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*Research article*

# Excess lifetime extropy for a mixed system at the system level

**Mohamed Kayid[1,*] and Mashael A. Alshehri[2]**

[1] Department of Statistics and Operations Research, College of Science, King Saud University, Riyadh 11451, Saudi Arabia

[2] Department of Quantitative Analysis, College of Business Administration, King Saud University, Riyadh 1362, Saudi Arabia

* **Correspondence:** Email: drkayid@ksu.edu.sa; Tel: +96614676258; Fax: +96614676274.

**Abstract:** We consider a mixed system with $n$ components, where at time $t$, all system components are functioning. We then use the system signature to evaluate the extropy of the excess lifetime of the mixed system, which is a useful criterion for predicting the lifetime of the system. We give several results including expressions, bounds, and order conditions for the above measure. Finally, based on the relative extropy, we establish a criterion for selecting a preferred system that is closely related to the parallel system.

**Keywords:** coherent system; residual extropy entropy; Shannon entropy; system signature
**Mathematics Subject Classification:** 94A15, 62B10, 60E15

## 1. Introduction

Analysis of distribution functions based on partial information is one of the most important topics in various fields, including biology, survival analysis, reliability engineering, econometrics, statistics, and demography. Model selection, estimation, hypothesis testing, inequality/poverty assessment, and portfolio analysis are some examples of relevant activities. The entropy induced by a probability distribution fulfills numerous applications in information science, physics, probability, statistics, communication theory, and economics since its full introduction in Shannon's extensive article [16]. If $X$ is a nonnegative random variable (RV) with an absolutely continuous distribution with density function (DF) $f(x)$, the Shannon entropy is given by $H(X) = H(f) = -E[\log f(X)]$, assuming that the expected value exists. Recently, a new measure of uncertainty has been proposed by Lad et al. as a complementary dual of entropy, called extropy [7]. If $X$ has survival function (SF) $S(x) = P(X > x)$, the extropy of $X$ is defined as

$$J(X) = J(f) = -\frac{1}{2}\int_0^\infty f^2(x)dx = -\frac{1}{2}E[f(S^{-1}(U))], \tag{1.1}$$

where $U$ is uniformly distributed on $[0,1]$, $S^{-1}(u) = \inf\{x; S(x) \geq u\}$, for $u \in [0,1]$, denotes its quantile function. In contrast, Shannon's measure has been enclosed in a fundamental question since its inception.

For engineers, the performance and quantification of uncertainties over the lifetime of a system is quite necessary. The reliability of a system decreases as uncertainty increases, and systems with longer lifetimes and lower uncertainty are better systems (see, e.g., Ebrahimi and Pellery [2]). If $X$ denotes the lifetime of a system, then $J(X)$ is used to evaluate its uncertainty. Occasionally, information about the current age of the system is available. For example, the system may be known to be functioning at time $t$ and it is therefore interesting to measure how uncertain the residual lifetime of $X$ is after $t$, i.e., $X_t = X - t|X > t$. In such situations, $J(X)$ can rarely be useful. Therefore, the residual extropy is defined as

$$J(X_t) = -\frac{1}{2}\int_0^\infty f_t^2(x)dx = -\frac{1}{2}\int_t^\infty \left(\frac{f(x)}{S(t)}\right)^2 dx, \tag{1.2}$$

$$= -\frac{1}{2}\int_0^1 f_t(S_t^{-1}(u))du, \tag{1.3}$$

in which

$$f_t(x) = \frac{f(x+t)}{S(t)}, \ x, t > 0,$$

is the DF of $X_t$, and $S_t^{-1}(u) = \inf\{x; S_t(x) \geq u\}$ is the right-continuous inverse function of $S_t(x) = S(x+t)/S(t)$, $x, t > 0$. Various properties and applications of extropy are studied by Lad et al. [7], Qiu [9], Qiu and Jia [10,11], and their references. In this case, Qiu et al. [12] explored a formula for the lifetime extropy of a mixed system, see also Kayal [3] and Toomaj and Doostparast [18]. In this study, we consider a mixed system with $n$ components, where at time $t$, all system components are running. The system signature is then used to evaluate the extropy in the remaining lifetime of a coherent system. Recently, Kayid and Alshehri [4] studied the Tsallis entropy of the lifetime of a coherent system with $n$ components, where at time $t$, all system components are running. The purpose of this paper is to study, on the basis of extropy, some variability properties of the lifetime of a mixed system which consists of $n$ components, where at time $t$, all system components are in operation. These variability properties help to investigate uncertainty aspects in the excess lifetime of the system. To achieve this goal, the system signature is used as a tool to establish the extropy of the residual lifetime of a mixed system. We give an explanation of the extropy of the lifetime of a mixed system as a function of some events. On the basis of the obtained expression, stochastic orders of the residual lifetime of mixed systems and limits are determined.

The results reported in this paper are organized as follows: In Section 2, we give an explanation of the extropy of the lifetime of a mixed system for the case where the system components are in operation at time $t$. The signature vector induced by a system is applied for the case where the lifetimes of the components in a mixed system are independent and also identically distributed. In Section 3, the residual entropy is also ordered, after establishing some ordering conditions for the system signature. In

Section 4, some useful thresholds are presented. In Section 5, a new criterion for selecting a preferred mixed system is presented. Some concluding remarks are given in Section 6.

Throughout the paper, "$\leq_{st}$", "$\leq_{hr}$", "$\leq_{lr}$" and "$\leq_d$" shall represent the usual stochastic order, hazard rate order, likelihood ratio order, and dispersive order, respectively. The reader is referred to Shaked and Shanthikumar [15] for the formal definitions and various properties of these stochastic orders.

## 2. Excess lifetime extropy

Here, the system signature is adopted to establish the extropy of the excess lifetime of a mixed system with any system-level structure taking into account that system components are all in operation at time $t$. Mixed systems are recognized as a combination of multiple coherent structures. A coherent system is one if it has no immaterial components and the corresponding structure function is monotonic. The probability vector $\mathbf{p} = (p_1, \ldots, p_n)$ in which the $i$th point is derived as $p_i = P(T = X_{i:n})$, $i = 1, 2, \ldots, n$; is identified as the signature of system (see [14]). Consider a mixed system having components with independent and identically distributed (i.i.d.) lifetimes $X_1, \ldots, X_n$, and a deterministic signature vector $\mathbf{p} = (p_1, \ldots, p_n)$. If $T_t^{1,n} = [T - t | X_{1:n} > t]$, stands for the excess lifetime of the system provided that at time $t$, system components are all running, then from [5] the SF of $T_t^{1,n}$ can be acquired as follows:

$$P(T_t^{1,n} > x) = \sum_{i=1}^{n} p_i P(T_t^{1,i,n} > x), \tag{2.1}$$

where $P(T_t^{1,i,n} > x) = P(X_{i:n} - t > x | X_{1:n} > t)$, $i = 1, 2, \cdots, n$, signifies the SF of the excess lifetime of an $i$-out-of-$n$ system provided that all of the components are operating at time $t$. The SF of $T_t^{1,i,n}$ is

$$P(T_t^{1,i,n} > x) = \sum_{k=0}^{i-1} \binom{n}{k} (1 - S_t(x))^k (S_t(x))^{n-k}, \ x, t > 0.$$

It follows from (2.1) that

$$f_{T_t^{1,n}}(x) = \sum_{i=1}^{n} p_i f_{T_t^{1,i,n}}(x), \ x, t > 0, \tag{2.2}$$

where

$$f_{T_t^{1,i,n}}(x) = \frac{\Gamma(n+1)}{\Gamma(i)\Gamma(n-i+1)} (1 - S_t(x))^{i-1} (S_t(x))^{n-i} f_t(x), \ x, t > 0 \tag{2.3}$$

is the DF of $T_t^{1,i,n}$ and $\Gamma(\cdot)$ denotes the full gamma function. In the following, we will emphasis on the learning of the extropy of the RV $T_t^{1,n}$, which determines the uncertainty degree produced by the DF of $[T - t | X_{1:n} > t]$, in terms of the anticipatory of the excess lifetime of the system. To reach our goal, the transformation $V = S_t(T_t^{1,n})$ is an important tool. It is not hard to observe that $U_{i:n} = S_t(T_t^{1,i,n})$ has beta distribution with parameters $n - i + 1$ and $i$ with the DF

$$g_i(u) = \frac{\Gamma(n+1)}{\Gamma(i)\Gamma(n-i+1)} (1 - u)^{i-1} u^{n-i}, \ \ 0 < u < 1, \ i = 1, \cdots, n. \tag{2.4}$$

We now provide an expression for the extropy of $T_t^{1,n}$.

**Theorem 2.1.** *The extropy of ${T_t}^{1,n}$ can be stated as*

$$J({T_t}^{1,n}) = -\frac{1}{2}\int_0^1 g_V^2(u) f_t(S_t^{-1}(u))du, \tag{2.5}$$

*for all $t > 0$.*

*Proof.* By making the change of $u = S_t(x)$, in spirit of (1.1) and (2.2), we get

$$\begin{aligned} J({T_t}^{1,n}) &= -\frac{1}{2}\int_0^\infty \left(f_{{T_t}^{1,n}}(x)\right)^2 dx \\ &= -\frac{1}{2}\int_0^\infty \left(\sum_{i=1}^n p_i f_{{T_t}^{1,i,n}}(x)\right)^2 dx \\ &= -\frac{1}{2}\int_0^1 \left(\sum_{i=1}^n p_i g_i(u)\right)^2 \left(f_t(S_t^{-1}(u))\right) dx \\ &= -\frac{1}{2}\int_0^1 g_V^2(u) \left(f_t(S_t^{-1}(u))\right) du. \end{aligned}$$

In the final equality $g_V(u) = \sum_{i=1}^n p_i g_i(u)$ is the DF of $V$ denotes the lifetime of the system with i.i.d. uniform distribution.

In the particular case, if an $i$-out-of-$n$ system with the system signature $\mathbf{p} = (0, \ldots, 0, 1_i, 0, \ldots, 0)$, $i = 1, 2, \cdots, n$ is considered, then Eq (2.5) reduces to

$$J(T_t^{1,i,n}) = -\frac{1}{2}\int_0^1 g_i^2(u) f_t(S_t^{-1}(u))du, \tag{2.6}$$

for all $t > 0$.

The following result is a direct conclusion of Theorem 2.1 dealing with the aging paths of system components. It is known that $X$ has an increasing (decreasing) failure rate (IFR(DFR)) if $S_t(x)$ is decreasing (increasing) in $x$ for all $t > 0$.

**Theorem 2.2.** *If $X$ is IFR (DFR), then $J({T_t}^{1,n})$ is decreasing (increasing) in $t$.*

*Proof.* We can verify that $f_t(S_t^{-1}(u)) = u\lambda_t(S_t^{-1}(u))$, $0 < u < 1$. This implies that Eq (2.5) can be rephrased as

$$-2J({T_t}^{1,n}) = \int_0^1 g_V^2(u) u \lambda_t(S_t^{-1}(u))du, \tag{2.7}$$

for all $t$. On the other hand,

$$\lambda_t(S_t^{-1}(u)) = \lambda(S^{-1}(uS(t))), \ 0 < u < 1. \tag{2.8}$$

If $t_1 \leq t_2$, then $S^{-1}(uS(t_1)) \leq S^{-1}(uS(t_2))$ and so when $X$ is IFR(DFR), from (2.8), we have

$$\begin{aligned}\int_0^1 g_V^2(u)u\lambda_{t_1}(S_{t_1}^{-1}(u))du &= \int_0^1 g_V^2(u)u\lambda(S^{-1}(uS(t_1)))du\\ &\leq (\geq) \int_0^1 g_V^2(u)u\lambda(S^{-1}(uS(t_2)))du\\ &= \int_0^1 g_V^2(u)u\lambda_{t_2}(S_{t_2}^{-1}(u))du.\end{aligned}$$

Using (2.7), we get

$$-2J(T_{t_1}{}^{1,n}) \leq (\geq) - 2J(T_{t_2}{}^{1,n}).$$

This implies that $J(T_{t_1}{}^{1,n}) \geq (\leq) J(T_{t_2}{}^{1,n})$ for all $t_1 \leq t_2$ and this completes the proof.

The example below explains Theorems 2.1 and 2.2.

**Example 2.1.** Consider a bridge system shown in Figure 1 with $\mathbf{p} = (0, 1/5, 3/5, 1/5, 0)$ as its signature vector.

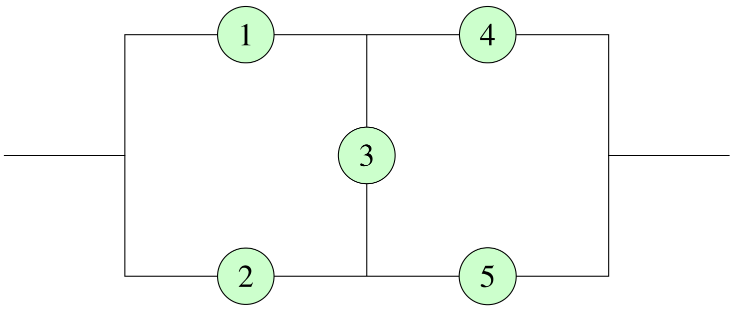


**Figure 1.** A coherent system with signature $\mathbf{p} = (0, 1/5, 3/5, 1/5, 0)$.

The component lifetimes are assumed to follow Weibull distribution with SF

$$S(t) = e^{-t^k},\ k, t > 0. \tag{2.9}$$

By some routine calculation, we get

$$J(T_t{}^{1,5}) = -\frac{k}{2}\int_0^1 \left(t^k - \log u\right)^{(1-\frac{1}{k})} ug_V^2(u)du,\ t > 0.$$

It is a heavy task to establish a plain statement for the foregoing identity. Thus one may need to calculate it numerically. In Figure 2, the extropy of $T_t{}^{1,5}$ is plotted with respect to time $t$ for $k > 0$. In such situation, $X$ is DFR if $0 < k < 1$ and $X$ is IFR when $k > 1$. As expected from Theorem 2.2, it is obvious that $J(T_t{}^{1,5})$ increases(decreases) for $t$ when $0 < k < 1(k > 1)$. The observations are exhibited in Figure 2.

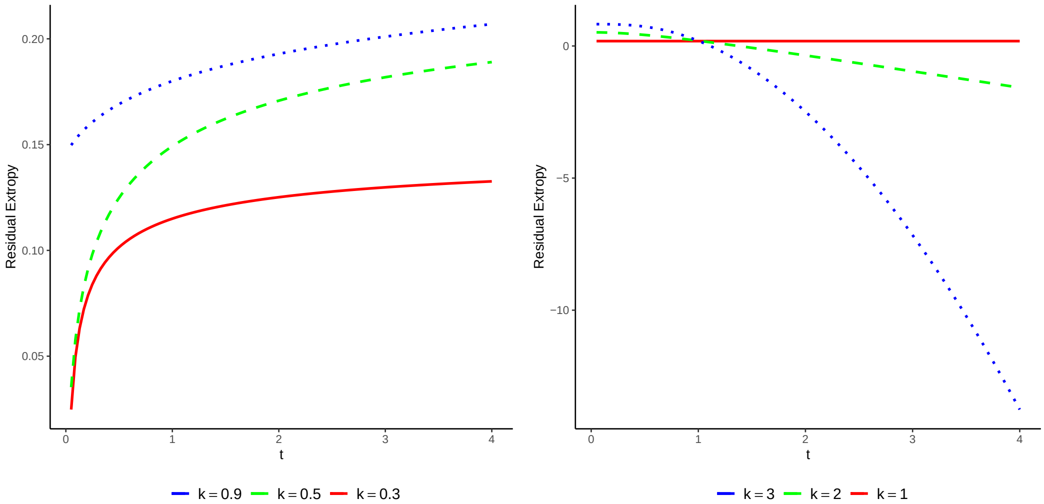


**Figure 2.** The accurate values of $J(T_t^{1,5})$ in terms of $t$ for the Weibull distribution for various values of $k > 0$.

We compare the extropy entropies of two mixed system's lifetimes and their excess lifetimes.

**Theorem 2.3.** *Consider a mixed system having component lifetimes which are i.i.d. and they have IFR (DFR) property. Then* $J(T_t^{1,n}) \leq J(T)$, *for all* $t > 0$.

*Proof.* Since $X$ is IFR (DFR), Theorem 3.B.25 of Shaked and Shanthikumar [15] provides that $X \geq (\leq)_d X_t$, for all $t > 0$, that is

$$f_t(S_t^{-1}(u)) \geq (\leq) f(S^{-1}(u)),\ 0 < u < 1,$$

for all $t > 0$. So, we have

$$\int_0^1 g_V^2(u) f_t(S_t^{-1}(u))du \geq (\leq) \int_0^1 g_V^2(u) f(S^{-1}(u))du,\ t > 0. \tag{2.10}$$

Thus, from (2.5) and (2.10), we obtain

$$\begin{aligned} J(T_t^{1,n}) &= -\frac{1}{2}\int_0^1 g_V^2(u) f_t(S_t^{-1}(u))du \\ &\leq (\geq) -\frac{1}{2}\int_0^1 g_V^2(u) f(S^{-1}(u))du = J(T). \end{aligned}$$

The proof is thus obtained.

A useful concept in technical reliability is the duality of the system, which reduces the computations of the signatures of all coherent systems of a given size approximately to half (see, e.g., Kochar et al. [6]). If $\mathbf{p} = (p_1, \cdots, p_n)$ is the signature of a given mixed system with lifetime $T_t^{1,n}$, then the signature of its dual system with lifetime $T_t^{D,1,n}$ is $\mathbf{p}^D = (p_n, \cdots, p_1)$. In the next theorem, we use the concept of duality to reduce the computation of the residual extropy of mixed systems. First, we need the following lemma.

**Lemma 2.1.** *If* $\phi(x)$ *is a continuous function on* $[0, 1]$ *such that* $\int_0^1 x^n \phi(x)dx = 0$ *for all* $n \geq 0$, *then* $\phi(x) = 0$ *for any* $x \in [0, 1]$.

**Theorem 2.4.** *Let $T_t^{1,n}$ be the lifetime of a mixed system with signature* $\mathbf{p}$ *consisting of n i.i.d. components. Then* $J(T_t^{1,n}) = J(T_t^{D,1,n})$ *for all* $\mathbf{p}$ *and all n if and only if* $f_t(S_t^{-1}(v)) = f_t(S_t^{-1}(1-v))$ *satisfies for all* $0 < v < 1$ *and t.*

*Proof.* For the sufficiency, assume that $f_t(S_t^{-1}(v)) = f_t(S_t^{-1}(1-v))$ for all $0 < v < 1$. Note that $g_i(1-v) = g_{n-i+1}(v)$ for all $i = 1, \ldots, n$ and $0 < v < 1$. Thus, from (2.5) we have that

$$\begin{aligned} J(T_t^{D,1,n}) &= -\frac{1}{2}\int_0^1 \left(\sum_{i=1}^n p_{n-i+1} g_i(u)\right)^2 f_t(S_t^{-1}(u))du \\ &= -\frac{1}{2}\int_0^1 \left(\sum_{r=1}^n p_r g_{n-r+1}(u)\right)^2 f_t(S_t^{-1}(u))du \\ &= -\frac{1}{2}\int_0^1 \left(\sum_{r=1}^n p_r g_r(1-u)\right)^2 f_t(S_t^{-1}(1-u))du \\ &= -\frac{1}{2}\int_0^1 \left(\sum_{r=1}^n p_r g_r(v)\right)^2 f_t(S_t^{-1}(v))dv \\ &= J(T_t^{1,n}). \end{aligned}$$

For the necessity, $J(T_t^{1,n}) = J(T_t^{D,1,n})$ holds for all $\mathbf{p}$ and all $n$. So, let $\mathbf{p} = (1, 0, \ldots, 0)$, then it follows from (2.5) that the assumption $J(T_t^{1,n}) = J(T_t^{D,1,n})$ is equivalent to

$$\begin{aligned} -\frac{1}{2}\int_0^1 g_n^2(u) f_t(S_t^{-1}(u))du &= -\frac{1}{2}\int_0^1 g_1^2(u) f_t(S_t^{-1}(u))du \\ &= -\frac{1}{2}\int_0^1 g_n^2(1-u) f_t(S_t^{-1}(u))du, \end{aligned}$$

where the identity in the last line is acquired from $g_1(u) = g_n(1-u),\ 0 < u < 1$. Putting $v = 1 - u$ in the right side of the foregoing equation yields

$$-\frac{1}{2}\int_0^1 g_n^2(v) f_t(S_t^{-1}(v))dv = -\frac{1}{2}\int_0^1 g_n^2(v) f_t(S_t^{-1}(1-v))dv.$$

Thus, we get

$$\begin{aligned} \int_0^1 g_n^2(v)[f_t(S_t^{-1}(v)) - f_t(S_t^{-1}(1-v))]dv &= \int_0^1 (1-v)^{n-1}[f_t(S_t^{-1}(v)) - f_t(S_t^{-1}(1-v))]dv \\ &= \int_0^1 u^{n-1}[f_t(S_t^{-1}(1-u)) - f_t(S_t^{-1}(u))]du = 0. \end{aligned}$$

Hence, $f_t(S_t^{-1}(1-u)) = f_t(S_t^{-1}(u))$ due to Lemma 2.1 and this completes the proof.

## 3. Extropy comparison

Given the uncertainties of two mixed systems, this section addresses the stochastic ordering of the conditional lifetimes of systems. Given several stochastic ordering properties arising from the component lifetimes of the system and the corresponding signature vector, we present some results to select one of the two mixed systems in terms of extropy. Some recent work that has appeared in the literature on stochastic orders of the residual lifetime of coherent systems can be found in Li and Zhang [8], Zhang [19], Salehi and Tavangar [13] and the references therein. The next result presents a comparison of the extropies of the excess lifetimes in two mixed systems.

**Theorem 3.1.** *Let* $T_t^{X,1,n} = [T - t|X_{1:n} > t]$ *and* $T_t^{Y,1,n} = [T - t|Y_{1:n} > t]$ *denote the excess lifetimes of two mixed systems having a common signature vector and component lifetimes* $X_1, \ldots, X_n$ *and* $Y_1, \ldots, Y_n$ *which are i.i.d. from SF's* $S_X(x)$ *and* $S_Y(x)$*, respectively. If* $X \leq_d Y$ *and* $X$ *or* $Y$ *is IFR, then* $J(T_t^{X,1,n}) \leq J(T_t^{Y,1,n})$ *for all* $t$.

*Proof.* Relying on the relation (2.5), it suffices to prove that $X_t \leq_d Y_t$. Given the assumption that $X \leq_d Y$ and $X$ or $Y$ is IFR, the proof of Theorem 5 in Ebrahimi and Kirmani [1] yields that $X_t \leq_d Y_t$ and so the proof ends.

We compare below the residual extropies of two mixed systems having common component lifetimes but distinct structures.

**Theorem 3.2.** *Let* $T_{1,t}^{1,n} = [T_1 - t|X_{1:n} > t]$ *and* $T_{2,t}^{1,n} = [T_2 - t|X_{1:n} > t]$ *signifies the excess lifetimes of two mixed systems associated with signature vectors* $\mathbf{p_1}$ *and* $\mathbf{p_2}$*, respectively, so that* $\mathbf{p_1} \leq_{lr} \mathbf{p_2}$*. The system components are considered i.i.d. with SF* $S$*. Then,*

**(i)** *if* $f_t(S_t^{-1}(u))$ *increases in* $u$ *for all* $t > 0$*, then* $J(T_{1,t}^{1,n}) \geq J(T_{2,t}^{1,n})$ *for all* $t > 0$*;*

**(ii)** *if* $f_t(S_t^{-1}(u))$ *decreases in* $u$ *for all* $t > 0$*, then* $J(T_{1,t}^{1,n}) \leq J(T_{2,t}^{1,n})$ *for all* $t > 0$.

*Proof.* Note that the Eq (2.5) can be reformulate as below:

$$-2J(T_{t_i}{}^{1,n}) = \int_0^1 g_{V_i}^2(u)du \int_0^1 g_{V_i}^{\star}(u) f_t(S_t^{-1}(u))du \ (i = 1, 2), \tag{3.1}$$

where $V_i^{\star}$ has the DF as

$$g_{V_i}^{\star}(u) = \frac{g_{V_i}^2(u)}{\int_0^1 g_{V_i}^2(u)du}, \ 0 < u < 1.$$

Assumption $\mathbf{p_1} \leq_{lr} \mathbf{p_2}$ implies $V_1 \leq_{lr} V_2$ and this gives that $V_1^{\star} \leq_{lr} V_2^{\star}$ which means that $V_1^{\star} \leq_{st} V_2^{\star}$. So, we obtain

$$\int_0^1 g_{V_1}^{\star}(u) f_t(S_t^{-1}(u))du \quad \leq (\geq) \quad \int_0^1 g_{V_2}^{\star}(u) f_t(S_t^{-1}(u))du, \tag{3.2}$$

in which the inequality in (3.2) is derived from the property that $V_1^{\star} \leq_{st} V_2^{\star}$ implies $\mathbb{E}[\pi(V_1^{\star})] \leq (\geq)\mathbb{E}[\pi(V_2^{\star})]$ for all increasing (decreasing) function $\pi$. Therefore, relation (3.1) gives

$$-2J(T_{t_1}{}^{1,n}) \leq (\geq) - 2J(T_{t_2}{}^{1,n}),$$

or equivalently $J(T_{1,t}{}^{1,n}) \geq (\leq) J(T_{2,t}{}^{1,n})$ for all $t > 0$.

The following example provides a situation where Theorem 3.2 is applicable.

**Example 3.1.** Think about two mixed systems of order 4 having excess lifetimes $T_{1,t}^{1,4} = [T_1 - t|X_{1:4} > t]$ and $T_{2,t}^{1,4} = [T_2 - t|X_{1:4} > t]$ and system signatures $\mathbf{p}_1 = (\frac{1}{2}, \frac{1}{2}, 0, 0)$ and $\mathbf{p}_2 = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, 0)$, respectively. It is uncomplicate to observe and verify that $\mathbf{p_1} \leq_{lr} \mathbf{p_2}$. Suppose that the component lifetimes are i.i.d. with the next SF

$$S(t) = (1 + t)^{-2}, \ t > 0.$$

Appealing to some routine calculation, we obtain $f_t(S_t^{-1}(u)) = \frac{2u\sqrt{u}}{1+t}$, $t > 0$, which is an increasing function in $u$, for all $t > 0$. Hence, by Theorem 3.2, one can claim that $J(T_{1,t}^{1,4}) \geq J(T_{2,t}^{1,4})$, for all $t > 0$. The accurate values of extropy for such systems are reported in Figure 3.

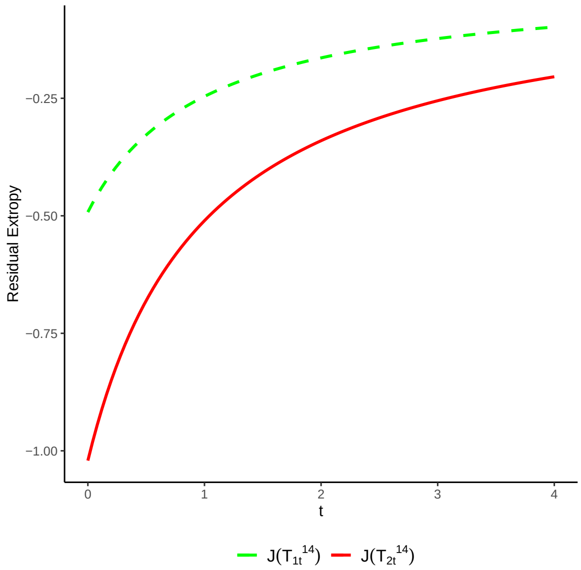


**Figure 3.** Residual extropy comparison of $T_{1,t}^{1,4}$ and $T_{2,t}^{1,4}$ with respect to $t$ given in Example 3.1.

## 4. Some bounds

In situations where systems are complex and components are numerous, it is not easy to determine $J(T_t^{1,n})$ for a mixed system. This situation is common in practice. Under these circumstances, a bound on the residual extropy may be beneficial in predicting the lifetime of a mixed system. For several recent papers on boundary conditions for the uncertainty introduced by the lifetime of mixed systems, the reader is referred to the following sites, for example, [17] and [12] and the references therein. The next result shows bounds on the extropy of survival of mixed systems based on the extropy of survival of $J(X_t)$.

**Theorem 4.1.** *Let $T_t^{1,n} = [T - t|X_{1:n} > t]$ represent the excess lifetime of a mixed system as described before. Suppose that $J(T_t^{1,n}) < \infty$ for all $t$. We have*

$$J(T_t^{1,n}) \geq (B_n(\boldsymbol{p}))^2 \, J(X_t), \tag{4.1}$$

*where $B_n(\boldsymbol{p}) = \sum_{i=1}^{n} p_i g_i(p_i)$, and $p_i = \frac{n-i}{n-1}$.*

*Proof.* The beta distribution with parameters $n-i+1$ and $i$ has a mode equal with $p_i = \frac{n-i}{n-1}$. Therefore,

$$g_V(v) \leq \sum_{i=1}^{n} p_i g_i(p_i) = B_n(\mathbf{p}),\ 0 < v < 1.$$

Thus,

$$\begin{aligned} -2J(T_t^{1,n}) &= \int_0^1 g_V^2(v) f_t(S_t^{-1}(v))dv \\ &\leq (B_n(\mathbf{p}))^2 \int_0^1 f_t(S_t^{-1}(v))dv \\ &= -2\,(B_n(\mathbf{p}))^2\, J(X_t). \end{aligned}$$

The identity in the last line above is derived in view of (1.3). The result is thus proved.

The bound given in (4.1) can be very advantageous when the number of components in systems increases or the structure of the system becomes more complicated. We now derive a general lower bound from mathematical aspects of extropy as a measure of information.

**Theorem 4.2.** *Considering the requirements given in Theorem 4.1,*

$$J(T_t^{1,n}) \geq \sum_{i=1}^{n} p_i J({T_t}^{1,i,n}), \tag{4.2}$$

*for all t.*

*Proof.* Applying Jensen's inequality, one derives

$$-\frac{1}{2}\left(\sum_{i=1}^{n} p_i f_{T_t^{1,i,n}}(x)\right)^2 \geq -\frac{1}{2}\sum_{i=1}^{n} p_i f^2_{T_t^{1,i,n}}(x),\ x, t > 0,$$

and thus one obtains

$$\begin{aligned} J(T_t^{1,n}) &= -\frac{1}{2}\left(\int_0^\infty f^2_{T_t^{1,n}}(x)dx\right) \\ &\geq -\frac{1}{2}\left(\sum_{i=1}^{n} p_i \int_0^\infty f^2_{T_t^{1,i,n}}(x)dx\right) \\ &= \sum_{i=1}^{n} p_i J({T_t}^{1,i,n}), \end{aligned}$$

which finalizes the proof of the theorem.

Note that the equality in (4.2) holds for $i$-out-of-$n$ systems, i.e. $p_j = 0$, for $j \neq i$, and $p_j = 1$, for $j = i$ and then $J(T_t^{1,n}) = J(T_t^{1,i,n})$. The lower bounds of the two assertions in the Theorems 4.1 and 4.2, if they are true and can be computed, their maximum can then be recognized as a sharper lower bound.

**Example 4.1.** Let $T_t^{1,5} = [T - t|X_{1:5} > t]$ speak for the excess lifetime in a mixed system with signature $\mathbf{p} = (0, \frac{3}{10}, \frac{5}{10}, \frac{2}{10}, 0)$ consisting of $n = 5$ i.i.d. component lifetimes with uniform distribution in $[0, 1]$. It is not very complicated to show that $B_5(\mathbf{p}) = 2.22$. Thus, by Theorem 4.1, the extropy of $T_t^{1,5}$ is bounded as follows:

$$J(T_t^{1,5}) \geq \frac{1.11}{(t-1)}, \ 0 < t < 1. \tag{4.3}$$

In addition, the lower bound acquired in (4.2) is found as

$$J(T_t^{1,5}) \geq \frac{[\Gamma(n+1)]^2}{2(t-1)\Gamma(2n)} \sum_{i=1}^{n} p_i \frac{\Gamma(2i-1)\Gamma(2n-2i+1)}{[\Gamma(i)\Gamma(n-i+1)]^2}, \tag{4.4}$$

for all $0 < t < 1$. Considering the uniform distribution for the random lifetimes of the components in the system, we have the limits in (4.3) (dashed line) and also the strict value $J(T_t^{1,5})$ given by (2.5). Moreover, the limits in (4.4) (dashed line) are also given and the associated observations are summarized in Figure 4. In this example, the lower bound in (4.4) is found to be more informative than the lower bound in (4.3).

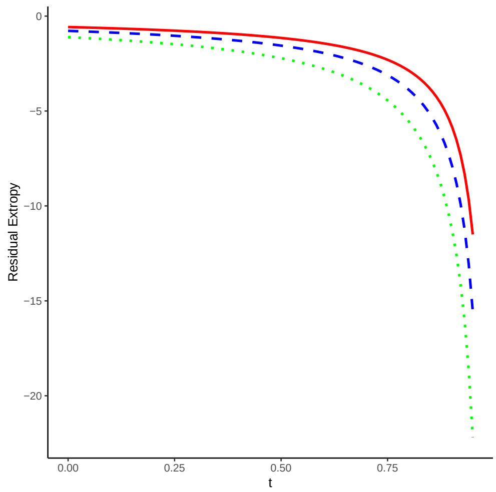


**Figure 4.** Accurate value of $J(T_t^{1,5})$ (solid line) and the lower bounds (4.3) (dotted line) and (4.4) (dashed line) for the standard uniform distribution in terms of time $t$.

## 5. Preferable system

In pairwise comparisons, the physical nature of certain system structures often makes it impossible to use the usual stochastic arrangement. There are many pairs of systems that are not comparable under any of the usual stochastic indices. We explore several metrics for comparing system performance with respect to this type of constraint. In what follows, we present an innovative approach to comparing information measures. In general, engineers agree that a system that performs over time is best. It is important that the characteristics of the competing systems be similar. Thus, given the same characteristics, the parallel system design is found to be more suitable because it has better performance and longer remaining life among all the systems. As for reliability, from (2.1) we have the following property:

$$P(T_t^{1,1,n} > x) \leq P(T_t^{1,n} > x) \leq P(T_t^{1,n,n} > x), \ x > 0,$$

for all $t > 0$. That is, instead of performing a study to compare systems pairwise, a system can be recognized that has a similar structure to the parallel system. In other words, we are looking for an answer to the following question: which of these systems is more similar (or closer) to the configuration of the parallel system and more distant from the configuration of the serial system? To answer this question, we will use the relative extropy distinction. Recall that the relative extropy in a DF $f(x)$ relative to $g(x)$ is defined as follows (see Lad et al. [7]):

$$D(X:Y)=\frac{1}{2}\int_0^\infty [f(x)-g(x)]^2\mathrm{dx}\geq 0, \tag{5.1}$$

provided the integral is finite. The equality is satisfied if, and only if, $f(x)=g(x)$ almost surely. Based on this measure,

$$\begin{aligned} D(T_t^{1,n}:T_t^{1,1,n}) &= \frac{1}{2}\int_0^\infty \left[f_{T_t^{1,n}}(x)-f_{T_t^{1,1,n}}(x)\right]^2 dx, &(5.2)\\ D(T_t^{1,n}:T_t^{1,n,n}) &= \frac{1}{2}\int_0^\infty \left[f_{T_t^{1,n}}(x)-f_{T_t^{1,n,n}}(x)\right]^2 dx. &(5.3)\end{aligned}$$

Large (small) values of $D(T_t^{1,n}:T_t^{1,1,n})$ ($D(T_t^{1,n}:T_t^{1,n,n})$) show that residual lifetime's distribution of the mixed system $T_t^{1,n}$ is far from the residual lifetime's distribution of the parallel system, and is therefore less preferable. Let us define the preferable system criterion as

$$\mathcal{J}(T_t^{1,n})=\frac{D(T_t^{1,n}:T_t^{1,1,n})-D(T_t^{1,n}:T_t^{1,n,n})}{D(T_t^{1,n}:T_t^{1,1,n})+D(T_t^{1,n}:T_t^{1,n,n})}, \tag{5.4}$$

for all $t > 0$. It is obvious that $-1\leq \mathcal{J}(T_t^{1,n})\leq 1$ for all $t>0$. So we can say that $\mathcal{J}(T_t^{1,n})=1$ only if $T_t^{1,n}=T_t^{1,n,n}$ and $\mathcal{J}(T_t^{1,n})=-1$ only if $T_t^{1,n}=T_t^{1,1,n}$. In other words, if $\mathcal{J}(T_t^{1,n})$ is closer to 1, the distribution of $T_t^{1,n}$ is closer to the distribution of the parallel system, and if $\mathcal{J}(T_t^{1,n})$ is closer to $-1$, the distribution of $T_t^{1,n}$ is closer to the distribution of the serial system. It is useful to observe that (5.4) depends on the signature of the system and the parent distribution. We now propose the following definition for the selection of a preferred system.

**Definition 5.1.** *Let $T_{1,t}^{1,n}$ and $T_{2,t}^{1,n}$ be excess lifetimes of two mixed systems with n i.i.d. component lifetimes and dynamic signatures $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$, respectively. We say that $T_{2,t}^{1,n}$ is more preferable than $T_{1,t}^{1,n}$ in terms of the dynamic relative extropy (DRE) at time t, denoted by $T_{1,t}^{1,n}\leq_{DRE} T_{2,t}^{1,n}$, if and only if $\mathcal{J}(T_{1,t}^{1,n})\leq \mathcal{J}(T_{2,t}^{1,n})$ for all $t>0$.*

If $u=S_t(x)$, Eqs (5.2) and (5.3) can then be rewritten as follows:

$$\begin{aligned} D(T_t^{1,n}:T_t^{1,1,n}) &= \frac{1}{2}\int_0^1 [g_V(v)-g_1(v)]^2 f_t(S_t(v))dv,\\ D(T_t^{1,n}:T_t^{1,n,n}) &= \frac{1}{2}\int_0^1 [g_V(v)-g_n(v)]^2 f_t(S_t(v))dv.\end{aligned}$$

In the next example an application of the proposed measure is introduced.

**Example 5.1.** Let $T_{1,t}^{1,4}$ and $T_{2,t}^{1,4}$ denote the lifetimes of two systems with signatures $\mathbf{p}_1 = (1/4, 1/4, 1/2, 0)$ and $\mathbf{p}_2 = (0, 2/3, 1/3, 0)$, respectively. The lifetimes of the components are assumed to be i.i.d. with the standard exponential distribution with the SF $S(x) = e^{-x}, \ x > 0$. We are aware that if system components are all functioning, they are not comparable in the sense of the usual stochastic order. In this case, however, we obtain $\mathcal{J}(T_{1,t}^{1,4}) = 0.4117, \mathcal{J}(T_{2,t}^{1,4}) = 0.5172$ and then $T_{1,t}^{1,4} \leq_{DRE} T_{2,t}^{1,4}$ for all $t > 0$. Therefore, the system with lifetime $T_{2,t}^{1,4}$ is more close to the parallel system and is therefore the preferable one.

## 6. Conclusions

In recent years, there has been increasing interest in quantifying the uncertainty created by the lifetime of engineering systems. This criterion can be used to evaluate predictability with respect to the lifetime of a system. Extropy, as an evolution of Shannon entropy, is very attractive in these cases. In this work, we have found an expression for the extropy of the system lifetime under the condition that all system components are functional at time $t$. Various properties of the proposed measure were also investigated. Some limits were obtained and partial orderings between the excess lifetimes of two mixed systems based on their extropy uncertainties were studied using the concept of system signature. Several examples were also presented to illustrate the applicability of the results.Finally, based on the relative extropy, we introduced a criterion to choose a preferred system that is closely related to the parallel system.

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## Conflict of interest

The authors declare no conflicts of interest.

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