Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(7): 15950–15968.
DOI: 10.3934/math.2023814
Received: 09 December 2022
Revised: 15 March 2023
Accepted: 20 March 2023
Published: 04 May 2023

Research article

New developments in fractional integral inequalities via convexity with applications

Maimoona Karim[1], Aliya Fahmi[1], Shahid Qaisar[2], Zafar Ullah[3] and Ather Qayyum[4,*]

[1] Department of Mathematics, University of Faisalabad, Pakistan
[2] Department of Mathematics, COMSATS University Islamabad Sahiwal Campus, Pakistan
[3] Department of Mathematics, University of Education, DGK Campus, Pakistan
[4] Department of Mathematics, Institute of Southern Punjab Multan, Pakistan

* **Correspondence:** Email: inequalities1978@gmail.com; Tel: +923014342388.

Abstract: The main objective of this article is to build up a new integral equality related to Riemann Liouville fractional (RLF) operator. Based on this integral equality, we show numerous new inequalities for differentiable convex as well as concave functions which are similar to celebrated Hermite-Hadamard and Simpson's integral inequalities. The present outcomes of this paper are a unification and generalization of the comparable results in the literature on Hermite-Hadamard and Simpson's integral inequalities. Furthermore as applications in numerical analysis, we find some means, q-digamma function and modified Bessel function type inequalities.

Keywords: Simpson's inequality; convex functions; Riemann-Liouville fractional operator; Jensen inequality
Mathematics Subject Classification: 26A33, 26D07, 26D10, 26D15

1. Introduction

An important inequality for classical convex functions which has been extensively studied in recent decades is the Hermite-Hadamard's inequality, which was obtained by Hermite and Hadamard independently. This inequality gives lower and upper estimates for the integral average of any convex function formed on a compact interval encompassing the domain midpoint and endpoints. To, more precise, In [1] Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function on the interval I of real numbers and $\alpha_1, \alpha_2 \in I$ with $\alpha_1 < \alpha_2$. Then

$$f\left(\frac{\alpha_1 + \alpha_2}{2}\right) \leq \frac{1}{\alpha_2 - \alpha_1} \int_{\alpha_1}^{\alpha_2} f(\lambda)\, d\lambda \leq \frac{f(\alpha_1) + f(\alpha_2)}{2}.$$

In the field of analysis, numerous mathematicians have observed the significance of the double inequality and have also used it in various useful applications. Moreover, it has been extended to various structures utilizing the classical convex function. Fractional calculus has applications in a variety of engineering and science domains, including electromagnetic, photoelasticity, fluid mechanics, electrochemistry, biological population models, optics, and signal processing. Due to its vast variety of applications, many mathematicians employed fractional calculus concepts and studied in various areas, one of which is integral inequalities for different classes of functions. For example, some authors, [2–5] obtained the inequalities for Riemann-Liouville fractional integrals and AB-fractional integral operator. Dragomir et al. [6], proved some Hermite-Hadamard type inequalities for Riemann-Liouville fractional integrals. In [7], Dragomir used the generalized form of Riemann-Liouville fractional integrals and proved some new Ostrowski type inequalities for bounded functions. In [8], he used the generalized form of Riemann-Liouville fractional integrals and proved some new trapezoid type inequalities for bounded functions. He gave trapezoid and ostrowski type inequalities using the fractional integrals. Iqbal et al. [9] presented some fractional midpoint type inequalities for convex functions. In recent years much attention has been devoted to the theory of convex sets and theory of convex functions by generalizing and extending these concepts in different dimensions using innovative techniques. Here, we recall the important definitions related to convex function and left-right Riemann-Liouville fractional integrals.

Definition 1.1. *[1] Let I be an interval in $\mathbb{R}$. A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is said to be convex on $[\alpha_1, \alpha_2]$, with $\alpha_1 < \alpha_2$, where $\alpha_1, \alpha_2 \in I$, if,*

$$f(\lambda\alpha_1 + (1-\lambda)\alpha_2) \leq \lambda f(\alpha_1) + (1-\lambda) f(\alpha_2), \quad \lambda \in [0,1].$$

Definition 1.2. *[10] For $f \in L[\alpha_1, \alpha_2]$. The left-sided and right-sided Riemann-Liouville fractional integrals of order $\kappa \in \mathbb{R}^+$ that are defined by*

$$\left(J^{\kappa}_{(\alpha_1)^+} f\right)(x) = \frac{1}{\Gamma(\kappa)} \int_{\alpha_1}^{x} (x-t)^{\kappa-1} f(t)dt, \quad (0 \leq \alpha_1 < x \leq \alpha_2),$$

and

$$\left(J^{\kappa}_{(\alpha_2)^-} f\right)(x) = \frac{1}{\Gamma(\kappa)} \int_{x}^{\alpha_2} (t-x)^{\kappa-1} f(t)dt, \quad (0 \leq \alpha_1 < x < \alpha_2).$$

respectively, where $\Gamma(.)$ is Gamma function and its definition is $\Gamma(\kappa) = \int_0^\infty e^{-u} u^{\kappa-1} du$. It is to be noted that $J^{0}_{(\alpha_1)^+} f(x) = J^{\kappa}_{(\alpha_2)^-} f(x) = f(x)$.

If we put $\kappa = 1$, the fractional integral becomes the classical integral. The recent results and the properties concerning this operator can be found [11, 12].

An inequality which is notable as Simpson's inequality in [1]:

Theorem 1.1. *Suppose $f : [\alpha_1, \alpha_2] \to \mathbb{R}$ is four times continuously differentiable function on (α_1, α_2) and $\left\|f^{(4)}\right\|_\infty = \sup_{\theta \in (\alpha_1, \alpha_2)} \left|f^{(4)}(\theta)\right| < \infty$, then the following inequality holds:*

$$\left|\left[\frac{1}{6} f(\alpha_1) + \frac{2}{3} f\left(\frac{\alpha_1 + \alpha_2}{2}\right) + \frac{1}{6} f(\alpha_2)\right] - \frac{1}{\alpha_2 - \alpha_1} \int_{\alpha_1}^{\alpha_2} f(\theta)\, d\theta\right| \leq \frac{(\alpha_2 - \alpha_1)^4}{2880} \left\|f^{(4)}\right\|_\infty. \tag{1.1}$$

The accompanying ongoing improvements for Riemann-Liouville fractional integral on double and Simpson's inequalities are demonstrated by Hwang et al. (see [13]).

Theorem 1.2. *Let $f:[\alpha_1,\alpha_2]\to\mathbb{R}$ be a differentiable function on (α_1,α_2) and $0<\kappa\le 1$. If $|f'|$ is convex function on $[\alpha_1,\alpha_2]$, then the following inequality holds:*

$$\begin{aligned}&\left|\frac{f(\alpha_1)+f(\alpha_2)}{2}-\frac{\Gamma(\kappa+1)}{2(\alpha_2-\alpha_1)^\kappa}\left[J^\kappa_{(\alpha_2)^-}f(\alpha_1)+J^\kappa_{(\alpha_1)^+}f(\alpha_2)\right]\right|\\ &\le\frac{(\alpha_2-\alpha_1)(2^\kappa-1)}{2^{\kappa+1}(\kappa+1)}\left[|f'(\alpha_1)|+|f'(\alpha_2)|\right].\end{aligned}\tag{1.2}$$

Proposition 1.1. *Suppose that all the assumptions of Theorem 2, are satisfied. If we choose $\kappa=1$, we have trapezoid inequality:*

$$\left|\frac{f(\alpha_1)+f(\alpha_2)}{2}-\frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}f(x)dx\right|\le\frac{\alpha_2-\alpha_1}{8}\left[|f'(\alpha_1)|+|f'(\alpha_2)|\right],\tag{1.3}$$

which is obtained by Dragomir in [14].

Theorem 1.3. *Let $f:[\alpha_1,\alpha_2]\to\mathbb{R}$ be a differentiable function on (α_1,α_2) and $0<\kappa\le 1$. If $|f'|$ is convex function on $[\alpha_1,\alpha_2]$, then the following inequality holds:*

$$\begin{aligned}&\left|\frac{\Gamma(\kappa+1)}{2(\alpha_2-\alpha_1)^\kappa}\left[J^\kappa_{(\alpha_2)^-}f(\alpha_1)+J^\kappa_{(\alpha_1)^+}f(\alpha_2)\right]-f\left(\frac{\alpha_1+\alpha_2}{2}\right)\right|\\ &\le\frac{\alpha_2-\alpha_1}{4(\kappa+1)}\left(\frac{2^{\kappa-1}(\kappa-1)+1}{2^{\kappa-1}}\right)\left[|f'(\alpha_1)|+|f'(\alpha_2)|\right].\end{aligned}\tag{1.4}$$

Proposition 1.2. *Suppose that all the assumptions of Theorem 3, are satisfied. If we choose $\kappa=1$, we have midpoint inequality:*

$$\left|\frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}f(x)dx-f\left(\frac{\alpha_1+\alpha_2}{2}\right)\right|\le\frac{\alpha_2-\alpha_1}{8}\left[|f'(\alpha_1)|+|f'(\alpha_2)|\right],\tag{1.5}$$

which is obtained by Kirmaci in [15].

Theorem 1.4. *Let $f:[\alpha_1,\alpha_2]\to\mathbb{R}$ be a differentiable function on (α_1,α_2) and $0<\kappa\le 1$. If $|f'|$ is convex on $[\alpha_1,\alpha_2]$, then the following inequality holds:*

$$\begin{aligned}&\left|\frac{\Gamma(\kappa+1)}{2(\alpha_2-\alpha_1)^\kappa}\left[J^\kappa_{(\alpha_2)^-}f(\alpha_1)+J^\kappa_{(\alpha_1)^+}f(\alpha_2)\right]-\left[\frac{5^\kappa-1}{6^\kappa}f\left(\frac{\alpha_1+\alpha_2}{2}\right)+\frac{6^\kappa-5^\kappa+1}{6^\kappa}\frac{f(\alpha_1)+f(\alpha_2)}{2}\right]\right|\\ &\le\left[\frac{1}{\kappa+1}\left(\frac{2^\kappa+1}{2^{\kappa+1}}-\frac{5^{\kappa+1}+1}{6^{\kappa+1}}\right)+\left(\frac{5^\kappa-1}{12\times 6^\kappa}\right)\right](\alpha_2-\alpha_1)\left[|f'(\alpha_1)|+|f'(\alpha_2)|\right].\end{aligned}\tag{1.6}$$

Proposition 1.3. *Suppose that all the assumptions of Theorem 1.4, are satisfied. If we choose $\kappa=1$, we have Simpson's inequality:*

$$\begin{aligned}&\left|\frac{1}{6}\left[f(\alpha_1)+4f\left(\frac{\alpha_1+\alpha_2}{2}\right)+f(\alpha_2)\right]-\frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}f(x)\,dx\right|\\ &\le\frac{5(\alpha_2-\alpha_1)}{72}\left[|f'(\alpha_1)|+|f'(\alpha_2)|\right],\end{aligned}\tag{1.7}$$

which is obtained by Sarikaya in [16].

In [17], Lian et al. presented fractional integrals inequalities for concave function as follows,

Theorem 1.5. *Let $f : [\alpha_1, \alpha_2] \to \mathbb{R}$ be a differentiable function on (α_1, α_2) and $0 < \kappa \leq 1$. If $|f'|$ is convex function on $[\alpha_1, \alpha_2]$, then the following inequality holds:*

$$\begin{aligned}&\left|f\left(\frac{\alpha_1+\alpha_2}{2}\right)+\frac{\Gamma(\kappa+1)}{(\alpha_2-\alpha_1)}\left[\left(\frac{2}{\alpha_1-\alpha_2}\right)^{\kappa} J^{\kappa}_{\left(\frac{\alpha_1+\alpha_2}{2}\right)^+} f(\alpha_1)-\left(\frac{2}{\alpha_1-\alpha_2}\right)^{\kappa} J^{\kappa}_{\left(\frac{\alpha_1+\alpha_2}{2}\right)^-} f(\alpha_2)\right]\right| \\ &\leq \frac{\alpha_2-\alpha_1}{4(\kappa+1)}\left[\left|f'(\frac{(\kappa+3)\alpha_1+(\kappa+1)\alpha_2}{2(\kappa+2)})\right|+\left|f'(\frac{(\kappa+1)\alpha_1+(\kappa+3)\alpha_2}{2(\kappa+2)})\right|\right].\end{aligned} \tag{1.8}$$

Proposition 1.4. *Suppose that all the assumptions of Theorem 1.5, are satisfied. If we choose $\kappa = 1$, we have midpoint inequality:*

$$\left|\frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2} f(x)dx - f\left(\frac{\alpha_1+\alpha_2}{2}\right)\right| \leq \frac{\alpha_2-\alpha_1}{8}\left[\left|f'\left(\frac{\alpha_1+2\alpha_2}{3}\right)\right|+\left|f'\left(\frac{2\alpha_1+\alpha_2}{3}\right)\right|\right]. \tag{1.9}$$

The current study is organized in two sections. The first section is related to the introductory body, where ideas and the hypotheses that provides the foundation for the advancement of the work has been discussed. While the second section has been divided into three sub-sections which shows the outcomes acquired for each of the inequalities under investigation. Moreover, the purpose of this paper is to study Hermite-Hadamard and Simpson's-like integral inequalities for convex functions as well as concave functions by applying the fractional concept. We also discuss the relation of our results with comparable results existing in the literature. Furthermore as applications, we find some means, q-digamma function and modified Bessel function type inequalities. We expect that the study initiated in this paper may inspire new research in this area.

2. Main results

Here, we prove an important new Lemma for Riemann-Livouille fractional integrals, which plays a key role to prove our main results as follows:

Lemma 2.1. *Let $f : [\alpha_1, \alpha_2] \to \mathbb{R}$ be a differentiable function on (α_1, α_2) with $\alpha_1 < \alpha_2$. If $f' \in L[\alpha_1, \alpha_2]$ with $0 < \kappa \leq 1$, $\lambda \in [0, 1]$,and $\rho, \vartheta \in [0, 1]$, then the following equality holds:*

$$\begin{aligned}&\frac{2^{\kappa}-2+\rho+\vartheta}{2^{\kappa}}\left(\frac{f(\alpha_1)+f(\alpha_2)}{2}\right)+\frac{2-\rho-\vartheta}{2^{\kappa}} f\left(\frac{\alpha_1+\alpha_2}{2}\right)-\frac{\Gamma(\kappa+1)}{2(\alpha_2-\alpha_1)^{\kappa}}\left[J^{\kappa}_{(\alpha_2)^-} f(\alpha_1)+J^{\kappa}_{(\alpha_1)^+} f(\alpha_2)\right] \\ &= \frac{\alpha_2-\alpha_1}{2^{\kappa+2}}\left(Q_1+Q_2+Q_3+Q_4\right),\end{aligned}$$

where

$$\begin{aligned}Q_1 &= \int_0^1 \left[(1-\lambda)^{\kappa}-\rho\right] f'\left(\lambda\alpha_1+(1-\lambda)\frac{\alpha_1+\alpha_2}{2}\right)d\lambda, \\ Q_2 &= \int_0^1 \left[\vartheta-(1-\lambda)^{\kappa}\right] f'\left(\lambda\alpha_2+(1-\lambda)\frac{\alpha_1+\alpha_2}{2}\right)d\lambda,\end{aligned}$$

$$Q_3 = \int_0^1 \left[(2-\lambda)^\kappa + \rho - 2\right] f'\left(\lambda \frac{\alpha_1+\alpha_2}{2} + (1-\lambda)\,\alpha_2\right) d\lambda,$$

$$Q_4 = \int_0^1 \left[2-\vartheta - (2-\lambda)^\kappa\right] f'\left(\lambda \frac{\alpha_1+\alpha_2}{2} + (1-\lambda)\,\alpha_1\right) d\lambda.$$

Proof. Integrating by parts successively, in order to compute each integral, one obtain □

$$\begin{aligned} Q_1 &= \int_0^1 \left[(1-\lambda)^\kappa - \rho\right] f'\left(\lambda\alpha_1 + (1-\lambda)\frac{\alpha_1+\alpha_2}{2}\right) d\lambda \\ &= \left.\frac{2\left[(1-\lambda)^\kappa - \rho\right] f\left(\lambda\alpha_1 + (1-\lambda)\frac{\alpha_1+\alpha_2}{2}\right) d\lambda}{\alpha_1-\alpha_2}\right|_0^1 \\ &\quad + \frac{2\kappa}{\alpha_1-\alpha_2}\int_0^1 (1-\lambda)^{\kappa-1} f\left(\lambda\alpha_1 + (1-\lambda)\frac{\alpha_1+\alpha_2}{2}\right) d\lambda \\ &= \frac{2\rho}{\alpha_2-\alpha_1} f(\alpha_1) + \frac{2(1-\rho)}{\alpha_2-\alpha_1} f\left(\frac{\alpha_1+\alpha_2}{2}\right) \\ &\quad - \frac{\kappa\left(2^{\kappa+1}\right)}{(\alpha_2-\alpha_1)^{\kappa+1}} \int_{\alpha_1}^{\frac{\alpha_1+\alpha_2}{2}} (x-\alpha_1)^{\kappa-1} f(x)\,dx. \end{aligned} \tag{2.1}$$

Simple calculations analogously

$$\begin{aligned} Q_2 &= \int_0^1 \left[\vartheta - (1-\lambda)^\kappa\right] f'\left(\lambda\alpha_2 + (1-\lambda)\frac{\alpha_1+\alpha_2}{2}\right) d\lambda \\ &= \left.\frac{2\left[\vartheta - (1-\lambda)^\kappa\right] f\left(\lambda\alpha_2 + (1-\lambda)\frac{\alpha_1+\alpha_2}{2}\right) d\lambda}{\alpha_2-\alpha_1}\right|_0^1 \\ &\quad + \frac{2\kappa}{\alpha_1-\alpha_2}\int_0^1 (1-\lambda)^{\kappa-1} f\left(\lambda\alpha_2 + (1-\lambda)\frac{\alpha_1+\alpha_2}{2}\right) d\lambda \\ &= \frac{2\vartheta}{\alpha_2-\alpha_1} f(\alpha_2) + \frac{2(1-\vartheta)}{\alpha_2-\alpha_1} f\left(\frac{\alpha_1+\alpha_2}{2}\right) \\ &\quad - \frac{\kappa\left(2^{\kappa+1}\right)}{(\alpha_2-\alpha_1)^{\kappa+1}} \int_{\frac{\alpha_1+\alpha_2}{2}}^{\alpha_2} (\alpha_2-x)^{\kappa-1} f(x)\,dx. \end{aligned} \tag{2.2}$$

$$\begin{aligned} Q_3 &= \int_0^1 \left[(2-\lambda)^\kappa + \rho - 2\right] f'\left(\lambda\frac{\alpha_1+\alpha_2}{2} + (1-\lambda)\,\alpha_2\right) d\lambda \\ &= \left.\frac{2\left[(2-\lambda)^\kappa + \rho - 2\right] f\left(\lambda\frac{\alpha_1+\alpha_2}{2} + (1-\lambda)\,\alpha_2\right) d\lambda}{\alpha_1-\alpha_2}\right|_0^1 \\ &\quad + \frac{2\kappa}{\alpha_1-\alpha_2}\int_0^1 (2-\lambda)^{\kappa-1} f\left(\lambda\frac{\alpha_1+\alpha_2}{2} + (1-\lambda)\,\alpha_2\right) d\lambda \\ &= \frac{2\left(2^\kappa - 2 + \rho\right)}{\alpha_2-\alpha_1} f(\alpha_2) + \frac{2(1-\rho)}{\alpha_2-\alpha_1} f\left(\frac{\alpha_1+\alpha_2}{2}\right) \\ &\quad - \frac{\kappa\left(2^{\kappa+1}\right)}{(\alpha_2-\alpha_1)^{\kappa+1}} \int_{\frac{\alpha_1+\alpha_2}{2}}^{\alpha_2} (x-\alpha_1)^{\kappa-1} f(x)\,dx. \end{aligned} \tag{2.3}$$

$$
\begin{aligned}
Q_4 &= \int_0^1 \left[2-\vartheta-(2-\lambda)^{\kappa}\right] f'\left(\lambda\frac{\alpha_1+\alpha_2}{2}+(1-\lambda)\,\alpha_1\right) d\lambda \\
&= \left.\frac{2\left[2-\vartheta-(2-\lambda)^{\kappa}\right] f\left(\lambda\frac{\alpha_1+\alpha_2}{2}+(1-\lambda)\,\alpha_1\right) d\lambda}{\alpha_2-\alpha_1}\right|_0^1 \\
&\quad +\frac{2\kappa}{\alpha_1-\alpha_2}\int_0^1 (2-\lambda)^{\kappa-1} f\left(\lambda\frac{\alpha_1+\alpha_2}{2}+(1-\lambda)\,\alpha_1\right) d\lambda \\
&= \frac{2\left(2^{\kappa}-2+\vartheta\right)}{\alpha_2-\alpha_1} f(\alpha_1)+\frac{2(1-\vartheta)}{\alpha_2-\alpha_1} f\left(\frac{\alpha_1+\alpha_2}{2}\right) \\
&\quad -\frac{\kappa\left(2^{\kappa+1}\right)}{(\alpha_2-\alpha_1)^{\kappa+1}}\int_{\alpha_1}^{\frac{\alpha_1+\alpha_2}{2}} (\alpha_2-x)^{\kappa-1} f(x)\,dx.
\end{aligned}
\tag{2.4}
$$

Hence, by adding (2.1)–(2.4), and multiplying the resultant one by $\frac{\alpha_2-\alpha_1}{2^{\kappa+2}}$, we obtain the resultant equality.

Theorem 2.1. *Let $f:[\alpha_1,\alpha_2]\to\mathbb{R}$ be a differentiable function on (α_1,α_2) with $\alpha_1<\alpha_2$. If $f'\in L^1[\alpha_1,\alpha_2]$ with $0<\kappa\le 1$ and $\rho,\vartheta\in[0,1]$, and $|f'|$ is a convex on $[\alpha_1,\alpha_2]$, then the following inequality holds:*

$$
\begin{aligned}
&\left|\frac{2^{\kappa}-2+\rho+\vartheta}{2^{\kappa}}\left(\frac{f(\alpha_1)+f(\alpha_2)}{2}\right)+\frac{2-\rho-\vartheta}{2^{\kappa}} f\left(\frac{\alpha_1+\alpha_2}{2}\right)\right. \\
&\left.-\frac{\Gamma(\kappa+1)}{2(\alpha_2-\alpha_1)^{\kappa}}\left[J^{\kappa}_{(\alpha_2)^-} f(\alpha_1)+J^{\kappa}_{(\alpha_1)^+} f(\alpha_2)\right]\right| \\
&\le \frac{(\alpha_2-\alpha_1)(U_1+U_2+U_3+U_4+U_5+U_6+U_7+U_8)}{2^{\kappa+2}}\left(|f'(\alpha_1)|+|f'(\alpha_2)|\right),
\end{aligned}
\tag{2.5}
$$

where,

$$
\begin{aligned}
U_1 &= \int_0^1 \left|\left[(1-\lambda)^{\kappa}-\rho\right]\right| \lambda\, d\lambda \\
&= \left[\frac{1-2\left(1-\left(1-\rho^{\frac{1}{\kappa}}\right)\right)^{\kappa+2}}{(\kappa+1)(\kappa+2)}-\frac{2\left(1-\rho^{\frac{1}{\kappa}}\right)\left(1-\left(1-\rho^{\frac{1}{\kappa}}\right)\right)^{\kappa+1}}{(\kappa+1)}\right. \\
&\quad \left.+\frac{\rho}{2}\left(1-2\left(\left(1-\rho^{\frac{1}{\kappa}}\right)\right)^2\right)\right].
\end{aligned}
$$

$$
\begin{aligned}
U_2 &= \int_0^1 \left|\left[(1-\lambda)^{\kappa}-\rho\right]\right| (1-\lambda)\, d\lambda \\
&= \left[\frac{2\left(\rho^{\frac{1}{\kappa}}\right)^{\kappa+1}\left(1-\rho^{\frac{1}{\kappa}}\right)}{(\kappa+1)}+\frac{1-2\left(\rho^{\frac{1}{\kappa}}\right)^{\kappa+1}}{(\kappa+1)}-\frac{1-2\left(\rho^{\frac{1}{\kappa}}\right)^{\kappa+2}}{(\kappa+1)(\kappa+2)}\right. \\
&\quad \left.+\rho\left(1-2\left(1-\rho^{\frac{1}{\kappa}}\right)\right)-\frac{\rho}{2}\left(1-2\left(1-\rho^{\frac{1}{\kappa}}\right)^2\right)\right].
\end{aligned}
$$

$$
\begin{aligned}
U_3 &= \int_0^1 \left|\left[\vartheta-(1-\lambda)^\kappa\right]\right| \lambda d\lambda \\
&= \left[\frac{1-2\left(1-\left(1-\vartheta^{\frac{1}{\kappa}}\right)\right)^{\kappa+2}}{(\kappa+1)(\kappa+2)}-\frac{2\left(1-\vartheta^{\frac{1}{\kappa}}\right)\left(1-\left(1-\vartheta^{\frac{1}{\kappa}}\right)\right)^{\kappa+1}}{(\kappa+1)}\right. \\
&\quad \left.+\frac{\vartheta}{2}\left(1-2\left(\left(1-\vartheta^{\frac{1}{\kappa}}\right)\right)^2\right)\right].
\end{aligned}
$$

$$
\begin{aligned}
U_4 &= \int_0^1 \left|\left[\vartheta-(1-\lambda)^\kappa\right]\right| (1-\lambda)\, d\lambda \\
&= \left[\frac{2\left(\vartheta^{\frac{1}{\kappa}}\right)^{\kappa+1}\left(1-\vartheta^{\frac{1}{\kappa}}\right)}{(\kappa+1)}+\frac{1-2\left(\vartheta^{\frac{1}{\kappa}}\right)^{\kappa+1}}{(\kappa+1)}-\frac{1-2\left(\vartheta^{\frac{1}{\kappa}}\right)^{\kappa+2}}{(\kappa+1)\,(\kappa+2)}\right. \\
&\quad \left.+\,\vartheta\left(1-2\left(1-\vartheta^{\frac{1}{\alpha}}\right)\right)-\frac{\vartheta}{2}\left(1-2\left(1-\vartheta^{\frac{1}{\alpha}}\right)^2\right)\right].
\end{aligned}
$$

$$
\begin{aligned}
U_5 &= \int_0^1 \left|\left[(2-\lambda)^\kappa+\rho-2\right]\right| \lambda d\lambda \\
&= \frac{1+2^{\kappa+2}-2(2-(2-(2-\rho))^{\frac{1}{\kappa}})^{\kappa+2}}{\kappa^2+3\kappa+2} \\
&\quad +\frac{1-2\,(2-(2-\rho))^{\frac{1}{\kappa}}\left((2-(2-(2-\rho))^{\frac{1}{\kappa}})^{\kappa+1}\right)}{(\kappa+1)} \\
&\quad +(2-\rho)\left(\frac{1}{2}-\left((2-(2-\rho))^{\frac{1}{\kappa}}\right)^2\right).
\end{aligned}
$$

$$
\begin{aligned}
U_6 &= \int_0^1 \left|\left[(2-\lambda)^\kappa+\rho-2\right]\right| (1-\lambda)\, d\lambda \\
&= \frac{1}{2}\rho\left(-\frac{4(\kappa(\rho-3)+\rho-2)(2-\rho)^\kappa}{(\kappa+1)(\kappa+2)}-2\rho^2+8\rho-9\right) \\
&\quad +\frac{\kappa\left(-4(2-\rho)^\kappa+2^{\kappa+1}+\kappa+3\right)+1}{(\kappa+1)(\kappa+2)}.
\end{aligned}
$$

$$
\begin{aligned}
U_7 &= \int_0^1 \left|\left[2-\vartheta-(2-\lambda)^\kappa\right]\right| \lambda d\lambda \\
&= \frac{1+2^{\kappa+2}-2(2-(2-(2-\vartheta))^{\frac{1}{\kappa}})^{\kappa+2}}{\kappa^2+3\kappa+2} \\
&\quad +\frac{1-2\,(2-(2-\vartheta))^{\frac{1}{\kappa}}\left((2-(2-(2-\vartheta))^{\frac{1}{\kappa}})^{\kappa+1}\right)}{(\kappa+1)} \\
&\quad +(2-\vartheta)\left(\frac{1}{2}-\left((2-(2-\vartheta))^{\frac{1}{\kappa}}\right)^2\right).
\end{aligned}
$$

$$\begin{aligned}
U_8 &= \int_0^1 \left|[2-\vartheta-(2-\lambda)^\kappa]\right|(1-\lambda)\,d\lambda \\
&= \frac{1}{2}\vartheta\left(-\frac{4(\kappa(\vartheta-3)+\vartheta-2)(2-\vartheta)^\kappa}{(\kappa+1)(\kappa+2)}-2\vartheta^2+8\vartheta-9\right) \\
&\quad+\frac{\kappa\left(-4(2-\vartheta)^\kappa+2^{\kappa+1}+\kappa+3\right)+1}{(\kappa+1)(\kappa+2)}.
\end{aligned}$$

Proof. From Lemma 2.1 and convexity, it follows that, □

$$\begin{aligned}
&\left|\frac{2^\kappa-2+\rho+\vartheta}{2^\kappa}\left(\frac{f(\alpha_1)+f(\alpha_2)}{2}\right)+\frac{2-\rho-\vartheta}{2^\kappa}f\left(\frac{\alpha_1+\alpha_2}{2}\right)\right. \\
&\left.-\frac{\Gamma(\kappa+1)}{2(\alpha_2-\alpha_1)^\kappa}\left[J^\kappa_{(\alpha_2)^-}f(\alpha_1)+J^\kappa_{(\alpha_1)^+}f(\alpha_2)\right]\right| \\
&=\left|\frac{\alpha_2-\alpha_1}{2^{\kappa+2}}\int_0^1[(1-\lambda)^\kappa-\rho]\,f'\left(\lambda\alpha_1+(1-\lambda)\,\frac{\alpha_1+\alpha_2}{2}\right)d\lambda\right| \\
&+\left|\frac{\alpha_2-\alpha_1}{2^{\kappa+2}}\int_0^1[\vartheta-(1-\lambda)^\kappa]\,f'\left(\lambda\alpha_2+(1-\lambda)\,\frac{\alpha_1+\alpha_2}{2}\right)d\lambda\right| \\
&+\left|\frac{\alpha_2-\alpha_1}{2^{\kappa+2}}\int_0^1[(2-\lambda)^\kappa+\rho-2]\,f'\left(\lambda\frac{\alpha_1+\alpha_2}{2}+(1-\lambda)\,\alpha_2\right)d\lambda\right| \\
&+\left|\frac{\alpha_2-\alpha_1}{2^{\kappa+2}}\int_0^1[2-\vartheta-(2-\lambda)^\kappa]\,f'\left(\lambda\frac{\alpha_1+\alpha_2}{2}+(1-\lambda)\,\alpha_1\right)d\lambda\right|
\end{aligned}$$

$$\begin{aligned}
\leq\ & \frac{\alpha_2-\alpha_1}{2^{\kappa+2}}\int_0^1\left|[(1-\lambda)^\kappa-\rho]\right||f'\left(\lambda\alpha_1+(1-\lambda)\,\frac{\alpha_1+\alpha_2}{2}\right)|d\lambda \\
&+\frac{\alpha_2-\alpha_1}{2^{\kappa+2}}\int_0^1\left|[\vartheta-(1-\lambda)^\kappa]\right||f'\left(\lambda\alpha_2+(1-\lambda)\,\frac{\alpha_1+\alpha_2}{2}\right)|d\lambda \\
&+\frac{\alpha_2-\alpha_1}{2^{\kappa+2}}\int_0^1\left|[(2-\lambda)^\kappa+\rho-2]\right||f'\left(\lambda\frac{\alpha_1+\alpha_2}{2}+(1-\lambda)\,\alpha_2\right)|d\lambda \\
&+\frac{\alpha_2-\alpha_1}{2^{\kappa+2}}\int_0^1\left|[2-\vartheta-(2-\lambda)^\kappa]\right||f'\left(\lambda\frac{\alpha_1+\alpha_2}{2}+(1-\lambda)\,\alpha_1\right)|d\lambda
\end{aligned}$$

$$\begin{aligned}
\leq\ & \frac{\alpha_2-\alpha_1}{2^{\kappa+2}}\int_0^1\left|[(1-\lambda)^\kappa-\rho]\right|\left[\lambda\,|f'(\alpha_1)|+(1-\lambda)\left|f'(\frac{\alpha_1+\alpha_2}{2})\right|\right]d\lambda \\
&+\frac{\alpha_2-\alpha_1}{2^{\kappa+2}}\int_0^1\left|[\vartheta-(1-\lambda)^\kappa]\right|\left[\lambda\,|f'(\alpha_2)|+(1-\lambda)\left|f'(\frac{\alpha_1+\alpha_2}{2})\right|\right]d\lambda \\
&+\frac{\alpha_2-\alpha_1}{2^{\kappa+2}}\int_0^1\left|[(2-\lambda)^\kappa+\rho-2]\right|\left[\lambda\left|f'(\frac{\alpha_1+\alpha_2}{2})\right|+(1-\lambda)\,|f'(\alpha_2)|\right]d\lambda \\
&+\frac{\alpha_2-\alpha_1}{2^{\kappa+2}}\int_0^1\left|[2-\vartheta-(2-\lambda)^\kappa]\right|\left[\lambda\left|f'(\frac{\alpha_1+\alpha_2}{2})\right|+(1-\lambda)\,|f'(\alpha_1)|\right]d\lambda
\end{aligned}$$

$$
\begin{aligned}
&\left|\frac{2^{\kappa}-2+\rho+\vartheta}{2^{\kappa}}\left(\frac{f(\alpha_1)+f(\alpha_2)}{2}\right)+\frac{2-\rho-\vartheta}{2^{\kappa}}f\left(\frac{\alpha_1+\alpha_2}{2}\right)\right.\\
&\left.-\frac{\Gamma(\kappa+1)}{2(\alpha_2-\alpha_1)^{\kappa}}\left[J^{\kappa}_{(\alpha_2)^-}f(\alpha_1)+J^{\kappa}_{(\alpha_1)^+}f(\alpha_2)\right]\right|\\
\leq\ &\frac{(\alpha_2-\alpha_1)(U_1+U_2+U_3+U_4+U_5+U_6+U_7+U_8)}{2^{\kappa+2}}(\,|f'(\alpha_1)|+|f'(\alpha_2)|\,).
\end{aligned}
$$

The proof is completed.

Example 2.1. *Let* $[\alpha_1,\alpha_2]=[0,1]$ *and define the function* $f:[0,1]\to\mathbb{R}$ *as* $f(\lambda)=\lambda^3+3$. *Let us consider the right-hand side of the inequality (2.5) as follows:*

$$
\begin{aligned}
&\frac{(\alpha_2-\alpha_1)(U_1+U_2+U_3+U_4+U_5+U_6+U_7+U_8)}{2^{\kappa+2}}\\
=\ &\frac{2\left(-\rho^{\kappa+1}+\kappa\left(2\rho^2-3\rho+2\right)-2(2-\rho)^{\kappa}+\rho(2-\rho)^{\kappa}+2^{\kappa+1}+2\rho^2\right)}{(\kappa+1)}\\
&+\frac{2\left(-3\rho-\vartheta^{\kappa+1}+2(\kappa+1)\vartheta^2-2(2-\vartheta)^{\kappa}+\vartheta\left(-3\kappa+(2-\vartheta)^{\kappa}-3\right)+4\right)}{(\kappa+1)}
\end{aligned}
$$

From the definitions of fractional integrals, the equalities

$$
\begin{aligned}
&\left|\frac{2^{\kappa}-2+\rho+\vartheta}{2^{\kappa}}\left(\frac{f(0)+f(1)}{2}\right)+\frac{2-\rho-\vartheta}{2^{\kappa}}f\left(\frac{1}{2}\right)-\frac{\Gamma(\kappa+1)}{2(\alpha_2-\alpha_1)^{\kappa}}\left[J^{\kappa}_{(\alpha_2)^-}f(\alpha_1)+J^{\kappa}_{(\alpha_1)^+}f(\alpha_2)\right]\right|\\
&\qquad=\frac{2-\rho-\vartheta}{2^{\kappa}}\left(\frac{33}{8}\right)+\frac{2^{\kappa}-2+\rho+\vartheta}{2^{\kappa}}\left(\frac{9}{2}\right)-\frac{1}{2}\left(\frac{33\kappa^2+99\kappa+72}{8\kappa^2+24\kappa+16}\right)
\end{aligned}
$$

are valid. Finally, we have the following inequality:

$$
\begin{aligned}
&\frac{2-\rho-\vartheta}{2^{\kappa}}\left(\frac{33}{8}\right)-\frac{1}{2}\left(\frac{33\kappa^2+99\kappa+72}{8\kappa^2+24\kappa+16}\right)\\
\leq\ &\frac{1}{2^{\kappa+2}}\left[\frac{2\left(\kappa^2-8.2^{\kappa}+2^{\kappa+2}+4\kappa+5\right)}{(\kappa+1)(\kappa+2)}+\frac{2\left(-4.2^{\kappa}+2^{\kappa+1}+\kappa+3\right)}{(\kappa+1)(\kappa+2)}+\frac{2(\kappa+1)+2}{(\kappa+1)(\kappa+2)}\right].
\end{aligned}
\tag{2.6}
$$

As one can see in Figure 1, (2.6) in Example 1 shows the correctness of this inequality for all values of $\kappa\in(0,1]$ and special choices of ρ,ϑ. The Figure 1 represents the Graphical description of inequality (2.6) and their difference.

Remark 2.1. *If we choose* $\rho=\vartheta=1$, $\kappa=1$, *in Theorem 2.1, then inequality* (2.5) *reduces to inequality* (1.3).

Remark 2.2. *If we choose* $\rho=\vartheta=0$, $\kappa=1$, *in Theorem 2.1, then inequality* (2.5) *reduces to inequality* (1.5).

Remark 2.3. *If we choose* $\rho=\vartheta=\frac{1}{3}$, $\kappa=1$, *in Theorem 2.1, then inequality* (2.5) *reduces to inequality* (1.7).

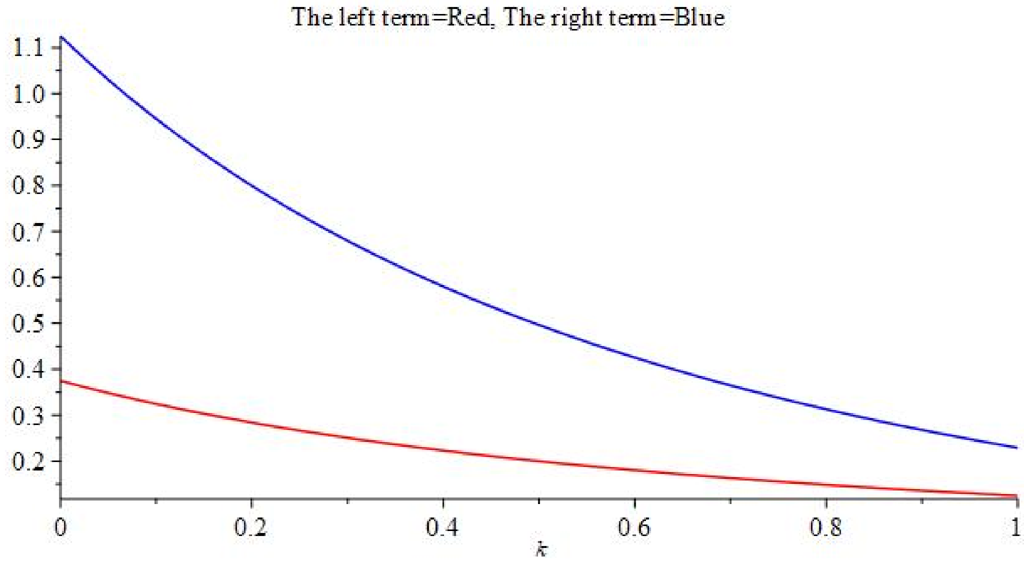

Figure 1. Graphical description of inequality (2.6).

Theorem 2.2. *Let $f:[\alpha_1,\alpha_2]\to\mathbb{R}$ be a differentiable function on (α_1,α_2) with $\alpha_1<\alpha_2$ with $0<\kappa\le 1$, $\lambda\in[0,1]$, and $\rho,\vartheta\in[0,1]$. If $f'\in L[\alpha_1,\alpha_2]$ and mapping $|f'|^q$ with $q\ge 1$, is convex on $[\alpha_1,\alpha_2]$, then the following inequality holds:*

$$\begin{aligned}&\Bigg|\frac{2^\kappa-2+\rho+\vartheta}{2^\kappa}\left(\frac{f(\alpha_1)+f(\alpha_2)}{2}\right)+\frac{2-\rho-\vartheta}{2^\kappa}f\left(\frac{\alpha_1+\alpha_2}{2}\right)\\&-\frac{\Gamma(\kappa+1)}{2(\alpha_2-\alpha_1)^\kappa}\left[J^\kappa_{(\alpha_2)^-}f(\alpha_1)+J^\kappa_{(\alpha_1)^+}f(\alpha_2)\right]\Bigg|\le\frac{\alpha_2-\alpha_1}{2^{\kappa+2}}\times\\&\Bigg[\Bigg\{(U_9)^{1-1/q}\left(U_1|f'(\alpha_1)|^q+U_2|f'(\frac{\alpha_1+\alpha_2}{2})|^q\right)^{1/q}+(U_{10})^{1-1/q}\left(U_3|f'(\alpha_1)|^q+U_4|f'(\frac{\alpha_1+\alpha_2}{2})|^q\right)^{1/q}\Bigg\}\\&+\Bigg\{(U_{11})^{1-1/q}\left(U_5|f'(\frac{\alpha_1+\alpha_2}{2})|^q+U_6|f'(\alpha_2)|^q\right)^{1/q}+(U_{12})^{1-1/q}\left(U_7|f'(\frac{\alpha_1+\alpha_2}{2})|^q+U_8|f'(\alpha_2)|^q\right)^{1/q}\Bigg\}\Bigg],\end{aligned}\tag{2.7}$$

where,

$$\begin{aligned}U_9&=\int_0^1\left|[(1-\lambda)^\kappa-\rho]\right|d\lambda\\&=\frac{1-2(1-\eta_1)^{\kappa+1}}{\kappa+1}+(1-2\eta_1)\rho,\quad \eta_1=1-\rho^{\frac{1}{\kappa}}.\\U_{10}&=\int_0^1\left|[\vartheta-(1-\lambda)^\kappa]\right|d\lambda\\&=\frac{1+2(1-\eta_2)^{\kappa+1}}{\kappa+1}-(1-2\eta_2)\vartheta,\quad \eta_2=1-\vartheta^{\frac{1}{\kappa}}.\end{aligned}$$

$$
\begin{aligned}
U_{11} &= \int_0^1 \left|[(2-\lambda)^\kappa - (2-\rho)]\right| d\lambda \\
&= \frac{1-2^{\kappa+1}+2(2-\eta_3)^{\kappa+1}}{\kappa+1} - (2^\kappa-\rho)(1-2\eta_2), \quad \eta_3 = 2-(2^\kappa-\rho)^{\frac{1}{\kappa}}.
\end{aligned}
$$

$$
\begin{aligned}
U_{12} &= \int_0^1 \left|[2-\vartheta-(2-\lambda)^\kappa]\right| d\lambda \\
&= \frac{1+2^{\kappa+1}-2(2-\eta_4)^{\kappa+1}}{\kappa+1} + (2^\kappa-\vartheta)(1-2\eta_4), \quad \eta_4 = 2-(2^\kappa-\vartheta)^{\frac{1}{\kappa}}.
\end{aligned}
$$

Proof. Utilizing the Lemma 2.1 and Power-mean inequality, we obtain □

$$
\begin{aligned}
&\left|\frac{2^\kappa-2+\rho+\vartheta}{2^\kappa}\left(\frac{f(\alpha_1)+f(\alpha_2)}{2}\right)+\frac{2-\rho-\vartheta}{2^\kappa}f\left(\frac{\alpha_1+\alpha_2}{2}\right)\right.\\
&\left.-\frac{\Gamma(\kappa+1)}{2(\alpha_2-\alpha_1)^\kappa}\left[J^\kappa_{(\alpha_2)^-}f(\alpha_1)+J^\kappa_{(\alpha_1)^+}f(\alpha_2)\right]\right|\\
\le\ & \frac{\alpha_2-\alpha_1}{2^{\kappa+2}}\left(\int_0^1 |(1-\lambda)^\kappa-\rho|\,d\lambda\right)^{1-1/q}\\
&\times\left(\int_0^1 |(1-\lambda)^\kappa-\rho|\left|f'\left(\lambda\alpha_1+(1-\lambda)\frac{\alpha_1+\alpha_2}{2}\right)\right|^q d\lambda\right)^{1/q}\\
&+\frac{\alpha_2-\alpha_1}{2^{\kappa+2}}\left(\int_0^1 [\vartheta-(1-\lambda)^\kappa]\,d\lambda\right)^{1-1/q}\\
&\times\left(\int_0^1 [\vartheta-(1-\lambda)^\kappa]\left|f'\left(\lambda\alpha_2+(1-\lambda)\frac{\alpha_1+\alpha_2}{2}\right)\right|^q d\lambda\right)^{1/q}\\
&+\frac{\alpha_2-\alpha_1}{2^{\kappa+2}}\left(\int_0^1 |(2-\lambda)^\kappa+\rho-2|\,d\lambda\right)^{1-1/q}\\
&\times\left(\int_0^1 |(2-\lambda)^\kappa+\rho-2|\left|f'\left(\lambda\frac{\alpha_1+\alpha_2}{2}+(1-\lambda)\alpha_2\right)\right|^q d\lambda\right)^{1/q}\\
&+\frac{\alpha_2-\alpha_1}{2^{\kappa+2}}\left(\int_0^1 [2-\vartheta-(2-\lambda)^\kappa]\,d\lambda\right)^{1-1/q}\\
&\times\left(\int_0^1 [(2-\lambda)^\kappa+\rho-2]\left|f'\left(\lambda\frac{\alpha_1+\alpha_2}{2}+(1-\lambda)\alpha_1\right)\right|^q d\lambda\right)^{1/q}
\end{aligned}
$$

$$
\begin{aligned}
\le\ & \frac{\alpha_2-\alpha_1}{2^{\kappa+2}}\left(\int_0^1 |(1-\lambda)^\kappa-\rho|\,d\lambda\right)^{1-1/q}\\
&\times\left(\int_0^1 |(1-\lambda)^\kappa-\rho|\left(\lambda\,|f'(\alpha_1)|^q+(1-\lambda)\left|f'\left(\frac{\alpha_1+\alpha_2}{2}\right)\right|^q\right)d\lambda\right)^{1/q}
\end{aligned}
$$

$$+\frac{\alpha_2-\alpha_1}{2^{\kappa+2}}\left(\int_0^1 |\vartheta-(1-\lambda)^{\kappa}|\,d\lambda\right)^{1-1/q}$$
$$\times\left(\int_0^1 |\vartheta-(1-\lambda)^{\kappa}|\left(\lambda\,|f'(\alpha_2)|^q+(1-\lambda)\left|f'\left(\frac{\alpha_1+\alpha_2}{2}\right)\right|^q\right)d\lambda\right)^{1/q}$$

$$+\frac{\alpha_2-\alpha_1}{2^{\kappa+2}}\left(\int_0^1 |(2-\lambda)^{\kappa}+\rho-2|\,d\lambda\right)^{1-1/q}$$
$$\times\left(\int_0^1 |(2-\lambda)^{\kappa}+\rho-2|\left(\lambda\left|f'\left(\frac{\alpha_1+\alpha_2}{2}\right)\right|^q+(1-\lambda)\,|f'(\alpha_2)|^q\right)d\lambda\right)^{1/q}$$
$$+\frac{\alpha_2-\alpha_1}{2^{\kappa+2}}\left(\int_0^1 [|2-\vartheta-(2-\lambda)^{\kappa}|]\,d\lambda\right)^{1-1/q}$$
$$\times\left(\int_0^1 |(2-\lambda)^{\kappa}+\rho-2|\left(\lambda\left|f'\left(\frac{\alpha_1+\alpha_2}{2}\right)\right|^q+(1-\lambda)\,|f'(\alpha_1)|^q\right)d\lambda\right)^{1/q}.$$

$$\left|\frac{2^{\kappa}-2+\rho+\vartheta}{2^{\kappa}}\left(\frac{f(\alpha_1)+f(\alpha_2)}{2}\right)+\frac{2-\rho-\vartheta}{2^{\kappa}}f\left(\frac{\alpha_1+\alpha_2}{2}\right)\right.$$
$$\left.-\frac{\Gamma(\kappa+1)}{2(\alpha_2-\alpha_1)^{\kappa}}\left[J^{\kappa}_{(\alpha_2)^-}f(\alpha_1)+J^{\kappa}_{(\alpha_1)^+}f(\alpha_2)\right]\right|\leq\frac{\alpha_2-\alpha_1}{2^{\kappa+2}}\times$$
$$\left[\left\{(U_9)^{1-1/q}\left(U_1|f'(\alpha_1)|^q+U_2|f'(\frac{\alpha_1+\alpha_2}{2})|^q\right)^{1/q}+(U_{10})^{1-1/q}\left(U_3|f'(\alpha_1)|^q+U_4|f'(\frac{\alpha_1+\alpha_2}{2})|^q\right)^{1/q}\right\}\right.$$
$$\left.+\left\{(U_{11})^{1-1/q}\left(U_5|f'(\frac{\alpha_1+\alpha_2}{2})|^q+U_6|f'(\alpha_2)|^q\right)^{1/q}+(U_{12})^{1-1/q}\left(U_7|f'(\frac{\alpha_1+\alpha_2}{2})|^q+U_8|f'(\alpha_2)|^q\right)^{1/q}\right\}\right]. \tag{2.8}$$

The proof is completed.

Now we discuss the particular inequalities which generalize inequalities in classical sense.

Corollary 2.1. *Suppose that all the assumptions of Theorem 2.2, are satisfied. If we choose $\rho=\vartheta=1$, $\kappa=1$, the following inequality holds:*

$$\left|\frac{f(\alpha_1)+f(\alpha_2)}{2}-\frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}f(u)du\right|$$
$$\leq\frac{\alpha_2-\alpha_1}{8}\left[\left(\frac{2}{3}\,|f'(\alpha_1)|^q+\frac{1}{3}\left|f'\left(\frac{\alpha_1+\alpha_2}{2}\right)\right|^q\right)^{\frac{1}{q}}+\left(\frac{2}{3}\,|f'(\alpha_2)|^q+\frac{1}{3}\left|f'\left(\frac{\alpha_1+\alpha_2}{2}\right)\right|^q\right)^{\frac{1}{q}}\right].$$

Corollary 2.2. *Suppose that all the assumptions of Theorem 2.2, are satisfied. If we choose $\rho=\vartheta=0$, $\kappa=1$, the following inequality holds:*

$$\left|f\left(\frac{\alpha_1+\alpha_2}{2}\right)-\frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2} f(u)du\right|$$
$$\leq \frac{\alpha_2-\alpha_1}{8}\left[\left(\frac{1}{3}\left|f'(\alpha_1)\right|^q+\frac{2}{3}\left|f'\left(\frac{\alpha_1+\alpha_2}{2}\right)\right|^q\right)^{\frac{1}{q}}+\left(\frac{1}{3}\left|f'(\alpha_2)\right|^q+\frac{2}{3}\left|f'\left(\frac{\alpha_1+\alpha_2}{2}\right)\right|^q\right)^{\frac{1}{q}}\right].$$

Corollary 2.3. *Suppose that all the assumptions of Theorem 2.2, are satisfied. If we choose $\rho=\vartheta=\frac{1}{3}$, $\kappa=1$, the following inequality holds:*

$$\left|\frac{1}{3}\left\{2f\left(\frac{\alpha_1+\alpha_2}{2}\right)+\frac{f(\alpha_1)+f(\alpha_2)}{2}\right\}-\frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2} f(u)du\right|$$
$$\leq \frac{5(\alpha_2-\alpha_1)}{72}\times\left[\left(\frac{8}{81}\left|f'(\alpha_1)\right|^q+\frac{29}{162}\left|f'\left(\frac{\alpha_1+\alpha_2}{2}\right)\right|^q\right)^{\frac{1}{q}}+\left(\frac{8}{81}\left|f'(\alpha_2)\right|^q+\frac{29}{162}\left|f'\left(\frac{\alpha_1+\alpha_2}{2}\right)\right|^q\right)^{\frac{1}{q}}\right].$$

Now we obtain some estimates of Simpson's and Hermite-Hadamard-inequalities for concavity.

Theorem 2.3. *Let $f:[\alpha_1,\alpha_2]\to\mathbb{R}$ be a differentiable mapping on (α_1,α_2) with $\alpha_1<\alpha_2, 0<\kappa\leq 1$, and $\rho,\vartheta\in[0,1]$, $q\geq 1$. If $|f'|^q$ is concave on $[\alpha_1,\alpha_2]$, then the following inequality holds:*

$$\begin{aligned}&\left|\frac{2^\kappa-2+\rho+\vartheta}{2^\kappa}\left(\frac{f(\alpha_1)+f(\alpha_2)}{2}\right)+\frac{2-\rho-\vartheta}{2^\kappa}f\left(\frac{\alpha_1+\alpha_2}{2}\right)\right.\\&\left.-\frac{\Gamma(\kappa+1)}{2(\alpha_2-\alpha_1)^\kappa}\left[J^\kappa_{(\alpha_2)^-}f(\alpha_1)+J^\kappa_{(\alpha_1)^+}f(\alpha_2)\right]\right|\\&\leq\frac{\alpha_2-\alpha_1}{2^{\kappa+2}}\left[\left\{U_9\times\left|f'\left(\left\{\frac{U_1\times(\alpha_1)+U_2\times\left(\frac{\alpha_1+\alpha_2}{2}\right)}{U_9}\right\}\right)\right|+U_{10}\times\left|f'\left(\left\{\frac{U_3\times(\alpha_2)+U_4\times\left(\frac{\alpha_1+\alpha_2}{2}\right)}{U_{10}}\right\}\right)\right|\right\}\right.\\&\left.+U_{11}\times\left|f'\left(\left\{\frac{(U_5\times\left(\frac{\alpha_1+\alpha_2}{2}\right)+U_6\times(\alpha_2)}{U_{11}}\right\}\right)\right|+U_{12}\times\left|f'\left(\left\{\frac{(U_7\times\left(\frac{\alpha_1+\alpha_2}{2}\right)+U_8\times(\alpha_1)}{U_{12}}\right\}\right)\right|\right].\end{aligned}\tag{2.9}$$

Proof. Using the concavity of $|f'|^q$ and the power-mean inequality, we know that for $\lambda\in[0,1]$, □

$$\begin{aligned}|f'(\lambda\alpha_1+(1-\lambda)\alpha_2)|^q&>\lambda|f'(\alpha_1)|^q+(1-\lambda)|f'(\alpha_2)|^q\\&\geq(\lambda|f'(\alpha_1)|+(1-\lambda)|f'(\alpha_2)|)^q\end{aligned}$$

Hence
$$|f'(\lambda\alpha_1+(1-\lambda)\alpha_2)|\geq\lambda|f'(\alpha_1)|+(1-\lambda)|f'(\alpha_2)|.$$

By the concavity and Jensen integral inequality, we have

$$
\begin{aligned}
&\left|\frac{2^{\kappa}-2+\rho+\vartheta}{2^{\kappa}}\left(\frac{f(\alpha_1)+f(\alpha_2)}{2}\right)+\frac{2-\rho-\vartheta}{2^{\kappa}}f\left(\frac{\alpha_1+\alpha_2}{2}\right)\right.\\
&\left.-\frac{\Gamma(\kappa+1)}{2(\alpha_2-\alpha_1)^{\kappa}}\left[J^{\kappa}_{(\alpha_2)^-}f(\alpha_1)+J^{\kappa}_{(\alpha_1)^+}f(\alpha_2)\right]\right|\\
&\leq\frac{\alpha_2-\alpha_1}{2^{\kappa+2}}\left(\int_0^1\left|[(1-\lambda)^{\kappa}-\rho]\right|d\lambda\right)\left|f'\left(\frac{\int_0^1[(1-\lambda)^{\kappa}-\rho]\left|\left(\lambda\alpha_1+(1-\lambda)\frac{\alpha_1+\alpha_2}{2}\right)\right|d\lambda}{\int_0^1\left|[(1-\lambda)^{\kappa}-\rho]\right|d\lambda}\right)\right|\\
&+\frac{\alpha_2-\alpha_1}{2^{\kappa+2}}\left(\int_0^1\left|[\vartheta-(1-\lambda)^{\kappa}]\right|d\lambda\right)\left|f'\left(\frac{\int_0^1[\vartheta-(1-\lambda)^{\kappa}]\left|\left(\lambda\alpha_2+(1-\lambda)\frac{\alpha_1+\alpha_2}{2}\right)\right|d\lambda}{\int_0^1\left|[\vartheta-(1-\lambda)^{\kappa}]\right|d\lambda}\right)\right|\\
&+\frac{\alpha_2-\alpha_1}{2^{\kappa+2}}\left(\int_0^1\left|[(2-\lambda)^{\kappa}+\rho-2]\right|d\lambda\right)\left|f'\left(\frac{\int_0^1[(2-\lambda)^{\kappa}+\rho-2]\left|\left(\lambda\frac{\alpha_1+\alpha_2}{2}+(1-\lambda)\alpha_2\right)\right|d\lambda}{\int_0^1\left|[(2-\lambda)^{\kappa}+\rho-2]\right|d\lambda}\right)\right|\\
&+\frac{\alpha_2-\alpha_1}{2^{\kappa+2}}\left(\int_0^1\left|[2-\vartheta-(2-\lambda)^{\kappa}]\right|d\lambda\right)\left|f'\left(\frac{\int_0^1[2-\vartheta-(2-\lambda)^{\kappa}]\left|\left(\lambda\frac{\alpha_1+\alpha_2}{2}+(1-\lambda)\alpha_1\right)\right|d\lambda}{\int_0^1\left|[2-\vartheta-(2-\lambda)^{\kappa}]\right|d\lambda}\right)\right|\\
&\leq\frac{\alpha_2-\alpha_1}{2^{\kappa+2}}(U_9)\left|f'\left(\frac{U_1(\alpha_1)+U_2\left(\frac{\alpha_1+\alpha_2}{2}\right)}{U_9}\right)\right|+\frac{\alpha_2-\alpha_1}{2^{\kappa+2}}(U_{10})\left|f'\left(\frac{U_3(\alpha_2)+U_4\left(\frac{\alpha_1+\alpha_2}{2}\right)}{U_9}\right)\right|\\
&+\frac{\alpha_2-\alpha_1}{2^{\kappa+2}}(U_{11})\left|f'\left(\frac{(U_5\left(\frac{\alpha_1+\alpha_2}{2}\right)+U_6(\alpha_2)}{U_6}\right)\right|+\frac{\alpha_2-\alpha_1}{2^{\kappa+2}}(U_{12})\left|f'\left(\frac{(U_7\left(\frac{\alpha_1+\alpha_2}{2}\right)+U_8(\alpha_1)}{U_6}\right)\right|
\end{aligned}
$$

$$
\begin{aligned}
&\left|\frac{2^{\kappa}-2+\rho+\vartheta}{2^{\kappa}}\left(\frac{f(\alpha_1)+f(\alpha_2)}{2}\right)+\frac{2-\rho-\vartheta}{2^{\kappa}}f\left(\frac{\alpha_1+\alpha_2}{2}\right)\right.\\
&\left.-\frac{\Gamma(\kappa+1)}{2^{\kappa+1}(\alpha_2-\alpha_1)^{\kappa}}\left[J^{\kappa}_{(\alpha_2)^-}f(\alpha_1)+J^{\kappa}_{(\alpha_1)^+}f(\alpha_2)\right]\right|\\
&\leq\frac{\alpha_2-\alpha_1}{2^{\kappa+2}}\left[\left\{U_9\times\left|f'\left(\left\{\frac{U_1\times(\alpha_1)+U_2\times\left(\frac{\alpha_1+\alpha_2}{2}\right)}{U_9}\right\}\right)\right|+U_{10}\times\left|f'\left(\left\{\frac{U_3\times(\alpha_2)+U_4\times\left(\frac{\alpha_1+\alpha_2}{2}\right)}{U_{10}}\right\}\right)\right|\right\}\right.\\
&\left.+U_{11}\times\left|f'\left(\left\{\frac{(U_5\times\left(\frac{\alpha_1+\alpha_2}{2}\right)+U_6\times(\alpha_2)}{U_{11}}\right\}\right)\right|+U_{12}\times\left|f'\left(\left\{\frac{(U_7\times\left(\frac{\alpha_1+\alpha_2}{2}\right)+U_8\times(\alpha_1)}{U_{12}}\right\}\right)\right|\right],
\end{aligned}
$$

which completes the proof.

As a special case of Theorem 2.3, we obtain the following result,

Corollary 2.4. *Suppose that all the assumptions of Theorem 2.3, are satisfied. If we choose* $\rho=\vartheta=\frac{1}{3}$, $\kappa=1$, *we have Simpson's inequality:*

$$\left|\frac{1}{3}\left\{2f\left(\frac{\alpha_1+\alpha_2}{2}\right)+\frac{f(\alpha_1)+f(\alpha_2)}{2}\right\}-\frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}f(u)du\right| \\ \leq \frac{5(\alpha_2-\alpha_1)}{72}\times\left[\left|f'\left(\frac{16\alpha_1+29\alpha_2}{45}\right)\right|+\left|f'\left(\frac{29\alpha_1+16\alpha_2}{45}\right)\right|\right]. \tag{2.10}$$

Remark 2.4. *Our inequality* (2.10) *is an improvement of Alomari inequality as obtained in [18].*

2.1. Applications to special means:

Let consider the following special means for $\alpha_1 \neq \alpha_2$.

The arithmetic mean:

$$A(\alpha_1,\alpha_2)=\frac{\alpha_1+\alpha_2}{2},\ \ \alpha_1,\alpha_2\in\mathbb{R}.$$

The logarithmic-mean:

$$L(\alpha_1,\alpha_2)=\frac{\alpha_2-\alpha_1}{\ln|\alpha_2|-\ln|\alpha_1|},|\alpha_1|\neq|\alpha_2|,\ \alpha_1,\alpha_2\neq 0,\ \alpha_1,\alpha_2\in\mathbb{R}.$$

The generalized logarithmic-mean:

$$L_r(\alpha_1,\alpha_2)=\left[\frac{(\alpha_2)^{r+1}-(\alpha_1)^{r+1}}{(r+1)(\alpha_2-\alpha_1)}\right]^{\frac{1}{r}};\ r\in\mathbb{R}\setminus\{-1,0\},\ \alpha_1,\alpha_2>0.$$

Proposition 2.1. *Suppose* $r\in\mathbb{R}\backslash\{-1,0\}$ *and* $\alpha_1,\alpha_2\in\mathbb{R}$ *such that* $0<\alpha_1<\alpha_2$ *with* $q\geq 1$*, then the following inequality holds:*

$$\left|A(\alpha_1,\alpha_2)-L_r^r(\alpha_1,\alpha_2)\right| \\ \leq \frac{r(\alpha_2-\alpha_1)}{8}\left[\left(\frac{2}{3}|\alpha_1|^q+\frac{1}{3}\left|\left(\frac{\alpha_1+\alpha_2}{2}\right)^{r-1}\right|^q\right)^{\frac{1}{q}}+\left(\frac{2}{3}|\alpha_2|^q+\frac{1}{3}\left|\left(\frac{\alpha_1+\alpha_2}{2}\right)^{r-1}\right|^q\right)^{\frac{1}{q}}\right].$$

Proof. The assertion follows from Corollary 2.1 for the function $f(x)=x^r$ and r as specified above. □

Proposition 2.2. *Suppose* $q\geq 1$ *and* $\alpha_1,\alpha_2\in\mathbb{R}$*, such that* $0<\alpha_1<\alpha_2$*, then the following inequality holds:*

$$\left|A^{-1}(\alpha_1,\alpha_2)-L^{-1}(\alpha_1,\alpha_2)\right| \\ \leq \frac{(\alpha_2-\alpha_1)}{8}\left[\left(\frac{1}{3}|\alpha_1|^q+\frac{2}{3}\left|A^{-2}(\alpha_1,\alpha_2)\right|^q\right)^{\frac{1}{q}}+\left(\frac{1}{3}|\alpha_2|^q+\frac{2}{3}\left|A^{-2}(\alpha_1,\alpha_2)\right|^q\right)^{\frac{1}{q}}\right].$$

Proof. The assertion follows from Corollary 2.2 for the function $f(x)=\frac{1}{x}$. □

2.2. q-digamma function

Suppose $0 < \mathbf{q} < 1$, the $\mathbf{q}$-digamma function $\varphi_{\mathbf{q}}$, is the $\mathbf{q}$-analogue of the digamma function φ defined by (see [19, 20]).

$$\begin{aligned}\varphi_{\mathbf{q}} &= -\ln(1-\mathbf{q}) + \ln\mathbf{q}\sum_{k=0}^{\infty}\frac{\mathbf{q}^{k+x}}{1-\mathbf{q}^{k+x}} \\ &= -\ln(1-\mathbf{q}) + \ln\mathbf{q}\sum_{k=0}^{\infty}\frac{\mathbf{q}^{kx}}{1-\mathbf{q}^{kx}}\end{aligned}$$

For $\mathbf{q} > 1$ and $x > 0$, $\mathbf{q}$-digamma function $\varphi_{\mathbf{q}}$defined by

$$\begin{aligned}\varphi_{\mathbf{q}} &= -\ln(\mathbf{q}-1) + \ln\mathbf{q}\left[x - \frac{1}{2} - \sum_{k=0}^{\infty}\frac{\mathbf{q}^{-(k+x)}}{1-\mathbf{q}^{-(k+x)}}\right] \\ &= -\ln(\mathbf{q}-1) + \ln\mathbf{q}\left[x - \frac{1}{2} - \sum_{k=0}^{\infty}\frac{\mathbf{q}^{-kx}}{1-\mathbf{q}^{-kx}}\right]\end{aligned}$$

Proposition 2.3. *Suppose α_1, α_2 be real numbers such that $0 < \alpha_1 < \alpha_2$, with $q \geq 1$, then the following inequality holds:*

$$\begin{aligned}&\left|A\left(\varphi_{\mathbf{q}}(\alpha_1), \varphi_{\mathbf{q}}(\alpha_2)\right) - \frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\varphi_{\mathbf{q}}(u)\,du\right| \\ &\leq \frac{(\alpha_2-\alpha_1)}{8}\left[\left(\frac{2}{3}\left|\varphi'_{\mathbf{q}}(\alpha_1)\right|^q + \frac{1}{3}\left|\varphi'_{\mathbf{q}}\left(\frac{\alpha_1+\alpha_2}{2}\right)\right|^q\right)^{\frac{1}{q}} + \left(\frac{2}{3}\left|\varphi'_{\mathbf{q}}(\alpha_2)\right|^q + \frac{1}{3}\left|\varphi'_{\mathbf{q}}\left(\frac{\alpha_1+\alpha_2}{2}\right)\right|^q\right)^{\frac{1}{q}}\right].\end{aligned}$$

Proof. The assertion can be obtained immediately by using Corollary 2.1 to $f(\varepsilon) = \varphi_{\mathbf{q}}(\varepsilon)$ and $\varepsilon > 0$, $f'(\varepsilon) = \varphi'_{\mathbf{q}}(\varepsilon)$ is convex on $(0, +\infty)$. □

Proposition 2.4. *Suppose α_1, α_2 be real numbers such that $0 < \alpha_1 < \alpha_2$, with $q \geq 1$, then the following inequality holds:*

$$\begin{aligned}&\left|\frac{1}{3}\left\{2\varphi_q\left(\frac{\alpha_1+\alpha_2}{2}\right) + A\left(\varphi_{\mathbf{q}}(\alpha_1), \varphi_{\mathbf{q}}(\alpha_2)\right)\right\} - \frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\varphi_{\mathbf{q}}(u)\,du\right| \\ &\leq \frac{5(\alpha_2-\alpha_1)}{72} \\ &\times\left[\left(\frac{8}{81}\left|\varphi'_{\mathbf{q}}(\alpha_1)\right|^q + \frac{29}{162}\left|\varphi'_{\mathbf{q}}\left(\frac{\alpha_1+\alpha_2}{2}\right)\right|^q\right)^{\frac{1}{q}} + \left(\frac{8}{81}\left|\varphi'_{\mathbf{q}}(\alpha_2)\right|^q + \frac{29}{162}\left|\varphi'_{\mathbf{q}}\left(\frac{\alpha_1+\alpha_2}{2}\right)\right|^q\right)^{\frac{1}{q}}\right].\end{aligned}$$

Proof. The assertion can be obtained immediately by using Corollary 2.3 to $f(\varepsilon) = \varphi_{\mathbf{q}}(\varepsilon)$ and $\varepsilon > 0$, $f'(\varepsilon) = \varphi'_{\mathbf{q}}(\varepsilon)$ is convex on $(0, +\infty)$. □

2.3. Modified Bessel function:

Recall the *first kind of modified Bessel function* I_ρ , which has the series representation ([19], p.77).

$$I_\rho(x) = \sum_{m\geq 0} \frac{\left(\frac{x}{2}\right)^{\rho+2m}}{m!\Gamma(\rho+m+1)},$$

where $x \in \mathbb{R}$ and $\rho > -1$, while the second kind modified Bessel function K_ρ ([19], p.78) is usually defined as

$$K_\rho(x) = \frac{\pi}{2}\frac{I_{-\rho}(x) - I_\rho(x)}{\sin\rho\pi}.$$

Here, we consider the function $\Omega_\rho(x) : \mathbb{R} \to [1, \infty)$ defined by

$$\Omega_\rho(x) = 2^\rho\Gamma(\rho+1)\, x^{-\rho} I_\rho(x),$$

where Γ is the Gamma function.

Proposition 2.5. *Suppose* $\rho > -1$ *and* $0 < \alpha_1 < \alpha_2$*. Then*

$$\begin{aligned}
&\left|\Omega_\rho\left(\frac{\alpha_1+\alpha_2}{2}\right) - \frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\Omega_\rho(\varepsilon)\,d\varepsilon\right| \\
&\leq \frac{(\alpha_2-\alpha_1)}{16(\rho+1)}\left[\left(\frac{1}{3}\alpha_1\left|\Omega_{\rho+1}(\alpha_1)\right|^q + \frac{\alpha_1+\alpha_2}{3}\left|\Omega_{\rho+1}\left(\frac{\alpha_1+\alpha_2}{2}\right)\right|^q\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\frac{1}{3}\alpha_2\left|\Omega_{\rho+1}(\alpha_2)\right|^q + \frac{\alpha_1+\alpha_2}{3}\left|\Omega_{\rho+1}\left(\frac{\alpha_1+\alpha_2}{2}\right)\right|^q\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Specifically, if $\rho = -\frac{1}{2}$*, then*

$$\begin{aligned}
&\left|\cosh\left(\frac{\alpha_1+\alpha_2}{2}\right) - \frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\cosh(\varepsilon)\,d\varepsilon\right| \\
&\leq \frac{(\alpha_2-\alpha_1)}{8}\left[\left(\frac{1}{3}\alpha_1\left|\frac{\sinh(\alpha_1)}{\alpha_1}\right|^q + \frac{\alpha_1+\alpha_2}{3}\left|\frac{\sinh\left(\frac{\alpha_1+\alpha_2}{2}\right)}{\frac{\alpha_1+\alpha_2}{2}}\right|^q\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\frac{1}{3}\alpha_2\left|\frac{\sinh(\alpha_2)}{\alpha_2}\right|^q + \frac{\alpha_1+\alpha_2}{3}\left|\frac{\sinh\left(\frac{\alpha_1+\alpha_2}{2}\right)}{\frac{\alpha_1+\alpha_2}{2}}\right|^q\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Proof. The assertion can be obtained immediately by using Corollary 2.2 to $f(\varepsilon) = \Omega_\rho(\varepsilon)$, $\varepsilon > 0$, and $\Omega'_\rho(\varepsilon) = \frac{\varepsilon}{\rho+1}\Omega_{\rho+1}(\varepsilon)$. Now taking into account the relations $\Omega_{-\frac{1}{2}}(\varepsilon) = \cosh(\varepsilon)$ and $\Omega_{\frac{1}{2}}(\varepsilon) = \frac{\sinh(\varepsilon)}{\varepsilon}$. □

3. Conclusions

In this article, we have established an integral identity via Riemann-Livouille fractional integral. Based on this identity, we present several midpoint, trapezoid and Simpson's-type inequalities whose absolute values are convex and concave. It is also shown that several results are given by special cases of the main results. We deduce that the findings proved in this work are naturally universal, contribute to the theory of inequalities. Finally, we have presented some applications to special means, q-digamma and modifies Bessel functions with respect to our deduced results. In future studies, researchers can obtain generalized versions of our results by utilizing other kinds of convex function classes or different types of generalized fractional integral operators.

Acknowledgments

We would like to thank you for following the instructions above very closely in advance. It will definitely save us lot of time and expedite the process of your paper's publication.

Conflict of interest

There is no conflict of interest among the authors.

References

1. S. S. Dragomir, C. E. M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs Victoria University, 2000.
2. M. Z. Sarikaya, E. Set, H. Yaldiz, N. Başak, Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities, *Math. Comput. Model.*, **57** (2013), 2403–2407. https://doi.org/10.1016/j.mcm.2011.12.048
3. J. Nasir, S. Qaisar, S. I. Butt, A. Qayyum, Some Ostrowski type inequalities for mappings whose second derivatives are preinvex function via fractional integral operator, *AIMS Mathematics*, **7** (2022), 3303–3320. https://doi.org/10.3934/math.2022184
4. M. Karim, A. Fahmi, Z. Ullah, M. A. T. Bhatti, A. Qayyum, On certain Ostrowski type integral inequalities for convex function via AB-fractional integral operator, *AIMS Mathematics*, **8** (2023), 9166–9184. https://doi.org/10.3934/math.2023459
5. A. Qayyum, I. Faye, M. Shoaib, On new generalized inequalities via Riemann-Liouville fractional integration, *J. Fract. Calc. Appl.*, **6** (2015), 91–100.
6. S. S. Dragomir, M. I. Bhatti, M. Iqbal, M. Muddassar, Some new Hermite-Hadamard's type inequalities, *J. Comput. Anal. Appl.*, **18** (2015), 655–660.
7. S. S. Dragomir, Ostrowski type inequalities for generalized Riemann-Liouville fractional integrals of functions with bounded variation, *RGMIA Res. Rep. Coll.*, **20** (2017).
8. S. S. Dragomir, Trapezoid type inequalities for generalized Riemann-Liouville fractional integrals of functions with bounded variation, *Acta Univ. Sapientiae, Math.*, **12** (2020), 30–53. https://doi.org/10.2478/ausm-2020-0003

9. M. Iqbal, S. Qaisar, S. Hussain, On Simpson's type inequalities utilizing fractional integral, *J. Comput. Anal. Appl.*, **23** (2017), 1137–1145. https://doi.org/10.1186/s13660-019-2160-1
10. R. Gorenflo, F. Mainardi, *Fractional Calculus: Integral and Differential Equations of Fractional Order*, Vienna: Springer, 1997.
11. S. Belarbi, Z. Dahmani, On some new fractional integral inequalities, *J. Ineq. Pure Appl. Math.*, **10** (2009), 86.
12. Z. Dahmani, New inequalities in fractional integrals, *Int. J. Nonlinear Sci.*, **9** (2010), 493–497.
13. S. R. Hwang, K. L. Tseng, K. C. Hsu, New inequalities for fractional integrals and their applications, *Turk. J. Math.*, **40** (2016), 471–486. https://doi.org/10.3906/mat-1411-61
14. S. S. Dragomir, R. P. Agarwal, Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula, *Appl. Math. Lett.*, **11** (1998), 91–95. https://doi.org/10.1016/S0893-9659(98)00086-X
15. U. S. Kırmacı, Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula, *Appl. Math. Comput.*, **147** (2004), 137–146. https://doi.org/10.1016/S0096-3003(02)00657-4
16. M. Z. Sarikaya, E. Set, M. E. Ozdemir, On new inequalities of Simpson's type for s-convex functions, *Comput. Math. Appl.*, **60** (2010), 2191–2199.
17. T. Lian, W. Tang, R. Zhou, Fractional Hermite-Hadamard inequalities for (s, m)-convex or s-concave functions, *J. Inequal. Appl.*, **240** (2018). https://doi.org/10.1186/s13660-018-1829-1
18. M. Alomari, M. Darus, S. S. Dragomir, New inequalities of Simpson's type for s-convex functions with applications, *RGMIA Res. Rep. Coll.*, **12** (2009), 13–20.
19. G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, 1995.
20. S. Jain, K. Mehrez, D. Baleanu, P. Agarwal, Certain Hermite-Hadamard inequalities for logarithmically convex functions with applications, *Mathematics*, **7** (2019). https://doi.org/10.3390/math7020163

AIMS Press