Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(7): 16175–16190.
DOI: 10.3934/math.2023827
Received: 09 March 2023
Revised: 20 April 2023
Accepted: 24 April 2023
Published: 06 May 2023

Research article

Consensus of double integrator multiagent systems under nonuniform sampling and changing topology

Ufuk Sevim*and Leyla Goren-Sumer

Department of Control and Automation Engineering, Istanbul Technical University, Ayazaga, Istanbul, Turkey

* **Correspondence:** Email: ufuk.sevim@itu.edu.tr.

Abstract: This article considers a consensus problem of multiagent systems with double integrator dynamics under nonuniform sampling. In the considered problem, the maximum sampling time can be selected arbitrarily. Moreover, the communication graph can change to any possible topology as long as its associated graph Laplacian has eigenvalues in an arbitrarily selected region. Existence of a controller that ensures consensus in this setting is shown when the changing topology graphs are undirected and have a spanning tree. Also, explicit bounds for controller parameters are given. A sufficient condition is given to solve the consensus problem based on making the closed loop system matrix a contraction using a particular coordinate system for general linear dynamics. It is shown that the given condition immediately generalizes to changing topology in the case of undirected topology graphs. This condition is applied to double integrator dynamics to obtain explicit bounds on the controller.

Keywords: multiagent systems; consensus; nonuniform sampling; changing topology
Mathematics Subject Classification: 93D50, 93C57

1. Introduction

The consensus problem of multi agent systems has received great attention in the last decade [22], mostly due to the broad application areas, such as mobile robot coordination [4], sensor network time synchronization [5], frequency synchronization in a microgrid [10], mitigating cybersecurity attacks [17] and many more [13]. Although general agent dynamics has received attention recently [9, 27, 28], the study of double integrator agent dynamics is common [6, 11, 15, 16, 21, 24, 26], partly because of their applicability to a broad range of applications.

Many different aspects of the real world challenges are also studied extensively, such as consensus under switching topologies [28], consensus with communication delays [24], nonuniformly sampled-data consensus [26], asynchronous consensus [8], event-based consensus [29], partial state feedback consensus [2], saturated input consensus [1], and consensus under uncertainties and nonlinearities [18], to name a few.

The study of nonuniform sampling and changing topology is particularly related to the topic of this paper, where sampling intervals and the communication topology cannot be determined reliably due to real-world constraints such as energy saving requirements, unreliable communication links and roaming agents. While there are numerous studies that deal with either nonuniform sampling or changing topology, there are relatively few that address both simultaneously. In [26] it is shown that consensus can be reached if the sampling intervals are constrained in a region that is determined by the max in-degree of the communication graph. However, the feasible sampling interval region gets smaller significantly when the max in-degree gets larger. In [3] an average dwell time approach was used to ensure the leader-following consensus. However, the set of switchable communication graphs must be known a priori, which limits the usability of the method, especially when the number of agents gets larger. There are also some works that tackle these aspects simultaneously in an event-triggered consensus framework, such as [7, 25]. However, these methods rely on continuous monitoring of triggering conditions on the agents, which may not be suitable for some applications. Also, the time between event triggers tends to get smaller as the network gets larger.

This paper proposes a new approach to the double integrator consensus problem that can address the nonuniform sampling and changing topology constraints simultaneously. Assuming all changing topology graphs have a spanning tree and undirected at every sampling interval, this method has the following advantages: (1) Minimum and maximum sampling intervals can be selected arbitrarily, (2) it is not necessary to know the graphs of the switchable topologies beforehand as long as the Laplacian eigenvalues fall within a given interval, (3) the Laplacian eigenvalue interval can be selected arbitrarily, (4) switching between all possible topologies at any sampling time is feasible, (5) a state feedback controller exists for any given sampling intervals and Laplacian eigenvalue intervals, and (6) the controller gain inequalities are given explicitly. To the best of our knowledge, there is no study in the literature on double integrator consensus that achieves these results.

We use the equivalent problem of stabilizing a subsystem of the overall system to solve the consensus problem for general linear dynamics, which was proven in [9]. Therefore, existing stabilization results can be utilized to solve the consensus problem. In particular, we use a coordinate transformation that makes the overall system matrix a contraction for all possible sampling intervals. We also show that this approach immediately generalizes to changing topology in the case of undirected topology graphs. The obtained results for the general linear dynamics are then applied to the double integrator dynamics to find explicit bounds for the controller.

The rest of the paper is organized as follows. In section 2, we give necessary definitions and basic results that are used throughout the paper. In section 3, a sufficient condition is given to solve the consensus problem for general linear dynamics. In section 4, the given condition is applied to the double integrator dynamics to obtain explicit bounds for the controller parameters. In section 5, numerical examples are given to demonstrate the accuracy of the method. Finally, conclusions are given in section 6.

We use subscripts i, j for the indices and $k \in \mathbb{N}$ for the discrete time instance, where $\mathbb{N}$ represents the nonnegative integers. For a matrix $A \in \mathbb{C}^{n\times n}$, $\sigma(A)$ denotes the set of singular values. $\|A\| = \bar{\sigma}(A)$ denotes the maximum singular value of A. We say A is a contraction if $\bar{\sigma}(A) < 1$ and $|\cdot|$ denotes the Euclidean vector norm.

2. Preliminaries

2.1. Graph theory

The network topology of a multiagent system with N agents is represented by a graph which is defined as a pair $G = (V, E)$ where $V = \{v_1, \dots, v_N\}$ is the set of nodes, and $E \subseteq V \times V$ is the set of edges. The nodes of the graph correspond to the agents, and an edge $(v_i, v_j) \in E$ denotes a directed information flow from agent j to agent i. We assume that the topology graph of the multiagent system is simple, i.e., there are no self-loops, that is, $(v_i, v_i) \notin E, \forall i$, and there are no multiple edges between pairs of nodes.

The neighbor set of node i is the index set defined as $\mathcal{N}_i := \{j \mid (v_i, v_j) \in E\}$, that is, only the nodes with edges incoming to v_i are considered neighbors. A directed tree is a graph where every node has exactly one neighbor except the so-called root node, which has no neighbors. If a subset of the edges of a graph forms a directed tree, then the graph is said to have a spanning tree. If a graph has a spanning tree, then all of the nodes are reachable from the root by following the (directed) edges.

The weighted adjacency matrix $W = (w_{ij}) \in \mathbb{R}^{N\times N}$ represents the nonnegative weights associated with edges, where $w_{ij} > 0$ if $(v_i, v_j) \in E$ and $w_{ij} = 0$ if $(v_i, v_j) \notin E$. A graph is called undirected if $w_{ij} = w_{ji}, \forall i, j$, that is, $W = W^T$ is symmetric.

The weighted in-degree of node i is defined as $d_i := \sum_{j\in\mathcal{N}_i} w_{ij}$. The Laplacian matrix of a graph is defined as $L := D - W$ where $D := \mathrm{diag}\{d_i\}$. Obviously, $L\mathbf{1} = 0$ where $\mathbf{1}^T := \begin{bmatrix}1 & \dots & 1\end{bmatrix}$. Therefore, 0 is an eigenvalue of L. By the Gershgorin circle theorem, all eigenvalues of L lie in the disc $\{z \in \mathbb{C} \mid |z - d_{\max}| \le d_{\max}\}$ where $d_{\max} := \max_i\{d_i\}$. Therefore, $\mathrm{Re}(\lambda) \ge 0$ for all eigenvalues λ of L.

Theorem 1 (Lewis et al. [12])**.** *The zero eigenvalue of L is simple, i.e., it has algebraic multiplicity 1, if and only if graph G has a spanning tree.*

2.2. Stability of time-varying discrete systems

It is well-known that the stability conditions for nonuniformly sampled systems are not trivial [14]. Stability of nonuniformly sampled systems can be analyzed by investigating the time-varying discrete systems. Such an investigation can be found in [23]. A sufficient stability result is given in this subsection.

Definition 1. *The system*

$$x_{k+1} = F_k x_k \tag{2.1}$$

is uniformly exponentially stable if there exist $\gamma > 0$ *and* $0 \le \alpha < 1$ *such that for any* $r \in \mathbb{N}$ *and* $x_r \in \mathbb{R}^n$

$$|x_k| \le \gamma\alpha^{k-r} |x_r|, \quad \forall k \ge r.$$

Theorem 2. *If there exist an invertible* $T \in \mathbb{R}^{n\times n}$ *and a constant* $\alpha < 1$ *such that* $\left\|T^{-1}F_kT\right\| \le \alpha, \forall k \in \mathbb{N}$, *then the system* (2.1) *is uniformly exponentially stable.*

Proof. Select $\gamma := \|T\| \left\|T^{-1}\right\|$. Therefore,

$$\begin{aligned}|x_k| &\leq \|T\| \left\|T^{-1}F_{k-1}T\right\| \left\|T^{-1}F_{k-2}T\right\| \dots \left\|T^{-1}F_rT\right\| \left\|T^{-1}\right\| |x_r| \\ &\leq \gamma\alpha^{k-r} |x_r| .\end{aligned}$$

□

2.3. Stability of nonuniformly sampled systems

Consider the continuous time system

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{2.2}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, rank $B = m \leq n$, and (A, B) is stabilizable.

Define the sequence of sampling time instances $\{t_k\}_{k\in\mathbb{N}}$ where

$$0 = t_0 < t_1 < \cdots < t_k < \dots$$

with $\lim_{k\to\infty} t_k = \infty$. We assume that sampling intervals $h_k := t_{k+1}-t_k$ are bounded, i.e., $h_k \in [\underline{h}, \bar{h}], \forall k \in \mathbb{N}$, for some $\bar{h} > \underline{h} > 0$.

Let $u(t) := Kx(t_k), \forall t \in [t_k, t_{k+1})$ where $K \in \mathbb{R}^{m\times n}$ is the state feedback controller. Then, the closed loop system becomes

$$\dot{x}(t) = Ax(t) + BKx(t_k), \quad \forall t \in [t_k, t_{k+1}). \tag{2.3}$$

The following result is standard:

Lemma 1. *Let $x_0 := x(t_0)$. Then, the solutions of* (2.3) *and*

$$x_{k+1} = (F(h_k) + G(h_k)K)\, x_k \tag{2.4}$$

are the same at the sampling instances, i.e., $x_k = x(t_k), \forall k \in \mathbb{N}$, where

$$F(h) := e^{Ah} \text{ and } G(h) := \left(\int_0^h e^{A\tau} d\tau\right) B. \tag{2.5}$$

Proof. We prove with induction. $x(t_0) = x_0$ by assumption. Assume that $x(t_k) = x_k$, so

$$\begin{aligned}x(t_{k+1}) &= e^{A(t_{k+1}-t_k)}x(t_k) + \int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-\eta)}Bu(\eta)d\eta \\ &= e^{Ah_k}x(t_k) + \left(\int_0^{h_k} e^{A\tau}d\tau\right)Bu(t_k) \\ &= F(h_k)x_k + G(h_k)Kx_k \\ &= x_{k+1}.\end{aligned}$$

□

Theorem 3. *The closed loop system* (2.3) *is uniformly exponentially stable for an arbitrary selection of sampling intervals $h_k \in [\underline{h}, \bar{h}], \forall k \in \mathbb{N}$, if there exists an invertible $T \in \mathbb{R}^{n\times n}$ such that*

$$\bar{\sigma}\left(T^{-1}\left(F(h) + G(h)K\right)T\right) < 1, \quad \forall h \in [\underline{h}, \bar{h}]. \tag{2.6}$$

Proof. It is well-known that norm of a continuous matrix-valued function is also continuous. Due to the Extreme Value Theorem, if a real-valued function is continuous on a closed interval, it attains a maximum. Let

$$\alpha := \max_h \bar{\sigma}\left(T^{-1}\left(F(h) + G(h)K\right)T\right) < 1.$$

Then, the result follows from Theorem 2 and Lemma 1. □

Note that if T satisfies (2.6), then $\overline{T} := TV$ also satisfies (2.6) for any orthogonal matrix $V^TV = I$.

2.4. Bounds on singular values

A Gershgorin type bound is given in [20] along with some other simple estimates for singular values. Since we are only interested in the maximum singular value of a square matrix, a corollary is given as follows:

Corollary 1 (of Theorem 2 in [20]). *Let $A = (a_{ij}) \in \mathbb{C}^{n\times n}$. Then,*

$$\bar{\sigma}(A) \leq \max_i \{s_i\} \tag{2.7}$$

where $s_i := \max\{r_i, c_i\}$, $r_i := \sum_{j=1}^n |a_{ij}|$ and $c_i := \sum_{j=1}^n |a_{ji}|$.

Corollary 1 is a fairly standard result, and one can easily show this using other methods such as Hölder's inequality. We extend this result to block matrices using the proof idea in [20].

Lemma 2. *Let $A = (A_{ij}) \in \mathbb{C}^{nN\times nN}$ be a block matrix where each block is the same size, i.e., $A_{ij} \in \mathbb{C}^{n\times n}$. Then,*

$$\bar{\sigma}(A) \leq \max_i \{s_i\} \tag{2.8}$$

where $s_i := \max\{r_i, c_i\}$, $r_i := \sum_{j=1}^n \bar{\sigma}(A_{ij})$ and $c_i := \sum_{j=1}^n \bar{\sigma}(A_{ji})$.

Proof. Let σ be a singular value of A. Then, there exist nonzero vectors

$$x = \begin{bmatrix} x_1^T & \dots & x_N^T \end{bmatrix}^T \in \mathbb{C}^{nN} \text{ and } y = \begin{bmatrix} y_1^T & \dots & y_N^T \end{bmatrix}^T \in \mathbb{C}^{nN}$$

such that

$$\begin{bmatrix} A_{11} & \dots & A_{1N} \\ \vdots & \ddots & \vdots \\ A_{N1} & \dots & A_{NN} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} = \sigma \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix} \text{ and } \begin{bmatrix} A_{11}^* & \dots & A_{N1}^* \\ \vdots & \ddots & \vdots \\ A_{1N}^* & \dots & A_{NN}^* \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix} = \sigma \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}$$

where $x_i, y_i \in \mathbb{C}^n, i = 1, \dots, N$, and A^* is the conjugate transpose of A. Let $\alpha := \max\{|x_1|, \dots, |x_N|, |y_1|, \dots, |y_N|\}$ where $|\cdot|$ is the Euclidean norm. If $\alpha = |y_i|$ for some i, then

$$\sigma y_i = \sum_{j=1}^N A_{ij}x_j \text{ implies } \sigma \leq \sum_{j=1}^n \bar{\sigma}(A_{ij}) = r_i.$$

If $\alpha = |x_i|$ for some i, then

$$\sigma x_i = \sum_{j=1}^N A_{ji}^* y_j \text{ implies } \sigma \leq \sum_{j=1}^n \bar{\sigma}(A_{ji}) = c_i.$$

In any case $\sigma \leq \max\{r_i, c_i\} = s_i$. Since this holds for any singular value of A, we can conclude the proof. □

Lemma 3. *Let $A, B \in \mathbb{R}^{n\times n}$ and*

$$C := \begin{bmatrix} A & -B \\ B & A \end{bmatrix}.$$

Then, $\sigma(C) = \sigma(A - jB) \cup \sigma(A + jB)$ where $\sigma(\cdot)$ is the set of singular values, and $j = \sqrt{-1}$.

Proof. It is well-known that

$$\det(C) = \det(A - jB)\det(A + jB).$$

Noting that CC^T has the same form as C, we have

$$\begin{aligned} \det(CC^T - \lambda I) &= \det\left(\begin{bmatrix} A & -B \\ B & A \end{bmatrix}\begin{bmatrix} A^T & B^T \\ -B^T & A^T \end{bmatrix} - \lambda I\right) \\ &= \det\begin{bmatrix} AA^T + BB^T - \lambda I & AB^T - BA^T \\ BA^T - AB^T & AA^T + BB^T - \lambda I \end{bmatrix} \\ &= \det\left((AA^T + BB^T - \lambda I) - j(BA^T - AB^T)\right) \\ &\quad \det\left((AA^T + BB^T - \lambda I) + j(BA^T - AB^T)\right) \\ &= \det\left((A - jB)(A^T + jB^T) - \lambda I\right) \\ &\quad \det\left((A + jB)(A^T - jB^T) - \lambda I\right). \end{aligned}$$

The result follows from the definition of the singular values. □

3. Consensus of general linear dynamics

We first define the problem for fixed network topology, and then we show that our approach immediately generalizes to changing topology in the case of undirected topology graphs. Consider the N-agent system on the network topology graph G with identical dynamics

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t), \quad i = 1, 2, \ldots, N, \tag{3.1}$$

where $x_i(t) \in \mathbb{R}^n$ is the state of agent i, and $u_i(t) \in \mathbb{R}^m$ is its control input. The control input $u_i(t)$ is said to solve the consensus problem asymptotically if they drive all the states to the same values for any initial states, i.e.,

$$\lim_{t\to\infty}\left(x_i(t) - x_j(t)\right) = 0, \quad \forall i, j = 1, \ldots, N.$$

We select the control input as distributed local state feedback law, i.e., every agent only uses the information from its neighbors, under nonuniform sampling with controller matrix $K \in \mathbb{R}^{m\times n}$

$$u_i(t) = K\sum_{j\in\mathcal{N}_i} w_{ij}(x_j(t_k) - x_i(t_k)), \quad \forall t \in [t_k, t_{k+1}), \tag{3.2}$$

where w_{ij} are the elements of the weighted adjacency matrix of the topology graph.

The overall closed-loop graph dynamics can be written as [see 12]

$$\dot{x}(t) = (I_N \otimes A)x(t) - (L \otimes BK)x(t_k) \tag{3.3}$$

where $x := \begin{bmatrix} x_1^T & x_2^T & \dots & x_N^T \end{bmatrix}^T \in \mathbb{R}^{nN}$ is the overall state vector, L is the graph Laplacian matrix and $\otimes$ is the Kronecker product.

Let $z(t) := (M^{-1} \otimes I_n)x(t)$ where $M := \begin{bmatrix} \mathbf{1} & \overline{M} \end{bmatrix}$ so that

$$M^{-1}LM = \begin{bmatrix} 0 & \ell^T \\ 0 & \bar{L} \end{bmatrix}$$

where $\mathbf{1} := \begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix}^T$. So, the closed loop system dynamics can be separated as

$$\dot{z}_1(t) = Az_1(t) - (\ell^T \otimes BK)\xi(t_k) \tag{3.4}$$

$$\dot{\xi}(t) = (I_{N-1} \otimes A)\xi(t) - (\bar{L} \otimes BK)\xi(t_k) \tag{3.5}$$

where $z = \begin{bmatrix} z_1^T & \xi^T \end{bmatrix}^T$ and $\xi := \begin{bmatrix} z_2^T & \dots & z_N^T \end{bmatrix}^T$.

Lemma 4 (Gao et al. [9])**.** *Control law* (3.2) *solves the consensus problem asymptotically if and only if system* (3.5) *is asymptotically stable. In other words,*

$$\lim_{t\to\infty} \xi(t) = 0 \iff \lim_{t\to\infty}(x_i(t) - x_j(t)) = 0 \quad i, j = 1, 2, \dots, N.$$

Using Lemma 1, stability of (3.5) can be analyzed using its discretized model.

$$\begin{aligned}
\xi_{k+1} &= e^{(I_{N-1}\otimes A)h_k}\xi_k - \left(\int_0^{h_k} e^{(I_{N-1}\otimes A)\tau} d\tau\right)\left(\bar{L} \otimes BK\right)\xi_k \\
&= \left(I_{N-1} \otimes e^{Ah_k}\right)\xi_k - \left(I_{N-1} \otimes \left(\int_0^{h_k} e^{A\tau} d\tau\right)\right)\left(\bar{L} \otimes BK\right)\xi_k \\
&= \left(I_{N-1} \otimes e^{Ah_k} - \bar{L} \otimes \left(\int_0^{h_k} e^{A\tau} d\tau\right) BK\right)\xi_k \\
&= \left(I_{N-1} \otimes F(h_k) - \bar{L} \otimes G(h_k)K\right)\xi_k.
\end{aligned} \tag{3.6}$$

Using Lemma 4 and Theorem 3, we can conclude that if there exists an invertible $T \in \mathbb{R}^{n\times n}$ such that the transformed closed loop system matrix

$$\begin{aligned}
\hat{\Phi}(h) &:= \left(I_{N-1} \otimes T^{-1}\right)\left(I_{N-1} \otimes F(h) - \bar{L} \otimes G(h)K\right)(I_{N-1} \otimes T) \\
&= I_{N-1} \otimes T^{-1}F(h)T - \bar{L} \otimes T^{-1}G(h)KT \\
&= I_{N-1} \otimes \hat{F}(h) - \bar{L} \otimes \hat{G}(h)\hat{K}
\end{aligned}$$

is a contraction, i.e., $\bar{\sigma}(\hat{\Phi}(h)) < 1$, for all $h \in [\underline{h}, \bar{h}]$, then control law (3.2) solves the consensus problem asymptotically for arbitrary selection of sampling intervals, where

$$\hat{F}(h) := T^{-1}F(h)T, \;\; \hat{G}(h) := T^{-1}G(h) \;\text{ and }\; \hat{K} := KT.$$

Without losing generality, we can assume that $\bar{L}$ is in Jordan form whose off-diagonals are arbitrarily small. Therefore $\hat{\Phi}(h) = \text{diag}\{\hat{J}_i(h)\}$ is in the block diagonal form where each block is as follows:

$$\hat{J}_i(h) := \begin{bmatrix} \hat{S}(h, \lambda_i) & \delta\hat{G}(h)\hat{K} & 0 & \dots & 0 \\ 0 & \hat{S}(h, \lambda_i) & \delta\hat{G}(h)\hat{K} & \dots & 0 \\ 0 & 0 & \hat{S}(h, \lambda_i) & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \hat{S}(h, \lambda_i) \end{bmatrix}$$

where λ_i is a possibly complex eigenvalue of $\bar{L}$ (see Lemma 3 for real block diagonal case), $\hat{S}(h, \lambda_i) := \hat{F}(h) - \lambda_i \hat{G}(h)\hat{K}$ and $\delta > 0$ is arbitrarily small. Using Lemma 2, we can see that

$$\bar{\sigma}\left(\hat{\Phi}(h)\right) = \max_i \left\{\bar{\sigma}\left(\hat{J}_i(h)\right)\right\} \leq \max_i \left\{\bar{\sigma}\left(\hat{S}(h, \lambda_i)\right) + \delta\bar{\sigma}\left(\hat{G}(h)\hat{K}\right)\right\} < 1$$

is sufficient to solve the consensus problem. Since δ is arbitrarily small and does not depend on the selection of T or K, we can conclude that

$$\max_i \left\{\bar{\sigma}\left(\hat{S}(h, \lambda_i)\right)\right\} < 1 \tag{3.7}$$

is sufficient. We summarize all the discussions above with the following theorem:

Theorem 4. *Let (A, B) be stabilizable system dynamics of the agents, and the topology graph has a spanning tree. Then, control law* (3.2) *solves the consensus problem asymptotically if there exists an invertible matrix $T \in \mathbb{R}^{n\times n}$ such that*

$$\bar{\sigma}\left(T^{-1}S(h, \lambda_i)T\right) < 1, \quad \forall h \in [\underline{h}, \bar{h}], \quad \forall i \in \{2, \ldots, N\}, \tag{3.8}$$

where

$$S(h, \lambda) := e^{Ah} - \lambda\left(\int_0^h e^{A\tau} d\tau\right) BK, \tag{3.9}$$

and λ_i are the eigenvalues of the Laplacian matrix of the topology graph.

Consider the case of changing topology at each sampling instance with graph Laplacian matrices L_k. From the proof given in [9], it is easy to see that Lemma 4 still applies. In the case of undirected graphs, i.e., $L_k = L_k^T$, one can select M such that $\bar{L}_k = \bar{L}_k^T$ for all k. Indeed, this is true when $\overline{M}^T\overline{M} = I$ and $\overline{M}^T\mathbf{1} = 0$.

Any symmetric matrix is orthogonally diagonalizable, so there exists orthogonal matrices $U_k^{-1} = U_k^T$ such that $U_k^T\bar{L}_k U_k$ is diagonal. Also, multiplication by orthogonal matrices does not change the singular values of a matrix. Since $U_k \otimes I_n$ is also orthogonal, we can assume $\bar{L}_k$ is diagonal for all k. Therefore, we can reach the following result.

Theorem 5. *Let (A, B) be stabilizable system dynamics of the agents. Assume that the changing topology graphs are undirected and have a spanning tree at each nonuniform sampling instance. Then, control law* (3.2) *solves the consensus problem asymptotically if there exists an invertible matrix $T \in \mathbb{R}^{n\times n}$ such that*

$$\bar{\sigma}\left(T^{-1}S(h, \lambda)T\right) < 1, \quad \forall h \in [\underline{h}, \bar{h}], \quad \forall \lambda \in [\lambda_2, \lambda_N], \tag{3.10}$$

where $\lambda_2 = \inf_{k,i} \lambda_i(\bar{L}_k)$, and $\lambda_N = \sup_{k,i} \lambda_i(\bar{L}_k)$.

4. Consensus of double integrator dynamics

Although Theorems 4 and 5 are given for general system dynamics, it is hard to find T and K that satisfy it in general. However, we give explicit bounds on the controller parameters in the case of double integrator dynamics and undirected network topologies. The explicit bounds depend on the maximum sampling interval and minimum and maximum eigenvalues of all possible Laplacian matrices. These parameters can be selected arbitrarily depending on the application. Also, it is shown that a controller always exists for any selection of these parameters, as long as the network graph has a spanning tree, which is known to be a necessary condition for consensusability [19].

Consider double integrator agent dynamics, i.e.,

$$A := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ and } B := \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Then,

$$S(h, \lambda) = \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} \frac{h^2}{2} \\ h \end{bmatrix} K.$$

Also, let the nonzero eigenvalues of the Laplacian matrix of the changing topology graphs be in the interval $[\lambda_2, \lambda_N]$.

We propose the following similarity transformation matrix:

$$T := \begin{bmatrix} -\mu_1 & -\mu_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} \mu_2 - \mu_1 & -(\mu_2 + \mu_1) \\ 0 & 2 \end{bmatrix} \tag{4.1}$$

where $0 < \mu_1 < \mu_2$. Therefore,

$$\hat{S} := T^{-1} S(h, \lambda) T = \begin{bmatrix} 1 - \frac{h\lambda\gamma k_1}{2} & \frac{2h}{\mu_2 - \mu_1} - \frac{h\lambda\gamma k_2}{2} \\ -\frac{h\lambda k_1}{2} & 1 - \frac{h\lambda k_2}{2} \end{bmatrix}$$

where

$$\gamma := \frac{\mu_1 + \mu_2 + h}{\mu_2 - \mu_1} \text{ and } \hat{K} := KT = \begin{bmatrix} k_1 & k_2 \end{bmatrix}.$$

Using Corollary 1, we need

$$\max\{|s_{11}| + |s_{12}|, |s_{11}| + |s_{21}|, |s_{22}| + |s_{12}|, |s_{22}| + |s_{21}|\} < 1 \tag{4.2}$$

to solve the consensus problem where $\hat{S} = (s_{ij})$. From (4.2), we have 4 inequalities to be satisfied, since each of the sums must be smaller than 1.

First note that we have $h, \lambda > 0$, $\gamma > 1$ and $\mu_2 > \mu_1 > 0$. We also need $k_1, k_2 > 0$, as otherwise $s_{11}, s_{22} > 1$, so we cannot use the condition (4.2).

We also require $s_{11}, s_{22} > 0$, since otherwise the lower limit for k_1 and k_2 can be large for small values of h. So, immediately we have the following conditions to be satisfied:

$$s_{11} = 1 - \frac{h\lambda\gamma k_1}{2} > 0 \qquad \Rightarrow \frac{2}{h\lambda\gamma} > k_1,$$

$$s_{22} = 1 - \frac{h\lambda k_2}{2} > 0 \qquad \Rightarrow \frac{2}{h\lambda} > k_2.$$

Since s_{21} is already negative, we need to satisfy

$$|s_{11}| + |s_{21}| = 1 - \frac{h\lambda(\gamma - 1)k_1}{2} < 1 \qquad \Rightarrow \text{Already satisfied},$$

$$|s_{22}| + |s_{21}| = 1 - \frac{h\lambda}{2}(k_2 - k_1) < 1 \qquad \Rightarrow k_2 - k_1 > 0.$$

For $s_{12} > 0$ one can show that the resulting inequalities are inconsistent. Assuming $s_{12} < 0$, we have the final set of conditions to be satisfied as follows:

$$s_{12} = \frac{2h}{\mu_2 - \mu_1} - \frac{h\lambda\gamma k_2}{2} < 0 \qquad \Rightarrow k_2 > \frac{4}{\lambda(\mu_1 + \mu_2 + h)},$$
$$|s_{11}| + |s_{12}| = 1 - \frac{2h}{\mu_2 - \mu_1} + \frac{h\lambda\gamma}{2}(k_2 - k_1) < 1 \qquad \Rightarrow \frac{4}{\lambda(\mu_1 + \mu_2 + h)} > k_2 - k_1,$$
$$|s_{22}| + |s_{12}| = 1 - \frac{2h}{\mu_2 - \mu_1} + \frac{h\lambda(\gamma - 1)k_2}{2} < 1 \qquad \Rightarrow \frac{4}{\lambda(2\mu_1 + h)} > k_2.$$

To solve the problem for all possible (h, λ), we must minimize the left sides and maximize the right sides of the inequalities. This can be done by writing $h = \bar{h}$ and $\lambda = \lambda_N$ for the left sides and $h = \underline{h}$ and $\lambda = \lambda_2$ for the right sides. Putting all of them together we finally obtain the following inequalities:

$$\frac{2(\mu_2 - \mu_1)}{\bar{h}\lambda_N(\mu_1 + \mu_2 + \bar{h})} > k_1 > 0 \tag{4.3}$$
$$\min\left\{\frac{2}{\bar{h}\lambda_N}, \frac{4}{\lambda_N(2\mu_1 + \bar{h})}\right\} > k_2 > \frac{4}{\lambda_2(\mu_1 + \mu_2 + \underline{h})} \tag{4.4}$$
$$\frac{4}{\lambda_N(\mu_1 + \mu_2 + \bar{h})} > k_2 - k_1 > 0. \tag{4.5}$$

We can conclude that the controller $K = \begin{bmatrix} k_1 & k_2 \end{bmatrix} T^{-1}$ solves the consensus problem for the double integrator agent dynamics for arbitrary selection of sampling intervals and changing undirected topology graphs such that $h_k \in [\underline{h}, \bar{h}]$ and $\lambda \in [\lambda_2, \lambda_N]$ when k_1, k_2 satisfy (4.3)–(4.5).

Now, we can analyze the consistency of these inequalities.

Lemma 5. *Let $a, b, c, d > 0$, and*

$$a > k_1 > 0$$
$$b > k_2 > c$$
$$d > k_2 - k_1 > 0.$$

Then, the inequalities are consistent, i.e., there exist k_1, k_2 satisfying them, if and only if $b > c$ and $a + d > c$.

Proof. ($\Rightarrow$) Assume the inequalities are consistent, and then obviously $b > c$. By adding the first and third one we obtain $a + d > k_2 > c$.

($\Leftarrow$) Let $b > c$ and $a + d > c$. If $d \geq c$, select $k_1 = \alpha$ where $\min\{a, c\} > \alpha > 0$. Then, by the second and third inequalities, we have

$$\min\{d + \alpha, b\} > k_2 > \max\{c, \alpha\} = c,$$

and the inequalities are consistent. If $c > d$, select $k_2 - k_1 = d - \epsilon$ where $a + d - c > \epsilon > 0$. Then, by the first and second inequalities, we have

$$\min\{a, b - d + \epsilon\} > k_1 > \max\{c - d + \epsilon, 0\} = c - d + \epsilon,$$

and the inequalities are consistent. □

Corollary 2. *Inequalities* (4.3)–(4.5) *are consistent if and only if*

$$\frac{\mu_1 + \mu_2 + \underline{h}}{\bar{h} + \max\{\bar{h}, 2\mu_1\}} > \frac{\lambda_N}{\lambda_2}. \tag{4.6}$$

Proof. Let

$$a := \frac{2(\mu_2 - \mu_1)}{\bar{h}\lambda_N(\mu_1 + \mu_2 + \bar{h})}, \qquad b := \frac{4}{\lambda_N\left(\bar{h} + \max\{\bar{h}, 2\mu_1\}\right)},$$

$$c := \frac{4}{\lambda_2(\mu_1 + \mu_2 + \underline{h})}, \qquad d := \frac{4}{\lambda_N(\mu_1 + \mu_2 + \bar{h})}.$$

According to Lemma 5, inequalities (4.3)–(4.5) are consistent if and only if $b > c$ and $a + d > c$. It is easy to see that $b > c$ is equivalent to (4.6). We are going to show that $a + d \geq b$ to conclude the proof. Therefore, we need

$$\begin{aligned}
\frac{4\bar{h} + 2(\mu_2 - \mu_1)}{\bar{h}\lambda_N(\mu_1 + \mu_2 + \bar{h})} &\geq \frac{4}{\lambda_N\left(\bar{h} + \max\{\bar{h}, 2\mu_1\}\right)} \\
\max\{\bar{h}, 2\mu_1\} &\geq \frac{2\bar{h}(\mu_1 + \mu_2 + \bar{h})}{2\bar{h} + \mu_2 - \mu_1} - \bar{h} \\
&= \frac{\bar{h}(3\mu_1 + \mu_2)}{2\bar{h} + \mu_2 - \mu_1}.
\end{aligned}$$

First, assume $\bar{h} \geq 2\mu_1$, and then

$$\begin{aligned}
2\bar{h}^2 &\geq 4\bar{h}\mu_1 \\
2\bar{h}^2 + \bar{h}\mu_2 - \bar{h}\mu_1 &\geq 3\bar{h}\mu_1 + \bar{h}\mu_2 \\
\bar{h} &\geq \frac{\bar{h}(3\mu_1 + \mu_2)}{2\bar{h} + \mu_2 - \mu_1}.
\end{aligned}$$

Now, assume $2\mu_1 \geq \bar{h}$, and then

$$\begin{aligned}
2\mu_1(\mu_2 - \mu_1) &\geq \bar{h}\mu_2 - \bar{h}\mu_1 \\
2\mu_1(2\bar{h} + \mu_2 - \mu_1) &\geq \bar{h}\mu_2 + 3\bar{h}\mu_1 \\
2\mu_1 &\geq \frac{\bar{h}(3\mu_1 + \mu_2)}{2\bar{h} + \mu_2 - \mu_1}.
\end{aligned}$$

□

From this corollary, it can be inferred that a controller exists for arbitrary choices of $\bar{h} > \underline{h} > 0$ and $\lambda_N \geq \lambda_2 > 0$. So, the designer should only know the limits of λ_N, λ_2 for the changing topologies and not the specific topologies themselves. They can also select the sampling time intervals arbitrarily.

5. Numerical examples

Given $\bar{h}$, $\underline{h}$, λ_2 and λ_N, let a, b, c, d represents the limit values of inequalities (4.3)–(4.5) as defined in the proof of Corollary 2. The following algorithm is used to calculate K that solves the consensus problem for the double integrator agent dynamics:

Algorithm 1 Stabilizing controller design algorithm

$\mu_1 \leftarrow \bar{h}/2$

$\mu_2 \leftarrow -\mu_1 - \underline{h} + 2\bar{h}\lambda_N/\lambda_2 + 1$ ▹ See (4.6)

$a \leftarrow \dfrac{2(\mu_2 - \mu_1)}{\bar{h}\lambda_N(\mu_1 + \mu_2 + \bar{h})}$ ▹ See (4.3)

$b \leftarrow \dfrac{2}{\bar{h}\lambda_N}$ ▹ Simplified by the selection of μ_1 above. See (4.4)

$c \leftarrow \dfrac{4}{\lambda_2(\mu_1 + \mu_2 + \underline{h})}$ ▹ See (4.4)

$d \leftarrow \dfrac{4}{\lambda_N(\mu_1 + \mu_2 + \bar{h})}$ ▹ See (4.5)

$\Delta k \leftarrow 0.9d$

$k_1 \leftarrow (\min\{a, b - \Delta k\} + \max\{0, c - \Delta k\})/2$

$k_2 \leftarrow k_1 + \Delta k$

$T \leftarrow \begin{bmatrix} \mu_2 - \mu_1 & -(\mu_2 + \mu_1) \\ 0 & 2 \end{bmatrix}$

$K \leftarrow \begin{bmatrix} k_1 & k_2 \end{bmatrix} T^{-1}$

Example 1. *We consider the random switching between the following topologies. The sampling intervals are generated randomly from* $[0.1,\ 3]$*.* $\lambda_2 = 0.3$ *and* $\lambda_N = 6$ *are selected, which contains all eigenvalues of the Laplacian matrices of the graphs in Figure 1. Using the algorithm above, the controller is calculated to be* $K = \begin{bmatrix} 0.0009 & 0.1093 \end{bmatrix}$*. Each curve in Figure 2 represents a simulation of 400 time steps, done by selecting the sampling intervals randomly. At every 50th time step one of the following topologies are selected randomly to show that no unstable behavior exist for any particular topology. A total of 100 simulations were done. Only the maximum state difference between agents, i.e.,* $\max_{ij} \left| x_i(t_k) - x_j(t_k) \right|$, $i, j = 1, \ldots, N$, *is plotted to be able to show many simulations simultaneously.*

Example 2. *We consider switching between randomly created graph topologies for 100 agents. The sampling intervals are generated randomly from* $[0.1,\ 1]$*.* $\lambda_2 = 5$ *and* $\lambda_N = 60$ *are selected. Using the algorithm above, the controller is calculated to be* $K = \begin{bmatrix} 0.0013 & 0.032 \end{bmatrix}$*. Each curve in Figure 3 represents a simulation of 300 time steps, done by selecting the sampling intervals randomly. At every 50th time step a new topology graph is generated randomly to show that no unstable behavior exist for any particular topology. A total of 100 simulations were done. Only the maximum state* *difference between agents, i.e.* $\max_{ij}\left|x_i(t_k)-x_j(t_k)\right|$, *i*, $j = 1, \ldots, N$, *is plotted to be able to show many simulations simultaneously.*

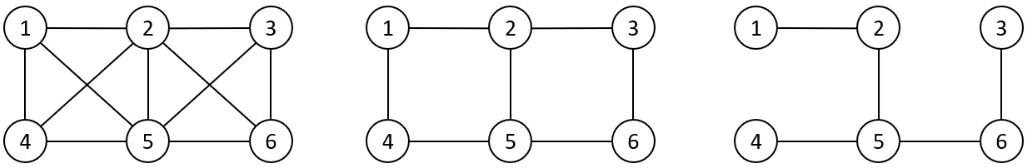

Figure 1. Switching graph topologies.

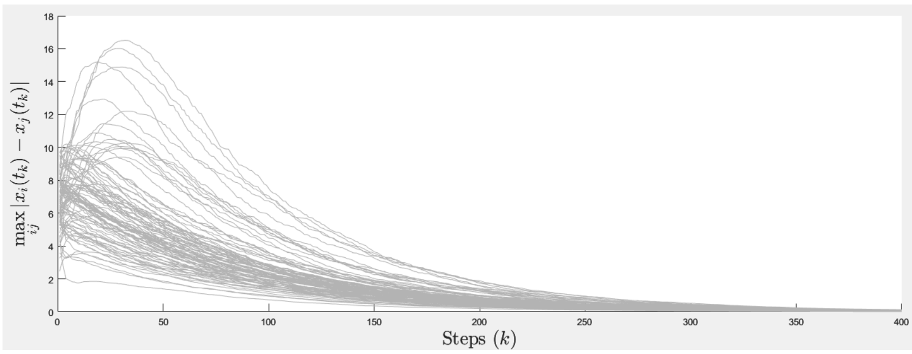

Figure 2. $\max_{ij}\left|x_i(t_k)-x_j(t_k)\right|$ for the controller $K = \begin{bmatrix}0.0009 & 0.1093\end{bmatrix}$.

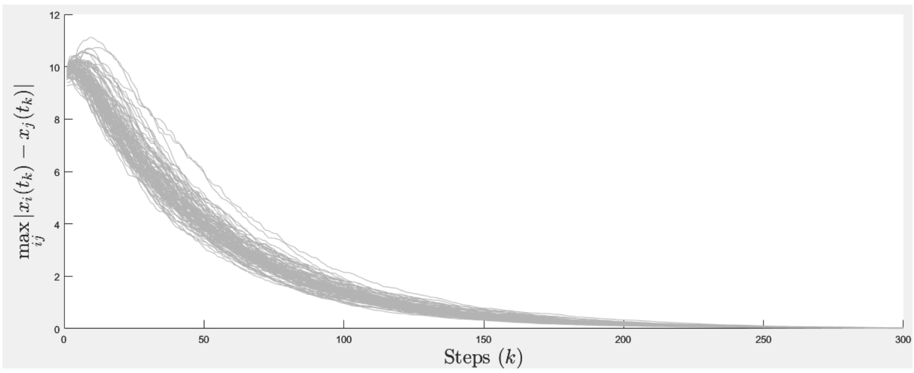

Figure 3. $\max_{ij}\left|x_i(t_k)-x_j(t_k)\right|$ for the controller $K = \begin{bmatrix}0.0013 & 0.032\end{bmatrix}$.

6. Conclusions

A sufficient condition is given for checking whether a distributed state feedback control law solves the asymptotic consensus problem for the general linear dynamics in the case of nonuniform sampling. The condition is based on the stabilization problem of a specific subsystem, which is equivalent to the consensus problem. It depends on transforming the state coordinates, in which the amplitude of the state vector gets smaller at every step. It is shown that the given condition immediately generalizes to the changing topology in the case of undirected networks.

Simple and explicit inequalities are given to design a controller in the case of double integrator dynamics and changing topology with undirected network graphs. These inequalities depend on the maximum sampling time and the interval that contains all eigenvalues of the changing topology graph Laplacian eigenvalues. So, the designer should only know the limits of the Laplacian eigenvalues and not the specific topologies themselves. It is also shown that such a controller always exists as long as the topology graphs have a spanning tree. Numerical examples are given to demonstrate the accuracy of the theoretical results.

Conflict of interest

All authors declare no conflicts of interest in this paper.

References

1. A. Abdessameud, A. Tayebi, On consensus algorithms for double-integrator dynamics without velocity measurements and with input constraints, *Syst. Control Lett.*, **59** (2010), 812–821. https://doi.org/10.1016/j.sysconle.2010.06.019

2. A. Abdessameud, A. Tayebi, On consensus algorithms design for double integrator dynamics, *Automatica*, **49** (2013), 253–260. https://doi.org/10.1016/j.automatica.2012.08.044

3. S. A. Ajwad, E. Moulay, M. Defoort, T. Menard, P. Coirault, Leader-following consensus of second-order multi-agent systems with switching topology and partial aperiodic sampled data, *IEEE Control Syst. Lett.*, **5** (2021), 1567–1572. https://doi.org/10.1109/LCSYS.2020.3041566

4. M. Andreasson, D. V. Dimarogonas, H. Sandberg, K. H. Johansson, Distributed control of networked dynamical systems: Static feedback, integral action and consensus, *IEEE T. Automat. Control*, **59** (2014), 1750–1764. https://doi.org/10.1109/TAC.2014.2309281

5. R. Carli, A. Chiuso, L. Schenato, S. Zampieri, A PI consensus controller for networked clocks synchronization, *IFAC Proc. Vol.*, **41** (2008), 10289–10294. https://doi.org/10.3182/20080706-5-KR-1001.01741

6. L. Cheng, Z. G. Hou, M. Tan, X. Wang, Necessary and sufficient conditions for consensus of double-integrator multi-agent systems with measurement noises, *IEEE T. Automat. Control*, **56** (2011), 1958–1963. https://doi.org/10.1109/TAC.2011.2139450

7. G. Dong, C. Yang, W. Zhu, Event-triggered average consensus of multiagent systems with switching topologies, *Discrete Dyn. Nat. Soc.*, **2020** (2020), 8742014. https://doi.org/10.1155/2020/8742014

8. Y. Gao, L. Wang, Consensus of multiple double-integrator agents with intermittent measurement, *Int. J. Robust Nonlin. Control*, **20** (2010), 1140–1155. https://doi.org/10.1002/rnc.1496

9. Y. Gao, B. Liu, M. Zuo, T. Jiang, J. Yu, Consensus of continuous-time multiagent systems with general linear dynamics and nonuniform sampling, *Math. Prob. Eng.*, **2013** (2013), 718759. https://doi.org/10.1155/2013/718759

10. D. Goldin, Double integrator consensus systems with application to power systems, *IFAC Proc. Vol.*, **46** (2013), 206–211. https://doi.org/10.3182/20130925-2-DE-4044.00023

11. T. A. Jesus, L. C. A. Pimenta, L. A. B. Tôrres, E. M. A. M. Mendes, Consensus for double-integrator dynamics with velocity constraints, *Int. J. Control Autom. Syst.*, **12** (2014), 930–938. https://doi.org/10.1007/s12555-013-0309-0

12. F. L. Lewis, H. Zhang, K. H. Movric, A. Das, *Cooperative control of multi-agent systems*, London: Springer, 2014.

13. Y. Li, C. Tan, A survey of the consensus for multi-agent systems, *Syst. Sci. Control Eng.*, **7** (2019), 468–482. https://doi.org/10.1080/21642583.2019.1695689

14. D. Liberzon, A. S. Morse, Basic problems in stability and design of switched systems, *IEEE Control Syst.*, **19** (1999), 59–70.

15. P. Lin, Y. Jia, Further results on decentralised coordination in networks of agents with second-order dynamics, *IET Control Theor. Appl.*, **3** (2009), 957–970. https://doi.org/10.1049/iet-cta.2008.0263

16. H. Liu, G. Xie, L. Wang, Necessary and sufficient conditions for solving consensus problems of double-integrator dynamics via sampled control, *Int. J. Robust Nonlin. Control*, **20** (2010), 1706–1722. https://doi.org/10.1002/rnc.1543

17. X. Liu, Y. Xie, F. Li, W. Gui, Consensus tracking of singular multiagent systems via sliding mode approach, *IEEE T. Automat. Control*, 2022, 1–7. https://doi.org/10.1109/TAC.2022.3214472

18. X. Liu, Y. Xie, F. Li, W. Gui, Sliding-mode-based admissible consensus tracking of nonlinear singular multiagent systems under jointly connected opologies, *IEEE T. Cybernetics*, **52** (2022), 12491–12500. https://doi.org/10.1109/TCYB.2021.3081801

19. C. Q. Ma, J. F. Zhang, Necessary and sufficient conditions for consensusability of linear multi-agent systems, *IEEE T. Automat. Control*, **55** (2010), 1263–1268. https://doi.org/10.1109/TAC.2010.2042764

20. L. Qi, Some simple estimates for singular values of a matrix, *Linear Algebra Appl.*, **56** (1984), 105–119.

21. J. Qin, H. Gao, A sufficient condition for convergence of sampled-data consensus for double-integrator dynamics with nonuniform and time-varying communication delays, *IEEE T. Automat. Control*, **57** (2012), 2417–2422. https://doi.org/10.1109/TAC.2012.2188425

22. J.Qin, Q. Ma, Y. Shi, L. Wang, Recent advances in consensus of multi-agent systems: a brief survey, *IEEE T. Ind. Electron.*, **64** (2017), 4972–4983. https://doi.org/10.1109/TIE.2016.2636810

23. W. J. Rugh, *Linear system theory*, New York: John Wiley and Sons, 1996.

24. A. Seuret, D. V. Dimarogonas, K. H. Johansson, Consensus of double integrator multi-agents under communication delay, *IFAC Proc, Vol.*, **42** (2009), 376–381. https://doi.org/10.3182/20090901-3-RO-4009.00062

25. Z. G. Wu, Y. Xu, R. Lu, Y. Wu, T. Huang, Event-triggered control for consensus of multiagent systems with fixed/switching topologies, *IEEE T. Syst. Man Cybernet. Syst.*, **48** (2018), 1736–1746. https://doi.org/10.1109/TSMC.2017.2744671

26. J. Zhan, X. Li, Consensus of multiple double-integrators with aperiodic sampled-data and switching topology, In: *34th Chinese Control Conference (CCC)*, 2015, 7113–7117. https://doi.org/10.1109/ChiCC.2015.7260765

27. X. Y. Zhang, J. Zhang, Sampled-data consensus of multi-agent systems with general linear dynamics based on a continuous-time model, *Acta Automat. Sin.*, **40** (2014), 2549–2555. https://doi.org/10.1016/S1874-1029(14)60400-6

28. X. Y. Zhang, J. Zhang, Sampled-data consensus of general linear multi-agent systems under switching topologies: averaging method, *Int. J. Control*, **90** (2017), 275–288. https://doi.org/10.1080/00207179.2016.1177776

29. L. Zou, Z. D. Wang, D. H. Zhou, Event-based control and filtering of networked systems: a survey, *Int. J. Autom. Comput.*, **14** (2017), 239–253. https://doi.org/10.1007/s11633-017-1077-8