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Research article

Fixed point theorem on an orthogonal extended interpolative $\psi\mathcal{F}$-contraction

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Abstract: In this paper, we establish the fixed point results for an orthogonal extended interpolative Ciric Reich-Rus type $\psi\mathcal{F}$-contraction mapping on an orthogonal complete $\mathfrak{b}$-metric spaces and give an example to strengthen our main results. Furthermore, we present an application to fixed point results to find analytical solutions for functional equation.

Keywords: complete $\mathfrak{b}$-metric spaces; fixed point; orthogonal extended interpolative Ciric Reich-Rus type $\psi\mathcal{F}$-contraction
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1. Introduction

The study of fixed point theory (FPT) is an important area of research tool in the nonlinear phenomena and one of the most important results of mathematics and other fields. The contraction mapping theorem was introduced by polish mathematician S.Banach [1]. Mathematicians have recently shown a strong interest in the field of fixed point theory. As a result, many generalizations and extension of the Banach contraction principle were reported in different directions [2–8]. The concept of $\mathfrak{b}$-metric spaces was introduced by Bakhtin [9] and Czerwik [10] in 1989. Since, then several fixed point results have been reported on $\mathfrak{b}$-metric spaces and its extensions, such as rectangular b-metric spaces, cone b-metric spaces etc. [11–18].

In 2012, Wardowski [19] initiated a new type of contractive mapping over a metric spaces. Subsequently, several researchers have generalized and extended the results of $\mathcal{F}$-contraction in various ways (for further reading, one may refer to [20–22]. In 2016, Nicolae-Adrian Secelean et al. [23] defined a new notion of $\psi\mathcal{F}$- contraction mapping. In 2017, Gordji et al. [24] initiated the concept of orthogonality in metric spaces. Recently, many authors improved the notion of orthogonality for metric and $\mathfrak{b}$-metric spaces see [25–34].

In 2018, Karapinar [35] initiated a new type of interpolative Kannan type contraction on a metric space and proved a fixed point theorem. Later, Karapinar et al. [36] introduced the notion of interpolative Ciric Reich-Rus (shortly i-CRR) type mapping for fixed point theorems on metric spaces. Recently in 2019, Mohammadi et al. [37] extended the result of i-CRR type map by using $\mathcal{F}$-contraction, namely extended i-CRR type $\mathcal{F}$-contraction. More recently, in 2021, Sayantan et al. [38] established the Fixed Point results for a class of extended interpolative $\psi\mathcal{F}$ -contraction maps over a b-metric space and presented an application to dynamical programming.

In this paper, we expand and present results by using i-CRR type $\psi\mathcal{F}$ -contraction, and introduce the notion of an orthogonal extended i-CRR type $\psi\mathcal{F}$-contraction mappings. We prove a fixed point theorem in the setting of an orthogonal complete $\mathfrak{b}$-metric space and suitable example is provided by using our obtained results. Finally, we give an application to solving the existence and uniqueness solution for functional equation.

The rest of the paper is organized as follows: In Section 2, we review some definitions and lemmas that will be used in our main results. In Section 3, we present our main result by establishing fixed point results supported with suitable non-trivial example, and in Section 4 we apply the derived result to find solution to functional equation. The analytical solution has been verified by using numerical simulation. Finally, we conclude our work with open problems for the future work.

2. Preliminaries

The main goal of this section is to remember some concepts and results used in the article. In 1993, Czerwik [10] initiated the concept of $\mathfrak{b}$-metric spaces which is defined as follows.

Definition 2.1. [10] Let $\mathcal{H}^*$ be a non-void set and $\mathfrak{s} \geq 1$ be a real number. Then the mapping $\mathfrak{d}^*_{\mathfrak{b}} : \mathcal{H}^* \times \mathcal{H}^* \longrightarrow \mathbb{R}^+$ is said to be $\mathfrak{b}$- metric if the given axioms are satisfied, for every $\vartheta, \ell, \varrho \in \mathcal{H}^*$:

(A_1) $\mathfrak{d}^*_{\mathfrak{b}}(\vartheta, \ell) = 0$ iff $\vartheta = \ell$.

(A_2) $\mathfrak{d}^*_{\mathfrak{b}}(\vartheta, \ell) = \mathfrak{d}^*_{\mathfrak{b}}(\ell, \vartheta)$.

(A_3) $\mathfrak{d}^*_{\mathfrak{b}}(\vartheta, \varrho) \leq \mathfrak{s}[\mathfrak{d}^*_{\mathfrak{b}}(\vartheta, \ell) + \mathfrak{d}^*_{\mathfrak{b}}(\ell, \varrho)]$.

Then, the pair $(\mathcal{H}^*, \mathfrak{d}^*_{\mathfrak{b}}, \mathfrak{s})$ is said to be $\mathfrak{b}$-metric space. It is clear that the class of metric spaces is smaller than the class of $\mathfrak{b}$-metric spaces.

Example 2.2. [10] Consider that space $\mathcal{H}^* = \{\mathfrak{p}, \mathfrak{q}, \mathfrak{r}, \mathfrak{s}\}$. Define $\mathfrak{d}_{\mathfrak{b}}^* : \mathcal{H}^* \times \mathcal{H}^* \to \mathbb{R}$ by

$$\mathfrak{d}_{\mathfrak{b}}^*(\delta, \mathfrak{y}) = \begin{cases} 0 & \textit{when } \delta = \mathfrak{y} \\ 5 & \textit{when } (\delta, \mathfrak{y}) \in \{(\mathfrak{p}, \mathfrak{q}), (\mathfrak{q}, \mathfrak{p})\} \\ 1 & \textit{when } (\delta, \mathfrak{y}) \in \{(\mathfrak{q}, \mathfrak{r}), (\mathfrak{r}, \mathfrak{q})\} \\ \frac{1}{4} & \textit{when } (\delta, \mathfrak{y}) \in \{(\mathfrak{p}, \mathfrak{r}), (\mathfrak{r}, \mathfrak{p})\} \\ 3 & \textit{when } (\delta, \mathfrak{y}) \in \{(\mathfrak{p}, \mathfrak{s}), (\mathfrak{s}, \mathfrak{p})\} \\ 2 & \textit{when } (\delta, \mathfrak{y}) \in \{(\mathfrak{q}, \mathfrak{s}), (\mathfrak{s}, \mathfrak{q}), (\mathfrak{r}, \mathfrak{s}), (\mathfrak{s}, \mathfrak{r})\}. \end{cases}$$

Then $(\mathcal{H}^*, \mathfrak{d}_{\mathfrak{b}}^*, 4)$ forms a $\mathfrak{b}$-metric space.

Wardowski [19] considered the following class of functions to define $\mathcal{F}$-contraction. Consider $\mathcal{F} : (0, \infty) \to \mathbb{R}$ to be a strictly increasing function fulfilled by the given axioms:

(1) $\mathcal{F}$ is strictly increasing.

(2) For every sequence $\{\delta_{\mathfrak{n}}\} \in (0, \infty)$, $\lim_{\mathfrak{n}\to\infty} \delta_{\mathfrak{n}} = 0 \iff \lim_{\mathfrak{n}\to\infty} \mathcal{F}(\delta_{\mathfrak{n}}) = -\infty$.

(3) There exists a constant $\mathfrak{k} \in (0, 1)$, such that $\delta^{\mathfrak{k}}\mathcal{F}(\delta) \to 0$, when $\delta \to 0^+$.

In 2021, Sayantan et al. [38] introduced the following lemma which will be used in the later in this article.

Lemma 2.3. *[38] If $\psi \in \Psi$, then $\psi(\mathfrak{h}) < \mathfrak{h}\ \ \forall\ \mathfrak{h} \in \mathbb{R}$.*

Proof. To contrary assume that, $\exists\ \mathfrak{h}' \in \mathbb{R}$ such that $\psi(\mathfrak{h}') \geq \mathfrak{h}'$. Since ψ is increasing so for every natural number $\mathfrak{n}$, successively we have

$$\psi^{\mathfrak{n}}(\mathfrak{h}') \geq \psi^{\mathfrak{n}-1}(\mathfrak{h}') \geq \psi^{\mathfrak{n}-2}(\mathfrak{h}')\psi^{\mathfrak{n}-1}(\mathfrak{h}') \geq \cdots \geq \psi^2(\mathfrak{h}') \geq \psi(\mathfrak{h}') \geq \mathfrak{h}',$$

which contradicts that $\psi^{\mathfrak{n}}(\mathfrak{h}) \to -\infty$ as $\mathfrak{n} \to \infty$. Hence the proof is done. □

In 2016, Secelean and Wardowski [23], introduced the new notion of a generalized $\psi\mathcal{F}$-contraction as follows.

Definition 2.4. [23] In a metric space $(\mathcal{H}^*, \mathfrak{d}_{\mathfrak{b}}^*)$ and a self maps $\mathcal{I}$ on $\mathcal{H}^*$ is called a $\psi\mathcal{F}$-contraction if $\exists\ \mathcal{F} \in \mathscr{F}$ and $\psi \in \Psi$ such that

$$\mathcal{F}(\mathfrak{d}_{\mathfrak{b}}^*(\mathcal{I}\vartheta, \mathcal{I}\ell)) \leq \psi(\mathcal{F}(\mathfrak{d}_{\mathfrak{b}}^*(\vartheta, \ell))), \tag{2.1}$$

$\forall\ \vartheta, \ell \in \mathcal{H}^*$ with $\mathcal{I}\vartheta \neq \mathcal{I}\ell$.

Karapinar et al. [36] introduced the notion of i-CRR type contraction which is defined as follows:

Definition 2.5. [36] In a metric space $(\mathcal{H}^*, \mathfrak{d}_{\mathfrak{b}}^*)$, a function $\mathcal{I} : \mathcal{H}^* \to \mathcal{H}^*$ is called an i-CRR type contraction if $\exists\ \mathfrak{k} \in [0, 1)$ and $\beta, \alpha \in (0, 1)$ with $\beta + \alpha < 1$ such that

$$\mathfrak{d}_{\mathfrak{b}}^*(\mathcal{I}\vartheta, \mathcal{I}\ell) \leq \mathfrak{k}(\mathfrak{d}_{\mathfrak{b}}^*(\vartheta, \ell))^{\alpha}(\mathfrak{d}_{\mathfrak{b}}^*(\vartheta, \mathcal{I}\vartheta))^{\beta}(\mathfrak{d}_{\mathfrak{b}}^*(\ell, \mathcal{I}\ell))^{1-\alpha-\beta}, \tag{2.2}$$

$\forall\ \vartheta, \ell \in \mathcal{H}^* \setminus Fix(\mathcal{I})$.

In 2019, Mohammadi et al. [37] introduced the notion of extended i-CRR type $\mathcal{F}$-contraction and proved some fixed-point theorems for such maps.

Definition 2.6. [37] In a metric space $(\mathcal{H}^*, \mathfrak{d}_\mathfrak{b}^*)$, a map $\mathcal{I} : \mathcal{H}^* \to \mathcal{H}^*$ is called an extended i-CRR type $\mathcal{F}$-contraction if there exists $\mathcal{F} \in \mathscr{F}$ such that $\forall\ \vartheta, \ell \in \mathcal{H}^* \setminus Fix(\mathcal{I})$ with $\mathcal{I}\vartheta \neq \mathcal{I}\ell$.

$$\lambda + \mathcal{F}(\mathfrak{d}_\mathfrak{b}^*(\mathcal{I}\vartheta, \mathcal{I}\ell)) \leq \alpha\mathcal{F}(\mathfrak{d}_\mathfrak{b}^*(\vartheta, \ell)) + \beta\mathcal{F}(\mathfrak{d}_\mathfrak{b}^*(\vartheta, \mathcal{I}\vartheta)) + (1 - \alpha - \beta)\mathcal{F}(\mathfrak{d}_\mathfrak{b}^*(\ell, \mathcal{I}\ell)), \tag{2.3}$$

for some $\alpha, \beta \in (0, 1)$ with $\beta + \alpha < 1$ and for some positive real λ.

In 2017, Gordji et al. [24] introduced some fundamental orthogonal concepts and its properties.

Definition 2.7. [24] Let $\mathcal{H}^* \neq \phi$ and $\perp \subseteq \mathcal{H}^* \times \mathcal{H}^*$ be a binary relation. If $\perp$ satisfies the given condition:

$$\exists\ \delta_0 \in \mathcal{H}^* : (\forall \delta \in \mathcal{H}^*, \delta \perp \delta_0) \quad \text{or} \quad (\forall \delta \in \mathcal{H}^*, \delta_0 \perp \delta),$$

then it is said to be an O-set . We denote this O-set by $(\mathcal{H}^*, \perp)$.

Example 2.8. [24] Suppose that $\mathcal{IU}_\wp(\mathcal{R})$ is a set of all $\wp \times \wp$ invertible matrices. Define relation $\perp$ on $\mathcal{IU}_\wp(\mathcal{R})$ by

$$\mathcal{G} \perp \mathcal{H} \iff \exists\ \mathcal{Y} \in \mathcal{IU}_\wp(\mathcal{R}) : \mathcal{GH} = \mathcal{HG}.$$

It is clear that $\mathcal{IU}_\wp(\mathcal{R})$ is an O-set.

Definition 2.9. [24] Let $(\mathcal{H}^*, \perp)$ be an O-set. A sequence $\{\delta_\mathfrak{n}\}$ is said to be an orthogonal sequence (briefly, O-sequence) if

$$(\forall \mathfrak{n} \in \mathrm{N}, \delta_\mathfrak{n} \perp \delta_{\mathfrak{n}+1}) \quad or \quad (\forall \mathfrak{n} \in \mathrm{N}, \delta_{\mathfrak{n}+1} \perp \delta_\mathfrak{n}).$$

Definition 2.10. [24] Let $(\mathcal{H}^*, \perp, \mathfrak{d}_\mathfrak{b}^*)$ is said to be an orthogonal $\mathfrak{b}$-metric space if $(\mathcal{H}^*, \perp)$ is an O-set and a $\mathfrak{b}$-metric $\mathfrak{d}_\mathfrak{b}^*$ on $\mathcal{H}^*$ with a real number $\mathfrak{s} \geq 1$.

Definition 2.11. [24] The triplet $(\mathcal{H}^*, \perp, \mathfrak{d}_\mathfrak{b}^*)$ be an orthogonal $\mathfrak{b}$-metric space. Then, a self map $\mathcal{I}$ on $\mathcal{H}^*$ is said to be an orthogonally continuous in $\delta \in \mathcal{H}^*$ if for each O-sequence $\{\delta_\mathfrak{n}\}$ in $\mathcal{H}^*$ with $\delta_\mathfrak{n} \to \delta$ we've $\mathcal{I}(\delta_\mathfrak{n}) \to \mathcal{I}(\delta)$. Also, $\mathcal{I}$ is called an $\perp$-continuous on $\mathcal{H}^*$ if $\mathcal{I}$ is $\perp$-continuous in for each $\delta \in \mathcal{H}^*$.

Definition 2.12. [24] The triplet $(\mathcal{H}^*, \perp, \mathfrak{d}_\mathfrak{b}^*)$ be an O-complete $\mathfrak{b}$-metric space Then, $\mathcal{H}^*$ is called an orthogonally complete, if every Cauchy O-sequence is convergent.

Definition 2.13. [24] The triplet $(\mathcal{H}^*, \perp, \mathfrak{d}_\mathfrak{b}^*)$ be an O-complete $\mathfrak{b}$-metric space. A map $\mathcal{I} : \mathcal{H}^* \to \mathcal{H}^*$ is called an $\perp$-preserving if $\mathcal{I}\delta \perp \mathcal{I}\mathfrak{y}$ whenever $\delta \perp \mathfrak{y}$ for all $\delta, \mathfrak{y} \in \mathcal{H}^*$.

In 2021, Sayantan et al. [38] introduced the notion of an extended i-CRR type $\psi\mathcal{F}$-contraction as follows:

Definition 2.14. [38] In a $\mathfrak{b}$-metric space $(\mathcal{H}^*, \mathfrak{d}_\mathfrak{b}^*)$ with $\mathfrak{s} \geq 1$, and a maps $\mathcal{I} : \mathcal{H}^* \to \mathcal{H}^*$ is said to be an extended i-CRR type $\psi\mathcal{F}$-contraction if $\exists\ \mathcal{F} \in \mathscr{F}$ and $\psi \in \Psi$ such that $\forall\ \vartheta, \ell \in \mathcal{H}^* \setminus Fix(\mathcal{I})$ with $\mathcal{I}\vartheta \neq \mathcal{I}\ell$.

$$\mathcal{F}(\mathfrak{s}\mathfrak{d}_\mathfrak{b}^*(\mathcal{I}\vartheta, \mathcal{I}\ell)) \leq \mathfrak{a}\psi(\mathcal{F}(\mathfrak{s}\mathfrak{d}_\mathfrak{b}^*(\vartheta, \ell))) + \mathfrak{b}\psi(\mathcal{F}(\mathfrak{s}\mathfrak{d}_\mathfrak{b}^*(\vartheta, \mathcal{I}\vartheta))) + \mathfrak{c}\psi(\mathcal{F}(\mathfrak{s}\mathfrak{d}_\mathfrak{b}^*(\ell, \mathcal{I}\ell))),$$

for some $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in [0, 1]$ with $0 < \mathfrak{a} + \mathfrak{b} + \mathfrak{c} \leq 1$.

3. Main results

In this section, we mainly focus on the existence and uniqueness of fixed point theorem on an orthogonal extended i-CRR type $\psi\mathcal{F}$-contraction mapping for on orthogonal complete $\mathfrak{b}$-metric spaces.

Definition 3.1. An orthogonal complete metric space $(\mathcal{H}^*, \perp, \mathfrak{d}_{\mathfrak{b}}^*)$ and a maps $\mathcal{I} : \mathcal{H}^* \to \mathcal{H}^*$ is called an orthogonal i-CRR type contraction if $\exists\, \mathfrak{k} \in [0, 1)$ and $\beta, \alpha \in (0, 1)$ such that

$$\begin{aligned} &\vartheta \perp \ell \ \text{ or } \ \ell \perp \vartheta,\ \mathfrak{d}_{\mathfrak{b}}^*(\mathcal{I}\vartheta, \mathcal{I}\ell) > 0 \\ &\Longrightarrow\ \mathfrak{d}_{\mathfrak{b}}^*(\mathcal{I}\vartheta, \mathcal{I}\ell) \leq \mathfrak{k}(\mathfrak{d}_{\mathfrak{b}}^*(\vartheta, \ell))^{\alpha}(\mathfrak{d}_{\mathfrak{b}}^*(\vartheta, \mathcal{I}\vartheta))^{\beta}(\mathfrak{d}_{\mathfrak{b}}^*(\ell, \mathcal{I}\ell))^{1-\alpha-\beta}, \end{aligned} \tag{3.1}$$

$\forall\, \vartheta, \ell \in \mathcal{H}^* \setminus Fix(\mathcal{I})$.

Definition 3.2. An orthogonal complete metric space $(\mathcal{H}^*, \perp, \mathfrak{d}_{\mathfrak{b}}^*)$, a map $\mathcal{I} : \mathcal{H}^* \to \mathcal{H}^*$ is called an orthogonal extended i-CRR type $\mathcal{F}$-contraction if there exists $\mathcal{F} \in \mathscr{F}$ such that $\forall\, \vartheta, \ell \in \mathcal{H}^* \setminus Fix(\mathcal{I})$ with $\mathcal{I}\vartheta \neq \mathcal{I}\ell$.

$$\begin{aligned} &\vartheta \perp \ell \ \text{ or } \ \ell \perp \vartheta,\ \mathfrak{d}_{\mathfrak{b}}^*(\mathcal{I}\vartheta, \mathcal{I}\ell) > 0 \\ &\Longrightarrow\ \lambda + \mathcal{F}(\mathfrak{d}_{\mathfrak{b}}^*(\mathcal{I}\vartheta, \mathcal{I}\ell)) \leq \alpha\mathcal{F}(\mathfrak{d}_{\mathfrak{b}}^*(\vartheta, \ell)) + \beta\mathcal{F}(\mathfrak{d}_{\mathfrak{b}}^*(\vartheta, \mathcal{I}\vartheta)) + (1 - \alpha - \beta)\mathcal{F}(\mathfrak{d}_{\mathfrak{b}}^*(\ell, \mathcal{I}\ell)), \end{aligned} \tag{3.2}$$

for some $\alpha, \beta \in (0, 1)$ with $\beta + \alpha < 1$ and for some positive real λ.

Definition 3.3. An orthogonal complete metric space $(\mathcal{H}^*, \perp, \mathfrak{d}_{\mathfrak{b}}^*)$ and a self maps $\mathcal{I}$ on $\mathcal{H}^*$ is called an orthogonal $\psi\mathcal{F}$-contraction if $\exists\, \mathcal{F} \in \mathscr{F}$ and $\psi \in \mathbf{\Psi}$ such that

$$\begin{aligned} &\vartheta \perp \ell \ \text{ or } \ \ell \perp \vartheta,\ \mathfrak{d}_{\mathfrak{b}}^*(\mathcal{I}\vartheta, \mathcal{I}\ell) > 0 \\ &\Longrightarrow\ \mathcal{F}(\mathfrak{d}_{\mathfrak{b}}^*(\mathcal{I}\vartheta, \mathcal{I}\ell)) \leq \psi(\mathcal{F}(\mathfrak{d}_{\mathfrak{b}}^*(\vartheta, \ell))), \end{aligned} \tag{3.3}$$

$\forall\, \vartheta, \ell \in \mathcal{H}^* \setminus Fix(\mathcal{I})$ with $\mathcal{I}\vartheta \neq \mathcal{I}\ell$.

Now, we define the new definition of an orthogonal extended i-CRR type $\psi\mathcal{F}$-contraction by using our main results.

Definition 3.4. An orthogonal complete $\mathfrak{b}$-metric space $(\mathcal{H}^*, \perp, \mathfrak{d}_{\mathfrak{b}}^*)$ with $\mathfrak{s} \geq 1$, and a maps $\mathcal{I} : \mathcal{H}^* \to \mathcal{H}^*$ is called an orthogonal extended i-CRR type $\psi\mathcal{F}$-contraction if $\exists\, \mathcal{F} \in \mathscr{F}$ and $\psi \in \mathbf{\Psi}$ such that $\forall\, \vartheta, \ell \in \mathcal{H}^* \setminus Fix(\mathcal{I})$ with $\mathcal{I}\vartheta \neq \mathcal{I}\ell$.

$$\begin{aligned} &\vartheta \perp \ell \ \text{ or } \ \ell \perp \vartheta,\ \mathfrak{d}_{\mathfrak{b}}^*(\mathcal{I}\vartheta, \mathcal{I}\ell) > 0 \\ &\Rightarrow \mathcal{F}(\mathfrak{s}\mathfrak{d}_{\mathfrak{b}}^*(\mathcal{I}\vartheta, \mathcal{I}\ell)) \leq \mathfrak{a}\psi(\mathcal{F}(\mathfrak{s}\mathfrak{d}_{\mathfrak{b}}^*(\vartheta, \ell))) + \mathfrak{b}\psi(\mathcal{F}(\mathfrak{s}\mathfrak{d}_{\mathfrak{b}}^*(\vartheta, \mathcal{I}\vartheta))) + \mathfrak{c}\psi(\mathcal{F}(\mathfrak{s}\mathfrak{d}_{\mathfrak{b}}^*(\ell, \mathcal{I}\ell))), \end{aligned} \tag{3.4}$$

for some $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in [0, 1]$ with $0 < \mathfrak{a} + \mathfrak{b} + \mathfrak{c} \leq 1$.

In the following theorem, we generalize and improve our fixed point results for an orthogonal extended i-CRR type $\psi\mathcal{F}$-contraction.

Theorem 3.5. *Let $(\mathcal{H}^*, \perp, \mathfrak{d}_{\mathfrak{b}}^*)$ be an O-complete $\mathfrak{b}$-metric space with an orthogonal element δ_0 and parameter $\mathfrak{s} \geq 1$ and Let $\mathcal{I} : \mathcal{H}^* \to \mathcal{H}^*$ be a self-maps satisfying:*

(i) $\mathcal{I}$ is an orthogonal preserving.

(ii) $\mathcal{I}$ is an orthogonal extended i-CRR type $\psi\mathcal{F}$-contraction.

Then $\mathcal{I}$ has a unique fixed point.

Proof. Consider $(\mathcal{H}^*, \perp)$ is an orthogonal set, there exists

$$\delta_0 \in \mathcal{H}^* : \forall \delta \in \mathcal{H}^*,\ \delta \perp \delta_0 \ \ (or) \ \ \forall \delta \in \mathcal{H}^*,\ \delta_0 \perp \delta.$$

It follows that $\delta_0 \perp \mathcal{I}\delta_0$ or $\mathcal{I}\delta_0 \perp \delta_0$. Let

$$\delta_1 = \mathcal{I}\delta_0,\ \delta_2 = \mathcal{I}\delta_1 = \mathcal{I}^2\delta_0 \cdots \delta_{\mathfrak{n}} = \mathcal{I}\delta_{\mathfrak{n}-1} = \mathcal{I}^{\mathfrak{n}}\delta_0 \ \ \forall \mathfrak{n} \in \mathbb{N}.$$

For any $\delta_0 \in \mathcal{H}^*$, set $\delta_{\mathfrak{n}} = \mathcal{I}\delta_{\mathfrak{n}-1}$. Now, we assume that the given cases:

(i) If $\exists\ \mathfrak{n} \in \mathbb{N} \cup \{0\}$ such that $\delta_{\mathfrak{n}} = \delta_{\mathfrak{n}+1}$, then we have $\mathcal{I}\delta_{\mathfrak{n}} = \delta_{\mathfrak{n}}$. It is clear that $\delta_{\mathfrak{n}}$ is a fixed point of $\mathcal{I}$. Therefore, the proof is finished.

(ii) If $\delta_{\mathfrak{n}} \neq \delta_{\mathfrak{n}+1}$, for any $\mathfrak{n} \in \mathbb{N} \cup \{0\}$, then we have $\mathfrak{d}_{\mathfrak{b}}^*(\delta_{\mathfrak{n}+1}, \delta_{\mathfrak{n}}) > 0$, for each $\mathfrak{n} \in \mathbb{N}$.

Since $\mathcal{I}$ is $\perp$-preserving, we obtain

$$\delta_{\mathfrak{n}} \perp \delta_{\mathfrak{n}+1} \ \ \text{(or)} \ \ \delta_{\mathfrak{n}+1} \perp \delta_{\mathfrak{n}}.$$

This implies that $\{\delta_{\mathfrak{n}}\}$ is an O-sequence. Since $\mathcal{I}$ is an orthogonal extended i-CRR type $\psi\mathcal{F}$-contraction, we get

$$\begin{aligned}
\mathcal{F}(\mathfrak{s}\mathfrak{d}_{\mathfrak{b}}^*(\delta_{\mathfrak{n}+1}, \delta_{\mathfrak{n}})) =& \mathcal{F}(\mathfrak{s}\mathfrak{d}_{\mathfrak{b}}^*(\mathcal{I}\delta_{\mathfrak{n}}, \mathcal{I}\delta_{\mathfrak{n}-1})) \\
\leq& \mathfrak{a}\psi(\mathcal{F}(\mathfrak{s}\mathfrak{d}_{\mathfrak{b}}^*(\delta_{\mathfrak{n}}, \delta_{\mathfrak{n}-1}))) + \mathfrak{b}\psi(\mathcal{F}(\mathfrak{s}\mathfrak{d}_{\mathfrak{b}}^*(\delta_{\mathfrak{n}}, \delta_{\mathfrak{n}+1}))) + \mathfrak{c}\psi(\mathcal{F}(\mathfrak{s}\mathfrak{d}_{\mathfrak{b}}^*(\delta_{\mathfrak{n}-1}, \delta_{\mathfrak{n}}))).
\end{aligned} \tag{3.5}$$

Now, we claim that $\mathfrak{d}_{\mathfrak{b}}^*(\delta_{\mathfrak{n}}, \delta_{\mathfrak{n}+1}) \leq \mathfrak{d}_{\mathfrak{b}}^*(\delta_{\mathfrak{n}-1}, \delta_{\mathfrak{n}})$. If not, then $\mathfrak{d}_{\mathfrak{b}}^*(\delta_{\mathfrak{n}-1}, \delta_{\mathfrak{n}}) < \mathfrak{d}_{\mathfrak{b}}^*(\delta_{\mathfrak{n}}, \delta_{\mathfrak{n}+1})$. As ψ and $\mathcal{F}$ both are increasing so we can write $\psi(\mathcal{F}(\mathfrak{s}\mathfrak{d}_{\mathfrak{b}}^*(\delta_{\mathfrak{n}-1}, \delta_{\mathfrak{n}}))) \leq \psi(\mathcal{F}(\mathfrak{s}\mathfrak{d}_{\mathfrak{b}}^*(\delta_{\mathfrak{n}}, \delta_{\mathfrak{n}+1})))$. Therefore from (3.5) we have,

$$\begin{aligned}
\mathcal{F}(\mathfrak{s}\mathfrak{d}_{\mathfrak{b}}^*(\delta_{\mathfrak{n}+1}, \delta_{\mathfrak{n}})) \leq& (\mathfrak{a} + \mathfrak{b} + \mathfrak{c})\psi(\mathcal{F}(\mathfrak{s}\mathfrak{d}_{\mathfrak{b}}^*(\delta_{\mathfrak{n}}, \delta_{\mathfrak{n}+1}))) \\
\leq& \psi(\mathcal{F}(\mathfrak{s}\mathfrak{d}_{\mathfrak{b}}^*(\delta_{\mathfrak{n}}, \delta_{\mathfrak{n}+1}))), \ \ as \ \ \mathfrak{a} + \mathfrak{b} + \mathfrak{c} \leq 1,
\end{aligned}$$

a contradiction, since $\psi(\mathfrak{h}) < \mathfrak{h}$ for all $\mathfrak{h} \in \mathbb{R}$. Thus our claim stands. Now by (3.5) we see that,

$$\begin{aligned}
\mathcal{F}(\mathfrak{s}\mathfrak{d}_{\mathfrak{b}}^*(\delta_{\mathfrak{n}+1}, \delta_{\mathfrak{n}})) \leq& (\mathfrak{a} + \mathfrak{b} + \mathfrak{c})\psi(\mathcal{F}(\mathfrak{s}\mathfrak{d}_{\mathfrak{b}}^*(\delta_{\mathfrak{n}}, \delta_{\mathfrak{n}-1}))) \\
\leq& \psi(\mathcal{F}(\mathfrak{s}\mathfrak{d}_{\mathfrak{b}}^*(\delta_{\mathfrak{n}}, \delta_{\mathfrak{n}-1}))) \\
\leq& \psi^2(\mathcal{F}(\mathfrak{s}\mathfrak{d}_{\mathfrak{b}}^*(\delta_{\mathfrak{n}-1}, \delta_{\mathfrak{n}-2}))) \\
\leq& \psi^3(\mathcal{F}(\mathfrak{s}\mathfrak{d}_{\mathfrak{b}}^*(\delta_{\mathfrak{n}-2}, \delta_{\mathfrak{n}-3}))) \\
&\vdots \\
\leq& \psi^{\mathfrak{n}}(\mathcal{F}(\mathfrak{s}\mathfrak{d}_{\mathfrak{b}}^*(\delta_1, \delta_0))) \to -\infty \ \ as \ \ \mathfrak{n} \to \infty.
\end{aligned} \tag{3.6}$$

Therefore $\lim_{\mathfrak{n}\to\infty} \mathcal{F}(\mathfrak{s}\mathfrak{d}_{\mathfrak{b}}^*(\delta_{\mathfrak{n}+1}, \delta_{\mathfrak{n}})) = -\infty$, which implies that

$$\lim_{\mathfrak{n}\to\infty} \mathfrak{s}\mathfrak{d}_{\mathfrak{b}}^*(\delta_{\mathfrak{n}+1}, \delta_{\mathfrak{n}}) = 0.$$

Define $\beta_{\mathfrak{n}} = \mathfrak{d}_{\mathfrak{b}}^*(\delta_{\mathfrak{n}}, \delta_{\mathfrak{n}+1})$. i.e., $\mathfrak{s}\beta_{\mathfrak{n}} \to 0$ as $\mathfrak{n} \to \infty$. Then there exists $\mathfrak{k} \in (0, 1)$ such that $(\mathfrak{s}\beta_{\mathfrak{n}})^{\mathfrak{k}}\mathcal{F}(\mathfrak{s}\beta_{\mathfrak{n}}) \to 0$ as $\mathfrak{n} \to \infty$. Again from (3.6), $\mathcal{F}(\mathfrak{s}\beta_{\mathfrak{n}}) \le \boldsymbol{\psi}^{\mathfrak{n}}(\mathcal{F}(\mathfrak{s}\beta_0))$. As $\boldsymbol{\psi}(\mathfrak{h}) < \mathfrak{h}$ for all $\mathfrak{h} \in \mathbb{R}$, we have the relation

$$(\mathfrak{s}\beta_{\mathfrak{n}})^{\mathfrak{k}}\mathcal{F}(\mathfrak{s}\beta_{\mathfrak{n}}) \le (\mathfrak{s}\beta_{\mathfrak{n}})^{\mathfrak{k}}\boldsymbol{\psi}^{\mathfrak{n}}(\mathcal{F}(\mathfrak{s}\beta_0)) \le (\mathfrak{s}\beta_{\mathfrak{n}})^{\mathfrak{k}}\mathcal{F}(\mathfrak{s}\beta_0).$$

Thus by Sandwitch Theorem, $\lim_{\mathfrak{n}\to\infty} \beta_{\mathfrak{n}}^{\mathfrak{k}}\boldsymbol{\psi}^{\mathfrak{n}}(\mathcal{F}(\mathfrak{s}\beta_0)) = 0$. So corresponding to $\epsilon = 1$, $\exists\ \mathfrak{n}_0 \in \mathbb{N}$ such that $\beta_{\mathfrak{n}} < |\boldsymbol{\psi}^{\mathfrak{n}}(\mathcal{F}(\mathfrak{s}\beta_0))|^{-1/\mathfrak{k}}$, whenever $\mathfrak{n} \ge \mathfrak{n}_0$. Hence,

$$\begin{aligned}
\mathfrak{d}_{\mathfrak{b}}^*(\delta_{\mathfrak{n}}, \delta_{\mathfrak{n}+\mathfrak{p}}) &\le \mathfrak{s}[\mathfrak{d}_{\mathfrak{b}}^*(\delta_{\mathfrak{n}}, \delta_{\mathfrak{n}+1}) + \mathfrak{d}_{\mathfrak{b}}^*(\delta_{\mathfrak{n}+1}, \delta_{\mathfrak{n}+\mathfrak{p}})]\\
&\le \mathfrak{s}\mathfrak{d}_{\mathfrak{b}}^*(\delta_{\mathfrak{n}}, \delta_{\mathfrak{n}+1}) + \mathfrak{s}^2[\mathfrak{d}_{\mathfrak{b}}^*(\delta_{\mathfrak{n}+1}, \delta_{\mathfrak{n}+2}) + \mathfrak{d}_{\mathfrak{b}}^*(\delta_{\mathfrak{n}+2}, \delta_{\mathfrak{n}+\mathfrak{p}})]\\
&\ \vdots\\
&\le \mathfrak{s}\mathfrak{d}_{\mathfrak{b}}^*(\delta_{\mathfrak{n}}, \delta_{\mathfrak{n}+1}) + \mathfrak{s}^2\mathfrak{d}_{\mathfrak{b}}^*(\delta_{\mathfrak{n}+1}, \delta_{\mathfrak{n}+2}) + \cdots + \mathfrak{s}^{\mathfrak{p}-1}\mathfrak{d}_{\mathfrak{b}}^*(\delta_{\mathfrak{n}+\mathfrak{p}-2}, \delta_{\mathfrak{n}+\mathfrak{p}-1})\\
&\quad + \mathfrak{s}^{\mathfrak{p}-1}\mathfrak{d}_{\mathfrak{b}}^*(\delta_{\mathfrak{n}+\mathfrak{p}-1}, \delta_{\mathfrak{n}+\mathfrak{p}})\\
&< \mathfrak{s}\beta_{\mathfrak{n}} + \mathfrak{s}^2\beta_{\mathfrak{n}+1} + \cdots + \mathfrak{s}^{\mathfrak{p}}\beta_{\mathfrak{n}+\mathfrak{p}-1}\\
&< \mathfrak{s}^{\mathfrak{p}} \sum_{\mathfrak{r}=\mathfrak{n}}^{\mathfrak{n}+\mathfrak{p}-1} \beta_{\mathfrak{r}} < \mathfrak{s}^{\mathfrak{p}} \sum_{\mathfrak{r}=\mathfrak{n}}^{\mathfrak{n}+\mathfrak{p}-1} |\boldsymbol{\psi}^{\mathfrak{r}}(\mathcal{F}(\mathfrak{s}\beta_0))|^{-1/\mathfrak{k}}.
\end{aligned}$$

Since the series $\sum_{\mathfrak{n}} |\boldsymbol{\psi}^{\mathfrak{n}}(\mathcal{F}(\mathfrak{s}\beta_0))|^{-1/\mathfrak{k}}$ is orthogonal convergent, so for every $\mathfrak{p} = 1, 2, \cdots$ we have $\mathfrak{d}_{\mathfrak{b}}^*(\delta_{\mathfrak{n}}, \delta_{\mathfrak{n}+\mathfrak{p}}) \to 0$ as $\mathfrak{n} \to \infty$ and consequently $\{\delta_{\mathfrak{n}}\}$ is an orthogonal Cauchy sequence in $\mathcal{H}^*$ and by an orthogonal completeness of $\mathcal{H}^*$, $\{\delta_{\mathfrak{n}}\}$ is an orthogonal convergent. Let $\lim_{\mathfrak{n}\to\infty} \delta_{\mathfrak{n}} = \delta \in \mathcal{H}^*$. Now, if $\mathcal{I}\delta = \delta$ then there is nothing to prove. So suppose $\mathcal{I}\delta \ne \delta$. Then

$$\begin{aligned}
\mathcal{F}(\mathfrak{s}\mathfrak{d}_{\mathfrak{b}}^*(\delta_{\mathfrak{n}+1}, \mathcal{I}\delta)) &= \mathcal{F}(\mathfrak{s}\mathfrak{d}_{\mathfrak{b}}^*(\mathcal{I}\delta_{\mathfrak{n}}, \mathcal{I}\delta))\\
&\le \mathfrak{a}\boldsymbol{\psi}(\mathcal{F}(\mathfrak{s}\mathfrak{d}_{\mathfrak{b}}^*(\delta_{\mathfrak{n}}, \delta))) + \mathfrak{b}\boldsymbol{\psi}(\mathcal{F}(\mathfrak{s}\mathfrak{d}_{\mathfrak{b}}^*(\delta_{\mathfrak{n}}, \delta_{\mathfrak{n}+1}))) + \mathfrak{c}\boldsymbol{\psi}(\mathcal{F}(\mathfrak{s}\mathfrak{d}_{\mathfrak{b}}^*(\delta, \mathcal{I}\delta)))\\
&\le \mathfrak{a}\mathcal{F}(\mathfrak{s}\mathfrak{d}_{\mathfrak{b}}^*(\delta_{\mathfrak{n}}, \delta)) + \mathfrak{b}\mathcal{F}(\mathfrak{s}\mathfrak{d}_{\mathfrak{b}}^*(\delta_{\mathfrak{n}}, \delta_{\mathfrak{n}+1})) + \mathfrak{c}\mathcal{F}(\mathfrak{s}\mathfrak{d}_{\mathfrak{b}}^*(\delta, \mathcal{I}\delta))\\
&\quad \to -\infty, \ \ as\ \mathfrak{n} \to \infty,
\end{aligned}$$

follows that $\lim_{\mathfrak{n}\to\infty} \mathfrak{d}_{\mathfrak{b}}^*(\delta_{\mathfrak{n}+1}, \mathcal{I}\delta) = 0$. i.e., $\mathfrak{d}_{\mathfrak{b}}^*(\delta, \mathcal{I}\delta) = 0$, proves that $\delta \in \mathcal{H}^*$ is a fixed point of $\mathcal{I}$.

Therefore, $\delta = \mathcal{I}\delta$. Now, to prove that the point $\varkappa \in \mathcal{H}^*$ is unique.

Assume that $\varkappa$ and σ are two distinct fixed points of $\mathcal{I}$. Suppose that, $\mathcal{I}^{\mathfrak{n}}\varkappa = \varkappa \ne \sigma = \mathcal{I}^{\mathfrak{n}}\sigma$ for all $\mathfrak{n} \in \mathbb{N}$. Hence, $\mathfrak{d}_{\mathfrak{b}}^*(\varkappa, \sigma) = \mathfrak{d}_{\mathfrak{b}}^*(\mathcal{I}\varkappa, \mathcal{I}\sigma) > 0$. Because $0 < \mathfrak{a} + \mathfrak{b} + \mathfrak{c} \le 1$. By choice of δ^*, we obtain

$$(\delta^* \perp \varkappa, \quad \delta^* \perp \sigma) \text{ or } (\varkappa \perp \delta^*, \quad \sigma \perp \delta^*).$$

Since $\mathcal{I}$ is $\perp$-preserving, we have

$$(\mathcal{I}^{\mathfrak{n}}\delta^* \perp \mathcal{I}^{\mathfrak{n}}\varkappa, \quad \mathcal{I}^{\mathfrak{n}}\delta^* \perp \mathcal{I}^{\mathfrak{n}}\sigma) \text{ or } (\mathcal{I}^{\mathfrak{n}}\varkappa \perp \mathcal{I}^{\mathfrak{n}}\delta^*, \quad \mathcal{I}^{\mathfrak{n}}\sigma \perp \mathcal{I}^{\mathfrak{n}}\delta^*),$$

for all $\mathfrak{n} \in \mathbb{N}$. Since $\mathcal{I}$ is an orthogonal extended i-CRR type $\boldsymbol{\psi}\mathcal{F}$-contraction, we get

$$\begin{aligned}\mathcal{F}(\mathfrak{s}\mathfrak{d}_{\mathfrak{b}}^{*}(\varkappa,\sigma)) &= \mathcal{F}(\mathfrak{s}\mathfrak{d}_{\mathfrak{b}}^{*}(\mathcal{I}^{\mathfrak{n}}\varkappa,\mathcal{I}^{\mathfrak{n}}\sigma))\\ &\le \mathfrak{a}\boldsymbol{\psi}(\mathcal{F}(\mathfrak{s}\mathfrak{d}_{\mathfrak{b}}^{*}(\varkappa,\sigma))) + \mathfrak{b}\boldsymbol{\psi}(\mathcal{F}(\mathfrak{s}\mathfrak{d}_{\mathfrak{b}}^{*}(\varkappa,\mathcal{I}^{\mathfrak{n}}\varkappa))) + \mathfrak{c}\boldsymbol{\psi}(\mathcal{F}(\mathfrak{s}\mathfrak{d}_{\mathfrak{b}}^{*}(\sigma,\mathcal{I}^{\mathfrak{n}}\sigma)))\\ &\le (\mathfrak{a}+\mathfrak{b}+\mathfrak{c})\boldsymbol{\psi}^{\mathfrak{n}}(\mathcal{F}(\mathfrak{s}\mathfrak{d}_{\mathfrak{b}}^{*}(\varkappa,\sigma)))\\ &\le \boldsymbol{\psi}^{\mathfrak{n}}(\mathcal{F}(\mathfrak{s}\mathfrak{d}_{\mathfrak{b}}^{*}(\varkappa,\sigma))).\end{aligned}$$

Taking limit as $\mathfrak{n}\to\infty$, we obtain

$$\mathfrak{s}\mathfrak{d}_{\mathfrak{b}}^{*}(\varkappa,\sigma) = 0. \tag{3.7}$$

Therefore, $\varkappa = \sigma$. Hence, $\mathcal{I}$ has a unique fixed point in $\mathcal{H}^*$. □

Now, the following result is a consequence of Theorem 3.5.

Corollary 3.6. *If we take* $\mathfrak{a} = 1, \mathfrak{b} = 0, \mathfrak{c} = 0$ *and* $\mathfrak{s} = 1$ *in* (3.4) *then it reduce to an orthogonal* $\boldsymbol{\psi}\mathcal{F}$*-contraction* (3.3).

There may be a mapping $\mathcal{I} : \mathcal{H}^* \to \mathcal{H}^*$ which is an orthogonal extended i-CRR type $\boldsymbol{\psi}\mathcal{F}$-contraction but not necessarily an orthogonal $\boldsymbol{\psi}\mathcal{F}$-contraction.

The following example supports our Theorem 3.5.

Example 3.7. Consider $\mathcal{H}^* = \mathcal{A}^* \cup \mathcal{B}^*$, where $\mathcal{A}^* = \{5, 6\}$ and $\mathcal{B}^* = [0, 4]$ equipped with an orthogonal $\mathfrak{b}$-metric $\mathfrak{d}_{\varkappa}^{*}$ of $\mathbb{R}$. Consider the binary relation $\perp$ on $\mathcal{H}^*$ by $\delta \perp \mathfrak{y}$ if $\delta, \mathfrak{y} \ge 0$ for any $\delta, \mathfrak{y} \in \mathcal{H}^*$.

We define $\mathcal{I} : \mathcal{H}^* \times \mathcal{H}^* \to \mathbb{R}^+$ by

$$\mathfrak{d}^*(\delta, \mathfrak{y}) = |\delta - \mathfrak{y}|^2.$$

Clearly, $(\mathcal{H}^*, \perp, \mathfrak{d}^*)$ is an orthogonal complete $\mathfrak{b}$-metric space with constant $\mathfrak{s} = 2$.

Define the map $\mathcal{I} : \mathcal{H}^* \to \mathcal{H}^*$ by

$$\mathcal{I}(\vartheta) = \begin{cases} 5, & if\ \vartheta \in [0,4] \cup \{5\},\\ 4, & if\ \vartheta = 6.\end{cases}$$

Clearly, $\mathcal{H}^*$ is an orthogonal preserving. Now, we show that $\mathcal{I}$ is an orthogonal extended i-CRR type $\boldsymbol{\psi}\mathcal{F}$-contraction. To verify this we choose $\ell = 6 \in \mathcal{A}^*$, $\vartheta = 4 \in \mathcal{B}^*$, $\mathcal{F}(\delta) = In\delta$, $\boldsymbol{\psi}(\mathfrak{h}) = \mathfrak{h} - 0.5$ and $\mathfrak{a} = 0.3, \mathfrak{b} = 0.3, \mathfrak{c} = 0.3$.

$$\begin{aligned}\mathcal{F}(\mathfrak{s}\mathfrak{d}_{\varkappa}^{*}(\mathcal{I}\vartheta,\mathcal{I}\ell)) &\le \mathfrak{a}\boldsymbol{\psi}(\mathfrak{s}\mathcal{F}(\mathfrak{d}_{\varkappa}^{*}(\vartheta,\ell))) + \mathfrak{b}\boldsymbol{\psi}(\mathfrak{s}\mathcal{F}(\mathfrak{d}_{\varkappa}^{*}(\vartheta,\mathcal{I}\vartheta))) + \mathfrak{c}\boldsymbol{\psi}(\mathfrak{s}\mathcal{F}(\mathfrak{d}_{\varkappa}^{*}(\ell,\mathcal{I}\ell)))\\ \mathfrak{s}\mathcal{F}(1) &= \mathfrak{a}\boldsymbol{\psi}(\mathfrak{s}\mathcal{F}(4)) + \mathfrak{b}\boldsymbol{\psi}(\mathfrak{s}\mathcal{F}(1)) + \mathfrak{c}\boldsymbol{\psi}(\mathfrak{s}\mathcal{F}(4))\\ &= 2[0.3(In4 - 0.5) + 0.3(In1 - 0.5) + 0.3(In4 - 0.5)]\\ 0 &\approx 0.76.\end{aligned}$$

Clearly, $\mathcal{I}$ is $\perp$-continuous with $\boldsymbol{\psi}(\mathfrak{h}) = \mathfrak{h} - 0.5$ and $\mathfrak{a} = \mathfrak{b} = \mathfrak{c} = 0.3$. Moreover it can be shown that $\mathcal{I}$ satisfies all the conditions of the Theorem 3.5. Also it is clear that $\vartheta = 5$ is a unique fixed point of $\mathcal{I}$.

In the sequel, we give some immediate corollaries of Theorem 3.5.

Corollary 3.8. *Assuming the constants* $\mathfrak{a} > 0$, $\mathfrak{b} > 0$ *such that* $\mathfrak{a} + \mathfrak{b} + \mathfrak{c} = 1$ *and* $\boldsymbol{\psi}(\mathfrak{h}) = \mathfrak{h} - \mathfrak{p}$ *for some constant* $\mathfrak{p} > 0$*, we consider* $\mathcal{F}(\delta) = In\delta$ *and* $\mathfrak{s} = 1$ *in* (3.4)*. Then we get,*

$$\vartheta \perp \ell \ \ or \ \ \ell \perp \vartheta, \ \mathfrak{d}_{\mathfrak{b}}^{*}(\mathcal{I}\vartheta, \mathcal{I}\ell) > 0$$
$$In\mathfrak{d}_{\mathfrak{b}}^{*}(\mathcal{I}\vartheta, \mathcal{I}\ell) \leq \mathfrak{a}In\mathfrak{d}_{\mathfrak{b}}^{*}(\vartheta, \ell) + \mathfrak{b}In\mathfrak{d}_{\mathfrak{b}}^{*}(\vartheta, \mathcal{I}\vartheta) + \mathfrak{c}In\mathfrak{d}_{\mathfrak{b}}^{*}(\ell, \mathcal{I}\ell) - \mathfrak{p},$$

from which it follows that $\mathcal{I}$ *is an orthogonal i-CRR type contraction* (3.1) *with* $\mathfrak{k} = e^{-\mathfrak{p}} \in [0, 1)$.

Corollary 3.9. *If we consider in particular,* $\boldsymbol{\psi}(\mathfrak{h}) = \mathfrak{h} - \mathfrak{k}$ *for some constant* $\mathfrak{k} > 0$ *and* $\mathfrak{s} = 1$ *then* (3.4) *turns into*

$$\vartheta \perp \ell \ \ or \ \ \ell \perp \vartheta, \ \mathfrak{d}_{\mathfrak{b}}^{*}(\mathcal{I}\vartheta, \mathcal{I}\ell) > 0$$
$$\mathcal{F}(\mathfrak{d}_{\mathfrak{b}}^{*}(\mathcal{I}\vartheta, \mathcal{I}\ell)) \leq \mathfrak{a}(\mathcal{F}(\mathfrak{d}_{\mathfrak{b}}^{*}(\vartheta, \ell))) + \mathfrak{b}(\mathcal{F}(\mathfrak{d}_{\mathfrak{b}}^{*}(\vartheta, \mathcal{I}\vartheta))) + \mathfrak{c}(\mathcal{F}(\mathfrak{d}_{\mathfrak{b}}^{*}(\ell, \mathcal{I}\ell))) - (\mathfrak{a} + \mathfrak{b} + \mathfrak{c})\mathfrak{k},$$

which is nothing but an orthogonal extended i-CRR type $\mathcal{F}$*-contraction* (3.2) *with* $\lambda = (\mathfrak{a} + \mathfrak{b} + \mathfrak{c})\mathfrak{k} > 0$.

4. Applications

As an application of Theorem 3.5, we find an existence and uniqueness of the solution of the following equation:

$$\vartheta(\mathfrak{h}) = \int_0^{\mathtt{A}} \Re(\mathfrak{h}, \beta)\Omega(\beta, \vartheta(\beta))d\beta, \quad \mathfrak{h} \in [0, \mathtt{A}]. \tag{4.1}$$

Let $\beta = C(\mathcal{H}, \mathbb{R})$ be the real valued continuous functions with $\mathcal{H}$, where $\mathcal{H} = [0, \mathtt{A}]$ and

(a) $\Omega : \mathcal{H} \times \mathbb{R} \to \mathbb{R}$ is continuous;

(b) $\Re : \mathcal{H} \times \mathbb{R} \to \mathbb{R}^{+}$ is continuous and measurable at $\beta \in \mathcal{H}$, $\forall\ \mathfrak{h} \in \mathcal{H}$;

(c) $\Re(\mathfrak{h}, \beta) \geq 0$, $\forall\ \mathfrak{h}, \beta \in \mathcal{H}$ and $\int_0^{\mathtt{A}} \Re(\mathfrak{h}, \beta)d\beta \leq 1$, $\forall\ \mathfrak{h} \in \mathcal{H}$.

Theorem 4.1. *Assume that the conditions* $(a) - (c)$ *hold. Suppose that there exists* $\iota > 0$ *such that*

$$|\Omega(v, \vartheta(\mathfrak{h})) - \Omega(v, \ell(\mathfrak{h}))| \leq e^{-\iota}|\vartheta(\mathfrak{h}) - \ell(\mathfrak{h})|,$$

for every $\mathfrak{h} \in \mathcal{H}$ *and* $\forall\ \vartheta, \ell \in C(\mathcal{H}, \mathbb{R})$*. Then* (4.1) *has a unique solution in* $C(\mathcal{H}, \mathbb{R})$.

Proof. Define the orthogonal relation $\perp$ on $\mathfrak{h}$ by

$$\vartheta \perp \ell \iff \vartheta(\mathfrak{h})\ell(\mathfrak{h}) \geq \vartheta(\mathfrak{h}) \quad or \quad \vartheta(\mathfrak{h})\ell(\mathfrak{h}) \geq \ell(\mathfrak{h}), \ \forall\ \mathfrak{h} \in \mathcal{H}.$$

Define a function $\wp : \beta \times \beta \to [0, \infty)$ by

$$\wp(\vartheta, \ell) = |\vartheta(\mathfrak{h}) - \ell(\mathfrak{h})|^2,$$

$\forall\ \vartheta, \ell \in \beta$. Thus, $(\beta, \perp, \wp)$ is a O-$\mathfrak{b}$-metric space and also a O-complete $\mathfrak{b}$-metric space.

Define $D : \beta \to \beta$ by

$$D\vartheta(\mathfrak{h}) = \int_0^{\mathtt{A}} \Re(\mathfrak{h}, \beta)\Omega(\beta, \vartheta(\beta)), \ \ \mathfrak{h} \in [0, \mathtt{A}].$$

Now, we show that D is $\perp$-preserving. For each $\vartheta, \ell \in \beta$ with $\vartheta \perp \ell$ and $\mathfrak{h} \in \mathcal{H}$, we have

$$D\vartheta(\mathfrak{h}) = \int_0^{\mathtt{A}} \mathfrak{K}(\mathfrak{h}, \beta)\Omega(\beta, \vartheta(\beta)) \geq 1.$$

It follows that $[(D\vartheta)(\mathfrak{h})][(D\ell)(\mathfrak{h})] \geq (D\ell)(\mathfrak{h})$ and so $(D\vartheta)(\mathfrak{h}) \perp (D\ell)(\mathfrak{h})$. Then, D is $\perp$-preserving.

Let $\vartheta, \ell \in \beta$ with $\vartheta \perp \ell$. Suppose that $\vartheta \neq \ell$. For every $\mathfrak{h} \in [0, \mathtt{A}]$, we have

$$\begin{aligned}
\wp(D\vartheta, D\ell) = |D\vartheta(\mathfrak{h}) - D\ell(\mathfrak{h})|^2 &= \left| \int_0^{\mathtt{A}} \mathfrak{K}(\mathfrak{h}, \beta)(\Omega(\beta, \vartheta(\beta)) - \Omega(\beta, \ell(\beta)))d\beta \right|^2 \\
&\leq \int_0^{\mathtt{A}} \mathfrak{K}(\mathfrak{h}, \beta)\Big|\Omega(\beta, \vartheta(\beta)) - \Omega(\beta, \ell(\beta))\Big|^2 d\beta \\
&\leq \int_0^{\mathtt{A}} \mathfrak{K}(\mathfrak{h}, \beta)e^{-\iota}\Big|\vartheta(\mathfrak{h}) - \ell(\mathfrak{h})\Big|^2 d\beta \\
&\leq e^{-\iota}\Big|\vartheta(\mathfrak{h}) - \ell(\mathfrak{h})\Big|^2 \int_0^{\mathtt{A}} \mathfrak{K}(\mathfrak{h}, \beta)d\beta \\
&\leq e^{-\iota}\Big|\vartheta(\mathfrak{h}) - \ell(\mathfrak{h})\Big|^2 \\
&= e^{-\iota}\wp(\vartheta, \ell).
\end{aligned}$$

Therefore,

$$\iota + \ln(\wp(D\vartheta, D\ell)) \leq \ln\left(\wp(\vartheta, \ell)\right).$$

Letting $\mathscr{F}(\delta) = \ln(\delta)$, we get

$$\iota + \mathscr{F}(\wp(D\vartheta, D\ell)) \leq \mathscr{F}\big(\wp(\vartheta, \ell)\big),$$

for all $\vartheta, \ell \in \beta$. Therefore, by Theorem 3.5, β has a unique fixed point. Hence there is a unique solution for (4.1). □

Example 4.2. Consider the integral equation

$$\mathscr{F}(\mathfrak{s}) = \int_0^1 K(\mathfrak{s}, \mathfrak{t})\mathfrak{x}(\mathfrak{t})\mathfrak{d}\mathfrak{t}, \quad 0 \leq \mathfrak{s} \leq 1, \tag{4.2}$$

where $K(\mathfrak{s}, \mathfrak{t}) = e^{\mathfrak{s}^2\mathfrak{t}}$ and $\mathscr{F}(\mathfrak{s}) = \dfrac{e^{(\mathfrak{s}^2+1)} - 1}{(\mathfrak{s}^2 + 1)}$, with exact solution $\mathfrak{x}(\mathfrak{s}) = e^{\mathfrak{s}}$.

From Table 1 the comparison of numerical results with analytic results.

Table 1. Numerical results for test problem, using the proposed method.

s	$\mathscr{F}(\mathfrak{s})$	$\mathfrak{x}(\mathfrak{s})$	Error
0.100	1.727	1.105	0.622
0.200	1.758	1.221	0.537
0.300	1.811	1.349	0.462
0.400	1.887	1.491	0.396
0.500	1.992	1.648	0.344
0.600	2.129	1.822	0.307
0.700	2.306	2.013	0.293
0.800	2.533	2.225	0.308
0.900	2.823	2.459	0.364
1.000	3.194	2.718	0.476

Table 1 shows that the error of an approximation compared to an exact solution is relatively different. Figure 1 shows that, there is no fixed point obtained in the Eq (4.2).

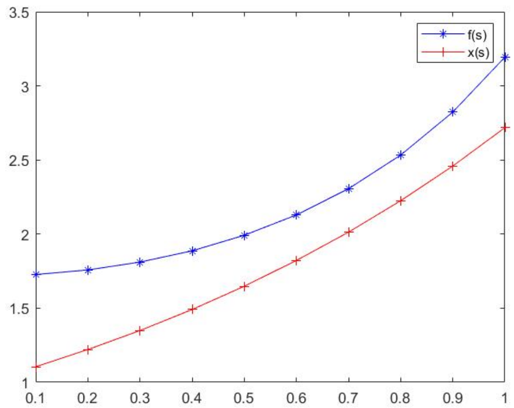

Figure 1. Graph of $f(s)$ compare to $x(s)$ with h=0.1.

5. Conclusions

In this paper, we demonstrated the fixed point theorem for an orthogonal extended i-CRR type $\psi\mathcal{F}$-contraction in an orthogonal complete $\mathfrak{b}$-metric spaces. The derived results have been supported with non-trivial examples. An application to find analytical solution to the functional equation with the numerical solution are presented. It is an open problem to extend the results using orthogonal extended $\mathfrak{b}$-metric spaces using i-CRR type $\psi\mathcal{F}$-contraction.

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Conflict of interest

The authors declare no conflicts of interest.

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