Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(7): 16278–16290.
DOI:10.3934/math.2023833
Received: 10 February 2023
Revised: 31 March 2023
Accepted: 05 April 2023
Published: 08 May 2023

Research article

Zermelo's navigation problem for some special surfaces of rotation

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Abstract: In this paper, we investigate Zermelo's navigation problem for some special rotation surfaces. In this respect, we find some Randers-type metrics for these rotation surfaces. Furthermore, we get the H-distortion for the metric induced by surfaces.

Keywords: Zermelo's problem; Frobenius norm
Mathematics Subject Classification: Primary 53C60; Secondary 53B40

1. Introduction

In an affine space A^3, we will consider the following proper affine surfaces of revolution by $(u, v) \in \mathbb{R}^2$, local coordinates. With $\Psi : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ a nondegenerate Blaschke immersion, we can consider a convenient coordinate system (x_1, x_2, x_3) of the following surfaces (for more details, see [1]).

Elliptic type: If Ψ is of elliptic type, in local coordinates there is the following representation:

$$\Psi(u, v) = (f(u)\cos v, f(u)\sin v, g(u)), \tag{1.1}$$

$$\Psi(u, v) = (f(u)\cos v, f(u)\sin v, u), \tag{1.2}$$

$$\Psi(u, v) = (u\cos v, u\sin v, g(u)). \tag{1.3}$$

Hyperbolic type: If Ψ is of hyperbolic type, in local coordinates there is the following representation:

$$\Psi(u, v) = (f(u)\cosh v, f(u)\sinh v, g(u)), \tag{1.4}$$

$$\Psi(u,v) = (f(u)\cosh v, f(u)\sinh v, u), \tag{1.5}$$

$$\Psi(u,v) = (u\cosh v, u\sinh v, g(u)). \tag{1.6}$$

Remark 1.1. *Also, there is the parabolic type of surface, but in this paper, we will choose just elliptic and hyperbolic surface types.*

In some recent papers [1–9], Zermelo's problem and the generalized Zermelo's problem were investigated. Under the assumption that the wind is a time-independent mild breeze for a Riemannian manifold (M, h), they found that the paths minimizing travel time are exactly the geodesics of a Randers metric.

$$F(x,y) = \alpha(x,y) + \beta(x,y) = \frac{\sqrt{\lambda\,|y|^2 + W_0^2}}{\lambda} - \frac{W_0}{\lambda},$$

where $W = W^i \frac{\partial}{\partial x^i}$ is the wind velocity, $|y|^2 = h(y,y)$, $\lambda = 1 - |W|^2$, $W_0 = h(W,y)$. According to [10], the Randers metric F is said to solve the Zermelo navigation problem in the case of a mild breeze, which means $h(W,W) < 1$.

The condition $h(W,W) < 1$ ensures that F is a positive definite metric.

For a positive function $f : \mathbb{R} \to \mathbb{R}^+$, in paper [10], the following rotation surface was considered.

$$M = \{(f(u)\cos v, f(u)\sin v, u), u \in I, 0 < v \leq 2\pi\},$$

with

$$ds^2 = (1 + f'^2)du^2 + f^2 dv^2.$$

The coefficients of the first fundamental form are

$$E = 1 + f'^2, F = 0, G = f^2.$$

The geodesics equation can be found by

$$\ddot{x}_k + \sum_{i,j} \Gamma^k_{ij} \dot{x}^i \dot{x}^j = 0, \tag{1.7}$$

that is,

$$\begin{cases} \frac{d^2u}{ds^2} + \frac{f'f''}{2(1+f'^2)}\left(\frac{du}{ds}\right)^2 - \frac{ff'}{2(1+f'^2)}\left(\frac{dv}{ds}\right)^2 = 0, \\ \frac{d^2v}{ds^2} + 2\frac{f'}{f}\frac{du}{ds}\frac{dv}{ds} = 0. \end{cases}$$

A function $f : [0, +\infty) \to [0, +\infty)$, defined in [10], is constructed as a rotational Randers metric on M, putting $W = \mu\frac{\partial}{\partial v}$ in the h-orthogonal system $\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right)$ of the tangent space $T_x(M)$. One obtains $W = \left(W^1, W^2\right) = (0, \mu) \Rightarrow$

$$h(W,W) = h\left(\mu\frac{\partial}{\partial v}, \mu\frac{\partial}{\partial v}\right) = \mu^2 h\left(\frac{\partial}{\partial v}, \frac{\partial}{\partial v}\right).$$

The navigation data (h, W) gives new data:

$$a_{ij} = \frac{\lambda h_{ij} + W_i W_j}{\lambda^2}, b_i = -\frac{W_i}{\lambda},$$

with $W_i = h_{ij}W^i$, $\lambda = 1 - h(W, W) = 1 - \mu^2 f^2 > 0$.

Finally, after computations (see [10]), we can obtain

$$a_{ij} = \begin{pmatrix} \frac{1}{1-f^2\mu^2} & 0 \\ 0 & \frac{f^2}{(1-f^2\mu^2)^2} \end{pmatrix}, \qquad b_i = \begin{pmatrix} 0 \\ \frac{-\mu f^2}{1-\mu^2} \end{pmatrix}.$$

Here, $i, j = 1, 2$. It can be easily observed that $\alpha(b, b) = a^{ij}b_ib_j = h(W, W) < 1$, and this ensures that F is a positive Randers metric (see [10]).

The Zermelo navigation problem, as the authors in [11] have remarked, is an important problem in Finsler geometry because it aims to find the minimum time trajectories in a Riemannian manifold (B, h) under the influence of a drift (wind), represented by the vector field W. In [12], authors proved that the trajectories that minimize the travel time are exactly the geodesics of the Randers metric which has the norm:

$$F(x, v) = \sqrt{a_{ij}(x)v^iv^j} + b_i(x)v^i,$$

where $v \in T_xB$.

In fact, it is known that every Randers metric F could be obtained as a solution to a Zermelo navigation problem defined by the data $\left\{h_{ij}, W^i\right\}$. This can be observed by (as we saw in [11]) the following

$$h_{ij} = \lambda(a_{ij} - b_ib_j), \lambda = 1 - a^{ij}b_ib_j,$$

$$W^i = -\frac{b^i}{\lambda}, b^i = a^{ij}b_j,$$

$$h^{ij} = \frac{\lambda a^{ij} + b^ib^j}{\lambda^2},$$

Then, it can be observed that there is a natural identification of Randers metrics with solutions to the Zermelo problem:

$$|W|^2 = h_{ij}W^iW^j = a_{ij}b^ib^j \equiv |b|^2 .$$

The Finsler condition $|b|^2 \leq 1$ ensures that F is positive and the metric is convex, i.e., $\partial_{v_i}\partial_{v_j}\left(\frac{F^2}{2}\right)$ is positive definite for all non-zero v.

Also, it is well known from the literature that the geodesic of a Finsler norm is obtained by taking F as the Finsler Lagrangian. If $F(x, v)$ represents a homogeneous Finsler Lagrangian of degree one, a Lagrangian could be defined as $L = \frac{1}{2}F^2$. The Hamiltonian will be on degree two in the momenta $p_i = \frac{\partial L}{\partial v^i} = F\frac{\partial F}{\partial v^i}$, and using [11], we can remark that we can choose a function $G(x, p)$ such that $H = \frac{1}{2}G^2$, $G(x, p) = F(x, v)$. For the Randers metric $F(x, v) = \sqrt{a_{ij}(x)v^iv^j} + b_i(x)v^i$, we have

$$p_i = F(n_i + b_i), n_i = \frac{a_{ij}v^j}{\sqrt{a_{kr}v^kv^r}}.$$

Since, $a^{ij}n_in_j = 1$, we get $G = \sqrt{h^{ij}p_ip_j} - W^ip_i$, with $\left\{h_{ij}, W^i\right\}$, the associated Zermelo data. Thus, the Legendre transformation maps the Randers (Lagrangian) to the Zermelo data (Hamiltonian). We can see G as the sum of the two moment maps $G = G_0 + G_1$, $G_0 = \sqrt{h^{ij}p_ip_j}$, $G_1 = -W^ip_i$.

Also, we can observe that these flows commute $\{G_0, G_1\} = 0$ if and only if W is a Killing vector field. We continue to present some other aspects from the paper [11]. The geodesic flow of the Randers metric can be seen as the null geodesic flow in a stationary space-time with a generic form

$$g_{\mu\nu}dx^\mu dx^\nu = -V^2\left(dt + w_i dx^i\right)^2 + \gamma_{ij}dx^i dx^j,$$

so that

$$g_{ij} = \gamma_{ij} - V^2 w_i w_j.$$

As is remarked in [11], the Fermat principle arises from the Randers structure given by

$$a_{ij} = V^{-2}\gamma_{ij}, a^{ij} = V^2\gamma^{ij}, b_i = -w_i.$$

Thus, the form $ds^2 = V^2\left[-\left(dt - b_i dx^i\right)^2 + a_{ij}dx^i dx^j\right]$, is called the Randers form of a stationary metric. According to [11], the Randers data

$$h_{ij} = \lambda(a_{ij} - b_i b_j), \lambda = 1 - a^{ij}b_i b_j,$$

$$W^i = -\frac{b^i}{\lambda}, b^i = a^{ij}b_j,$$

are equivalent to

$$h_{ij} = \frac{1}{1 + V^2 g^{rs}w_r w_s}\frac{g_{ij}}{V^2}, W^i = V^2 g^{ij}w_j,$$

with

$$\gamma_{ij} = g_{ij} + V^2 w_i w_j,$$

$$\gamma^{ij} = g^{ij} - \frac{V^2 w^i w^j}{1 + V^2 g^{rs}w_r w_s},$$

$$w^i = g^{ij}w_j, W^i = \frac{V^2\gamma^{ij}w_j}{1 - V^2 g^{rs}w_r w_s},$$

$$1 - V^2\gamma^{ij}w_i w_j = \frac{1}{1 + V^2 g^{ij}w_i w_j}.$$

With the above notations, the generic metric

$$g_{\mu\nu}dx^\mu dx^\nu = -V^2\left(dt + w_i dx^i\right)^2 + \gamma_{ij}dx^i dx^j$$

could be put (according to [11]) as follows:

$$ds^2 = \frac{V^2}{1 - h_{ij}W^i W^j}\left[-dt^2 + h_{ij}\left(dx^i - W^i dt\right)\left(dx^j - W^j dt\right)\right],$$

which represents the Zermelo form of stationary spacetime. The metric in square brackets from above is of Painlevé-Gullstrand form.

In this part of the paper, we recall some classical results regarding the Frobenius (Hilbert-Schmidt) norm of a matrix. For more results, please see [13]. The Frobenius norm of a matrix $A = (a_{ij})$ is defined as follows:

$$||A||_{HS} = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^2}.$$

Some of its properties are

$$||A \cdot B||_{HS} \leq ||A||_{HS} \cdot ||B||_{HS} \,,$$

$$||A||_{HS} \leq \frac{\sqrt{r}}{\sigma_{min}(A)},$$

where $\sigma_{min}(A)$ denotes the minimum singular value of A, and r is the rank of A.

2. Main results

Using a similar approach as in paper [10], we will try to solve Zermelo's navigation problem for the following two surfaces:

$$M_1 = \{(f(u)\cos v, f(u)\sin v, g(u)), u \in I, v \in [0, 2\pi]\}, \tag{2.1}$$

$$M_2 = \{(f(u)\cosh v, f(u)\sinh v, g(u)), u \in I, v \in [0, 2\pi]\}. \tag{2.2}$$

These two surfaces represent two revolution surfaces of elliptic type and hyperbolic type, respectively. We will start with surface (2.1). We will consider the Randers metric

$$F(x, y) = \alpha(x, y) + \beta(x, y) = \frac{\sqrt{\lambda\,|y|^2 + W_0^2}}{\lambda} - \frac{W_0}{\lambda},$$

with $W = W^i \frac{\partial}{\partial x^i}$ the wind velocity. Here, $|y|^2 = h(y, y)$, $\lambda = 1 - |W|^2$, $W_0 = h(W, y)$.

For two definite positive functions $f, g : \mathbb{R} \to \mathbb{R}^+$, surfaces (2.1) and (2.2) represent revolution surfaces.

For surface (2.1), after computations, we get the following coefficients for the first fundamental form: $E = f'^2 + g'^2$, $F = 0$, $G = f^2$. From this, we get

$$ds^2 = (f'^2 + g'^2)du^2 + f^2dv^2.$$

The coefficients of the second fundamental form, L, M, N, are

$$L = \frac{f'g'' - f''g'}{\sqrt{f'^2 + g'^2}}, M = 0, N = \frac{g'f}{\sqrt{f'^2 + g'^2}}$$

Finally, the Gauss curvature of this surface can be obtained as follows:

$$K = \frac{LN - M^2}{EG - F^2} = \frac{g'}{f'}(f'g'' - f''g').$$

The Christoffel symbols for this surface are

$$\Gamma_{11}^{1} = \frac{f'f'' + g'g''}{f'^2 + g'^2},$$

$$\Gamma_{12}^{1} = 0 = \Gamma_{11}^{2} = \Gamma_{22}^{2},$$

$$\Gamma_{12}^{2} = \frac{f'}{f},$$

$$\Gamma_{22}^{1} = \frac{ff'}{f'^2 + g'^2}.$$

Remark 2.1. *The geodesic equations for this surface are given by*

$$\begin{cases} \frac{d^2u}{ds^2} + \frac{f'f''+g'g''}{f'^2+g'^2}\left(\frac{du}{ds}\right)^2 - \frac{ff'}{f'^2+g'^2}\left(\frac{dv}{ds}\right)^2 = 0, \\ \frac{d^2v}{ds^2} + 2\frac{f'}{f}\frac{du}{ds}\frac{dv}{ds} = 0. \end{cases}$$

Next, inspired by [10], we will construct a rotational Randers metric on M_1, choosing $W = \mu\frac{\partial}{\partial v}$ which is in the h-orthogonal coordinate system $\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right)$ on T_xM_1. Also, we have $W = \left(W^1, W^2\right) = (0, \mu)$, $\lambda = 1 - h(W, W) = 1 - \mu f^2$. Therefore, we can obtain

$$h(W, W) = h\left(\mu\frac{\partial}{\partial v}, \mu\frac{\partial}{\partial v}\right) = \mu^2\left(\frac{\partial}{\partial v}, \frac{\partial}{\partial v}\right) = \mu^2 f^2 < 1.$$

The navigation data (h, W) induce the following:

$$a_{ij} = \frac{\lambda h_{ij} + W_iW_j}{\lambda^2}, b_i = -\frac{W_i}{\lambda},$$

$$W_1 = h_{11}W^1 + h_{12}W^2 = 0,$$

$$W_2 = h_{21}W^1 + h_{22}W^2 = \mu f^2,$$

with $h_{11} = f'^2 + g'^2$ and $h_{22} = f^2$. Next, one obtains the following:

Lemma 2.1. *The Riemannian metric (a_{ij}) and the functions (b_i) obtained through Zermelo's navigation problem from h and W, for the surface* (2.1)*, are given by*

$$a_{ij} = \begin{pmatrix} \frac{f'^2+g'^2}{1-f^2\mu^2} & 0 \\ 0 & \frac{f^2}{(1-f^2\mu^2)^2} \end{pmatrix}, b_i = \begin{pmatrix} 0 \\ \frac{-\mu f^2}{1-\mu^2} \end{pmatrix}. \tag{2.3}$$

We remark that $\alpha(b, b) = h(W, W) < 1$.

Remark 2.2. *We can observe that in a particular case, for $g(u) = u$, one can obtain the same metric as in Lemma 3.1 from paper [10].*

Let us now choose another coordinates system: $\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right)$ on T_xM_1. Next, $W = \mu\frac{\partial}{\partial u} + \epsilon\frac{\partial}{\partial v} \Rightarrow W = (W^1, W^2) = (\mu, \epsilon)$.

$$\begin{aligned}\lambda &= 1 - h(W, W) = 1 - h\left(\mu\frac{\partial}{\partial u} + \epsilon\frac{\partial}{\partial v}, \mu\frac{\partial}{\partial u} + \epsilon\frac{\partial}{\partial v}\right)\\ &= 1 - \mu^2 h\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial u}\right) - \epsilon^2 h\left(\frac{\partial}{\partial v}, \frac{\partial}{\partial v}\right) = 1 - \mu^2(f'^2 + g'^2) - \epsilon^2 f^2.\end{aligned}$$

Therefore, we get

$$h(W, W) = \mu^2(f'^2 + g'^2) + \epsilon^2 f^2 < 1.$$

Let f and g are bounded positive real-valued functions defined as $f, g : \mathbb{R} \to \mathbb{R}^+$, then there exist positive values $\mu > 0$ and $\epsilon > 0$ such that $f(z) < \frac{1}{\epsilon}$ and $\sqrt{f'(z)^2 + g'(z)^2} < \frac{1}{\mu}$. Using the previous statement we can formulate the following

Lemma 2.2. *The Riemannian metric (a_{ij}) and the functions (b_i) obtained through Zermelo's navigation problem from h and W, for the surface* (2.1)*, are given by*

$$a_{ij} = \begin{pmatrix} \frac{f'^2+g'^2}{1-f^2\epsilon^2-\mu^2(f'^2+g'^2)} & 0\\ 0 & \frac{f^2}{1-f^2\epsilon^2-\mu^2(f'^2+g'^2)}\end{pmatrix}, b_i = \begin{pmatrix} 0\\ \frac{-f^2\epsilon^2-\mu^2(f'^2+g'^2)}{1-f^2\epsilon^2-\mu^2(f'^2+g'^2)}\end{pmatrix}. \tag{2.4}$$

We remark that $\alpha(b, b) = h(W, W) < 1$.

Now, let us construct the rotational Randers metric on surface M_2 given in (2.2), choosing $W = \mu\frac{\partial}{\partial v}$ which is in the h-orthogonal coordinate system $\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right)$ on T_xM_1. Also, we have $W = \left(W^1, W^2\right) = (0, \mu)$, $\lambda = 1 - h(W, W) = 1 - \mu f^2$.

$$h(W, W) = h\left(\mu\frac{\partial}{\partial v}, \mu\frac{\partial}{\partial v}\right) = \mu^2 h\left(\frac{\partial}{\partial v}, \frac{\partial}{\partial v}\right) = \mu^2 f^2 < 1.$$

Let us recall that the surface M_2 is given by

$$M_2 = \{(f(u)\cosh v, f(u)\sinh v, g(u)), u \in I, v \in [0, 2\pi]\}.$$

This means that for the first fundamental form, we get the following results:

$$E = f'^2\cosh(2v) + g'^2, F = ff'\sinh(2v), G = f^2(1 + 2\sinh(v)^2) = f^2\cosh(2v).$$

Thus, we get

$$ds^2 = (f'^2\cosh(2v) + g'^2)du^2 + 2ff'\sinh(2v)dudv + f^2\cosh(2v)dv^2.$$

The coefficients of the second fundamental form, L, M, N, are

$$L = \frac{f'g'' - f''g'}{\sqrt{f'^2 + g'^2(\cosh(2v))^2}}, M = 0, N = \frac{g'f}{\sqrt{f'^2 + g'^2(\cosh(2v))^2}}.$$

Finally, the Gauss curvature of this surface can be obtained as follows:

$$K = \frac{LN - M^2}{EG - F^2} = \frac{g'(f'g'' - f''g')}{f'(f'^2 + g'^2(\cosh(2v))^2)}.$$

Using [10], following the same approach, we will construct a rotational Randers metric on M_2, (2.2), choosing $W = \mu\frac{\partial}{\partial v}$, which is in the h-orthogonal coordinates system $\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right)$ on T_xM_1. Also, we have $W = \left(W^1, W^2\right) = (0, \mu)$, $\lambda = 1 - h(W, W) = 1 - \mu f^2$.

$$h(W, W) = h\left(\mu\frac{\partial}{\partial v}, \mu\frac{\partial}{\partial v}\right) = \mu^2 h\left(\frac{\partial}{\partial v}, \frac{\partial}{\partial v}\right) = \mu^2 f^2 < 1,$$

for positive function $f : \mathbb{R} \to \mathbb{R}^+$, $f(z) < \frac{1}{\mu}$. Now, we can give the following lemma:

Lemma 2.3. *The Riemannian metric* (a_{ij}) *and the functions* (b_i)*, obtained through Zermelo's navigation problem from h and W, for surface* (2.2)*, are the following ones:*

$$a_{ij} = \begin{pmatrix} \frac{f'^2\cosh(2v)+g'^2}{1-\mu^2 f^2} & \frac{2ff'\sinh(2v)}{1-\mu^2 f^2} \\ \frac{2ff'\sinh(2v)}{1-\mu^2 f^2} & \frac{f^2}{(1-\mu^2 f^2)^2} \end{pmatrix}, b_i = \begin{pmatrix} 0 \\ \frac{-\mu f^2}{1-\mu^2 f^2} \end{pmatrix}. \tag{2.5}$$

Now, using Lemma 2.1, we will give the main result theorem for this paper.

Theorem 2.1. *The Zermelo data for a stationary space-time with respect to metric (2.3) take the following form:*

$$ds^2 = \frac{V^2}{1 - (f'^2 + g'^2)\mu - \epsilon f^2}\left[-dt^2 + (f'^2 + g'^2)(dx^1 - \mu dt)^2 + f^2\left(dx^2 - \epsilon dt\right)^2\right].$$

Proof. Before we begin to prove this theorem, we can remark that the metric in square brackets has the Painlevé-Gullstrand form, and this can be defined up to a conformal factor due to the fact that Zermelo's problem is interested only in the null geodesic flow. From Lemma 2.1, we deduce the following:

$$h_{11} = f'^2 + g'^2, h_{22} = f^2, W_1 = 0, W_2 = \mu f^2,$$

$$W^1 = \mu, W^2 = \epsilon, g^{11} = \frac{\mu}{V^2 w_1}, g_{11} = \frac{V^2 w_1}{\mu},$$

$$g^{22} = \frac{\epsilon}{V^2 w_2}, g_{11} = \frac{V^2 w_2}{\epsilon}.$$

Using all these above relations and also the fact that the form of Randers metric in stationary space-time is given by

$$g_{\mu\nu}dx^\mu dx^\nu = -V^2\left(dt + w_i dx^i\right)^2 + \gamma_{ij}dx^i dx^j,$$

the following form for the Randers metric:

$$ds^2 = \frac{V^2}{1 - h_{ij}W^iW^j}\left[-dt^2 + h_{ij}\left(dx^i - W^i dt\right)\left(dx^j - W^j dt\right)\right].$$

Thus, we get

$$ds^2 = \frac{V^2}{1 - (f'^2 + g'^2)\mu - \epsilon f^2}\left[-dt^2 + (f'^2 + g'^2)\left(dx^1 - \mu dt\right)^2 + f^2\left(dx^2 - \epsilon dt\right)^2\right].$$

Then, the proof is done. □

Remark 2.3. *The metric in the square represents the Painlevé-Gullstrand form, and the physical-geometrical interpretation is that the Zermelo data for the metric (2.3) can be encoded in a stationary space-time just by writing this metric in these coordinates. In quantum physics, these kinds of coordinates can be used for various metrics to describe the process of gravitational collapse. In this respect, the link with the Zermelo navigation problem is not an easy but a very challenging problem and could be considered an open problem.*

We are ready now to give another main theorem for this paper but this time using the Frobenius norms.

Theorem 2.2. *The following inequality takes place for the metric (2.3)*

$$\frac{\sqrt{r}}{\sigma_{min}\left(a_{ij}\right)} \geq \frac{1}{\sqrt{2}} \sqrt{\frac{f'^2 + g'^2}{1 - f^2\mu^2} + \frac{f^2}{(1 - f^2\mu^2)^2}}.$$

Here, $\sigma_{min}(a_{ij})$ represents the minimum singular eigenvalue for the matrix (a_{ij}) and r represents the rank of the matrix.

Proof. Starting with the matrix (2.3),

$$a_{ij} = \begin{pmatrix} \frac{f'^2+g'^2}{1-f^2\mu^2} & 0 \\ 0 & \frac{f^2}{(1-f^2\mu^2)^2} \end{pmatrix},$$

we can easily compute the Frobenius (or Hilbert-Schmidt) norm for this matrix as follows:

$$\left\|a_{ij}\right\|_{HS} = \sqrt{\frac{(f'^2 + g'^2)^2}{(1 - f^2\mu^2)^2} + \frac{f^4}{(1 - f^2\mu^2)^4}}.$$

Using the well-known inequality

$$\sqrt{A^2 + B^2} \geq \frac{A + B}{\sqrt{2}},$$

we get

$$\sqrt{\left(\frac{(f'^2 + g'^2)}{(1 - f^2\mu^2)}\right)^2 + \left(\frac{f^2}{(1 - f^2\mu^2)^2}\right)^2} \geq \frac{1}{\sqrt{2}} \sqrt{\frac{f'^2 + g'^2}{1 - f^2\mu^2} + \frac{f^2}{(1 - f^2\mu^2)^2}},$$

and this concludes the proof of the theorem. □

Now, we will investigate some properties involving the Frobenius (Hilbert-Schmidt) norm for the above-constructed Riemannian metrics. As we know, the Frobenius norm is computed for the matrix $A = (a_{ij})$ as follows:

$$\|A\|_F = \sqrt{\sum_{i,j} \left|a_{ij}\right|^2}.$$

In [14], we have introduced the notion of H-distortion for pseudo-Riemannian manifolds. From a geometrical point of view, the H-distortion is important because it is directly linked with the classical distortion in Finsler's geometry. We already know the importance of distortion not just in Finsler geometry but also in physics because the distortion could be linked with Tchebychev's potential. In this respect, please see [15].

Definition 2.1. *[14] For a pseudo-Riemannian manifold, we will denote by*

$$\sigma_{F,\nabla F} = \frac{\sigma_F(x)}{\sigma_{\nabla F}(x)} = \sqrt{\frac{det(g_{ij})}{det(\nabla_g^2 f)}},$$

the H-distortion, if and only if $\sigma^2_{F,\nabla F}(x) = constant$.

Using the same approach as in [14], we will compute for the constructed Riemannian metrics (2.3), (2.4), and the H-distortion for the surface (2.1). As we well know, for this surface, we already obtained the Christoffel coefficients:

$$\Gamma^1_{11} = \frac{f'f'' + g'g''}{f'^2 + g'^2},$$

$$\Gamma^1_{12} = 0 = \Gamma^2_{11} = \Gamma^2_{22} = \Gamma^1_{21},$$

$$\Gamma^2_{12} = \frac{f'}{f} = \Gamma^2_{21},$$

$$\Gamma^1_{22} = \frac{ff'}{f'^2 + g'^2}.$$

For a function $h : M_1 \to \mathbb{R}$, we know from [14], the following relations:

$$h_{,11} = \frac{\partial^2 h}{\partial u^2} - \left(\Gamma^1_{11}\frac{\partial h}{\partial u} + \Gamma^2_{12}\frac{\partial h}{\partial v}\right),$$

$$h_{,12} = \frac{\partial^2 h}{\partial u \partial v} - \left(\Gamma^1_{12}\frac{\partial h}{\partial u} + \Gamma^2_{12}\frac{\partial h}{\partial v}\right),$$

$$h_{,21} = \frac{\partial^2 h}{\partial v \partial u} - \left(\Gamma^1_{21}\frac{\partial h}{\partial u} + \Gamma^2_{21}\frac{\partial h}{\partial v}\right),$$

$$h_{,22} = \frac{\partial^2 h}{\partial v^2} - \left(\Gamma^1_{21}\frac{\partial h}{\partial u} + \Gamma^2_{22}\frac{\partial h}{\partial v}\right).$$

With the above Christoffel coefficients obtained before, we get the Hessian matrix

$$\nabla^2 h = \begin{pmatrix} \frac{\partial^2 h}{\partial u^2} - \left(\frac{f'f''+g'g''}{f'^2+g'^2}\frac{\partial h}{\partial u} + \frac{f'}{f}\frac{\partial h}{\partial v}\right) & \frac{\partial^2 h}{\partial u \partial v} - \frac{f'}{f}\frac{\partial h}{\partial v} \\ \frac{\partial^2 h}{\partial u \partial v} - \frac{f'}{f}\frac{\partial h}{\partial v} & \frac{\partial^2 h}{\partial v^2} \end{pmatrix}.$$

Using this matrix we will try to find the Hessian matrix for the function $h(u, v) = u^2 + v^2$, which represents a paraboloid. One obtains: $\frac{\partial h}{\partial u} = 2u$, $\frac{\partial h}{\partial v} = 2v$, $\frac{\partial^2 h}{\partial u^2} = 2$, $\frac{\partial^2 h}{\partial v^2} = 2$. Finally, we get

$$\nabla^2 h = \begin{pmatrix} 2 - \frac{(f'f''+g'g'')2u}{f'^2+g'^2} + \frac{2vf'}{f} & -\frac{2vf'}{f} \\ -\frac{2vf'}{f} & 2 \end{pmatrix}.$$

From this, we get

$$det(\nabla^2 h(u, v)) = 4\left(1 - \frac{(f'f'' + g'g'')u}{f'^2 + g'^2}\right),$$

$$det(a_{ij}) = \frac{(f'^2 + g'^2)f^2}{(1 - \mu^2 f^2)^3}.$$

Finally, we get the H-distortion for the function $h(u, v)$, as follows:

$$\sigma_{F,\nabla F} = \frac{f(f'^2 + g'^2)}{2(1 - \mu^2 f^2)} \sqrt{\frac{1}{(1 - \mu^2 f^2)(f'^2 + g'^2 - (f' f'' + g' g'')u)}}.$$

Imposing the condition from Definition 2.1, we get the following two conditions to have H-distortion for the function $h(u, v)$:

$$\begin{cases} \frac{f(f'^2+g'^2)}{2(1-\mu^2 f^2)} = constant, \\ (1 - \mu^2 f^2)(f'^2 + g'^2 - (f' f'' + g' g'')u) = constant. \end{cases}$$

Using these equalities, we can construct easily two functions that respect these conditions. For example, $f(u) = c = constant$ and $g(u) = u$, are two functions that fulfill these conditions. The H-distortion is directly linked with the classical distortion in Finsler's geometry. For this reason, the geometrical significance is huge. The distortion in Finsler geometry is directly linked with the S-curvature because in Finsler geometry the S-curvature was introduced to measure the rate of distortion along geodesics:

$$S(y) = \frac{d}{dt} \left[\tau\left(\dot{\gamma}(t)\right)\right]_{t=0},$$

where $\gamma(t)$ represent the geodesic with $\dot{\gamma}(0) = y$.

3. Conclusions

As we have proved in this paper. we successfully obtained some important results regarding the Zermelo navigation problem for some special manifolds. We have also obtained the conditions for the H-distortion for this kind of manifold. The importance of H-distortion and the distortion in Finsler geometry is huge because is in direct link with the S-curvature. The S-curvature is in Finsler geometry an important and unique quantity and geometrically, the S-curvature represents the study of the rate of change of distortion along geodesics. It is known that S-curvature is a non-Riemannian quantity, and this means that every Riemannian manifold has a vanishing S-curvature. For this reason, the results obtained by us in this paper underline the role of these metrics that we found for the rotations surfaces considered. The H-distortion could be computed for all these metrics and also we can compute the S-curvature for all these metrics. This will be done in a future paper. Another important result established by us in this paper was to express one of the constructed metrics in Painlevé-Gullstrand form and we have explained the physical-geometrical interpretation. The Zermelo data for that metric can be encoded in a stationary space-time just by writing this metric in this Painlevé-Gullstrand coordinates. In quantum physics, these kinds of coordinates can be used to describe the formation of the gravitational collapse. Therefore, we recommend to the readers new problems in [15–23] and references therein. In this respect, the link with the Zermelo navigation problem is not an easy but a very challenging problem and could be considered an open problem.

Acknowledgments

The author (Ali. H. AlKhaldi) extends his appreciation to the Deanship of Scientific Research at King Khalid University for funding this work through the Research Groups Program under grant number R. G. P2/199/43. This research was funded by the National Natural Science Foundation of China (Grant No. 12101168) and the Zhejiang Provincial Natural Science Foundation of China (Grant No. LQ22A010014).

Conflict of interest

The authors declare no conflict of interest.

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