^{\mu,\theta,\psi}\left(\rho_p\left({}^H\mathcal{D}_{0^+}^{\mu,\theta,\psi}z(s)\right)\right) = \lambda g(s, z(s)) + \frac{f(s)}{z^{\gamma}}, \quad s \in (0, T), \\ I_{0^+}^{\theta(\theta-1);\psi}(0) = I_T^{\theta(\theta-1);\psi}(T) = 0, \end{cases} \tag{1.3}$$

where $\lambda > 0$, $\|.\|$ denotes the well known $L^p(a, b)$-norm which is defined by

$$\|z\|_{L^r(a,b)} = \left(\int_a^b |z(s)|^r ds\right)^{\frac{1}{r}}.$$

μ, θ are such that

$$1 \leq \frac{1}{\mu} < p, \text{ and } 0 \leq \theta \leq 1.$$

${}^H\mathcal{D}_T^{\mu,\theta,\psi}$ and ${}^H\mathcal{D}_{0^+}^{\mu,\theta,\psi}$ are defined in Definition 2.2, $I_{0^+}^{\theta(\theta-1);\psi}$ and $I_T^{\theta(\theta-1);\psi}$ are given in Definition 2.1. The Kirchhoff term K is defined on $\mathbb{R}$ by

$$K(t) = a + bt^m,$$

for some $a \in [1,\infty)$ and $b \in (1,\infty)$.

In the rest of this work, we assume that the functions f, g are continuous on $[0,T] \times \mathbb{R}$, moreover for any $s > 0$ we have

$$g(y, sz) = s^{\delta-1} g(y,z), \text{ for all } (y,z) \in [0,T] \times \mathbb{R}.$$

Also, we assume the following hypotheses:
($\mathbf{H_1}$) Assume that $p < \delta < p^2$ and for some $C_0 > 0$, we have

$$|G(s,z)| \leq C_0 |z|^\delta, \tag{1.4}$$

here $G(s,z) = \int_0^z g(s,\xi)d\xi$.
($\mathbf{H_2}$) f is a positive function in $L^{\frac{p}{p+\gamma-1}}([0,T],(0,\infty))$.

Remark 1.1. *Since the function g is positively homogenuous of degree $\delta - 1$ and $G(s,0) = 0$, the the function G satisfies*

$$G(y, sz) = s^\delta G(y,z), \textit{ for all } (y,z) \in [0,T] \times \mathbb{R}.$$

If we differentiate the last equation with respect to s and taking $s = 1$, we get

$$zg(y,z) = \delta G(y,z) \textit{ for all } (y,z) \in [0,T] \times \mathbb{R}.$$

We note that problem (1.3) is a nonlocal problem since it contain a Kirchhoff term k, and this make a study of such problem more complicated and gives a higher difficulties in the manipulation of this type of problems. Problems of type (1.3) are related to the problem investigated by Kirchhoff [21]. Precisley, Kirchhoff [21] studied the following problem

$$\xi \frac{\partial^2 z}{\partial t^2} - \left(\frac{\xi_0}{h} + \delta E\right) \frac{\partial^2 z}{\partial x^2}, \tag{1.5}$$

where ξ and ξ_0 denote respectively the mass density and the initial tension, E is the Young modulus of the material, the area of the cross section is denoted by h and δ is the average which is given by

$$\delta = \frac{1}{2T} \int_0^T \left|\frac{\partial z}{\partial x}\right| dx.$$

here T denotes the length of the string.

Recently, Kirchhoff type problems have been extensively studied by several authors, we cite for example the works [4, 12, 31] and references therein.

Our main result of this paper is the following theorem.

Theorem 1.1. *Assume that hypotheses* ($\mathbf{H_1}$) *and* ($\mathbf{H_2}$) *are satisfied. Then problem* (1.3) *admits a nontrivial weak solution provided that $\lambda \in (0, \lambda_0)$ for some positive constant λ_0.*

2. Preliminaries and variational setting

In this part, we will give some lemmas and preliminaries on the fractional calculus, essentially, we introduce the ψ-Hilfer fractional operators and present some properties which will be used in the variational setting of our problem. We note that all results is this section can be found in [20,25]. In the rest of this work, the Euler gamma function is denoted by Γ, $\mu > 0$, $\theta > 0$ and $-\infty \leq a < b \leq \infty$. The function $\psi : [a, b] \to (0, \infty)$ is an increasing function with continuous derivative $\psi'(s) \neq 0$ on (a, b). Finally, for a simplicity, we denote by $d_\psi(x)$ the following expression:

$$d_\psi(x) = \frac{1}{\psi'(x)} \frac{d}{dx}.$$

Also for a given real number ξ and ρ, we adopt the following notation

$$\psi_\xi(\rho) = \psi(\xi) - \psi(\rho).$$

Next, we begin by presenting the notion of fractional integral operators.

Definition 2.1. *([20,25])* Let $z : [a, b] \to \mathbb{R}$ be an integrable function and assume that ψ is of class C^1. We define respectively the left and right fractional integrals of a function z with respect to another function ψ as follows:

$$I^{\theta,\psi}_{a^+} z(x) := \frac{1}{\Gamma(\theta)} \int_a^x \psi'(t)(\psi_x(t))^{\theta-1} z(t)dt,$$

and

$$I^{\theta,\psi}_{b^-} z(x) := \frac{1}{\Gamma(\theta)} \int_x^b \psi'(t)(\psi_t(x))^{\theta-1} z(t)dt.$$

Now, we give in the next definition the notion of ψ-Hilfer fractional derivative.

Definition 2.2. *([8,10])* Let $z : [a, b] \to \mathbb{R}$ be an integrable function and assume that $0 \leq \theta \leq 1$ and the function ψ is of class C^1. We define respectively the left and right sided ψ-Hilfer fractional derivatives of order μ and type θ as follows:

$${}^H\mathcal{D}^{\mu,\theta,\psi}_{a^+} z(s) := I^{\theta(n-\mu),\psi}_{a^+} \left(d_\psi(s)\right)^n I^{(1-\theta)(n-\mu),\psi}_{a^+} z(s),$$

and

$${}^H\mathcal{D}^{\mu,\theta,\psi}_{b^-} z(s) := I^{\theta(n-\mu),\psi}_{b^-} \left(d_\psi(s)\right)^n I^{(1-\theta)(n-\mu),\psi}_{b^-} z(s),$$

here n is an integer satisfying $\mu \in (n - 1, n]$.

We note that this new fractional derivative generalizes some other fractional derivatives in the literature. Precisely, we give three cases in the following remark.

Remark 2.1. *We have*

(i) If θ is very close to zero, then we get

$$\mathcal{D}^{\mu,\psi}_{a^+} z(x) = \left(d_\psi(x)\right)^n I^{n-\mu,\psi}_{a^+} z(x),$$

and

$$\mathcal{D}^{\mu,\psi}_{b^-} z(x) = \left(-d_\psi(x)\right)^n I^{n-\mu,\psi}_{b^-} z(x),$$

where $\mathcal{D}^{\mu,\psi}_{a^+}$ and $\mathcal{D}^{\mu,\psi}_{b^-}$ are respectively the left and right ψ-Riemann-Liouville fractional derivatives *(ii) If θ is very close to one, then we get*

$$ {}^{C}\mathcal{D}_{a^+}^{\mu,\psi} z(x) = I_{a^+}^{n-\mu,\psi} \left(d_\psi(x)\right)^n z(x), $$

and

$$ {}^{C}\mathcal{D}_{b^-}^{\mu,\theta,\psi} z(x) = I_{b^-}^{n-\mu,\psi} \left(-d_\psi(x)\right)^n z(x), $$

where ${}^{C}\mathcal{D}_{a^+}^{\mu,\psi}$ and ${}^{C}\mathcal{D}_{b^-}^{\mu,\theta,\psi}$ are respectively the left and right sided ψ-Caputo fractional derivatives
(iii) If $\delta = \mu + \theta(n - \mu)$, then we get

$$ {}^{H}\mathcal{D}_{a^+}^{\mu,\theta,\psi} z(x) = I_{a^+}^{\delta-\mu,\psi} \mathcal{D}_{a^+}^{\delta,\psi} z(x), $$

and

$$ {}^{H}\mathcal{D}_{b^-}^{\mu,\theta,\psi} z(x) = I_{b^-}^{\delta-\mu,\psi} \mathcal{D}_{b^-}^{\delta,\psi} z(x). $$

The principal result used in the variational formulation of integral equation is the integration by parts. So in this direction, we give the following lemma:

Lemma 2.1. *([9]) Assume that $0 < \mu \leq 1$, $0 \leq \theta \leq 1$ and ψ ia a function of class C^1. Let $z : [a, b] \to \mathbb{R}$, be an absolutely countinous function. If the function $\xi : [a, b] \to \mathbb{R}$ is of class C^1 with $\xi(a) = \xi(b) = 0$, then we have*

$$ \int_a^b {}^{H}\mathcal{D}_{a^+}^{\mu,\theta,\psi} z(s)\xi(s)\, ds = \int_a^b z(s)\psi'(s)\, {}^{H}\mathcal{D}_{b^-}^{\mu,\theta,\psi} \left(\frac{\xi(s)}{\psi'(s)}\right) ds. $$

Remark 2.2. *([9, 27]) If $r > 1 \geq \mu > 0$, and $q = \frac{r}{r-1}$, then we have:*

(i) If $z \in L^r(a, b)$, the $I_{a^+}^{\mu,\psi} z \in L^r(a, b)$, moreover

$$ \|I_{a^+}^{\mu,\psi} z\|_{L^r(a,b)} \leq \frac{(\psi_b(a))^\mu}{\Gamma(\mu + 1)} \|z\|_{L^r(a,b)}. $$

(ii) If $\frac{1}{r} < \mu < 1$, then $\lim_{t\to a} I_{a^+}^{\mu,\psi} z(t) = 0$. So, $I_{a^+}^{\mu,\psi} z$ is continuous on $[a, b]$, moreover, we get

$$ \|I_{a^+}^{\mu,\psi} z\|_\infty \leq \frac{(\psi_b(a))^{\mu-\frac{1}{r}}}{\Gamma(\mu)\left((\mu - 1)q + 1\right)^{\frac{1}{q}}} \|z\|_{L^r(a,b)}, $$

where $L^r(a, b)$ denotes the classique Lebesgue space and

$$ \|z\|_\infty = ess \sup_{a\leq s\leq b} |z(s)|. $$

3. Variational fromwork and proof of the main result

In this section, we will apply the min-max method in order to prove Theorem 1.1. So, we denoted by $E_p^{\mu,\theta,\psi}$ the closure of the set $C_0^\infty([0,T],\mathbb{R})$ endowed with the norm

$$\|z\|_{E_p^{\mu,\theta,\psi}} = \left(\|z\|^p_{L^p(0,T)} + \|{}_0\mathcal{D}_t^{\mu,\theta,\psi} z\|^p_{L^p(0,T)}\right)^{\frac{1}{p}}.$$

We collect from [27] that $E_p^{\mu,\theta,\psi}$ is also defined as:

$$E_p^{\mu,\theta,\psi} = \left\{\xi : [0,T] \to \mathbb{R} : \mathcal{D}_{0^+}^{\mu,\theta,\psi}\xi \in L^p([0,T]), I_{0^+}^{\theta(\theta-1);\psi}(0) = I_T^{\theta(\theta-1);\psi}(T) = 0\right\}.$$

Moreover, in the following remark, we collect some important properties about this space.

Remark 3.1. *(See [9, 27]) Assume that $\mu \in (0,1]$ and $\theta \in [0,1]$, then we have*

(i) The space $E_p^{\mu,\theta,\psi}$ is a separable Banach space which is also reflexive.

(ii) Assume further that either $\mu > \frac{1}{p}$ or $1-\mu > \frac{1}{p}$, then we get

$$\|z\|_{L^p(0,T)} \leq \frac{(\psi_T(0))^\mu}{\Gamma(\mu+1)} \|\mathcal{D}_{0^+}^{\mu,\theta,\psi} z\|_{L^p(0,T)}.$$

(iii) If $\frac{1}{p} < \mu$, and $q = \frac{p}{p-1}$, then we have

$$\|z\|_\infty \leq \frac{(\psi_T(a))^{\mu-\frac{1}{r}}}{\Gamma(\mu)\left((\mu-1)q+1\right)^{\frac{1}{q}}} \|\mathcal{D}_{0^+}^{\mu,\theta,\psi} z\|_{L^r(0,T)}.$$

We note that, from Remark 3.1, we can endowed the space $E_p^{\mu,\psi}$ by the following norm:

$$\|z\|_{\mu,\psi} = \|{}_0\mathcal{D}_t^{\mu,\theta,\psi} z\|_{L^p(0,T)},$$

moreover we have

$$\|z\|_\infty \leq \frac{(\psi_T(a))^{\mu-\frac{1}{r}}}{\Gamma(\mu)\left((\mu-1)q+1\right)^{\frac{1}{q}}} \|z\|_{\mu,\psi}. \tag{3.1}$$

Now, we are in a position to define the notion of solutions.

Definition 3.1. A function φ is said to be a weak solution for problem (1.3), if for every $\xi \in E_p^{\mu,\theta,\psi}$ we have:

$$K\left(\int_0^T \left|{}^H\mathcal{D}_{0^+}^{\mu,\theta,\psi}\varphi(s)\right|^p ds\right) \int_0^T |{}_0\mathcal{D}_t^{\mu,\theta,\psi}\varphi(t)|^{p-2}\, {}_0\mathcal{D}_t^{\mu,\theta,\psi}\varphi(t)\, {}_0\mathcal{D}_t^{\mu,\theta,\psi}\xi(t)dt$$
$$= \lambda \int_0^T g(t,\varphi(t)\xi(t)dt + \int_0^T f(t)\varphi^{-\gamma}(t)\xi(t)dt.$$

Let $\xi \in C_0^\infty([0,T],\mathbb{R})$ and let φ be a solution of problem (1.3). Multplying the first equation in system (1.3) by ξ and integrating over $[0,T]$ we obtain

$$K\left(\|\varphi\|^p_{\mu,\psi}\right) \int_0^T {}^H\mathcal{D}_T^{\mu,\theta,\psi}\left(\rho_p\left({}^H\mathcal{D}_{0^+}^{\mu,\theta,\psi}\varphi(s)\right)\right)\xi\, ds = \lambda \int_0^T g(s,\varphi(s))\xi\, ds + \int_0^T \frac{f(s)}{\varphi^\gamma}\xi\, ds. \tag{3.2}$$

Using the Lemma 2.1, we have

$$\int_0^T {}^H\mathcal{D}_T^{\mu,\theta,\psi}\left(\rho_p\left({}^H\mathcal{D}_{0^+}^{\mu,\theta,\psi}\varphi(s)\right)\right)\xi\,ds = \int_0^T \rho_p\left({}^H\mathcal{D}_{0^+}^{\mu,\theta,\psi}\varphi(s)\right)\psi'(s){}^H\mathcal{D}_{0^+}^{\mu,\theta,\psi}\left(\frac{\xi(s)}{\psi'(s)}\right)ds. \tag{3.3}$$

If for all $s \in [0,T]$ we have ${}^H\mathcal{D}_{0^+}^{\mu,\theta,\psi}\left(\frac{\xi(s)}{\psi'(s)}\right) = \frac{1}{\psi'(s)}{}^H\mathcal{D}_{0^+}^{\mu,\theta,\psi}\xi(s)$, then Eq (3.2) can be rewritten as

$$K\left(\|\varphi\|_{\mu,\psi}^p\right)\int_0^T \rho_p\left({}^H\mathcal{D}_{0^+}^{\mu,\theta,\psi}\varphi(s)\right){}^H\mathcal{D}_{0^+}^{\mu,\theta,\psi}\xi(s)\,ds = \lambda\int_0^T g(s,\varphi(s))\xi\,ds + \int_0^T \frac{f(s)}{\varphi^\gamma}\xi\,ds.$$

Now, if we take $\xi = \varphi$ then we get

$$K\left(\|\varphi\|_{\mu,\psi}^p\right)\int_0^T \left|{}^H\mathcal{D}_{0^+}^{\mu,\theta,\psi}\varphi(s)\right|^p\,ds = \lambda\int_0^T g(s,\varphi(s))\varphi(s)\,ds + \int_0^T f(s)\varphi^{1-\gamma}(s)\,ds.$$

From Remark 1.1, we can define the functional associate to problem (1.3), $\rho_\lambda : E_p^{\mu,\theta,\psi} \to \mathbb{R}$, as follows:

$$\rho_\lambda(\varphi) = \frac{1}{p}\widetilde{K}(\|\varphi\|_{\mu,\psi}^p) - \lambda\int_0^T G(t,\varphi(t))dt - \frac{1}{1-\gamma}\int_0^T f(t)|\varphi(t)|^{1-\gamma}\,dt,$$

where $\widetilde{K}(t) = \int_0^t K(s)\,ds$.

It is not difficult to show that for all $t \geq 0$, we have

$$K(t) \geq a, \text{ and } \widetilde{K}(t) \geq \frac{t}{m+1}K(t). \tag{3.4}$$

We note that ρ_λ is well defined, moreover, due to the singular term, it is not of class C^1. So we can not use the direct variational method to prove the existence of solutions. For this reason, we will apply the min-max method. So, we begin by proving the following result.

Lemma 3.1. *Under the hypotheses* $(\mathbf{H_1})$ *and* $(\mathbf{H_2})$*, if* $\delta < p^2$*, then* ρ_λ *is coercive and bounded from bellow on* $E_p^{\mu,\theta,\psi}$.

Proof. Let $\varphi \in E_p^{\mu,\theta,\psi}$, then from Eqs (1.4), (3.1), (3.4) hypothesis $(\mathbf{H_2})$ and the Hölder inequality, we obtain

$$\begin{aligned}
\rho_\lambda(\varphi) &= \frac{1}{p}\widetilde{K}(\|\varphi\|_{\mu,\psi}^p) - \lambda\int_0^T G(t,\varphi(t))dt - \frac{1}{1-\gamma}\int_0^T f(t)|\varphi(t)|^{1-\gamma}\,dt \\
&\geq \frac{1}{p(m+1)}\|\varphi\|_{\mu,\psi}^p K(\|\varphi\|_{\mu,\psi}^p) - \lambda C_0\int_0^T |\varphi(t)|^\delta\,dt \\
&\quad -\frac{1}{1-\gamma}\left(\int_0^T |f(t)|^{\frac{p}{p+\gamma-1}}dt\right)^{\frac{p+\gamma-1}{p}}\left(\int_0^T |\varphi(t)|^p\,dt\right)^{\frac{1-\gamma}{p}} \\
&\geq \frac{b}{p(m+1)}\|\varphi\|_{\mu,\psi}^{mp+1} - \frac{T^{\frac{1-\gamma}{p}}}{1-\gamma}\|f\|_{L^{\frac{p}{p+\gamma-1}}(0,T)}\|\varphi\|_\infty^{1-\gamma} \\
&\geq \frac{b}{p(m+1)}\|\varphi\|_{\mu,\psi}^{mp+1} - \lambda C_0 T\frac{(\psi_T(a))^{\delta(\mu-\frac{1}{r})}}{\Gamma^\delta(\mu)\left((\mu-1)q+1\right)^{\frac{\delta}{q}}}\|\varphi\|_{\mu,\psi}^\delta
\end{aligned}$$

$$-\frac{T^{\frac{1-\gamma}{p}}}{1-\gamma}\|f\|_{L^{\frac{p}{p+\gamma-1}}(0,T)}\frac{(\psi_T(a))^{(1-\gamma)(\mu-\frac{1}{r})}}{\Gamma^{1-\gamma}(\mu)\,((\mu-1)q+1)^{\frac{1-\gamma}{q}}}\|\varphi\|_{\mu,\psi}^{1-\gamma}.$$

Since $0<1-\gamma<1<\delta<mp+1$, then we see that ρ_λ is coercive and bounded from bellow on $E_p^{\mu,\theta,\psi}$. This ends the proof of Lemma 3.1.

Lemma 3.2. *Assume that hypotheses* $(\mathbf{H_1})$ *and* $(\mathbf{H_2})$ *hold. Then, there exists a nonnegative nontrivial function* $\varphi_0 \in E_p^{\mu,\theta,\psi}$ *such that* $\rho_\lambda(t\varphi_0)<0$*, provided that* $t>0$ *is small enough.*

Proof. Let $\varphi_0 \in C^\infty([0,T])$. Assume that for some bounded sub-intevals I_0 and I_1, we have $I_0 \subset supp(\varphi_0) \subset I_1 \subset (0,T)$, $0 \leq \varphi_0 \leq 1$ in z_1 and $\varphi_0 = 1$ in z_0.

$$\begin{aligned}
\rho_\lambda(t\varphi_0) &= \frac{1}{p}\widetilde{K}(\|t\varphi_0\|_{\mu,\psi}^p) - \lambda t^\delta \int_0^T G(s,\varphi_0)\,ds - \frac{1}{1-\gamma}t^{1-\gamma}\int_0^T f(s)|\varphi_0(s)|^{1-\gamma}\,ds \\
&= \frac{at^p}{p}\|\varphi\|_{\mu,\psi}^p + \frac{bt^{p(m+1)}}{p(m+1)}\|\varphi\|_{\mu,\psi}^{p(m+1)} - \lambda t^\delta \int_0^T G(s,\varphi_0)\,ds - \frac{t^{1-\gamma}}{1-\gamma}\int_0^T f(s)|\varphi_0(s)|^{1-\gamma}\,ds \\
&\leq t^p\left(\frac{a}{p}\|\varphi\|_{\mu,\psi}^p + \frac{b}{p(m+1)}\|\varphi\|_{\mu,\psi}^{p(m+1)}\right) - \frac{1}{1-\gamma}\int_0^T f(s)|\varphi_0(s)|^{1-\gamma}\,ds \\
&\leq t^{1-\gamma}\left[t^{p+\gamma-1}\left(\frac{a}{p}\|\varphi\|_{\mu,\psi}^p + \frac{b}{p(m+1)}\|\varphi\|_{\mu,\psi}^{p(m+1)}\right) - \frac{1}{1-\gamma}\int_0^T f(s)|\varphi_0(s)|^{1-\gamma}\,ds\right] \\
&< 0,\ \forall\, t\in(0,\nu^{\frac{1}{p+\gamma-1}}),
\end{aligned}$$

where

$$\nu = \min\left(1, \frac{\frac{1}{1-\gamma}\int_0^T f(s)|\varphi_0(s)|^{1-\gamma}\,ds}{\frac{a}{p}\|\varphi\|_{\mu,\psi}^p + \frac{b}{p(m+1)}\|\varphi\|_{\mu,\psi}^{p(m+1)}}\right).$$

According to Lemma 3.1, we can define the following expression:

$$m_\lambda = \inf_{u\in E_p^{\mu,\theta,\psi}} \rho_\lambda(u).$$

Moreover, from Lemma 3.2, we have $m_\lambda < 0$.

Lemma 3.3. *The functional* ρ_λ *attains it's global minimizer in E. That is, there exists* $u_\lambda \in E_p^{\mu,\theta,\psi}$*, such that*

$$\rho_\lambda(u_\lambda) = m_\lambda < 0.$$

Proof. Let $\{u_n\}$ be a minimizing sequence of ρ_λ, which means that $\rho_\lambda \to m_\lambda$ as $n\to\infty$. Since ρ_λ is coercive, then $\{u_n\}$ is bounded in E. endeed, if not, up to a subsequence, we can assume that $\|u_n\|\to\infty$. Therefore, the coercivity of ρ_λ, implies that $\rho_\lambda(u_n)\to\infty$, which is a contradicts. Hence, $\{u_n\}$ is bounded. Therefore, there exist $u_\lambda \in E_p^{\mu,\alpha,\psi}$, and a subsequence still denoted by $\{u_n\}$ such that, as n tends to infinity, we have

$$\begin{cases} u_k \rightharpoonup u_\lambda, \text{ weakly in } E_p^{\mu,\alpha,\psi}, \\ u_k \to u_\lambda, \text{ in } C([0,T],\mathbb{R}). \end{cases}$$

Since $\{u_n\}$ is bounded in $E_p^{\mu,\alpha,\psi}$, then, from the proof of Lemma 3.1, we have

$$\int_0^T f(t)|u_n|^{1-\gamma}\,dt \le T^{\frac{1-\gamma}{p}}\|f\|_{L^{\frac{p}{p+\gamma-1}}(0,T)} \frac{(\psi_T(a))^{(1-\gamma)(\mu-\frac{1}{r})}}{\Gamma^{1-\gamma}(\mu)\,((\mu-1)q+1)^{\frac{1-\gamma}{q}}}\|u_n\|_{\mu,\psi}^{1-\gamma}.$$

So, from the absolutely continuity of $\|f\|_{L^{\frac{p}{p+\gamma-1}}(0,T)}$, we deduce that

$$\left\{\int_0^T f(t)|u_n|^{1-\gamma}\,dt,\ n\in\mathbb{N}\right\},$$

is equi-absolutely-continuous. Therefore, using Vitali's convergence theorem (see [5]) and the fact that $\|u_n\|_{\mu,\psi}$ is bounded, we obtain

$$\lim_{n\to\infty}\int_0^T f(t)|u_n|^{1-\gamma}\,dt = \int_0^T f(t)|u_\lambda|^{1-\gamma}\,dt. \tag{3.5}$$

On the other hand, from (1.4), the continuity of the function G, and the dominated convergence theorem we get

$$\lim_{n\to\infty}\int_0^T G(t,u_n)\,dt = \int_0^T G(t,u_\lambda)\,dt. \tag{3.6}$$

Finally, combining Eqs (3.5), (3.6) and the weak lower semi-continuity of the norm, we deduce

$$m_\lambda \le \rho_\lambda(u_\lambda) \le \lim_{n\to\infty}\rho_\lambda(u_n) = m_\lambda.$$

Hence

$$\rho_\lambda(u_\lambda) = m_\lambda < 0. \tag{3.7}$$

This ends the proof of Lemma 3.3.

Now, we are in a position to prove the main result of this paper.

Proof of Theorem 1.1. From Lemma 3.3, we get the existence of a function u_λ which is a global minimum for the functional ρ_λ in $E_p^{\mu,\alpha,\psi}$. So u_λ satisfies the following inequality:

$$0 \le \rho_\lambda(u_\lambda + t\varphi) - \rho_\lambda(u_\lambda),\quad \forall\ (t,\varphi)\in(0,\infty)\times E_p^{\mu,\alpha,\psi}. \tag{3.8}$$

So

$$0 \le \lim_{t\to 0}\frac{\rho_\lambda(u_\lambda + t\varphi) - \rho_\lambda(u_\lambda)}{t}.$$

Which yelds to

$$K\Big(\int_0^T \left|{}^H\mathcal{D}_{0^+}^{\mu,\theta,\psi}u_\lambda(s)\right|^p ds\Big)\int_0^T |{}_0\mathcal{D}_t^{\mu,\theta,\psi}u_\lambda(t)|^{p-2}\,{}_0\mathcal{D}_t^{\mu,\theta,\psi}u_\lambda(t)\,{}_0\mathcal{D}_t^{\mu,\theta,\psi}\varphi(t)dt$$

$$-\lambda\int_0^T g(t,u_\lambda(t)\varphi(t)dt - \int_0^T f(t)u_\lambda^{-\gamma}(t)\varphi(t)dt \ge 0.$$

The fact that φ is arbitrary in $E_p^{\mu,\alpha,\psi}$, implies that we can replace φ by $-\varphi$ in the last inequality, which yields to

$$K\left(\int_0^T \left|{}^H\mathcal{D}_{0^+}^{\mu,\theta,\psi}u_\lambda(s)\right|^p ds\right)\int_0^T |{}_0\mathcal{D}_t^{\mu,\theta,\psi}u_\lambda(t)|^{p-2}\,{}_0\mathcal{D}_t^{\mu,\theta,\psi}u_\lambda(t)\,{}_0\mathcal{D}_t^{\mu,\theta,\psi}\varphi(t)dt$$
$$-\lambda\int_0^T g(t,u_\lambda(t)\varphi(t)dt-\int_0^T f(t)u_\lambda^{-\gamma}(t)\varphi(t)dt=0.$$

Therefore, from Definition 3.1, we can see that u_λ is a weak solution for problem (1.3). Moreover, from Eq (3.7), we see that u_λ is nontrivial.

The proof of Theorem 1.1 is now completed.

4. Conclusions

This paper considers some classes of Kirchhoff problems involving the ψ-Hilfer fractional derivative and a singular nonlinearity. Our main tools are based on the combination of the variational method with the min-max method. More precisely, some important properties of the associated functional energy are proved in order to ensure the existence of a nontrivial weak solution. This study can be generalized to similar problems involving the (k,ψ)-Hilfer derivative.

Acknowledgments

This work was funded by the University of Jeddah, Saudi Arabia, under grant No. UJ-22-DR-120. The authors, therefore, acknowledge with thanks the University's technical and financial support.

Conflict of interest

All authors declare no conflicts of interest in this paper.

References

1. R. P. Agarwal, M. Benchohra, S. Hamani, Boundary value problems for fractional differential equations, *Georgian Math. J.*, **16** (2009), 401–411. https://doi.org/10.1515/GMJ.2009.401
2. K. B. Ali, A. Ghanmi, K. Kefi, Existence of solutions for fractional differential equations with Dirichlet boundary conditions, *Electron. J. Differ. Eq.*, **2016** (2016), 116.
3. R. Almeida, A Caputo fractional derivative of a function with respect to another function, *Commun. Nonlinear Sci.*, **44** (2017), 460–481. https://doi.org/10.1016/j.cnsns.2016.09.006
4. R. Alsaedi, A. Ghanmi, Variational approach for the Kirchhoff problem involving the p-Laplace operator and the ψ-Hilfer derivative, *Math. Method. Appl. Sci.*, 2023. https://doi.org/10.1002/mma.9053
5. J. K. Brooks, Equicontinuous sets of measures and applications to Vitalis integral convergence theorem and control measures, *Adv. Math.*, **10** (1973), 165–171. https://doi.org/10.1016/0001-8708(73)90104-7

6. W. Chen, G. Pang, A new definition of fractional Laplacian with application to modeling three-dimensional nonlocal heat conduction, *J. Comput. Phys.*, **309** (2016), 350–367. https://doi.org/10.1016/j.jcp.2016.01.003

7. Y. Cho, I. Kim, D. Sheen, A fractional-order model for MINMOD millennium, *Math. Biosci.*, **262** (2015), 36–45. https://doi.org/10.1016/j.mbs.2014.11.008

8. J. V. da C. Sousa, E. C. de Oliveira, On the ψ-fractional integral and applications, *Comput. Appl. Math.*, **38** (2019), 4. https://doi.org/10.1007/s40314-019-0774-z

9. J. V. da C. Sousa, J. Zuo, D. O'Regand, The Nehari manifold for a ψ-Hilfer fractional p-Laplacian, *Appl. Anal.*, **101** (2022), 5076–5106. https://doi.org/10.1080/00036811.2021.1880569

10. J. V. da C. Sousa, E. C. de Oliveira, On the ψ-Hilfer fractional derivative, *Commun. Nonlinear Sci.*, **60** (2018), 72–91. https://doi.org/10.1016/j.cnsns.2018.01.005

11. J. V. da C. Sousa, L. S. Tavares, C. Torres, A variational approach for a problem involving a ψ-Hilfer fractional operator, *J. Appl. Anal. Comput.*, **11** (2021), 1610–1630. https://doi.org/10.11948/20200343

12. R. Ezati, N. Nyamoradi, Existence of solutions to a Kirchhoff ψ-Hilfer fractional p-Laplacian equations, *Math. Method. Appl. Sci.*, **44** (2021), 12909–12920. https://doi.org/10.1002/mma.7593

13. A. Ghanmi, M. Althobaiti, Existence results involving fractional Liouville derivative, *Bol. Soc. Paran. Mat.*, **39** (2021), 93–102. https://doi.org/10.5269/bspm.42010

14. A. Ghanmi, M. Kratou, K. Saoudi, A multiplicity results for a singular problem involving a Riemann-Liouville fractional derivative, *Filomat*, **32** (2018), 653–669. https://doi.org/10.2298/FIL1802653G

15. A. Ghanmi, S. Horrigue, Existence of positive solutions for a coupled system of nonlinear fractional differential equations, *Ukr. Math. J.*, **71** (2019), 39–49. https://doi.org/10.1007/s11253-019-01623-w

16. A. Ghanmi, Z. Zhang, Nehari manifold and multiplicity results for a class of fractional boundary value problems with p-Laplacian, *Bull. Korean Math. Soc.*, **56** (2019), 1297–1314. https://doi.org/10.4134/BKMS.b181172

17. N. M. Grahovac, M. M. Žigič, Modelling of the hamstring muscle group by use of fractional derivatives, *Comput. Math. Appl.*, **59** (2010), 1695–1700. https://doi.org/10.1016/j.camwa.2009.08.011

18. R. Hilfer, *Applications of fractional calculus in physics*, World Scientific Publishing Company, 2000.

19. F. Jiao, Y. Zhou, Existence results for fractional boundary value problem via critical point theory, *Int. J. Bifurcat. Chaos*, **22** (2012), 1250086. https://doi.org/10.1142/S0218127412500861

20. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, elsevier, 2006.

21. G. Kirchhoff, *Vorlesungen über mechanik*, Teubner, 1897.

22. R. L. Magin, M. Ovadia, Modeling the cardiac tissue electrode interface using fractional calculus, *J. Vib. Control*, **14** (2008), 1431–1442. https://doi.org/10.1177/1077546307087439

23. K. S. Miller, B. Ross, *An introduction to the fractional calculus and fractional differential equations*, Wiley, 1993.
24. Y. A. Rossikhin, M. Shitikova, Analysis of two colliding fractionally damped spherical shells in modeling blunt human head impacts, *Open Phys.*, **11** (2013), 760–778. https://doi.org/10.2478/s11534-013-0194-4
25. S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional integrals and derivatives, theory and functions*, Switzerland: Gordon and breach science publishers, 1993.
26. S. Saravanakumar, P. Balasubramaniam, Non-instantaneous impulsive Hilfer fractional stochastic differential equations driven by fractional Brownian motion, *Stoch. Anal. Appl.*, **39** (2021), 549–566. https://doi.org/10.1080/07362994.2020.1815545
27. C. Torres, Mountain pass solution for a fractional boundary value problem, *J. Fract. Calc. Appl.*, **5** (2014), 1–10.
28. C. Torres Ledesma, Boundary value problem with fractional p-Laplacian operator, *Adv. Nonlinear Anal.*, **5** (2016), 133–146. https://doi.org/10.1515/anona-2015-0076
29. C. Torres Ledesma, Existence and concentration of solutions for a nonlinear fractional Schödinger equation with steep potential well, *Commun. Pure Appl. Anal.*, **15** (2016), 535–547. https://doi.org/10.3934/cpaa.2016.15.535
30. C. Torres Ledesma, Existence of solution for a general fractional advection-dispersion equation, *Anal. Math. Phys.*, **9** (2019), 1303–1318. https://doi.org/10.1007/s13324-018-0234-8
31. N. H. Tuan, A. T. Nguyen, N. H. Can, Existence and continuity results for Kirchhoff parabolic equation with CaputoFabrizio operator, *Chaos Soliton. Fract.*, **167** (2023), 113028. https://doi.org/10.1016/j.chaos.2022.113028

AIMS Press