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Research article

Differences weighted composition operators acting between kind of weighted Bergman-type spaces and the Bers-type space -I-

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Abstract: Let $\mathcal{O}(\mathbb{D})$ denote the class of all analytic or holomorphic functions on the open unit disk $\mathbb{D}$ of $\mathbb{C}$. Let φ and ψ are an analytic self-maps of $\mathbb{D}$ and $u, v \in \mathcal{O}(\mathbb{D})$. The difference of two weighted composition operators is defined by

$$T_{\varphi,\psi} f(z) := \Big(W_{\varphi,\,u} f - W_{\psi,\,v} f\Big)(z) = u(z)(f\circ\varphi)(z) - v(z)(f\circ\psi)(z),\ \ f \in \mathcal{O}(\mathbb{D}) \text{ and } z \in \mathbb{D}.$$

The boundedness and compactness of the differences of two weighted composition operators from $\mathcal{H}_\alpha^\infty(\mathbb{D})$ spaces into $\mathcal{N}_K(\mathbb{D})$ spaces (resp. from $\mathcal{N}_K(\mathbb{D})$ into $\mathcal{H}_\alpha^\infty(\mathbb{D})$) are investigate in this paper.

Keywords: weighted composition operator; compact difference bounded operator; weighted analytic space
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1. Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} \;/\; |z| < 1\}$ be the unit disk in the complex space. $\mathcal{O}(\mathbb{D})$ denotes the space of functions that are holomorphic in $\mathbb{D}$ and $\mathcal{H}^\infty(\mathbb{D})$ denotes the Banach space of bounded holomorphic functions on $\mathbb{D}$ with the norm $\|f\|_\infty = \sup\limits_{z\in\mathbb{D}} |f(z)|$. For a holomorphic self-mapping φ of $\mathbb{D}$ ($\varphi(\mathbb{D}) \subset \mathbb{D}$) and a holomorphic function u: $\mathbb{D} \longrightarrow \mathbb{C}$, the pair (u, φ) induces the linear operator $W_{\varphi,\,u}$: $\mathcal{O}(\mathbb{D}) \longrightarrow \mathcal{O}(\mathbb{D})$ defined by

$$W_{\varphi,u}(f)(z) = u(z)(f\circ\varphi(z)),\ \ f \in \mathcal{O}(\mathbb{D}),\ \ z \in \mathbb{D}.$$

$W_{\varphi,\,u}$ which is called weighted composition operator with symbols u and φ. Observe that $W_{\varphi,\,u}(f) = M_u C_\varphi(f)$, where $M_u(f) = u.f$, is the multiplication operator with symbol u, and $C_\varphi(f) = f\circ\varphi$, is the composition operator with symbol φ.

If $u \equiv 1$, then $W_{\varphi,\,u} = C_\varphi$, and if φ is the identity ($\varphi(z) = z$), then $W_{\varphi,\,u} = M_u$.

During the past few decades, composition operators and weighted composition operators have been studied extensively on spaces of holomorphic functions on various domains in $\mathbb{C}$ or $\mathbb{C}^n$. We refer the readers to the monographs [1, 3, 5, 13, 18, 20, 23] for detailed information and the references therein.

For $a \in \mathbb{D}$ the Möbius transformation $\varphi_a(z)$ is defined by

$$\varphi_a(z) = \frac{a - z}{1 - \bar{a}z}, \text{ for } z \in \mathbb{D}.$$

For each $a \in \mathbb{D}$, the Green's function with logarithmic singularity at $a \in \mathbb{D}$ is denoted by

$$g(z, a) = \log\left(\frac{1}{|\varphi_a(z)|}\right).$$

The pseudohyperbolic distance ρ: $\mathbb{D} \times \mathbb{D} \longrightarrow [0, 1)$ is defined by

$$\rho(a, z) = |\varphi_a(z)| = \left|\frac{a - z}{1 - \bar{a}z}\right| \text{ for } a, z \in \mathbb{D}.$$

We will denote by

$$\rho(\varphi(z), \psi(z)) = \left|\frac{\varphi(z) - \psi(z)}{1 - \overline{\varphi(z)}\psi(z)}\right|.$$

It is easy to check that $\rho(a, z)$ satisfies the following inequalities:

$$\frac{1 - \rho(a, z)}{1 + \rho(a, z)} \leq \frac{1 - |z|^2}{1 - |a|^2} \leq \frac{1 + \rho(a, z)}{1 - \rho(a, z)}, \quad z, a \in \mathbb{D}.$$

For $0 < \alpha < \infty$, recall that an $f \in \mathcal{O}(\mathbb{D})$ is said to belong to the α-Bloch space $\mathcal{B}^\alpha$ if

$$\mathcal{B}_\alpha(f) = \sup_{z \in \mathbb{D}} ((1 - |z|^2)^\alpha |f'(z)|) < \infty.$$

With the norm $\|f\| = |f(0)| + \mathcal{B}_\alpha(f)$, $\mathcal{B}^\alpha$ is a Banach space. When $\alpha = 1$, $\mathcal{B}^1 = \mathcal{B}$ is the well-known Bloch space. For more information on Bloch spaces we refer the interested reader to [19]. Let $\mathcal{B}_0^\alpha$ be the space which consists of all $f \in \mathcal{B}$ satisfying

$$\lim_{|z| \longrightarrow 0} (1 - |z|^2)^\alpha |f'(z)| = 0.$$

This space is called the little Bloch space. See [19] for more information on Bloch spaces.

Let $\alpha \geq 0$. The Bers-type space, denoted by $\mathcal{H}_\alpha^\infty(\mathbb{D})$, is a Banach space defined by

$$\mathcal{H}_\alpha^\infty(\mathbb{D}) := \{ f \in \mathcal{O}(\mathbb{D}) \ / \sup_{z \in \mathbb{D}} ((1 - |z|^2)^\alpha |f(z)|) < \infty\},$$

$$\mathcal{H}_{(\alpha,\, 0)}^\infty(\mathbb{D}) := \{ f \in \mathcal{O}(\mathbb{D}) \ / \ \lim_{|z| \longrightarrow 1^-} ((1 - |z|^2)^\alpha |f(z)|) = 0 \}$$

equipped with the norm

$$\|f\|_{\mathcal{H}_\alpha^\infty(\mathbb{D})} := \sup_{z \in \mathbb{D}} ((1 - |z|^2)^\alpha |f(z)|) \quad \text{for } f \in \mathcal{H}_\alpha^\infty(\mathbb{D}).$$

Note that, $\mathcal{H}_\alpha^\infty(\mathbb{D})$ is a Banach space with the norm $\|\|_{\mathcal{H}_\alpha^\infty(\mathbb{D})}$.

When $\alpha = 0$, $\mathcal{H}_0^\infty(\mathbb{D})$ is just the bounded analytic function space $\mathcal{H}^\infty(\mathbb{D})$. For more information about several studied on Bers-type spaces we refer to [3, 20].

Let K: $[0, \infty) \longrightarrow (0, \infty)$ be right continuous and nondecreasing function. The authors Ahmed and Bakhit in [7] introduced the $\mathcal{N}_K(\mathbb{D})$ spaces as follows:

The analytic $\mathcal{N}_K(\mathbb{D})$-space is defined by

$$\mathcal{N}_K(\mathbb{D}) := \{ f \in O(\mathbb{D}) \ / \int_{\mathbb{D}} |f(z)|^2 K(g(z,a))dA(z) < \infty\},$$

$$\mathcal{N}_{(K,\,0)}(\mathbb{D}) := \{ f \in O(\mathbb{D}) \ / \ \lim_{|a| \longrightarrow 1^-} \int_{\mathbb{D}} |f(z)|^2 K(g(z,a))dA(z) = 0\}$$

equipped with the norm

$$||f||^2_{\mathcal{N}_K(\mathbb{D})} = \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f(z)|^2 K(g(z,a))dA(z), \ f \in \mathcal{N}_K(\mathbb{D}).$$

Remark 1.1. *We make the following observations:*
(1) *If* $K(t) = t^p$, *then* $\mathcal{N}_K(\mathbb{D}) = \mathcal{N}_p(\mathbb{D})$, *since* $g(z,a) \approx (1 - |\varphi_a|^2)$.

(2) *If* $K(t) \equiv 1$, *then* $\mathcal{N}_1(\mathbb{D}) = \mathcal{A}^2$ *(the Bergman space), where for* $0 < p < \infty$, *the Bergman space* $\mathcal{A}^p$ *is the set of analytic functions* f *in the unit disk* $\mathbb{D}$ *with*

$$||f||^p_{\mathcal{A}^p} = \frac{1}{p} \int_{\mathbb{D}} |f(z)|^p dA(z) < \infty.$$

Remark 1.2. *In the study of the space* $\mathcal{N}_K(\mathbb{D})$, *the authors in [7] assume that the following condition*

$$\sup_{0 \le t \le 1} \int_0^1 \frac{(1-t)^2}{(1-tr^2)^3} K(log(\frac{1}{r}))rdr < \infty \tag{1.1}$$

is satisfied, so that the $\mathcal{N}_K(\mathbb{D})$ *space is not trivial.*

Lemma 1.1. *([8, Lemma 2.2]) Assume that the function* K *satisfies* (1.1)*. For each* $w \in \mathbb{D}$, *let*

$$h_w(z) = \frac{1 - |w|^2}{(1 - \overline{w}z)^2},$$

for $z \in \mathbb{D}$. *Then* h_ω *satisfies the following conditions:*
(i) $h_w \in \mathcal{N}_K(\mathbb{D})$.
(ii) $||h_w||_{\mathcal{N}_K(\mathbb{D})} \lesssim 1$.
(iii) $\sup_{\omega \in \mathbb{D}} ||h_\omega||_{\mathcal{N}_K(\mathbb{D})} \le 1$.

Several important properties of the $\mathcal{N}_K(\mathbb{D})$-spaces and $H_\alpha^\infty(\mathbb{D})$ spaces and also of weighted composition operators from $\mathcal{N}_K(\mathbb{D})$-spaces to the spaces $H_\alpha^\infty(\mathbb{D})$ and from $H_\alpha^\infty(\mathbb{D})$-spaces to $\mathcal{N}_K(\mathbb{D})$ have been characterized in [7, 8, 15].

We cite here main results from [15] for the readers' convenience.

Theorem 1.1. *([15,22]) Let K:* $[0,\infty)\to[0,\infty)$*, be a nondecreasing function and* φ *be a holomorphic self-map of* $\mathbb{D}$*. For* $\alpha\in(0,\infty)$ *and* $u\in\mathcal{O}(\mathbb{D})$*. The weighted composition operator*

$$W_{\varphi,\,u} := uC_\varphi : \mathcal{N}_K(\mathbb{D}) \longrightarrow \mathcal{H}^\infty_\alpha(\mathbb{D})$$

(1) *is bounded if and only if*

$$\sup_{z\in\mathbb{D}}\left(\frac{|u(z)|(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)}\right)<\infty, \tag{1.2}$$

(2) *is compact if and only if*

$$\lim_{r\longrightarrow 1}\sup_{|\varphi(z)|>r}\left(\frac{|u(z)|(1-|z|^2)^\alpha}{1-|\varphi(z)|^2}\right)=0. \tag{1.3}$$

Remark 1.3. *When* $K(t)=t^p$*, Theorem 1.1 coincides with [22, Thoerem 3, Corollary 2].*

Theorem 1.2. *([15,22]) Let K:* $[0,\infty)\to[0,\infty)$*, be a nondecreasing function and* φ *be a holomorphic self-map of* $\mathbb{D}$*. For* $\alpha\in(0,\infty)$ *and* $u\in\mathcal{O}(\mathbb{D})$*. Then the following properties hold:*

(1) *The weighted composition operator* $W_{\varphi,\,u}=uC_\varphi$*:* $\mathcal{H}^\infty_\alpha(\mathbb{D})\longrightarrow\mathcal{N}_K(\mathbb{D})$ *is bounded.*

(2) *u and* φ *satisfy*

$$\sup_{z\in\mathbb{D}}\int_{\mathbb{D}}\frac{|u(z)|^2}{(1-|\varphi(z)|^2)^{2\alpha}}K(g(z,a))dA(z)<\infty. \tag{1.4}$$

(3) *u and* φ *satisfy*

$$\sup_{I\subset\partial\mathbb{D}}\int_{S(I)}\frac{|u(z)|^2}{(1-|\varphi(z)|^2)^{2\alpha}}K(1-z)dA(z)<\infty. \tag{1.5}$$

Remark 1.4. *When* $K(t)=t^p$*, Theorem 1.2 coincides with [22, Theorem 1].*

Theorem 1.3. *([15,22]) K:* $[0,\infty)\to[0,\infty)$*, be a nondecreasing function and* φ *be a holomorphic self-map of* $\mathbb{D}$*. For* $\alpha\in(0,\infty)$ *and* $u\in\mathcal{O}(\mathbb{D})$*, then the following are equivalent:*

(i) $W_{\varphi,\,u}$: $\mathcal{H}^\infty_\alpha(\mathbb{D})\longrightarrow\mathcal{N}_K(\mathbb{D})$ is compact operator.

(ii) u and φ satisfy

$$\lim_{r\longrightarrow 1}\sup_{a\in\mathbb{D}}\int_{\mathbb{D}_r}\frac{u(z)|^2}{(1-|\varphi(z)|^2)^{2\alpha}}K(g(z,a))dA(z)=0.$$

(iii) u and φ satisfy

$$\lim_{r\longrightarrow 1}\sup_{I\subset\mathbb{D}}\int_{S(I)\cap\mathbb{D}_r}\frac{u(z)|^2}{(1-|\varphi(z)|^2)^{2\alpha}}K(1-|z|)dA(z)=0.$$

Remark 1.5. *When* $K(t)=t^p$*, Theorem 1.3 coincides with [22, Corollary 1].*

Lemma 1.2. *([7, Proposition 2.1]) For each right continuous and nondecreasing function K:* $[0,\infty)\to[0,\infty)$*, the following inclusion holds:*

$$\mathcal{N}_K(\mathbb{D})\subset\mathcal{H}^\infty_1(\mathbb{D}).$$

Our goal here is to investigate the boundedness and compactness of the difference of two weighted composition operators acting from $\mathcal{N}_K(\mathbb{D})$ -spaces to $\mathcal{H}^\infty_\alpha(\mathbb{D})$-spaces and form $\mathcal{H}^\infty_\alpha(\mathbb{D})$-spaces to $\mathcal{N}_K(\mathbb{D})$-spaces. To this end we introduce analytic maps φ,ψ: $\mathbb{D}\longrightarrow\mathbb{D}$ and u,v: $\mathbb{D}\longrightarrow\mathbb{C}$ and look at the operator

$$T_{\varphi,\psi} := W_{\varphi,\,u}-W_{\psi,\,v}=uC_\varphi-vC_\psi.$$

2. Main results

2.1. Differences of weighted composition operators from $\mathcal{N}_K(\mathbb{D})$ into $\mathcal{H}_\alpha^\infty(\mathbb{D})$

In this section we study the boundedness and compactness of two differences weighted composition operators

$$T_{\varphi,\,\psi} := W_{\varphi,\,u} - W_{\psi,\,v} : \mathcal{N}_K(\mathbb{D}) \longrightarrow \mathcal{H}_\alpha^\infty(\mathbb{D}).$$

In fact, the following results corresponds to the results obtained in [2,4,6,9–12,16,21].

We are now ready to prove a necessary condition and a sufficient condition for the boundedness of $T_{\varphi,\psi}$: $\mathcal{N}_K(\mathbb{D}) \longrightarrow \mathcal{H}_\alpha^\infty(\mathbb{D})$.

For that purpose, consider the following three conditions:

$$\sup_{z\in\mathbb{D}}\left(\frac{|u(z)|(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)}\right) < \infty, \tag{2.1}$$

$$\sup_{z\in\mathbb{D}}\left(\frac{|v(z)|(1-|z|^2)^\alpha}{(1-|\psi(z)|^2)}\right) < \infty, \tag{2.2}$$

$$\sup_{z\in\mathbb{D}}\left|\frac{|u(z)|(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)} - \frac{|v(z)|(1-|z|^2)^\alpha}{(1-|\psi(z)|^2)}\right| < \infty. \tag{2.3}$$

In order to prove the main results of this paper, the following auxiliary lemma is needed.

Lemma 2.1. *([17, Lemma 2.3]) For $f \in \mathcal{H}_\alpha^\infty(\mathbb{D})$ and $z, w \in \mathbb{D}$,*

$$\left|(1-|z|^2)^\alpha f(z) - (1-|w|^2)^\alpha f(w)\right| \lesssim \|f\|_{\mathcal{H}_\alpha^\infty(\mathbb{D})}\rho(z,w).$$

Theorem 2.1. *Let K: $[0,\ \infty) \longrightarrow [0,\ \infty)$ be a nondecreasing function, φ and ψ are holomorphic self-maps from $\mathbb{D}$ to $\mathbb{D}$. For $u, v \in \mathcal{O}(\mathbb{D})$ and $\alpha > 0$. Then the following statements are equivalent:*

(1) *$T_{\varphi,\psi}$: $\mathcal{N}_K(\mathbb{D}) \longrightarrow \mathcal{H}_\alpha^\infty(\mathbb{D})$ is bounded.*

(2) *φ, ψ and u, v satisfy the conditions* (2.1) *and* (2.3).

(3) *φ, ψ and u, v satisfy the conditions* (2.2) *and* (2.3).

Proof. (3) ⇒ (1). Assume that the functions φ, ψ and u, v satisfy the conditions (2.2) and (2.3), and we need to prove that $T_{\varphi,\psi}$ is bounded. In fact, let $f \in \mathcal{N}_K(\mathbb{D})$, then we have

$$\begin{aligned}
\left\|T_{\varphi,\psi}(f)\right\|_{\mathcal{H}_\alpha^\infty}(\mathbb{D}) &= \sup_{z\in\mathbb{D}}\left((1-|z|^2)^\alpha\left|T_{\varphi,\psi}f(z)\right|\right)\\
&= \sup_{z\in\mathbb{D}}\left((1-|z|^2)^\alpha\left|(uC_\varphi - vC_\psi)f(z)\right|\right)\\
&= \sup_{z\in\mathbb{D}}\left|u(z)(1-|z|^2)^\alpha f(\varphi(z)) - v(z)(1-|z|^2)^\alpha f(\psi(z))\right|\\
&= \sup_{z\in\mathbb{D}}\Bigg|(1-|\varphi(z)|^2)f(\varphi(z))\left[\frac{u(z)(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)} - \frac{v(z)(1-|z|^2)^\alpha}{(1-|\psi(z)|^2)}\right]\\
&\quad + \frac{v(z)(1-|z|^2)^\alpha}{(1-|\psi(z)|^2)}\left[(1-|\varphi(z)|^2)f(\varphi(z)) - (1-|\psi(z)|^2)f(\psi(z))\right]\Bigg|
\end{aligned}$$

$$
\begin{aligned}
&\leq \sup_{z\in\mathbb{D}}\Bigg\{\Big|(1-|\varphi(z)|^2)f(\varphi(z))\Big|\Big|\Big[\frac{u(z)(1-|z|^2)^{\alpha}}{(1-|\varphi(z)|^2)}-\frac{v(z)(1-|z|^2)^{\alpha}}{(1-|\psi(z)|^2)}\Big]\Big|\\
&\quad+\Big|\frac{v(z)(1-|z|^2)^{\alpha}}{(1-|\psi(z)|^2)}\Big|\Big|\Big[(1-|\varphi(z)|^2)f(\varphi(z))-(1-|\psi(z)|^2)f(\psi(z))\Big]\Big|\Bigg\}\\
&\leq \|f\|_{\mathcal{N}_K(\mathbb{D})}\sup_{z\in\mathbb{D}}\Big|\frac{|u(z)|(1-|z|^2)^{\alpha}}{(1-|\varphi(z)|^2)}-\frac{|v(z)|(1-|z|^2)^{\alpha}}{(1-|\psi(z)|^2)}\Big|\\
&\quad+\sup_{z\in\mathbb{D}}\Big(\frac{|v(z)|(1-|z|^2)^{\alpha}}{(1-|\psi(z)|^2))}\Big)2\|f\|_{\mathcal{N}_K(\mathbb{D})}.
\end{aligned}
$$

Taking in to account that $\mathcal{N}_K(\mathbb{D}) \subset \mathcal{H}_1^\infty(\mathbb{D})$ ([7, Proposition 2.1]), it follows from conditions (2.2) and (2.3) that

$$\left\|T_{\varphi,\,\psi}(f)\right\|_{\mathcal{H}_\alpha^\infty(\mathbb{D})} \leq C\,\|f\|_{\mathcal{N}_K(\mathbb{D})} \quad \text{for all } \ f \in \mathcal{N}_K(\mathbb{D}),$$

where C is a positive constant. Therefore $T_{\varphi,\psi}$ is bounded form $\mathcal{N}_K(\mathbb{D})$ to $\mathcal{H}_\alpha^\infty(\mathbb{D})$ as required.

(2) ⇒ (3). Observe that

$$\Big(\frac{|v(z)|(1-|z|^2)^{\alpha}}{(1-|\psi(z)|^2)}\Big) \leq \Big(\frac{|u(z)|(1-|z|^2)^{\alpha}}{(1-|\varphi(z)|^2)}\Big)+\Big|\frac{|u(z)|(1-|z|^2)^{\alpha}}{(1-|\varphi(z)|^2)}-\frac{|v(z)|(1-|z|^2)^{\alpha}}{(1-|\psi(z)|^2)}\Big|,$$

which implies that (2.3) holds.

Finally we show the implication (1) ⇒ (2). Assume that $T_{\varphi,\,\psi}$ is bounded from $\mathcal{N}_K(\mathbb{D})$ to $\mathcal{H}_\alpha^\infty(\mathbb{D})$ and prove that (2.1) and (2.3) are hold. Since $T_{\varphi,\psi}$ is bounded, we have for all $f \in \mathcal{N}_K(\mathbb{D})$

$$\left\|T_{\varphi,\psi}(f)\right\|_{\mathcal{H}_\alpha^\infty(\mathbb{D})} \lesssim \|f\|_{\mathcal{N}_K(\mathbb{D})}\,.$$

For each $z \in \mathbb{D}$, set

$$h_\omega(z) = \frac{1-|\varphi(\omega)|^2}{(1-\overline{\varphi(w)}z)^2}$$

be the function test in Lemma 1.1.

By taking into account Lemma 1.1, we have $h_w \in \mathcal{N}_K$ and $\|h_w\|_{\mathcal{N}_K(\mathbb{D})} \lesssim 1$.

Fix $\omega \in \mathbb{D}$, and consider the function g_ω defined by

$$g_\omega(z) = \frac{1-|\varphi(\omega)|^2}{(1-\overline{\varphi(\omega)}z)^2} \times \frac{\varphi_{\psi(\omega)}(z)}{\varphi_{\psi(\omega)}(\varphi(\omega))},$$

for $z \in \mathbb{D}$. We have

$$\|g_\omega\|_{\mathcal{N}_K(\mathbb{D})} \leq C\|h_\omega\|_{\mathcal{N}_K(\mathbb{D})}.$$

Thus $g_\omega \in \mathcal{N}_K(\mathbb{D})$. Note that

$$g_\omega(\varphi(\omega)) = h_\omega(\varphi(\omega)) \text{ and } \ g_\omega(\psi(\omega)) = 0. \tag{2.4}$$

From the boundeness of

$$T_{\varphi,\,\psi} = W_{\varphi,\,u} - W_{\psi,\,v} : \mathcal{N}_K(\mathbb{D}) \longrightarrow \mathcal{H}_\alpha^\infty(\mathbb{D}),$$

it then follows that

$$\begin{aligned}\infty > \|T_{\varphi,\psi} g_\omega\|_{\mathcal{H}_\alpha^\infty(\mathbb{D})} &= \sup_{z\in\mathbb{D}}\Big((1-|z|^2)^\alpha\big|u(z)g_\omega(\varphi(z)-v(z)g_\omega(\psi(z))\big|\Big)\\ &\geq \Big((1-|\omega|^2)^\alpha\big|u(\omega)g_\omega(\varphi(\omega)-v(\omega)g_\omega(\psi(\omega))\big|\Big)\\ &\geq \frac{(1-|\omega|^2)^\alpha|u(\omega)|(1-|\varphi(\omega)|^2)}{(1-|\varphi(\omega)|^2)^2}\\ &\geq \frac{(1-|\omega|^2)^\alpha|u(\omega)|}{1-|\varphi(\omega)|^2}.\end{aligned}$$

Hence the condition (2.1) holds. On the other hand we have

$$\begin{aligned}\infty > \|T_{\varphi,\psi}(h_\omega)\|_{\mathcal{H}_\alpha^\infty(\mathbb{D})} &\geq (1-|w|^2)^\alpha\big|u(\omega)h_\omega(\varphi(\omega))-v(\omega)h_\omega(\psi(\omega))\big|\\ &\geq |A(\omega)+B(\omega)|,\end{aligned}$$

where

$$A(\omega)=\frac{(1-|\omega|^2)^\alpha u(\omega)}{(1-|\varphi(\omega)|^2)^2}-\frac{(1-|\omega|^2)^\alpha v(\omega)}{(1-|\psi(\omega)|^2)^2}$$

and

$$B(\omega)=\frac{(1-|\omega|^2)^\alpha u(\omega)}{(1-|\varphi(\omega)|^2)}\Big[(1-|w|^2)^\alpha u(\omega)h_\omega(\varphi(\omega))-(1-|w|^2)^\alpha v(\omega)h_\omega(\psi(\omega))\Big].$$

In view of Lemma 2.1 and the condition (2.1) we deduce that $|B(\omega)|<\infty$ for all $w\in\mathbb{D}$, which implies that $|A(\omega)|<\infty$ for all $w\in\mathbb{D}$. Thus, the condition (2.3) is proved. □

Remark 2.1. *the statement* (1) *of Theorem 1.1 follows easily for the simple case* $v\equiv 0$ *of Theorem 2.1.*

Corollary 2.1. *Let* K*:* $[0,\ \infty)\longrightarrow[0,\ \infty)$ *be a nondecreasing function,* φ *and* ψ *are holomorphic self-maps from* $\mathbb{D}$ *to* $\mathbb{D}$*. For* $u\in\mathcal{O}(\mathbb{D})$ *and* $\alpha>0$*, then,* $uC_\varphi-uC_\psi$*:* $\mathcal{N}_K(\mathbb{D})\longrightarrow\mathcal{H}_\alpha^\infty(\mathbb{D})$ *is bounded if and only if the following two conditions hold:*

$$\sup_{z\in\mathbb{D}}\Big(\frac{(1-|z|^2)^\alpha|u(z)|}{1-|\varphi(z)|^2}\Big)<\infty \tag{2.5}$$

and

$$\sup_{z\in\mathbb{D}}\Big(\frac{(1-|z|^2)^\alpha|u(z)|}{1-|\psi(z)|^2}\Big)<\infty. \tag{2.6}$$

Proof. Assume that $T_{\varphi,\psi}$ is bounded. Then by letting $v=u$ in Theorem 2.1 it follows that the conditions (2.5) and (2.6) hold.

Conversely, assume that the conditions (2.5) and (2.6) hold. To prove that $T_{\varphi,\psi}$ is bounded, it suffices in view of Theorem 2.1 to prove that

$$\sup_{z\in\mathbb{D}}\Big(\frac{(1-|z|^2)^\alpha|u(z)|}{1-|\varphi(z)|^2}-\frac{(1-|z|^2)^\alpha|u(z)|}{1-|\psi(z)|^2}\Big)<\infty.$$

We have

$$\begin{aligned}\left|\frac{(1-|z|^2)^\alpha |u(z)|}{1-|\varphi(z)|^2}\right. &- \left.\frac{(1-|z|^2)^\alpha |u(z)|}{1-|\psi(z)|^2}\right| \\ &= \frac{(1-|z|^2)^\alpha |u(z)|}{1-|\varphi(z)|^2}\left|1-\frac{(1-|\varphi(z)|^2)}{1-|\psi(z)|^2}\right| \\ &\leq \frac{(1-|z|^2)^\alpha |u(z)|}{1-|\varphi(z)|^2}\left|1-\frac{1+\rho(\varphi(z),\psi(z))}{1-\rho(\varphi(z),\psi(z))}\right| \\ &\leq \frac{(1-|z|^2)^\alpha |u(z)|}{1-|\varphi(z)|^2}\frac{2\rho(\varphi(z),\psi(z))}{1-\rho(\varphi(z),\psi(z))} < \infty.\end{aligned}$$

Using Theorem 2.1, we obtain the boundedness of $uC_\varphi - uC_\psi$: $\mathcal{N}_K(\mathbb{D}) \longrightarrow \mathcal{H}^\infty_\alpha(\mathbb{D})$. The proof of the corollary is complete. □

Remark 2.2. *There exist non-bounded weighted composition operators such that their difference is bounded.*

In the following example we give operators such that neither $W_{\varphi,\,u}$, $W_{\psi,\,v}$ and $T_{\varphi,\,\psi} = W_{\varphi,\,u} - W_{\psi,\,v}$ are bounded from $\mathcal{N}_K(\mathbb{D})$ to $\mathcal{H}^\infty_\alpha(\mathbb{D})$.

Example 2.1. *By choosing the maps u, v, φ and ψ as follows:*

$$u(z) = v(z) \equiv 1 \quad \textit{and} \quad \varphi(z) = \psi(z) = z, \quad 0 < \alpha < \frac{1}{2}.$$

A direct calculation shows

$$\sup_{z\in\mathbb{D}}\left(\frac{|u(z)|(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)}\right) = \sup_{z\in\mathbb{D}}\left(\frac{|v(z)|(1-|z|^2)^\alpha}{(1-|\psi(z)|^2)}\right) = \infty.$$

In view of Theorem 2.1, it follows that neither $W_{\varphi,\,u}$: $\mathcal{N}_K(\mathbb{D}) \longrightarrow \mathcal{H}^\infty_\alpha(\mathbb{D})$ nor $W_{\psi,\,v}$: $\mathcal{N}_K(\mathbb{D}) \longrightarrow H^\infty_\alpha(\mathbb{D})$ is bounded. However from condition (2.1) *or* (2.2) *it is clear that the difference operator $W_{\varphi,u} - W_{\psi,\,v}$: $\mathcal{N}_K(\mathbb{D}) \longrightarrow \mathcal{H}^\infty_\alpha(\mathbb{D})$ is not bounded.*

The following theorem characterize when the difference weighted composition operators $T_{\varphi,\,\psi}$ acting between weighted analytic type spaces $\mathcal{N}_K(\mathbb{D})$ and $\mathcal{H}^\infty_\alpha(\mathbb{D})$ are compact.

Theorem 2.2. *Let φ, ψ: $\mathbb{D} \longrightarrow \mathbb{D}$ be two holomorphic functions,u, v: $\mathbb{D} \longrightarrow \mathbb{C}$ two holomorphic functions. Let further $W_{\varphi,u}$ and $W_{\psi,v}$ be two weighted composition operators acting from $\mathcal{N}_K(\mathbb{D})$ into $H^\infty_{\alpha,}(\mathbb{D})$. Then the operators $T_{\varphi,\psi} = W_{\varphi,u} - W_{\psi,v}$ is compact if and only if the following conditions hold.*

$$\lim_{r\longrightarrow 1^-}\sup_{|\varphi(z)|>r}\left(\frac{|u(z)|(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)}\right) = 0, \tag{2.7}$$

$$\lim_{r\longrightarrow 1^-}\sup_{|\psi(z)|>r}\left(\frac{|v(z)|(1-|z|^2)^\alpha}{(1-|\psi(z)|^2)}\right) = 0, \tag{2.8}$$

$$\lim_{r\longrightarrow 1^-}\sup_{\min\{|\varphi(z)|,\,|\psi(z)|\,\}>r}(\Lambda(z)) = 0, \tag{2.9}$$

where

$$\Lambda(z) = |u(z) - v(z)| \min\left[\frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2}, \frac{(1-|z|^2)^\alpha}{(1-|\psi(z)|^2)}\right].$$

Proof. We omit the proof, since the techniques are similar to those of [14, Theorem 2.4]. □

Remark 2.3. *The statement* (2) *of Theorem 1.1 follows easily for the simple case* $v \equiv 0$ *of Theorem 2.2.*

Corollary 2.2. *Let* K*:* $[0, \infty) \longrightarrow [0, \infty)$ *be a nondecreasing function,* φ *and* ψ *are holomorphic self-maps from* $\mathbb{D}$ *to* $\mathbb{D}$*. For* $u \in O(\mathbb{D})$ *and* $\alpha > 0$*, then,* $uC_\varphi - uC_\psi$*:* $\mathcal{N}_K(\mathbb{D}) \longrightarrow \mathcal{H}_\alpha^\infty(\mathbb{D})$ *is compact if and only if the following two conditions hold:*

$$\lim_{r\longrightarrow 1^-} \sup_{|\varphi(z)|>r} \left(\frac{(1-|z|^2)^\alpha |u(z)|}{1-|\varphi(z)|^2}\right) = 0 \tag{2.10}$$

and

$$\lim_{r\longrightarrow 1^-} \sup_{|\psi(z)|>r} \left(\frac{(1-|z|^2)^\alpha |u(z)|}{1-|\psi(z)|^2}\right) = 0. \tag{2.11}$$

Proof. Assume that $T_{\varphi,\psi}$ is compact. Then by letting $v = u$ in Theorem 2.2 it follows that the conditions (2.10) and (2.11) hold.

Conversely, assume that the conditions (2.10) and (2.11) hold. To prove that $T_{\varphi,\psi}$ is compact, it suffices in view of Theorem 2.2 to prove that the condition (2.9) is holds. Since $u \equiv v$, then

$$\lim_{r\longrightarrow 1^-} \sup_{\min\{|\varphi(z)|,\ |\psi(z)|\ \}>r} (\Lambda(z)) = 0.$$

Using Theorem 2.2, we obtain the compactness of $uC_\varphi - uC_\psi$: $\mathcal{N}_K(\mathbb{D}) \longrightarrow \mathcal{H}_\alpha^\infty(\mathbb{D})$. The proof of the corollary is complete. □

2.2. Differences of weighted composition operators from $\mathcal{H}_\alpha^\infty(\mathbb{D})$ *into* $\mathcal{N}_K(\mathbb{D})$

In this section, we investigate the boundedness of differences weighted composition operators

$$T_{\varphi,\psi} := W_{\varphi,\,u} - W_{\psi,\,v} : \mathcal{H}_\alpha^\infty(\mathbb{D}) \longrightarrow \mathcal{N}_K(\mathbb{D}).$$

Theorem 2.3. *Let* K*:* $[0, \infty) \longrightarrow [0, \infty)$ *be a nondecreasing function,* φ *and* ψ *are holomorphic self-maps from* $\mathbb{D}$ *to* $\mathbb{D}$*. For* $u, v \in O(\mathbb{D})$ *and* $\alpha > 0$*. Then the operator* $T_{\varphi,\psi}$*:* $H_\alpha^\infty(\mathbb{D}) \longrightarrow \mathcal{N}_K(\mathbb{D})$ *is bounded if the following condition is satisfies* $\max(I,\ J) < \infty$*, where*

$$I = \sup_{a\in\mathbb{D}} \left(\int_{\mathbb{D}} \frac{|u(z)|^2}{(1-|\varphi(z)|^2)^{2\alpha}} K(g(z,a)) A(z) \right)$$

and

$$J = \sup_{a\in\mathbb{D}} \left(\int_{\mathbb{D}} \frac{|v(z)|^2}{(1-|\psi(z)|^2)^{2\alpha}} K(g(z,a)) dA(z) \right).$$

Proof. Assume that the condition in the statement (2) is holds and let $f \in H_\alpha^\infty(\mathbb{D})$. We have

$$\begin{aligned} \left\|T_{\varphi,\psi}(f)\right\|_{\mathcal{N}_K(\mathbb{D})} &= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left|T_{\varphi,\psi}(f)(z)\right|^2 K(g(z,a))dz \\ &= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left|uT_\varphi(f)(z) - vC_\psi f(z)\right|^2 K(g(z,a))dz \end{aligned}$$

$$
\begin{aligned}
&= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |u(z)f(\varphi(z)) - v(z)f(\psi(z))|^2\, K(g(z,a))dz \\
&\le \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \Big(|u(z)f(\varphi(z))| + |v(z)f(\psi(z))|\Big)^2 K(g(z,a))dz \\
&\le 2 \int_{\mathbb{D}} \Big(|u(z)f(\varphi(z))|^2 + |v(z)f(\psi(z))|^2\Big)K(g(z,a))dA(z) \\
&= 2 \sup \int_{\mathbb{D}} |u(z)f(\varphi(z))|^2 K(g(z,a))dz + 2 \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |v(z)f(\psi(z))|^2 K(g(z,a))dA(z) \\
&= 2 \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{|u(z)|^2}{(1-|\varphi(z)|^2)^{2\alpha}} (1-|\varphi(z)|^2)^{2\alpha} |f(\varphi(z))|^2 |K(g(z,a))dA(z) \\
&\quad +2 \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{|v(z)|^2}{(1-|\psi(z)|^2)^{2\alpha}} (1-|\psi(z)|^2)^{2\alpha} |f(\psi(z))|^2 K(g(z,a))dz \\
&\le 2\,\|f\|_{H^\infty_\alpha(\mathbb{D})} \Big(\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{|u(z)|^2}{(1-|\varphi(z)|^2)^{2\alpha}} K(g(z,a))dz \Big) \\
&\quad +2\,\|f\|_{H^\infty_\alpha(\mathbb{D})} \Big(\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{|v(z)|^2}{(1-|\psi(z)|^2)^{2\alpha}} K(g(z,a))dA(z) \Big) \\
&\le 2\,\|f\|_{H^\infty_{\alpha,\,\omega}(\mathbb{D})}\, I + 2\,\|f\|_{H^\infty_\alpha(\mathbb{D})}\, J \\
&\le 2\,\|f\|_{H^\infty_\alpha(\mathbb{D})}\, (I + J) \\
&\le C\,\|f\|_{H^\infty_\alpha(\mathbb{D})}\,,
\end{aligned}
$$

which shows that $T_{\varphi,\,\psi}$ is bounded form $H^\infty_\alpha(\mathbb{D})$ to $\mathcal{N}_K(\mathbb{D})$. □

Finally, it seems to be natural to enquire a necessary and sufficient conditions for the boundedness and compactness of difference weighted composition operator

$$T_{\varphi,\,\psi} : H^\infty_\alpha(\mathbb{D}) \longrightarrow \mathcal{N}_K(\mathbb{D}).$$

So it is left as an open question.

3. Conclusions

Firstly, the boundedness and compactness of two differences weighted composition operators

$$T_{\varphi,\,\psi} := W_{\varphi,\,u} - W_{\psi,\,v} : \mathcal{N}_K(\mathbb{D}) \longrightarrow \mathcal{H}^\infty_\alpha(\mathbb{D})$$

are obtained. Secondly, we have investigated the boundedness of differences weighted composition operators

$$T_{\varphi,\psi} := W_{\varphi,\,u} - W_{\psi,\,v} : \mathcal{H}^\infty_\alpha(\mathbb{D}) \longrightarrow \mathcal{N}_K(\mathbb{D}).$$

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Conflict of interest

The author declares that there is no conflict of interest.

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