AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(7): 16062–16072.
DOI:10.3934/math.2023819
Received: 25 February 2023
Revised: 23 April 2023
Accepted: 24 April 2023
Published: 05 May 2023

Research article

The Chen type of Hasimoto surfaces in the Euclidean 3-space

Hassan Al-Zoubi[1,*], **Bendehiba Senoussi**[2], **Mutaz Al-Sabbagh**[3] **and Mehmet Ozdemir**[3]

[1] Department of Mathematics, Al-Zaytoonah University of Jordan, P.O. Box 130, Amman, Jordan 11733

[2] Department of Mathematics, Ecole Normale Superieure, 45 rue d'Ulm, 75230, Mostaganem, Algeria

[3] Department of Basic Engineering Sciences, Imam Abdulrahman bin Faisal University, Dammam 31441, Saudi Arabia

* **Correspondence:** Email: dr.hassanz@zuj.edu.jo; Tel: +962788624525.

Abstract: A surface $\mathcal{M}^2$ with position vector $r = r(s,t)$ is called a Hasimoto surface if the relation $r_t = r_s \wedge r_{ss}$ holds. In this paper, we first define the Beltrami-Laplace operator according to the three fundamental forms of the surface, then we classify the J-harmonic Hasimoto surfaces and their Gauss map in $\mathbb{E}^3$, for $J = II$ and III.

Keywords: Hasimoto surface; Gauss map; Euclidean space; Beltrami-Laplace operator; surfaces of finite type
Mathematics Subject Classification: 14J25, 53Z05

1. Introduction

In the theory of curves in Riemannian manifolds, one of the important and interesting problems is the characterizations of a regular curve. The authors in [1] examined the curvatures of the Hasimoto surface according to Bishop's frame and gave some characterization of the parameter curves for these surfaces. The geometric properties of Hasimoto surfaces are investigated by [2]. In [3], Schief and Rogers studied the binormal motion of curves with constant curvatures. In [4], the authors studied the intrinsic geometry of the nonlinear Schrödinger (NLS) equation in $\mathbb{E}^3$.

Hasimoto surfaces, also known as Frenet frames surfaces, are a class of surfaces in Euclidean 3-space that arise from solutions to the Schrödinger equation in quantum mechanics. Specifically, they are associated with the motion of a charged particle in a magnetic field.

The construction of Hasimoto surfaces involves the use of the Frenet-Serret frame, which is a set of orthonormal vectors that describe the local geometry of a curve or surface. In the case of Hasimoto surfaces, the Frenet-Serret frame is used to describe the motion of a curve in Euclidean 3-space under the influence of a magnetic field. This leads to the construction of a surface in 3-space that has certain interesting properties, such as having a constant mean curvature.

In [5], authors studied Hasimoto surfaces in Minkowski 3-space. On the other hand, M. Elzawy in [6] investigated Hasimoto surfaces in Galilean space G_3.

In this work, we briefly give the geometric properties of Hasimoto surfaces in the Euclidean 3-space. Especially, we obtain the curvatures of Hasimoto surface according to Bishop's frame. Then, we investigate the second Laplace operator for the first, second, and third fundamental forms of Hasimoto surfaces.

Further, one can follow the idea in [7] by defining the first and second Beltrami operators using the definition of the fractional vector operators. It is also interesting, as future work, to do interdisciplinary research, apply a mix or a blend of the techniques followed in [8–11] and combine them with the methods of this paper to obtain more new properties of Hasimoto surfaces.

2. Preliminaries

Let $\phi : I \to \mathcal{M}^2$ be a regular unit speed curve on the orientable surface $\mathcal{M}^2$. Let $\{\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}\}$ be the orthonormal moving Frenet frame along the curve ϕ in $\mathcal{M}^2$ such that $\boldsymbol{T} = \phi'$ is the unit vector field tangent to ϕ, $\boldsymbol{N}$ is the unit vector field in the direction $\boldsymbol{T}'$ normal to ϕ (principal normal) and $\boldsymbol{B} = \boldsymbol{T} \wedge \boldsymbol{N}$ (binormal vector). Then we have the following Frenet equations

$$\begin{pmatrix} \boldsymbol{T}' \\ \boldsymbol{N}' \\ \boldsymbol{B}' \end{pmatrix} = \begin{pmatrix} 0 & k & 0 \\ -k & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{T} \\ \boldsymbol{N} \\ \boldsymbol{B} \end{pmatrix}. \tag{2.1}$$

Functions k and τ are the curvature and the torsion of ϕ.

Introduce a new frame, called Darboux frame $\{\boldsymbol{T},\ \boldsymbol{\eta},\ \boldsymbol{g}\}$ with

$$\begin{pmatrix} \boldsymbol{T} \\ \boldsymbol{\eta} \\ \boldsymbol{g} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & \sin\beta \\ 0 & -\sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} \boldsymbol{T} \\ \boldsymbol{N} \\ \boldsymbol{B} \end{pmatrix}, \tag{2.2}$$

where $\boldsymbol{g} = \boldsymbol{\eta} \wedge \boldsymbol{T}$ and β is the angle between the vector fields $\boldsymbol{N}$ and $\boldsymbol{\eta}$.

The derivative formulas of (2.2) can be given as follows:

$$\begin{pmatrix} \boldsymbol{T}_s \\ \boldsymbol{\eta}_s \\ \boldsymbol{g}_s \end{pmatrix} = \begin{pmatrix} \boldsymbol{T}' \\ \boldsymbol{\eta}' \\ \boldsymbol{g}' \end{pmatrix} = \begin{pmatrix} 0 & k_\eta & k_g \\ -k_\eta & 0 & -t_r \\ -k_g & t_r & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{T} \\ \boldsymbol{\eta} \\ \boldsymbol{g} \end{pmatrix}, \tag{2.3}$$

where k_g is the geodesic curvature, k_η is the normal curvature, t_r is the geodesic torsion of the curve ϕ and $\boldsymbol{T}_s = \frac{d\boldsymbol{T}}{ds}$. From now on we will use the prime $'$ to denote the derivative with respect to the parameter s.

Here Darboux curvatures are defined by

$$k_\eta(s) = k(s)\cos\beta(s),\ \ k_g(s) = -k(s)\sin\beta(s),\ \ t_r(s) = -\tau(s) - \beta'(s). \tag{2.4}$$

Theorem 1. *[12] Suppose $r = r(s,t)$ is an NLS surface such that $r = r(s,t)$ is a unit speed curve with a normal vector field for all t. Then the following is satisfied:*

$$\begin{pmatrix} \boldsymbol{T}_t \\ \boldsymbol{\eta}_t \\ \boldsymbol{g}_t \end{pmatrix} = \begin{pmatrix} 0 & \alpha & \lambda \\ -\alpha & 0 & -\gamma \\ -\lambda & \gamma & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{T} \\ \boldsymbol{\eta} \\ \boldsymbol{g} \end{pmatrix}, \tag{2.5}$$

where α, λ and γ are smooth functions given by

$$\begin{cases} \alpha = k_g' - k_\eta t_r, \\ \lambda = -k_\eta' - k_g t_r, \\ k^2\gamma = (kk')' - \alpha^2 - \lambda^2 + \delta, \end{cases} \tag{2.6}$$

where $\delta = k_{g_t} k_\eta - k_{\eta_t} k_g$ and $\boldsymbol{T}_t = \frac{d\boldsymbol{T}}{dt}$.

From (2.4) and (2.6) we obtain

$$\delta = -\beta_t k^2,$$

$$\alpha^2 + \lambda^2 = k^2\tau^2 + k'^2,$$

$$\alpha k_g - \lambda k_\eta = kk'.$$

Using compatibility conditions $\boldsymbol{T}_{st} = \boldsymbol{T}_{ts}$, $\boldsymbol{\eta}_{st} = \boldsymbol{\eta}_{ts}$ and $\boldsymbol{g}_{st} = \boldsymbol{g}_{ts}$, we get

$$\begin{pmatrix} \alpha' \\ \lambda' \\ \gamma' \end{pmatrix} = \begin{pmatrix} 0 & -t_r & k_g \\ t_r & 0 & -k_\eta \\ -k_g & k_\eta & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \lambda \\ \gamma \end{pmatrix} + \begin{pmatrix} k_{\eta_t} \\ k_{g_t} \\ t_{r_t} \end{pmatrix}.$$

The mean curvature H_{mean} and the Gaussian curvature K_G are, respectively, defined by

$$H_{mean} = \frac{EN + GL - 2FM}{2(EG - F^2)},$$

and

$$K_G = \frac{LN - M^2}{EG - F^2}.$$

The Laplace-Beltrami operator of a smooth function $\varphi : \mathcal{M}^2 \to \mathbb{R}$, with respect to the first fundamental form I of the surface $\mathcal{M}^2$ is the operator Δ^I, defined in [13–15] as follows:

$$\Delta^I \varphi = \frac{-1}{\sqrt{EG - F^2}} \left[\frac{\partial}{\partial s} \left(\frac{G\varphi_s - F\varphi_t}{\sqrt{EG - F^2}} \right) - \frac{\partial}{\partial t} \left(\frac{F\varphi_s - E\varphi_t}{\sqrt{EG - F^2}} \right) \right]. \tag{2.7}$$

The second differential parameter of Beltrami of a function $\varphi : \mathcal{M}^2 \to \mathbb{R}, (s,t) \to \varphi(s,t)$ with respect to the second fundamental form II of $\mathcal{M}^2$ is the operator Δ^{II} which is defined by [13–15]

$$\Delta^{II} \varphi = \frac{-1}{\sqrt{|LN - M^2|}} \left[\frac{\partial}{\partial s} \left(\frac{N\varphi_s - M\varphi_t}{\sqrt{|LN - M^2|}} \right) + \frac{\partial}{\partial t} \left(\frac{L\varphi_t - M\varphi_s}{\sqrt{|LN - M^2|}} \right) \right], \tag{2.8}$$

where $LN - M^2 \neq 0$ since the surface has no parabolic points.

In classical literature, one writes the third fundamental form as

$$III(s,t) = e_{11}ds^2 + 2e_{12}ds\,dt + e_{22}dt^2,$$

where

$$e_{11} =< \boldsymbol{N_s}, \boldsymbol{N_s} >= \frac{EM^2 - 2FLM + GL^2}{EG - F^2},$$

$$e_{12} =< \boldsymbol{N_s}, \boldsymbol{N_t} >= \frac{EMN - FLN + GLM - FM^2}{EG - F^2},$$

$$e_{22} =< \boldsymbol{N_t}, \boldsymbol{N_t} >= \frac{GM^2 - 2FNM + EN^2}{EG - F^2}.$$

The second Beltrami differential operator with respect to the third fundamental form III is defined by [13–15]

$$\Delta^{III} = -\frac{1}{\sqrt{|e|}}\left(\frac{\partial}{\partial x^i}\left(\sqrt{|e|}e^{ij}\frac{\partial}{\partial x^j}\right)\right), \tag{2.9}$$

where $e = \det(e_{ij})$ and e^{ij} denote the components of the inverse tensor of e_{ij}.

3. Hasimoto surfaces

In this section, Hasimoto surfaces are investigated by using the Darboux frame and discussing the geometric properties of Hasimoto surfaces. We find the Gaussian K_G and mean curvatures H_{mean} of this surface.

Let $r = r(s,t)$ be the position vector of a curve ϕ moving on a surface $\mathcal{M}^2$ in $\mathbb{E}^3$ such that $r(s,t)$ is a unit speed curve for all t. If the surface $\mathcal{M}^2$ is a Hasimoto surface, then, the position vector r satisfies the following condition

$$r_t = r_s \wedge r_{ss}. \tag{3.1}$$

This is called the vortex filament or smoke ring equation.

Lemma 1. *[16] The evolution equations for curvature and torsion are*

$$\begin{aligned} k_t &= k\tau' + 2\tau k', \\ \tau_t &= -\left(\frac{k''}{k} - \tau^2\right)' - kk', \\ \beta_t &= \frac{k''}{k} - \tau^2 - \gamma. \end{aligned}$$

The coefficients of the first fundamental form of the surface $r = r(s,t)$ are

$$E = 1, \quad F = 0, \quad G = k^2. \tag{3.2}$$

The unit normal vector of the Hasimoto surface is given by

$$\boldsymbol{N} = -\cos\beta(s)\,\boldsymbol{\eta} + \sin\beta(s)\,\boldsymbol{g}. \tag{3.3}$$

Lemma 2. *The components b_{ij} and e_{ij} of the second and the third fundamental tensors in coordinates are the following*

$$
\begin{aligned}
b_{11} = L &= -k, \quad b_{12} = M = -k\tau, \quad b_{22} = N = -k\tau^2 + k'', \qquad (3.4)\\
e_{11} &= k^2 + \tau^2, \quad e_{12} = \tau(k^2 + \tau^2) - \frac{k''\tau}{k},\\
e_{22} &= k^2\tau^2 + \left(\frac{k''}{k} - \tau^2\right)^2. \qquad (3.5)
\end{aligned}
$$

From (3.2) and (3.4) we have

$$K_G = -\frac{k''}{k}, \tag{3.6}$$

$$H_{mean} = \frac{k'' - k(k^2 + \tau^2)}{2k^2}, \tag{3.7}$$

where $k \neq 0$, since the surface has no parabolic points.

4. J-Hasimoto surface in the Euclidean 3-space

We consider a surface $\mathcal{M}^2$ in $\mathbb{E}^3$ parametrized by

$$r(s,t) = (r_1(s,t),\ r_2(s,t),\ r_3(s,t)). \tag{4.1}$$

Definition 1. *A surface in the three-dimensional Euclidean space is said to be J-harmonic if it satisfies the condition $\Delta^J r = 0$, where Δ^J denotes the Laplace operator with respect to the fundamental forms I–III.*

4.1. I-harmonic Hasimoto surfaces in $\mathbb{E}^3$

Theorem 2. *[16] The Laplacian Δ^I of the Hasimoto surface $r = r(s,t)$ can be expressed as*

$$\Delta^I r(s,t) = \frac{-1}{k}\left[Q(s,t)\boldsymbol{\eta} + P(s,t)\boldsymbol{g}\right], \tag{4.2}$$

where

$$Q(s,t) = -\frac{k_t k_g}{k^2} + kk_\eta + \frac{k_{g_t}}{k} - \frac{\gamma k_\eta}{k}, \quad P(s,t) = \frac{k_t k_\eta}{k^2} + kk_g - \frac{k_{\eta_t}}{k} - \frac{\gamma k_g}{k},$$

$k_{g_t} = \frac{\partial k_g}{\partial t}$, $k_{\eta_t} = \frac{\partial k_\eta}{\partial t}$.

Remark 1.

$$k_\eta Q(s,t) + k_g P(s,t) = k(k^2 + \tau^2) - k''.$$

Corollary 1. *[16] Therefore, r is I-harmonic if and only if $H_{mean} = 0$.*

4.2. II-harmonic Hasimoto surfaces in $\mathbb{E}^3$

In this section, we classify Hasimoto surface with non-degenerate second fundamental form in $\mathbb{E}^3$ satisfying the equation

$$\Delta^{II} r = 0. \tag{4.3}$$

By a straightforward computation, the Laplacian Δ^{II} of the second fundamental form II on $\mathcal{M}^2$ with the help of (2.8) and (3.4) turns out to be

$$\Delta^{II} = \frac{-\varepsilon}{2k^2k''^2}\Big[\Lambda_1(s,t)\frac{\partial}{\partial t} + \Lambda_2(s,t)\frac{\partial}{\partial s} + \Lambda_3(s,t)\frac{\partial^2}{\partial s^2} + \Lambda_4(s,t)\frac{\partial^2}{\partial s\partial t} + \Lambda_5(s,t)\frac{\partial^2}{\partial t^2}\Big], \tag{4.4}$$

where

$$\begin{aligned}
\Lambda_1(s,t) &= 6k^2k''\tau' + \tau k^2k''' + k^3\tau''' - \tau kk'k'' + 4k^2k'\tau'',\\
\Lambda_2(s,t) &= -4k^2\tau(k''\tau' + \tau''k') + k'k''^2 + kk'k''\tau^2 - k^3\tau\tau''' - \tau^2k^2k''' - kk''k''' - 2k^3k'k'',\\
\Lambda_3(s,t) &= 2kk''(k'' - k\tau^2),\\
\Lambda_4(s,t) &= 4\tau k^2k'', \quad \Lambda_5(s,t) = -2k^2k''.
\end{aligned}$$

Theorem 3. *The formula of the Laplacian Δ^{II} takes the following form*

$$\Delta^{II} r(s,t) = \frac{-\varepsilon}{2k^2k''^2}\left[\Gamma_1(s,t)\boldsymbol{T} + \Gamma_2(s,t)\boldsymbol{\eta} + \Gamma_3(s,t)\boldsymbol{g}\right], \tag{4.5}$$

where

$$\begin{aligned}
\Gamma_1(s,t) &= -4k^2\tau(k''\tau' + \tau''k') - k^2\tau(k\tau''' + \tau k''') + k'k''^2 + kk'k''\tau^2 - kk''k''',\\
\Gamma_2(s,t) &= (-4k^3(k''\tau' + \tau''k') + \tau k^2(k'k'' - kk''') - k^4\tau''')\sin\beta + (4k^2k''^2)\cos\beta,\\
\Gamma_3(s,t) &= (-4k^3(k''\tau' + \tau''k') + \tau k^2(k'k'' - kk''') - k^4\tau''')\cos\beta - (4k^2k''^2)\sin\beta,
\end{aligned}$$

and $|LN - M^2| = \varepsilon kk'' \neq 0$, $\varepsilon = \pm 1$ *since the surface has no parabolic points.*

Proof. From (3.1), we have

$$\begin{aligned}
r_s &= \boldsymbol{T},\\
r_t &= -k\sin\beta\,\boldsymbol{\eta} - k\cos\beta\,\boldsymbol{g},\\
r_{ss} &= k\cos\beta\,\boldsymbol{\eta} - k\sin\beta\,\boldsymbol{g},\\
r_{tt} &= -(kk')\boldsymbol{T} + ((k\tau^2 - k'')\cos\beta - (k\tau' + 2\tau k')\sin\beta)\boldsymbol{\eta} - ((k\tau^2 - k'')\sin\beta + (k\tau' + 2\tau k')\cos\beta)\boldsymbol{g},\\
r_{st} &= (k\tau\cos\beta - k'\sin\beta)\boldsymbol{\eta} - (k'\cos\beta + k\tau\sin\beta)\boldsymbol{g}.
\end{aligned}$$

Substituting the last equations into (4.4) gives (4.5). □

If $\Delta^{II} r(s,t) = 0$, then we get $k^2k''^2 = 0$. It contradicts the non-degeneracy of the second fundamental form on $\mathcal{M}^2$.

Theorem 4. *There do not exist Hasimoto surfaces in $\mathbb{E}^3$ which satisfy the condition $\Delta^{II} r(s,t) = 0$.*

4.3. III-harmonic Hasimoto surfaces in $\mathbb{E}^3$

In this section, we classify Hasimoto surface with non-degenerate third fundamental form in $\mathbb{E}^3$ satisfying the equation

$$\Delta^{III} r = 0. \tag{4.6}$$

Theorem 5. *The formula of the Laplacian Δ^{III} takes the following form*

$$\Delta^{III} r(s,t) = \frac{-1}{k''^3}\left[\Theta_1(s,t)\boldsymbol{T} + \Theta_2(s,t)\boldsymbol{\eta} + \Theta_3(s,t)\boldsymbol{g}\right], \tag{4.7}$$

where

$$\begin{aligned}
\Theta_1(s,t) &= (k^2+\tau^2)(\tau^2 k''' + k\tau\tau''' + 3\tau\tau' k'' + 4\tau k'\tau'') \\
&+ kk''k''' - 2k'k''^2 - \frac{k'k''^3}{k^3} - \frac{3\tau\tau' k''^2}{k} - \frac{3\tau^2 k' k''^2}{k^2} + \frac{3\tau^2 k'' k'''}{k},
\end{aligned}$$

$$\begin{aligned}
\Theta_2(s,t) &= \sin\Big(\beta(k^2+\tau^2)(3k\tau' k'' + k\tau k''' + k^2\tau''' + 4kk'\tau'') - \tau' k''^2 \\
&+ 2\tau k'' k''' - \frac{3\tau k' k''^2}{k}\Big) - \cos\beta\left(k''^2(k^2+\tau^2) - \frac{k''^3}{k}\right),
\end{aligned}$$

$$\begin{aligned}
\Theta_3(s,t) &= \cos\beta\Big((k^2+\tau^2)(3k\tau' k'' + k\tau k''' + k^2\tau''' + 4kk'\tau'') - \tau' k''^2 \\
&+ 2\tau k'' k''' - \frac{3\tau k' k''^2}{k}\Big) + \sin\beta\left(k''^2(k^2+\tau^2) - \frac{k''^3}{k}\right).
\end{aligned}$$

Proof. By (2.9), the Laplacian operator Δ^{III} of r can be expressed as

$$\begin{aligned}
\Delta^{III} r(s,t) &= -\frac{1}{k''^3}\left[(k''((e_{22})_s - (e_{12})_t) - k''' e_{22} + (k'')_t e_{12}) r_s \right. \\
&+ (k''((e_{11})_t - (e_{12})_s) + k''' e_{12} - (k'')_t e_{11}) r_t \\
&+ \left.(k'' e_{22}) r_{ss} - 2(k'' e_{12}) r_{st} + (k'' e_{11}) r_{tt}\right].
\end{aligned}$$

Using (3.5), we have (4.7).

□

Remark 2. *We observe that*

$$(k\sin\beta)\,\Theta_1(s,t) - \tau\,\Theta_2(s,t) = k''^3\tau\left(\frac{2\,H_{mean}}{K_G}\right)\cos\beta - k''^3\left(\frac{2\,H_{mean}}{K_G}\right)'\sin\beta, \tag{4.8}$$

$$(\cos\beta)\,\Theta_2(s,t) - (\sin\beta)\,\Theta_3(s,t) = 2k^3 H_{mean} K_G^2. \tag{4.9}$$

S. Stamatakis, H. Al-Zoubi proved in [14] the relation

$$\Delta^{III} r = \nabla^{III}\left(\frac{2H_{mean}}{K_G}, \boldsymbol{N}\right) - \frac{2H_{mean}}{K_G}\boldsymbol{N}. \tag{4.10}$$

From (4.8)–(4.10) we have

Theorem 6. *Let $\mathcal{M}^2$ be a Hasimoto surface in $\mathbb{E}^3$. Then $\mathcal{M}^2$ is III-harmonic if and only if $\mathcal{M}^2$ has zero mean curvature.*

5. Hasimoto surface in the Euclidean 3-space satisfying $\Delta^J \boldsymbol{N} = 0$

In this section, we consider the Gauss map $\boldsymbol{N}$ of the surface $\mathcal{M}^2$ with parametric representation (3.3).

5.1. Hasimoto surface in the Euclidean 3-space satisfying $\Delta^I \boldsymbol{N} = 0$

Let $r = r(s,t)$ be a Hasimoto surface. From (2.7), (3.3) and (3.2), we write the Laplacian operator of the Gauss map as [16]

$$\Delta^I \boldsymbol{N} = -\frac{1}{k^3}\Lambda_1\, \boldsymbol{T} - \frac{1}{k^4}\Lambda_2\, \boldsymbol{\eta} - \frac{1}{k^4}\Lambda_3\, \boldsymbol{g}, \tag{5.1}$$

where

$$\begin{aligned}
\Lambda_1 &= kk'(k^2+\tau^2) - kk'\tau + 3k^2\tau\tau' - kk''' + 2k'k'', \\
\Lambda_2 &= (k^2\tau''' - k^4\tau' - k'k'' + kk'\tau^2 + 4kk''\tau' + 4kk'\tau'' - 4k'k''\tau + \\
&\quad 4k\tau k''' - 4k^2\tau^2\tau')\sin\beta + (k^4(k^2+2\tau^2) + (k''-k\tau^2)^2)\cos\beta, \\
\Lambda_3 &= (k^2\tau''' - k^4\tau' - k'k'' + kk'\tau^2 + 4kk''\tau' + 4kk'\tau'' - 4k'k''\tau + \\
&\quad 4k\tau k''' - 4k^2\tau^2\tau')\cos\beta - (k^4(k^2+2\tau^2) + (k''-k\tau^2)^2)\sin\beta.
\end{aligned}$$

Theorem 7. *[16] Let $r = r(s,t)$ be a Hasimoto surface. There are no Hasimoto surfaces in $\mathbb{E}^3$, satisfying the condition $\Delta^I \boldsymbol{N} = 0$.*

5.2. Hasimoto surface in the Euclidean 3-space satisfying $\Delta^{II} \boldsymbol{N} = 0$

Using (3.3) and (4.4), we obtain

Theorem 8.

$$\Delta^{II} \boldsymbol{N} = \frac{\varepsilon}{2k^2k''}\hat{\Lambda}_1\, \boldsymbol{T} + \frac{\varepsilon}{2k^2k''}\hat{\Lambda}_2\, \boldsymbol{\eta} + \frac{\varepsilon}{2k^2k''}\hat{\Lambda}_3\, \boldsymbol{g}, \tag{5.2}$$

where

$$\begin{aligned}
\hat{\Lambda}_1 &= k(k'k'' - kk''') = k\,K_G', \\
\hat{\Lambda}_2 &= \hat{\mathcal{D}}_0 \sin\beta + \hat{\mathcal{D}}_1\cos\beta, \\
\hat{\Lambda}_3 &= \hat{\mathcal{D}}_0 \cos\beta - \hat{\mathcal{D}}_1\sin\beta, \\
\hat{\mathcal{D}}_0 &= 4k(\tau'k'' + k'\tau'') + 2\tau(kk''' - k'k'') + k^2\tau''' = -k^2(K_G)_t, \\
\hat{\mathcal{D}}_1 &= 2k''(k'' - k(\tau^2+k^2)) = 4k''k^2 H_{mean}.
\end{aligned}$$

Proof. Using (4.4) and

$$\begin{aligned}
\boldsymbol{N}_s &= k\,\boldsymbol{T} - \tau\sin\beta\,\boldsymbol{\eta} - \tau\cos\beta\,\boldsymbol{g}, \\
\boldsymbol{N}_t &= k\tau\,\boldsymbol{T} + \sin\beta\left(\frac{k''}{k} - \tau^2\right)\boldsymbol{\eta} + \cos\beta\left(\frac{k''}{k} - \tau^2\right)\boldsymbol{g}, \\
\boldsymbol{N}_{ss} &= k'\,\boldsymbol{T} + ((k^2+\tau^2)\cos\beta - \tau'\sin\beta)\,\boldsymbol{\eta} - ((k^2+\tau^2)\sin\beta + \tau'\cos\beta)\,\boldsymbol{g}, \\
\boldsymbol{N}_{tt} &= \mathcal{D}_0\,\boldsymbol{T} + (\mathcal{D}_1\cos\beta + \mathcal{D}_2\sin\beta)\,\boldsymbol{\eta} - (\mathcal{D}_1\sin\beta - \mathcal{D}_2\cos\beta)\,\boldsymbol{g}, \\
\boldsymbol{N}_{st} &= \mathcal{D}_3\,\boldsymbol{T} + (\mathcal{D}_4\sin\beta + \mathcal{D}_5\cos\beta)\,\boldsymbol{\eta} - (\mathcal{D}_5\sin\beta - \mathcal{D}_4\cos\beta)\,\boldsymbol{g},
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{D}_0 &= 3k\tau\tau' + k'(\tau^2 - k^2) - k''' + \frac{2k'k''}{k}, \\
\mathcal{D}_1 &= \tau^2(\tau^2 + k^2) + \frac{k''^2}{k^2} - \frac{2k''\tau^2}{k}, \\
\mathcal{D}_2 &= -4\tau'\tau^2 + \tau kk' + \tau''' + \frac{4(\tau'k'' + k'\tau'')}{k} + \frac{4\tau(kk''' - k'k'')}{k^2}, \\
\mathcal{D}_3 &= k'\tau + \tau'k, \\
\mathcal{D}_4 &= \frac{k'''}{k} - \frac{k'k''}{k^2} - 2\tau\tau', \\
\mathcal{D}_5 &= -\frac{\tau k''}{k} + \tau(\tau^2 + k^2) = -2\tau k H_{mean}.
\end{aligned}$$

we have (5.2). □

Suppose that the Hasimoto surface has a II- harmonic Gauss map. Then, the vector $\Delta^{II}\boldsymbol{N}$ given from (5.2) is zero. Thus, we have

$$K_G' = 0, \quad (K_G)_t = 0, \quad H_{mean} = 0.$$

Therefore

$$-\frac{k''}{k} = c, \quad c \in \mathbb{R}; \qquad k'' = k(k^2 + \tau^2).$$

Case I. Suppose $c > 0$. Then from $-\frac{k''}{k} = c$ we have $k^2 + \tau^2 < 0$, it is impossible.
Case II. Suppose $c < 0$. Then $k'' = \alpha^2 k$, where $c = -\alpha^2$.

The solution of the last equation is

$$k = \delta_1(t)\cos\alpha s + \delta_2(t)\sin\alpha s,$$

where δ_1 and δ_2 are smooth functions on open set of $\mathbb{R}$.

Hence, the equation $k'' = k(k^2 + \tau^2)$ implies that

$$\tau = \varepsilon\sqrt{\alpha^2 - k^2}, \qquad \varepsilon = \mp 1.$$

Theorem 9. *Let $r = r(s,t)$ be a Hasimoto surface. $\Delta^{II}\boldsymbol{N} = 0$ if and only if*

$$k = \delta_1(t)\cos\alpha s + \delta_2(t)\sin\alpha s,$$

and

$$\tau = \varepsilon\sqrt{\alpha^2 - k^2},$$

where $\varepsilon = \mp 1$ and δ_1 and δ_2 are smooth functions on open set of $\mathbb{R}$.

5.3. Hasimoto surface in the Euclidean 3-space satisfying $\Delta^{III}\boldsymbol{N} = 0$

In [14] S. Stamatakis, H. Al-Zoubi proved the relation

$$\Delta^{III}\boldsymbol{N} = 2\boldsymbol{N}. \tag{5.3}$$

From (5.3), it can be seen that the Gauss map $\boldsymbol{N}$ of $\mathcal{M}^2$ in $\mathbb{E}^3$ is of finite III-type 1, the corresponding eigenvalue is 2. Then

Theorem 10. *The Gauss map $\boldsymbol{N}$ of a Hasimoto surface $\mathcal{M}^2$ in $\mathbb{E}^3$ is of finite III-type 1, the corresponding eigenvalue is 2.*

6. Conclusions

In the beginning, a brief introduction and definition for the Hasimoto surfaces were given in the Euclidean 3-space. Then, we investigate Hasimoto surfaces by using the Darboux frame and discuss its geometric properties of it. Consequentially, we define a formula for the Laplace operator regarding the first, second, and third fundamental forms. Finally, we classify the Hasimoto surfaces satisfying the relations $\Delta^J \boldsymbol{r} = 0$, and $\Delta^J \boldsymbol{N} = 0$ for $J = I, II$ and III, where $\boldsymbol{N}$ is the Gauss map of $\mathcal{M}^2$ in $\mathbb{E}^3$. We distinguish three types according to whether these surfaces are determined, with each type investigated in a subsection of section 4 and for the Gauss map in Section 5. An interesting study can be drawn, if this type of study can be applied to the general definition of surfaces of finite Chen k-type.

Acknowledgments

Most of this work was done by Dr. Bendehiba Senoussi and I the correspondence author would like to thank him for the great work he did.

Conflict of interest

The authors declare that they have no conflict of interest.

References

1. A. Kelleci, M. Bektaş, M. Ergüt, The Hasimoto surface according to bishop frame, *Adıyaman University Journal of Science*, **9** (2019), 13–22.
2. N. Abdel-All, R. Hussien, T. Youssef, Hasimoto surfaces, *Life Sci. J.*, **9** (2012) 556–560.
3. W. Schief, C. Rogers, Binormal motion of curves of constant curvature and torsion. generation of soliton surfaces, *Proc. R. Soc. Lond.*, **455** (1999), 3163–3188. http://dx.doi.org/10.1098/rspa.1999.0445
4. C. Rogers, W. Schief, Intrinsic geometry of the NLS equation and its backlund transformation, *Stud. Appl. Math.*, **101** (1998), 267–287. http://dx.doi.org/10.1111/1467-9590.00093
5. M. Erdogdu, M. Özdemir, Geometry of Hasimoto surfaces in Minkowski 3-space, *Math. Phys. Anal. Geom.*, **17** (2014), 169–181. http://dx.doi.org/10.1007/s11040-014-9148-3
6. M. Elzawy, Hasimoto surfaces in Galilean space G_3, *J. Egypt. Math. Soc.*, **29** (2021), 5. http://dx.doi.org/10.1186/s42787-021-00113-y
7. M. Mhailan, M. Abu Hammad, M. Al Horani, R. Khalil, On fractional vector analysis, *Journal of Mathematical and Computational Science*, **10** (2020), 2320–2326.
8. Y. Li, Z. Chen, S. Nazra, R. Abdel-Baky, Singularities for timelike developable surfaces in Minkowski 3-space, *Symmetry*, **15** (2023), 277. http://dx.doi.org/10.3390/sym15020277
9. Y. Li, M. Aldossary, R. Abdel-Baky, Spacelike circular surfaces in Minkowski 3-space, *Symmetry*, **15** (2023), 173. http://dx.doi.org/10.3390/sym15010173

10. Y. Li, O. Tuncer, On (contra)pedals and (anti)orthotomics of frontals in de Sitter 2-space, *Math. Method. Appl. Sci.*, in press. http://dx.doi.org/10.1002/mma.9173

11. Y. Li, A. Alkhaldi, A. Ali, R. Abdel-Baky, M. Khalifa Saad, Investigation of ruled surfaces and their singularities according to Blaschke frame in Euclidean 3-space, *AIMS Mathematics*, **8** (2023), 13875–13888. http://dx.doi.org/10.3934/math.2023709

12. M. Erdoğdu, A. Yavuz, Differential geometric aspects of nonlinear Schrödinger equation, *Commun. Fac. Sci. Univ.*, **70** (2021), 510–521. http://dx.doi.org/10.31801/cfsuasmas.724634

13. B. Senoussi, M. Bekkar, Helicoidal surfaces with $\Delta^J r = Ar$ in 3-dimensional Euclidean space, *Stud. Univ. Babes-Bol. Math.*, **60** (2015), 437–448.

14. S. Stamatakis, H. Al-Zoubi, On surfaces of finite Chen-type, *Results Math.*, **43** (2003), 181–190. http://dx.doi.org/10.1007/BF03322734

15. S. Stamatakis, H. Al-Zoubi, Surfaces of revolution satisfying $\Delta^{III} x = Ax$, *J. Geom. Graph.*, **14** (2010), 181–186.

16. B. Senoussi, A. Akbay, Characterizations of Hasimoto surfaces in Euclidean 3-spaces $\mathbb{E}^3$, *Appl. Math. Inf. Sci.*, **16** (2022), 689–694. http://dx.doi.org/10.18576/amis/160504