Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(7): 16127–16136.
DOI: 10.3934/math.2023823
Received: 12 February 2023
Revised: 20 April 2023
Accepted: 24 April 2023
Published: 05 May 2023

Research article

Completely independent spanning trees in some Cartesian product graphs

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Abstract: Let $T_1, T_2, \ldots, T_k$ be spanning trees of a graph G. For any two vertices u, v of G, if the paths from u to v in these k trees are pairwise openly disjoint, then we say that $T_1, T_2, \ldots, T_k$ are completely independent. Hasunuma showed that there are two completely independent spanning trees in any 4-connected maximal planar graph, and that given a graph G, the problem of deciding whether there exist two completely independent spanning trees in G is NP-complete. In this paper, we consider the number of completely independent spanning trees in some Cartesian product graphs such as $W_m \square P_n$, $W_m \square C_n$, $K_{m,n} \square P_r$, $K_{m,n} \square C_r$, $K_{m,n,r} \square P_s$, $K_{m,n,r} \square C_s$.

Keywords: completely independent spanning tree; Cartesian product graph; path; cycle; wheel; complete bipartite graph; complete tripartite graph
Mathematics Subject Classification: 05C05

1. Introduction

The graphs considered in this paper are finite, undirected, and simple (no loops or multiple edges). The *vertex set* and the *edge set* of G are denoted by $V(G)$ and $E(G)$, respectively. For a vertex $v \in V(G)$, the *neighbor set* $N_G(v)$ is the set of vertices adjacent to v, $d_G(v) = |N_G(v)|$ is the *degree* of v. For a subgraph H of G, $N_H(v)$ is the set of the neighbours of v which are in H, and $d_H(v) = |N_H(v)|$ is the degree of v in H. When no confusion arises, we shall write $N(v)$ and $d(v)$, instead of $N_G(v)$ and $d_G(v)$.

$$\delta(G) = min\{d(v) : v \in V(G)\}$$

is the *minimum degree* of G. For a subset $U \subseteq V(G)$, the subgraph induced by U is denoted by $G[U]$, which is the graph on U whose edges are precisely the edges of G with both ends in U.

A tree T of G is a *spanning tree* of G if $V(T) = V(G)$. A *leaf* is a vertex of degree 1. An *internal vertex* is a vertex of degree at least 2. A *wheel graph* W_m is a graph with $m(m \geq 4)$ vertices, formed by connecting a single vertex u_0 to all vertices of cycle $C_{m-1} = u_1 u_2 \cdots u_{m-1}$. Its vertex set is $\{u_0, u_1, \cdots, u_{m-1}\}$ and edge set $\{u_0u_i, u_iu_{i+1(mod\ m-1)}|1 \leq i \leq m-1\}$. We called u_0 is a center and $u_0u_i(1 \leq i \leq m-1)$ is a spoke. A graph G is a *complete k-partite graph* if there is a partition $V_1 \cup \cdots \cup V_k$ of the vertex set $V(G)$, such that $uv \in E(G)$ if and only if u and v are in different parts of the partition. If $|V_i| = n_i(1 \leq i \leq k)$, then G is denoted by $K_{n_1,\cdots,n_k}$. Particularly, if $k = 2, 3$, then call it a complete bipartite graph and complete tripartite graph, respectively. Denote P_n and C_n to be path and cycle with n vertices. Given two graphs G and H, the Cartesian product of G and H, denoted by $G\Box H$, is the graph with vertex set

$$V(G\Box H) = V(G) \times V(H),$$

and edge set

$$\{(u, u^{'})(v, v^{'})|(u = v \wedge u^{'}v^{'} \in E(H)) \vee (u^{'} = v^{'} \wedge uv \in E(G))\}.$$

Let x, y be two vertices of G. An (x, y)*-path* is a path with the two ends x and y. Two (x, y)-paths P_1, P_2 are openly disjoint if they have no common edge and no common vertex except for the two ends x and y. Let $T_1, T_2, \ldots, T_k$ be spanning trees in a graph G. If for any two vertices u, v of G, the paths from u to v in $T_1, T_2, \ldots, T_k$ are pairwise openly disjoint, then we say that $T_1, T_2, \ldots, T_k$ are completely independent spanning trees (CISTs) in G. The concept of completely independent spanning trees (CISTs) was proposed by Hasunuma [6] and he gave a characterization for CISTs. It can be seen from the definition that a completely independent spanning tree is an independent spanning tree rooted at any vertex. That is to say, in the study of fault-tolerant broadcasting problems in parallel computing, if we construct a completely independent spanning tree, then when the source vertex becomes any other vertex, there is no need to re-construct the independent spanning tree. In fact, completely independent spanning trees have been studied from not only the theoretical point of view but also the practical point of view because of their applications to fault-tolerant broadcasting in parallel computers [14].

It is well known [16] that every $2k$-edge-connected graph has k edge-disjoint spanning trees. Similarly, Hasunuma [7] conjectured that every $2k$-connected graph has k CISTs. Ten years later, Péterfalvi [18] disproved the conjecture by constructing a k-connected graph, for each $k \geq 2$, which does not have two CISTs. Based on the question raised by Araki [1], in recent years, a specific relationship has been given between Hamilton's sufficient condition and the existence of a completely independent spanning tree, such as Fleischner's condition [1], Dirac's condition [1], Ore's condition [5] and Neighborhood union and intersection condition [13]. Moreover, the Dirac's condition has been generalized to $k(\geq 2)$ completely independent spanning trees [3, 9, 10] and has been independently improved [9, 10] for two completely independent spanning trees. Yuan et al. [20] show that a degree condition for the existence of k CISTs in bipartite graphs. Wang et al. [19] established the number of CISTs that can be constructed in the line graph of the complete graph. Chen et al. [4] proved that if G is a $\{claw, hourglass, P_6^2\}$-free graph with $\delta(G) \geq 4$, then G contains two CISTs if and only if $cl(G)$ has two CISTs. For the result of completely independent spanning trees in the k-th power graph, see [11, 12]. In [7], it has been proved that it is NP-complete to find the number of completely independent spanning trees for a general graph. Therefore, it is meaningful to study the existence of completely independent spanning trees for special graphs. In [8], Hasunuma showed that there are two completely independent spanning trees in the Cartesian product $C_m\Box C_n$ for all $m \geq 3, n \geq 3$. Also, Masayoshi [15] considered the number of completely independent spanning trees in any k-trees. In this paper, we consider the number of completely independent spanning trees in some Cartesian product graphs such as $W_m\Box P_n, W_m\Box C_n, K_{m,n}\Box P_r, K_{m,n}\Box C_r, K_{m,n,r}\Box P_s, K_{m,n,r}\Box C_s$.

2. Preliminaries

Definition 2.1. *([6]) Let $T_1, T_2, \ldots, T_k$ be spanning trees in a graph G. If for any two vertices u, v of G, the paths from u to v in $T_1, T_2, \ldots, T_k$ are pairwise openly disjoint, then we say that $T_1, T_2, \ldots, T_k$ are completely independent spanning trees(CISTs) in G.*

Given a graph G, let mcist(G) be the maximum integer k such that there exist k completely independent spanning trees in G. The following result obtained by Hasunuma [6] plays a key role in our proof.

Lemma 2.1. *([6]) Let $k \geq 2$ be an integer. $T_1, T_2, \cdots, T_k$ are completely independent spanning trees in a graph G if and only if they are edge disjoint spanning trees of G and for any $v \in V(G)$, there is at most one T_i such that $d_{T_i}(v) > 1$.*

It is easy to see from the lemma that there are two completely independent spanning trees in the following graph as Figure 1.

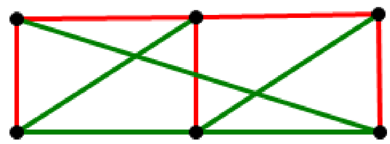

Figure 1. T_1, T_2 are two completely independent spanning trees (red and green).

Lemma 2.2. *([7]) There are two completely independent spanning trees in any 4-connected maximal plane graph.*

Pai et al. [17] showed that the following results.

Lemma 2.3. *([17]) There are $\lfloor \frac{n}{2} \rfloor$ completely independent spanning trees in complete graph K_n for all $n \geq 4$.*

Lemma 2.4. *([17]) There are $\lfloor \frac{n}{2} \rfloor$ completely independent spanning trees in complete bipartite graph $K_{m,n}$ for all $m \geq n \geq 4$.*

Lemma 2.5. *([17]) There are $\lfloor \frac{n_2+n_1}{2} \rfloor$ completely independent spanning trees in complete tripartite graph K_{n_3,n_2,n_1} for all $n_3 \geq n_2 \geq n_1$ and $n_2 + n_1 \geq 4$.*

In 2012, Hasunuma [8] showed that the following result holds.

Lemma 2.6. *([8]) There are two completely independent spanning trees in the Cartesian product of any 2-connected graphs.*

Darties [2] determined the maximum number of completely independent spanning trees in Cartesian product $K_m \Box C_n$.

Lemma 2.7. *([2]) Let $m \geq 3$ and $n \geq 3$ be integers. We have*

$$mcist(K_m \Box C_n) = \begin{cases} \lceil \frac{m}{2} \rceil, & \text{if } (m = 3, 5 \vee (m = 7 \wedge n = 3, 4) \vee (m = 9 \wedge n = 4, 5)); \\ \lfloor \frac{m}{2} \rfloor, & \text{otherwise.} \end{cases}$$

In 2015, Matsushita et al. [15] consider the maximum number of completely independent spanning trees in any k-trees and proved the following result.

Lemma 2.8. *([15]) If $k \geq 3$, then $\lfloor \frac{k+1}{2} \rfloor \leq mcist(G) \leq k-1$ for any k-trees G.*

3. Main results

Theorem 3.1. *Let m and n be integers and $m \geq 4,\ n \geq 2$. We have*

$$mcist(W_m \square P_n) = 2.$$

Proof. Denote wheel

$$V(W_m) = \{u_0, u_1, \ldots, u_{m-1}\}$$

and

$$E(W_m) = \{u_0u_i, u_iu_{i+1(mod\ m-1)} | 1 \leq i \leq m-1\}.$$

The Cartesian product $W_m \square P_n$ is denoted by as follows:

$$V(W_m \square P_n) = \{u_i^j | 0 \leq i \leq m-1, 0 \leq j \leq n-1\},$$

$$E(W_m \square P_n) = \{u_0^j u_k^j | 1 \leq k \leq m-1, 0 \leq j \leq n-1\} \cup \{u_i^j u_{i+1(mod\ m-1)}^j | 1 \leq i \leq m-1, 0 \leq j \leq n-1\}$$
$$\cup \{u_i^j u_i^{j+1} | 0 \leq i \leq m-1, 0 \leq j \leq n-2\}.$$

Note that

$$|V(W_m \square P_n)| = mn, |E(W_m \square P_n)| = 3mn - 2n - m.$$

On the one hand, completely independent spanning trees are edge disjoint by Lemma 2 and every spanning tree has $mn-1$ edges, and combining with $m \geq 4,\ n \geq 2$, we have

$$mcist(W_m \square P_n) \leq \lfloor \frac{3mn - 2n - m}{mn - 1} \rfloor \leq 2.$$

On the other hand, we give the lower bound of $mcist(W_m \square P_n)$ by constructing two completely independent spanning trees in $W_m \square P_n$.

We construct two completely independent spanning trees T_1 ,T_2 as follows:

$$E(T_1) = \{u_0^j u_k^j | 2 \leq k \leq m-1, 0 \leq j \leq n-1\} \cup \{u_1^j u_2^j | 0 \leq j \leq n-1\} \cup \{u_0^j u_0^{j+1} | 0 \leq j \leq n-2\},$$

and

$$E(T_2) = \{u_i^j u_{i+1(mod\ m-1)}^j, u_0^j u_1^j | 2 \leq i \leq m-1, 0 \leq j \leq n-1\} \cup \{u_1^j u_1^{j+1} | 0 \leq j \leq n-2\}.$$

It is easy to see that T_1 and T_2 are edge disjoint. Note that the spanning tree T_1 contains $2n$ internal vertices $\{u_0^j, u_2^j | 0 \leq j \leq n-1\}$ which are leaves in T_2. And T_2 contains $(m-2)n$ internal vertices $\{u_1^j, u_i^j | 3 \leq i \leq m-1, 0 \leq j \leq n-1\}$ which are leaves in T_1. Hence, by Lemma 2, T_1 and T_2 are two completely independent spanning trees as Figure 2. Therefore, $mcist(W_m \square P_n) \geq 2$ and further we have $mcist(W_m \square P_n) = 2$.

□

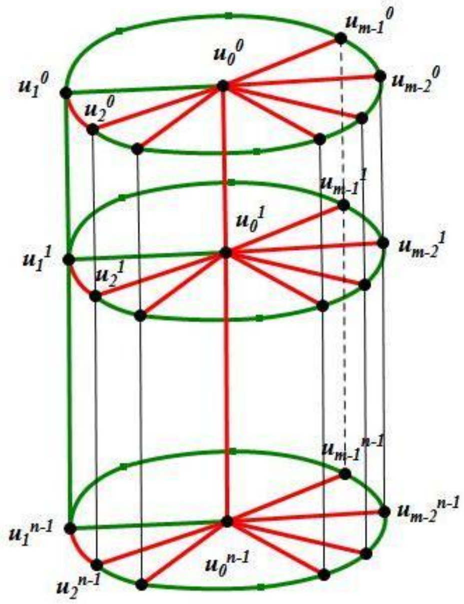

Figure 2. T_1, T_2 are two completely independent spanning trees in $W_m \square P_n$(red and green).

Corollary 3.1. *Let m and n be integers and* $m \geq 4,\ n \geq 2$. *We have*

$$mcist(W_m \square C_n) = 2.$$

Proof. Denote wheel

$$V(W_m) = \{u_0, u_1, \cdots, u_{m-1}\}$$

and

$$E(W_m) = \{u_0 u_i, u_i u_{i+1 (mod\ m-1)} | 1 \leq i \leq m-1\}.$$

The Cartesian product $W_m \square C_n (n \geq 3)$ is denoted by as follows:

$$V(W_m \square C_n) = \{u_i^j | 0 \leq i \leq m-1, 0 \leq j \leq n-1\},$$
$$E(W_m \square C_n) = E(W_m \square P_n) \cup \{u_i^0 u_i^{n-1} | 0 \leq i \leq m-1\}.$$

Note that

$$|V(W_m \square C_n)| = mn, |E(W_m \square C_n)| = 3mn - 2n.$$

On the one hand, completely independent spanning trees are edge disjoint by Lemma 2 and every spanning tree has $mn - 1$ edges, and combining with $m \geq 4, n \geq 2$, we have

$$mcist(W_m \square C_n) \leq \lfloor \frac{3mn - 2n}{mn - 1} \rfloor \leq 2.$$

On the other hand, we give the lower bound of $mcist(W_m \square C_n)$ by constructing two completely independent spanning trees in $W_m \square C_n$. To obtain the lower bound, the construction is similar with Theorem 3.1. Therefore, $mcist(W_m \square C_n) = 2$.

□

Theorem 3.2. *Let m, n, r be integers and $m \geq n \geq 4,\ r \geq 2$. We have*

$$\lfloor \frac{n}{2} \rfloor \leq mcist(K_{m,n} \square P_r) \leq \lfloor \frac{mn + n + m}{m + n - 1} \rfloor.$$

Proof. Denote $K_{m,n}$ is complete bipartite graph with

$$V(K_{m,n}) = \{u_i, v_k | 1 \leq i \leq m, 1 \leq k \leq n\}.$$

We denote the Cartesian product graphs $K_{m,n} \square P_r$ and $K_{m,n} \square C_r (r \geq 3)$ as follows:

$$\begin{aligned} V(K_{m,n} \square P_r) &= V(K_{m,n} \square C_r) = \{u_i^j, v_k^j | 1 \leq i \leq m, 1 \leq k \leq n, 1 \leq j \leq r\}, \\ E(K_{m,n} \square P_r) &= \{u_i^j v_k^j | 1 \leq i \leq m, 1 \leq k \leq n, 1 \leq j \leq r\} \\ &\quad \cup \{u_i^j u_i^{j+1}, v_k^j v_k^{j+1} | 1 \leq i \leq m, 1 \leq k \leq n, 1 \leq j \leq r - 1\}, \\ E(K_{m,n} \square C_r) &= E(K_{m,n} \square P_r) \cup \{u_i^1 u_i^r, v_k^1 u_k^r | 1 \leq i \leq m, 1 \leq k \leq n\}. \end{aligned}$$

Note that

$$|V(K_{m,n} \square P_r)| = mr + nr, |E(K_{m,n} \square P_r)| = mnr + mr + nr - n - m.$$

On one hand, completely independent spanning trees are edge disjoint by Lemma 2 and every spanning tree has $mr + nr - 1$ edges, and combining with $m \geq n \geq 4,\ r \geq 2$, we have

$$mcist(K_{m,n} \square P_r) \leq \lfloor \frac{mnr + mr + nr - n - m}{mr + nr - 1} \rfloor \leq \lfloor \frac{mn + m + n}{m + n - 1} \rfloor.$$

On the other hand, we give the lower bound of $mcist(K_{m,n} \square P_r)$ by constructing $\lfloor \frac{n}{2} \rfloor$ completely independent spanning trees in $K_{m,n} \square P_r$.

We construct $\lfloor \frac{n}{2} \rfloor$ completely independent spanning trees $T_1, \cdots, T_{\lfloor \frac{n}{2} \rfloor}$ as follows:

$$\begin{gathered} E(T_i) = \{u_i^j v_k^j, v_i^j u_l^j | i \leq k \leq \tfrac{n}{2} + i, i \leq l \leq \tfrac{n}{2} + i, 1 \leq j \leq r\} \\ \cup \{u_{i+\frac{n}{2}}^j v_p^j, v_{i+\frac{n}{2}}^j u_q^j | (\tfrac{n}{2} + i < p \leq n) \wedge (1 \leq p < i), \\ (\tfrac{n}{2} + i < q \leq m) \wedge (1 \leq q < i), 1 \leq j \leq r\} \cup \{u_i^j u_i^{j+1} | 1 \leq j \leq r - 1\}\}, i = 1, \cdots, \lfloor \tfrac{n}{2} \rfloor. \end{gathered}$$

It is easy to see that $T_1, T_2, \cdots, T_{\lfloor \frac{n}{2} \rfloor}$ are edge disjoint. Note that every spanning tree T_i contains $4r$ internal vertices $\{u_i^j, v_i^j, u_{\frac{n}{2}+i}^j, v_{\frac{n}{2}+i}^j | 1 \leq j \leq r\}$ which are leaves in $T_j (j \neq i)$. Hence, by Lemma 2, $T_1, T_2, \cdots, T_{\lfloor \frac{n}{2} \rfloor}$ are completely independent spanning trees. Therefore, $mcist(K_{m,n} \square P_r) \geq \lfloor \frac{n}{2} \rfloor$ and further it holds. □

An immediate consequence of the above theorem is the following corollary.

Corollary 3.2. *Let m, n, r be integers and $m \geq n \geq 4,\ r \geq 2$. We have*

$$\lfloor \frac{n}{2} \rfloor \leq mcist(K_{m,n} \square C_r) \leq \lfloor \frac{mn + n + m}{m + n - 1} \rfloor.$$

Theorem 3.3. *Let m, n, r be integers and $m \geq n \geq r,\ n + r \geq 4$. We have*

$$\lfloor \frac{n + r}{2} \rfloor \leq mcist(K_{m,n,r} \square P_s) \leq \lfloor \frac{mn + nr + mr + (m + n + r)}{m + n + r - 1} \rfloor.$$

Proof. Denote $K_{m,n,r}$ is complete tripartite graph with

$$V(K_{m,n,r}) = \{u_i, v_j, w_k | 1 \le i \le m, 1 \le j \le n, 1 \le k \le r\}.$$

We denote the Cartesian product $K_{m,n,r}\Box P_s(s \ge 3)$ and $K_{m,n,r}\Box C_s(s \ge 3)$ as follows:

$$\begin{aligned} V(K_{m,n,r}\Box P_s) &= V(K_{m,n,r}\Box C_s) = \{u_i^l, v_j^l, w_k^l | 1 \le i \le m, 1 \le j \le n, 1 \le k \le r, 1 \le l \le s\},\\ E(K_{m,n,r}\Box P_s) &= \{u_i^l v_j^l, v_j^l w_k^l, w_k^l u_i^l | 1 \le i \le m, 1 \le j \le n, 1 \le k \le r, 1 \le l \le s\}\\ &\quad \cup \{u_i^l u_i^{l+1}, v_j^l v_j^{l+1}, w_k^l w_k^{l+1} | 1 \le i \le m, 1 \le j \le n, 1 \le k \le r, 1 \le l \le s-1\},\\ E(K_{m,n,r}\Box C_s) &= E(K_{m,n,r}\Box P_s) \cup \{u_i^1 u_i^s, v_j^1 v_j^s, , w_k^1 w_k^s | 1 \le i \le m, 1 \le j \le n, 1 \le k \le r\}. \end{aligned}$$

Note that

$$\begin{aligned} |V(K_{m,n,r}\Box P_s)| &= (m+n+r)s,\\ |E(K_{m,n,r}\Box P_s)| &= (mn+mr+nr)s + (m+n+r)(s-1). \end{aligned}$$

On one hand, completely independent spanning trees are edge disjoint by Lemma 2 and every spanning tree has $(m+n+r)s-1$ edges, and combining with $m \ge n \ge r, n+r \ge 4$, we have

$$mcist(K_{m,n,r}\Box P_s) \le \lfloor \frac{(mn+mr+nr)s+(m+n+r)(s-1)}{(m+n+r)s-1} \rfloor \le \lfloor \frac{mn+nr+mr+(m+n+r)}{m+n+r-1} \rfloor.$$

On the other hand, we give the lower bound of $mcist(K_{m,n,r}\Box P_s)$ by constructing $\lfloor \frac{n+r}{2} \rfloor$ completely independent spanning trees in $K_{m,n,r}\Box P_s$.

We construct $\lfloor \frac{n+r}{2} \rfloor$ completely independent spanning trees $T_1, \cdots, T_{\lfloor \frac{n+r}{2} \rfloor}$ as follows:

If $i \le r$, then let

$$\begin{aligned} E(T_i) = &\{w_i^l u_k^l, w_i^l v_{2j}^l | 1 \le k \le m, k \ne i, r < 2j \le n, 1 \le l \le s\} \cup \{v_i^l w_p^l | 1 \le p \le r\}\\ &\cup \{u_i^l v_q^l, u_i^l v_{2j+1}^l | 1 \le q \le r, r < 2j+1 \le n, 1 \le l \le s\}\\ &\cup \{w_i^l w_i^{l+1} | 1 \le l \le s-1\}, i = 1, \cdots, \lfloor \tfrac{n+r}{2} \rfloor. \end{aligned}$$

If $i > r$, then let

$$\begin{aligned} E(T_i) = &\{v_i^l w_k^l, v_i^l u_{2a}^l, v_i^l u_{2b+1}^l | 1 \le k \le r, r < 2a \le i, i+1 < 2b+1 \le n, 1 \le l \le s\}\\ &\cup \{u_i^l v_{2c}^l, u_i^l v_{2d+1}^l | r < 2c \le i, i \le 2d+1 \le n, 1 \le l \le s\}\\ &\cup \{v_{i+1}^l u_t^l, v_{i+1}^l u_{2a+1}^l, v_{i+1}^l u_{2b}^l, v_{i+1}^l u_q^l | 1 \le t \le r,\\ &r < 2a+1 \le i, i \le 2b \le n, n \le q \le m, 1 \le l \le s\}\\ &\cup \{u_{i+1}^l v_k^l, u_{i+1}^l v_{2c+1}^l, u_{i+1}^l v_{2d}^l | 1 \le k \le r, r \le 2c+1 < i, i < 2d \le n, 1 \le l \le s\}\\ &\cup \{v_i^l v_i^{l+1} | 1 \le l \le s-1\}, i = 1, \cdots, \lfloor \tfrac{n+r}{2} \rfloor. \end{aligned}$$

It is easy to see that $T_1, T_2, \cdots, T_{\lfloor \frac{n+r}{2} \rfloor}$ are edge disjoint in Figure 1. Note that every spanning tree T_i contains $3s$ internal vertices $\{u_i^l, v_i^l, w_i^l | 1 \le i \le m, 1 \le l \le s\}$ (or $4s$ internal vertices $\{v_i^l, v_{i+1}^l, w_i^l, w_{i+1}^l | 1 \le l \le s\}$) which are leaves in $T_j(j \ne i)$. Hence, by Lemma 2, $T_1, T_2, \cdots, T_{\lfloor \frac{n+r}{2} \rfloor}$ are completely independent spanning trees as Figure 3. Therefore, $mcist(K_{m,n,r}\Box P_s) \ge \lfloor \frac{n+r}{2} \rfloor$ and further it holds. □

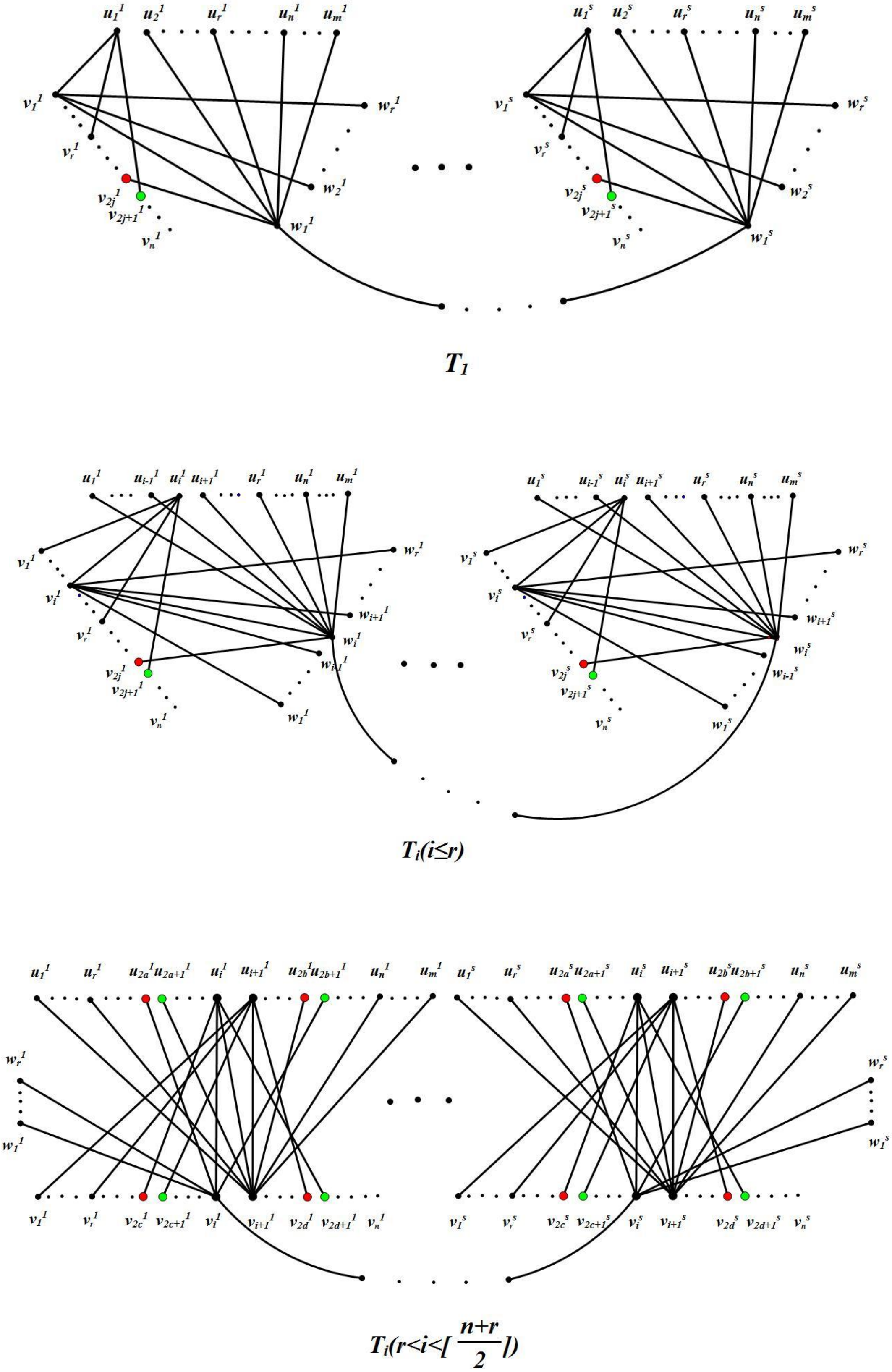

Figure 3. $T_1, T_2, \cdots, T_{\lfloor \frac{n+r}{2} \rfloor}$ are completely independent spanning trees.

An immediate consequence of the above theorem is the following corollary.

Corollary 3.3. *Let m, n, r be integers and $m \geq n \geq r, n + r \geq 4$. We have*

$$\lfloor \frac{n+r}{2} \rfloor \leq mcist(K_{m,n,r} \square C_s) \leq \lfloor \frac{mn + nr + mr + (m + n + r)}{m + n + r - 1} \rfloor.$$

4. Conclusions

Constructing CIST is has many applications on interconnection networks such as fault-tolerant broadcasting and secure message distribution. Hasunuma [7] proved that it is NP-complete to find the number of completely independent spanning trees for a general graph, and Hasunuma [8] showed also that there are two completely independent spanning trees in the Cartesian product $C_m \square C_n$ for all $m \geq 3,\ n \geq 3$. Therefore, it is meaningful to study the existence of completely independent spanning trees for special graphs. In this paper, we cleverly use the characterization of completely independent spanning trees to determine the number of completely independent spanning trees in Cartesian product graphs such as $W_m \square P_n,\ W_m \square C_n,\ K_{m,n} \square P_r,\ K_{m,n} \square C_r,\ K_{m,n,r} \square P_s,\ K_{m,n,r} \square C_s$. It is natural and interesting to consider the following problem, that is,

Problem 4.1. *How can we determine the number of completely independent spanning trees in the Cartesian product graph of any two connected graphs?*

Acknowledgments

The authors thank the referees for their remarks that allowed to improve the clarity of this paper. Xia Hong is financially supported by National Natural Science Foundation of China (Grant No. 12126336) and the Young backbone teachers in Luoyang Normal College (Grant No. 2021XJGGJS-07). Wei Feng is financially supported by Natural Science Foundation of Inner Mongolia autonomous regions (Grant No. 2022LHMS01006) and the Programs Foundation of basic scientific research of colleges and universities directly under Inner Mongolia Autonomous Region in 2022 (Grant No. GXKY22156).

Conflict of interest

The authors declare that they have no conflicts of interest.

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