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Research article

Integral inequalities of Hermite-Hadamard type via $q-h$ integrals

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Abstract: The well-known Hermite-Hadamard inequality for convex functions is extensively studied for different kinds of integrals and derivatives. This paper investigates some of its variants for $q-h$-integrals using properties of convex functions. Inequalities for q-integrals that have been published in recent years can be extracted from the main results of this paper.

Keywords: Hermite-Hadamard inequality; convex function; $q-h$-integral; $q-h$-derivative; q-integral
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1. Introduction and preliminaries

Inequalities are an important part of mathematical analysis, functional analysis, and optimization theory. There are several classical inequalities that are the outcomes of convex functions introduced at the start of the twentieth century. Convex functions are also frequently used in establishing new innovative results in statistics, economics, graph theory, and many other subjects of pure and applied nature.

A real valued function $f : C \to \mathbb{R}$, where C is a convex subset of $\mathbb{R}^n$, is called convex on C if the following inequality holds:

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y),$$

for $t \in [0,1]$, $x, y \in C$. The Hermite-Hadamard inequality is the best geometric interpretation of a convex function, it is given as follows:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}, \tag{1.1}$$

where f is convex function on an interval I of real numbers and $a, b \in I$, $a < b$.

The inequality (1.1) has been published in several variants by defining new classes of functions and integrals. For instance, it was studied for various kinds of convexities via fractional order derivatives/integrals in [1,2], and for its variants via quantum derivatives/integrals we refer the readers to [3–5].

Motivated by recent published work on the Hermite-Hadamard inequality for q-calculus, our aim in this article is to give inequalities of Hermite-Hadamard type by using $q-h$-integrals defined in [6] via properties of differentiable convex functions. These inequalities represent implicit results on h-integrals, we deduce some interesting outcomes as consequences of these inequalities in the form of q-integral inequalities.

In the next, we define q-derivatives, $q-h$-derivatives, q-derivatives on finite intervals, q-definite integrals, $q-h$-derivatives on finite intervals and $q-h$-definite integrals.

Definition 1. *The q-derivative of a continuous function $f : I \to \mathbb{R}$ is defined by;*

$$D_q f(x) = \frac{f(qx) - f(x)}{(q-1)x}, \tag{1.2}$$

where $0 < q < 1$.

Definition 2. *[6] The $q-h$-derivative of a continuous function $f : I \to \mathbb{R}$ is defined by;*

$$C_h D_q f(x) = \frac{{}_h d_q f(x)}{{}_h d_q x} = \frac{f(q(x+h)) - f(x)}{(q-1)x + qh}, \tag{1.3}$$

where $0 < q < 1$, $h \in \mathbb{R}$.

For $h = 0$ in (1.3), we get (1.2) i.e.,

$$C_0 D_q f(x) = D_q f(x).$$

Definition 3. *[4,7] The q_a-derivative and q_b-derivative of a continuous function $f : I = [a, b] \to \mathbb{R}$ at $x \in I$ are defined by;*

$${}_a D_q f(x) = \frac{f(x) - f(qx + (1-q)a)}{(1-q)(x-a)}, \; x \neq a \tag{1.4}$$

and

$${}_b D_q f(x) = \frac{f(x) - f(qx + (1-q)b)}{(1-q)(b-x)}, \; x \neq b \tag{1.5}$$

respectively. Also, ${}_a D_q f(a) = \lim_{x \longrightarrow a} {}_a D_q f(x)$ *and* ${}_b D_q f(b) = \lim_{x \longrightarrow b} {}_b D_q f(x)$.

Definition 4. *[4, 7] The q_a-definite integral and q_b-definite integral of a continuous function $f : I := [a, b] \to \mathbb{R}$ on $[a, b]$ are defined by;*

$$\int_a^x f(t)_a d_q t = (1-q)(x-a)\sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)a) \tag{1.6}$$

and

$$\int_x^b f(t)_b d_q t = (1-q)(b-x)\sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)b) \tag{1.7}$$

respectively, where $x \in I,\ q \in (0, 1)$.

Definition 5. *[8] The q_{a-h}-derivative, where $q \in (0, 1),\ h \in \mathbb{R}$ of a continuous function $f : I \to \mathbb{R}$ at $x \in [a, b] \subset I$ is defined by;*

$$C_h D_q^a f(x) = \frac{f(x) - f(qx + (1-q)a + qh)}{(1-q)(x-a) - qh}, x \neq \frac{a(1-q) + qh}{1-q} := x_0 \tag{1.8}$$

and q_{b-h}-derivative of f at $x \in [a, b]$ is given by;

$$C_h D_q^b f(x) = \frac{f(x) - f(qx + (1-q)b + qh)}{(1-q)(x-b) - qh}, x \neq \frac{b(1-q) + qh}{1-q} := y_0. \tag{1.9}$$

Also, $C_h D_q^a f(x_0) = \lim_{x \longrightarrow x_0} C_h D_q^a f(x)$ and $C_h D_q^b f(x_0) = \lim_{x \longrightarrow y_0} C_h D_q^b f(x)$.

If $h = 0$ in Definition 5, then it reduces to Definition 3.

Definition 6. *[8] The $q_a - h$-integral and $q_b - h$-integral of a continuous function $f : I = [a, b] \to \mathbb{R}$ on $[a, b]$ are defined by;*

$$\begin{aligned} I_{q-h}^a f(x) &:= \int_a^x f(t)_h d_q^a t \\ &= ((1-q)(x-a) + qh)\sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)a + nq^n h),\ x > a \end{aligned}$$

and

$$\begin{aligned} I_{q-h}^b f(x) &:= \int_x^b f(t)_h d_q^b t \\ &= ((1-q)(b-x) + qh)\sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)b + nq^n h),\ x < b \end{aligned}$$

respectively, where $0 < q < 1,\ h \in \mathbb{R}$.

If $h = 0$ in Definition 6, then it reduces to Definition 4.

By utilizing the above generalized definitions of integrals and derivatives, a new and novel theory called q-calculus has been established. This is used frequently in generalizing classical results based on ordinary calculus; for a detailed study, we refer the readers to [9,10]. In the field of integral inequalities, it is a natural phenomenon to convert integral inequalities into q-integral inequalities. Authors have been working massively in this direction, and many interesting articles have been published by them. Recently, in [3], authors proved the following q-Hermite-Hadamard inequalities for convex functions by using q-definite integrals.

Theorem 1. *Let f be a convex function on $[a,b]$. If it is differentiable on (a,b), then the following inequality for q_a-integrals holds:*

$$f\left(\frac{qa+b}{1+q}\right) \leq \frac{1}{b-a}\int_a^b f(x)d_q^a x \leq \frac{qf(a)+f(b)}{1+q}, \tag{1.10}$$

where $0 < q < 1$.

Here it is important to state that, firstly the above inequality was given by Marinković et al., see [11, Theorem 5.3].

Theorem 2. *With assumptions of the above theorem, the following inequality for q_a-integrals holds:*

$$f\left(\frac{a+qb}{1+q}\right) + f'\left(\frac{a+qb}{1+q}\right)\frac{(1-q)(b-a)}{1+q} \leq \frac{1}{b-a}\int_a^b f(x)d_q^a x \leq \frac{qf(a)+f(b)}{1+q}. \tag{1.11}$$

Theorem 3. *With assumptions of the above theorem, the following inequality for q_a-integrals holds:*

$$f\left(\frac{a+b}{2}\right) + f'\left(\frac{a+b}{2}\right)\frac{(1-q)(b-a)}{2(1+q)} \leq \frac{1}{b-a}\int_a^b f(x)d_q^a x \leq \frac{qf(a)+f(b)}{1+q}. \tag{1.12}$$

In [4], authors have proved the q-Hermite-Hadamard inequalities for convex functions stated in the following theorems:

Theorem 4. *Let f be a convex function on $[a,b]$. If it is differentiable on (a,b), then the following inequality for q_b-integrals holds:*

$$f\left(\frac{a+qb}{1+q}\right) \leq \frac{1}{b-a}\int_a^b f(x)d_q^b x \leq \frac{f(a)+qf(b)}{1+q}. \tag{1.13}$$

For a detailed study on q-integral inequalities, we refer the readers to [12–16] and the references therein. The following example is frequently used in proving the inequalities of this paper.

Example 1: Let $g(x) = x$, $x \in [a,b]$ and $0 < q < 1$. Then $I^a_{q-h}(g(x))$, $I^b_{q-h}(g(x))$ are calculated as fallows:

$$\begin{aligned} I^a_{q-h}(g(x)) &= \int_a^x g(x)_h d_q^a x = ((1-q)(x-a)+qh)\sum_{n=0}^{\infty} q^n g(q^n x + (1-q^n)a + nq^n h) \\ &= ((1-q)(x-a)+qh)\left(\sum_{n=0}^{\infty} q^{2n}x + \sum_{n=0}^{\infty}(q^n - q^{2n})a + h\sum_{n=0}^{\infty} nq^{2n}\right) \\ &= ((1-q)(x-a)+qh)\left(\frac{x+aq}{1-q^2} + h\sum_{n=0}^{\infty} nq^{2n}\right), \end{aligned} \tag{1.14}$$

$$\begin{aligned} I^b_{q-h}(g(x)) &= \int_x^b g(x)_h d_q^b x = ((1-q)(b-x)+qh)\sum_{n=0}^{\infty} q^n g(q^n x + (1-q^n)b + nq^n h) \\ &= ((1-q)(b-x)+qh)\left(\sum_{n=0}^{\infty} q^{2n}x + \sum_{n=0}^{\infty}(q^n - q^{2n})b + h\sum_{n=0}^{\infty} nq^{2n}\right) \\ &= ((1-q)(b-x)+qh)\left(\frac{x+bq}{1-q^2} + h\sum_{n=0}^{\infty} nq^{2n}\right). \end{aligned} \tag{1.15}$$

2. Generalized $q-h$-Hermite-Hadamard inequalities

In this section, we prove generalized $q-h$-Hermite-Hadamard type inequalities for convex functions. Several variants of the q-Hermite-Hadamard inequality are deducible in special cases. Throughout the paper, we consider the sum of the series $\sum_{n=0}^{\infty} nq^{2n}$ equal to $\mathcal{S}$.

Theorem 5. *Let $\zeta : [a,b] \to \mathbb{R}$ be a convex function on (a,b) such that $0 \leq a < b$. Then we have*

$$\begin{aligned}&\zeta(x_o)\left(\frac{(1-q)(b-x)+qh}{1-q}\right)+m\Big((1-q)(b-x)+qh\Big)\\&\left(\frac{(x-b)+q(b-a)}{(1-q)(1+q)}+h\mathcal{S}\right)\leq\int_x^b\zeta(x)_h d_q^b x\leq\Big((1-q)(b-x)+qh\Big)\\&\left(\frac{\zeta(a)}{1-q}+\frac{\zeta(b)-\zeta(a)}{b-a}\left(\frac{(x-a)+q(b-a)}{1-q^2}+h\mathcal{S}\right)\right),\end{aligned}\tag{2.1}$$

where $x_o = \frac{qa+b}{1+q}$ and $m \in \left[\zeta'_-(x_o), \zeta'_+(x_o)\right]$.

Proof. Since ζ is a convex function, it has at least one line of support at each $x_o = \frac{qa+b}{1+q}$ in (a,b). The lines of support for ζ are denoted by $\mathcal{L}$ and defined as follows:

$$\mathcal{L}(x) = \zeta(x_o) + m(x - x_o), \quad m \in \left[\zeta'_-(x_o), \zeta'_+(x_o)\right].$$

The lines of support always lie below the graph of a convex function; therefore we have

$$\zeta(x_o) + m(x - x_o) \leq \zeta(x), \quad \forall x \in [a,b].\tag{2.2}$$

By applying q_{b-h}-integral on both sides of (2.2), we have the following inequality:

$$\int_x^b \left(\zeta(x_o) + m(x - x_o)\right)\, {}_h d_q^b x \leq \int_x^b \zeta(x)_h d_q^b x.\tag{2.3}$$

The above inequality (2.3), can be written as follows:

$$\zeta(x_o)\int_x^b {}_h d_q^b x + m\left(\int_x^b x\, {}_h d_q^b x - x_o\int_x^b {}_h d_q^b x\right) \leq \int_x^b \zeta(x)_h d_q^b x.\tag{2.4}$$

For $\zeta(x) = 1$ in (1.15), we have the following identity:

$$\int_x^b {}_h d_q^b x = \frac{(1-q)(b-x)+qh}{1-q}.\tag{2.5}$$

By using (1.15) and (2.5) in (2.4), after simplification, the first inequality of (2.1) is obtained. Now, on the other hand, let $\mathcal{K}$ be a function expressing the line connecting points $(a, \zeta(a))$ and $(b, \zeta(b))$, then we have:

$$\mathcal{K}(x) = \zeta(a) + \frac{\zeta(b)-\zeta(a)}{b-a}(x-a).$$

Since ζ is a convex function on $[a, b]$, the inequality $\zeta(x) \leq \mathcal{K}(x)$ holds true. Therefore, the following inequality holds:

$$\zeta(x) \leq \zeta(a) + \frac{\zeta(b) - \zeta(a)}{b - a}(x - a), \quad \forall x \in [a, b]. \tag{2.6}$$

By applying q_{b-h}-integral on both sides of (2.6), we have the following inequality:

$$\int_x^b \zeta(x)\, {}_h d_q^b x \leq \int_x^b \left(\zeta(a) + \frac{\zeta(b) - \zeta(a)}{b - a}(x - a)\right) {}_h d_q^b x. \tag{2.7}$$

The above inequality (2.7), can be written as follows:

$$\int_x^b \zeta(x)\, {}_h d_q^b x \leq \zeta(a) \int_x^b {}_h d_q^b x + \frac{\zeta(b) - \zeta(a)}{b - a}\left(\int_x^b x\, {}_h d_q^b x - a \int_x^b {}_h d_q^b x\right). \tag{2.8}$$

By using (1.15) and (2.5) in (2.8), then after simplification, the second inequality of (2.1) is obtained. □

Corollary 1. *A convex differentiable function must satisfies the following inequalities:*

$$\begin{aligned} &\zeta(x_o)\left(\frac{1}{1-q}\right) + \zeta'(x_o)\left(\frac{(x-b) + q(b-a)}{(1-q)(1+q)} + h\mathcal{S}\right) \\ &\leq \frac{1}{((1-q)(b-x) + qh)} \int_x^b \zeta(x)_h d_q^b x \\ &\leq \left[\frac{\zeta(a)}{1-q} + \frac{\zeta(b) - \zeta(a)}{b-a}\left(\frac{(x-a) + q(b-a)}{1-q^2} + h\mathcal{S}\right)\right]. \end{aligned} \tag{2.9}$$

Proof. A convex differentiable function has a unique line of support at each point in the interior of its domain. Therefore, at point x_o, the slope of the line of support (tangent line) will be $m = \zeta'(x_o)$. By using this value of m in (2.1), we get the required inequality (2.9). □

Corollary 2. *For $h = 0$ in (2.1), the following inequalities must hold:*

$$\begin{aligned} &\zeta(x_o) + m\left(\frac{(x-b) + q(b-a)}{(1+q)}\right) \leq \frac{1}{b-x} \int_x^b \zeta(x)_0 d_q^b x \\ &\leq \zeta(a) + \frac{\zeta(b) - \zeta(a)}{b-a}\left(\frac{(x-a) + q(b-a)}{1+q}\right). \end{aligned} \tag{2.10}$$

Corollary 3. *For $x = a$ in (2.1), the following inequalities must hold:*

$$\begin{aligned} &\zeta(x_o)\left(\frac{1}{1-q}\right) + m\left(h\mathcal{S} - \frac{b-a}{1+q}\right) \leq \frac{1}{((1-q)(b-a) + qh)} \int_a^b \zeta(x)_h d_q^b x \\ &\leq \frac{\zeta(a) + q\zeta(b)}{(1-q)(1+q)} + \frac{\zeta(b) - \zeta(a)}{b-a} h\mathcal{S}. \end{aligned} \tag{2.11}$$

Theorem 6. *Let $\zeta : [a, b] \to \mathbb{R}$ be a convex function on (a, b) such that $0 \leq a < b$. Then we have*

$$\zeta(y_o)\left(\frac{1}{1-q}\right) + m\left(\frac{x-a}{(1-q)(1+q)} + h\mathcal{S}\right) \tag{2.12}$$

$$
\begin{aligned}
&\leq \frac{1}{((1-q)(b-x)+qh)} \int_x^b \zeta(x)_h d_q^b x \\
&\leq \left[\frac{\zeta(a)}{1-q} + \frac{\zeta(b)-\zeta(a)}{b-a} \left(\frac{(x-a)+q(b-a)}{1-q^2} + h\mathcal{S} \right) \right],
\end{aligned}
$$

where $y_o = \frac{qa+b}{1+q}$ *and* $m \in \left[\zeta'_-(y_o), \zeta'_+(y_o)\right]$.

Proof. Since ζ is a convex function, it has at least one line of support at each $y_o = \frac{a+qb}{1+q}$ in (a, b). The lines of support for ζ are denoted by $\mathcal{L}_1$ and defined as follows:

$$
\mathcal{L}_1(x) = \zeta(y_o) + m(x - y_o), \quad m \in \left[\zeta'_-(y_o), \zeta'_+(y_o)\right].
$$

The lines of support always lie below the graph of a convex function, therefore we have

$$
\zeta(y_o) + m(x - y_o) \leq \zeta(x), \quad \forall x \in [a, b]. \tag{2.13}
$$

By applying q_{b-h}-integral on both sides of (2.13), we have the following inequality:

$$
\int_x^b (\zeta(y_o) + m(x - y_o)) \ {}_h d_q^b x \leq \int_x^b \zeta(x)_h d_q^b x. \tag{2.14}
$$

The above inequality (2.14), can be written as follows:

$$
\zeta(y_o) \int_x^b {}_h d_q^b x + m\left(\int_x^b x \ {}_h d_q^b x - y_o \int_x^b {}_h d_q^b x \right) \leq \int_x^b \zeta(x)_h d_q^b x. \tag{2.15}
$$

By using (1.15) and (2.5) in (2.15), after simplification, the first inequality of (2.12) is obtained. The proof of the second inequality of (2.12) is similar to the proof of the second inequality given in Theorem 5. □

Corollary 4. *A convex differentiable function must satisfies the following inequalities:*

$$
\begin{aligned}
&\zeta(y_o)\left(\frac{1}{1-q}\right) + \zeta'(y_o)\left(\frac{x-a}{(1-q)(1+q)} + h\mathcal{S} \right) \\
&\leq \frac{1}{((1-q)(b-x)+qh)} \int_x^b \zeta(x)_h d_q^b x \\
&\leq \left[\frac{\zeta(a)}{1-q} + \frac{\zeta(b)-\zeta(a)}{b-a} \left(\frac{(x-a)+q(b-a)}{1-q^2} + h\mathcal{S} \right) \right].
\end{aligned} \tag{2.16}
$$

Proof. A convex differentiable function has a unique line of support at each point in the interior of its domain. Therefore, at point y_o, the slope of the line of support (the tangent line) will be $m = \zeta'(y_o)$. By using this value of m in (2.12), we get the required inequality (2.16). □

Corollary 5. *For* $h = 0$ *in (2.12), the following inequalities hold:*

$$
\begin{aligned}
&\zeta(y_o) + m\left(\frac{x-a}{1+q}\right) \leq \frac{1}{b-x} \int_x^b \zeta(x)_0 d_q^b x \\
&\leq \zeta(a) + \frac{\zeta(b)-\zeta(a)}{b-a}\left(\frac{(x-a)+q(b-a)}{1+q} \right).
\end{aligned} \tag{2.17}
$$

Corollary 6. *For $x = a$ in (2.12), the following inequalities hold:*

$$\zeta(y_o)\left(\frac{1}{1-q}\right) + mh\mathcal{S} \le \frac{1}{((1-q)(b-a)+qh)}\int_a^b \zeta(x)_h d_q^b x \le \frac{\zeta(a)+q\zeta(b)}{1-q^2} + \frac{\zeta(b)-\zeta(a)}{b-a}h\mathcal{S}. \tag{2.18}$$

Theorem 7. *Let $\zeta : [a,b] \to \mathbb{R}$ be a convex function on (a,b) such that $0 \le a < b$. Then we have*

$$\begin{aligned} &\zeta(z_o)\left(\frac{1}{1-q}\right) + m\left(\frac{2x-(b+a)+q(b-a)}{2(1-q)(1+q)} + h\mathcal{S}\right) \\ &\le \frac{1}{((1-q)(b-x)+qh)}\int_x^b \zeta(x)_h d_q^b x \\ &\le \left[\frac{\zeta(a)}{1-q} + \frac{\zeta(b)-\zeta(a)}{b-a}\left(\frac{(x-a)+q(b-a)}{1-q^2} + h\mathcal{S}\right)\right], \end{aligned} \tag{2.19}$$

where $z_o = \frac{a+b}{2}$ and $m \in \left[\zeta'_-(z_o), \zeta'_+(z_o)\right]$.

Proof. Since ζ is a convex function, it has at least one line of support at each $z_o = \frac{a+b}{2}$ in (a,b). The lines of support for ζ are denoted by $\mathcal{L}_2$ and defined as follows:

$$\mathcal{L}_2(x) = \zeta(z_o) + m(x - z_o), \quad m \in \left[\zeta'_-(z_o), \zeta'_+(z_o)\right].$$

The lines of support always lie below the graph of a convex function, therefore we have

$$\zeta(z_o) + m(x - z_o) \le \zeta(x), \quad \forall x \in [a,b]. \tag{2.20}$$

By applying q_{b-h}-integral on both sides of (2.20), we have the following inequality:

$$\int_x^b (\zeta(z_o) + m(x-z_o))\ {}_h d_q^b x \le \int_x^b \zeta(x)_h d_q^b x. \tag{2.21}$$

The above inequality (2.21), can be written as follows:

$$\zeta(z_o)\int_x^b {}_h d_q^b x + m\left(\int_x^b x\ {}_h d_q^b x - z_o\int_x^b {}_h d_q^b x\right) \le \int_x^b \zeta(x)_h d_q^b x. \tag{2.22}$$

By using (1.15) and (2.5) in (2.22), after simplification, the first inequality of (2.19) is obtained. The proof of the second inequality of (2.19) is similar to the proof of the second inequality given in Theorem 5. □

Corollary 7. *A convex differentiable function must satisfies the following inequalities:*

$$\begin{aligned} &\zeta(z_o)\left(\frac{1}{1-q}\right) + \zeta'(z_o)\left(\frac{2x-(b+a)+q(b-a)}{2(1-q)(1+q)} + h\mathcal{S}\right) \\ &\le \frac{1}{((1-q)(b-x)+qh)}\int_x^b \zeta(x)_h d_q^b x \\ &\le \left[\frac{\zeta(a)}{1-q} + \frac{\zeta(b)-\zeta(a)}{b-a}\left(\frac{(x-a)+q(b-a)}{1-q^2} + h\mathcal{S}\right)\right]. \end{aligned} \tag{2.23}$$

Proof. A convex differentiable function has a unique line of support at each point in the interior of its domain. Therefore, at point z_o, the slope of the line of support (the tangent line) will be $m = \zeta'(z_o)$. By using this value of m in (2.19), we get the required inequality (2.23). □

Corollary 8. *For $h = 0$ in (2.19), the following inequalities hold:*

$$\zeta(z_o) + m\left(\frac{2x-(b-a)+q(b-a)}{2(1+q)}\right) \le \frac{1}{b-x}\int_x^b \zeta(x)_0 d_q^b x \tag{2.24}$$
$$\le \zeta(a) + \frac{\zeta(b)-\zeta(a)}{b-a}\left(\frac{(x-a)+q(b-a)}{1+q}\right).$$

Corollary 9. *For $x = a$ in (2.19), the following inequalities hold:*

$$\zeta(z_o)\left(\frac{1}{1-q}\right) + m\left(h\mathcal{S} - \frac{b-a}{2(1+q)}\right) \le \frac{1}{((1-q)(b-a)+qh)}\int_a^b \zeta(x)_h d_q^b x \tag{2.25}$$
$$\le \frac{\zeta(a)+q\zeta(b)}{(1-q)(1+q)} + \frac{\zeta(b)-\zeta(a)}{b-a}h\mathcal{S}.$$

3. Conclusions

We studied Hermite-Hadamard type inequalities for convex functions by using $q-h$-integrals. Properties of convex functions are utilized in establishing these inequalities. Results are derived for h-integrals in implicit form, while the inequalities for q-integrals are given explicitly. In future work, we are interested in the utilization of other well-known classes of functions to get corresponding results for $q-h$-integrals. Also, other classical integral inequalities can be studied for $q-h$-integrals.

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Conflict of interest

The authors declare that they have no conflict of interest.

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