Mathematics

Research article

Existence and stability of fuzzy Pareto-Nash equilibria for fuzzy constrained multi-objective games with fuzzy payoffs

Wen Li[1], Deyi Li[1,2], Yuqiang Feng[1,2,*] and Du Zou[1]

[1] School of Science, Wuhan University of Science and Technology, Wuhan 430065, China

[2] Hubei Province Key Laboratory of Systems Science in Metallurgical Process, Wuhan 430081, China

* **Correspondence:** Email: yqfeng6@126.com.

Abstract: A new class of fuzzy constrained multi-objective games with fuzzy payoffs (FCMGFPs) is considered in this paper. First, Berge's maximum theorem for the fuzzy-vector-valued function is obtained. Based on this theorem and the Fan-Glicksberg fixed point theorem, the existence theorem of the fuzzy Pareto-Nash equilibrium for the FCMGFP is established. Second, the abstract rationality function for the FCMGFP is given by using a nonlinear scalarization function of interval vectors. Finally, a series of results, such as structural stability ((γ, ϵ)-stability) and robustness to ϵ-equilibrium ((γ, ϵ)-robustness), are obtained.

1. Introduction

Multi-objective games are games with vector-valued payoffs, which were first studied by Zeleny [56]. Since then, many scholars have begun to pay attention to multi-objective games; see [16, 44, 52, 54]. Yu and Yuan [52] used Ky Fan's minimax inequality and the Fan-Glicksberg fixed point theorem to study the existence of Pareto equilibria in multi-objective games. Considering the feasible strategy correspondence in the multi-objective game, the constrained multi-objective game can be obtained. Ding [15] first proved the existence of Pareto equilibria in constrained multi-objective games by using the quasi-equilibrium existence theorem, and the results were extended in [52]. Subsequently, some scholars have focused on the existence of equilibria and the stability of equilibrium sets in constrained multi-objective games; see [14, 29, 41] and references therein.

The main limitation of traditional non-cooperative games is that all data in the game are assumed

to be known accurately by the players. In real-world games, players often cannot accurately assess the payoffs of different strategies or their own and other players' preferences. Usually, the uncertainty involved in the game includes fuzziness, randomness and fuzzy randomness.

Zadeh [55] first introduced fuzzy set theory. Subsequently, Butnariu [8] introduced fuzzy sets in non-cooperative games. Campos [9] proposed a fuzzy linear programming method to solve the fuzzy two-player zero-sum matrix game. This method is more suitable for solving the problem that the payoff is a triangular fuzzy number. Based on the literature [23, 39], Clemente [12] introduced the standard fuzzy orders to compare the fuzzy payoffs and studied the Pareto equilibrium of the fuzzy matrix game. Buckley [7] used the decision principles of Bellman and Zadeh [4] in fuzzy environments to formulate multi-objective non-cooperative games under uncertainty. Kim and Lee [25] extended generalized games to generalized fuzzy games and established the existence of fuzzy equilibria for generalized fuzzy games. For generalized games and generalized multi-objective games, in addition to the ambiguity of payoffs, the ambiguity of strategy choice is also studied; see [19, 24]. For more literature on fuzzy matrix games and generalized fuzzy games; see [6, 26, 37, 40, 47] and references therein.

Since the α-cut of a fuzzy number is a compact interval, we need to investigate the properties of fuzzy games with the help of interval analysis. Moore [35] systematically introduced the theory of interval analysis for the first time. Subsequently, many scholars further developed the theory of interval arithmetic and interval-valued functions; see [1, 10, 36, 42] and references therein. Li et al. [28] discussed the weak Pareto-Nash equilibria of generalized interval-valued multi-objective games with fuzzy mappings and gave the existence theorem of weak Pareto-Nash equilibrium. In game theory, the current application of interval analysis is mainly to discuss the existence of equilibria of interval-valued matrix games and their calculation methods; see, e.g., [13, 27, 28, 30]. Stefanini and Bede [42] introduced the concept of gH-difference, which is a generalization of H-difference. The gH-difference is applied to several topics in interval mathematical analysis, such as interval optimization problems [20, 21, 38, 46], interval differential equations [31, 42, 43] and so on.

Anderlini and Canning [2] established a kind of model with abstract rational function in a general decision environment as follows:

Definition 1.1. A bounded rationality model $\mathcal{M}$ consists of a quadruple $\{\Gamma, X, K, R\}$ with the following interpretation.

(1) Γ is a nonempty parameter space;

(2) X is an action space;

(3) $K : \Gamma \times X \rightrightarrows X$ is a feasibility correspondence, and K induces a further correspondence $k : \Gamma \rightrightarrows X$ defined by $k(\gamma) = \{x \in X \mid x \in K(\gamma, x)\}$ for $\gamma \in \Gamma$;

(4) $R : Graph(k) \rightarrow \mathbb{R}_+$ is a rationality function with $R(\gamma, x) = 0$ corresponding to perfect rationality, where $Graph(k) = \{(\gamma, x) \in \Gamma \times X \mid x \in k(\gamma)\}$ is the graph of k.

For any $\gamma \in \Gamma$ and $\epsilon \geq 0$, the set of all ϵ-equilibria of γ is defined as $E(\gamma, \epsilon) = \{x \in k(\gamma) \mid R(\gamma, x) \leq \epsilon\}$, and $E(\gamma) = E(\gamma, 0) = \{x \in k(\gamma) \mid R(\gamma, x) = 0\}$ is the set of all equilibria of γ. The notions of structural stability and robustness to ϵ-equilibria for a bounded rationality model were introduced in [2]. A model is structurally stable if the equilibrium set (given fully rational agents) varies continuously with the parameter values of the model. A model is robust to ϵ-equilibria if small deviations from rationality

result can lead to only small changes in the equilibrium set. Since then, many researchers began to discuss structural stability and robustness to ϵ-equilibrium for the bounded rationality model. They presented a series of applications to non-cooperative games and fixed point problems; see, e.g., [33, 48, 49, 51, 53].

Hung et al. [24] discussed the existence of fuzzy equilibria in generalized multi-objective games with fuzzy mappings when the payoff space is a Hausdorff topological vector space and investigated the structural stability and robustness of ϵ-equilibrium for this class of games in the framework of bounded rationality. Furthermore, when the payoff space is a Banach space, with the help of an abstract rationality function, a bounded rationality model for generalized multi-objective games with fuzzy mappings was constructed. However, Wang et al. [45] pointed out that the space of fuzzy vectors is a quasi-linear space, not a linear space. Therefore, it is impossible to directly obtain the fuzzy constrained multi-objective game with fuzzy payoffs by using the conclusions in [24].

Motivated by [24, 28], we are concerned in this paper with a class of fuzzy constrained multi-objective games with fuzzy payoffs. With the help of the α-cuts of fuzzy vectors and the partial order of interval vectors, we establish the partial order of fuzzy vectors. Moreover, the definitions of fuzzy Pareto-Nash equilibrium and fuzzy α-Pareto-Nash equilibrium are introduced. Based on Berge's maximum theorem for fuzzy-valued functions (interval-valued functions) and the Fan-Glicksberg fixed point theorem, the existence of a fuzzy Pareto-Nash equilibrium (fuzzy α-Pareto-Nash equilibrium) for the FCMGFP is investigated. Furthermore, the bounded rationality model is applied to a new class of FCMGFPs.

This article is organized as follows. In Section 2, we review some basic terminology and related facts. In Section 3, we first discuss the semi-continuity and generalized quasi-concavity for fuzzy vector-valued functions. Moreover, we get Berge's maximum theorem for the fuzzy-vector-valued function. Second, we obtain two existence theorems of fuzzy Pareto-Nash equilibria and fuzzy α-Pareto-Nash equilibria for FCMGFPs. These new results directly yield the existence theorems of fuzzy Pareto-Nash equilibria and fuzzy α-Pareto-Nash equilibria for fuzzy constrained games with fuzzy payoffs (FCGFPs). In Section 4, we construct the rationality functions for FCMGFPs with the help of the nonlinear scalarization function of interval vectors. A further discussion approaches the structural stability ((γ, ϵ)-stability) and the robustness to ϵ-equilibria ((γ, ϵ)-robustness) for this class of games. Finally, the conclusion is made in Section 5.

2. Preliminaries and terminology

Throughout this paper, we write $\mathbb{R}$ for the set of all real numbers, $\mathbb{R}_+$ for the set of all nonnegative real numbers, $\mathbb{N}$ for the set of all natural numbers and $\mathbb{R}^d$ for d-dimensional Euclidean space. We recall basic terminology and some related conclusions in this section.

2.1. Interval vectors and interval-vector-valued functions

We now review the relevant properties of interval vectors and interval vector-valued functions.

2.1.1. Interval vectors

Let $\mathcal{I}(\mathbb{R})^d$ be the family of all d-dimensional interval vectors, and let

$$\mathcal{I}(\mathbb{R}_+)^d = \{(A_1, \ldots, A_d) \mid A_i \in \mathcal{I}(\mathbb{R}_+), i = 1, \ldots, d\},$$

where $\mathcal{I}(\mathbb{R}_+)$ is the family of all compact intervals contained in $[0, +\infty)$. For $\boldsymbol{A} = (A_1, \ldots, A_d), \boldsymbol{B} = (B_1, \ldots, B_d) \in \mathcal{I}(\mathbb{R})^d$ and $t \in \mathbb{R}$, $\boldsymbol{A} + \boldsymbol{B} = (A_1 + B_1, \ldots, A_d + B_d)$, and $t \cdot \boldsymbol{A} = (t \cdot A_1, \ldots, t \cdot A_d)$. The gH-difference between $\boldsymbol{A}$ and $\boldsymbol{B}$, $\boldsymbol{A} \ominus \boldsymbol{B}$, is defined by $\boldsymbol{A} \ominus \boldsymbol{B} = (A_1 \ominus B_1, \ldots, A_d \ominus B_d)$, where $A_i = [\underline{a}_i, \overline{a}_i]$, $B_i = [\underline{b}_i, \overline{b}_i]$, and $A_i \ominus B_i = [\min\{\underline{a}_i - \underline{b}_i, \overline{a}_i - \overline{b}_i\}, \max\{\underline{a}_i - \underline{b}_i, \overline{a}_i - \overline{b}_i\}]$. In particular, $(-1) \cdot (\boldsymbol{A} \ominus \boldsymbol{B}) = \boldsymbol{B} \ominus \boldsymbol{A}$.

Definition 2.1. (See [28]) For $\boldsymbol{A} = (A_1, \ldots, A_d) \in \mathcal{I}(\mathbb{R})^d$, the function $\|\cdot\|_{\mathcal{I}(\mathbb{R})^d} : \mathcal{I}(\mathbb{R})^d \to \mathbb{R}_+$ defined by $\|\boldsymbol{A}\|_{\mathcal{I}(\mathbb{R})^d} = \left(\sum_{i=1}^{d} \|A_i\|_{\mathcal{I}(\mathbb{R})}^2\right)^{1/2}$ is called a norm on $\mathcal{I}(\mathbb{R})^d$, where $\|A_i\|_{\mathcal{I}(\mathbb{R})} = \max\{|\underline{a}_i|, |\overline{a}_i|\}$.

The set of all interval vectors $\mathcal{I}(\mathbb{R})^d$ equipped with the norm $\|\cdot\|_{\mathcal{I}(\mathbb{R})^d}$ is a normed quasilinear space with respect to the operations $\{+, \ominus, .\}$, and $(\mathcal{I}(\mathbb{R})^d, \|\cdot\|_{\mathcal{I}(\mathbb{R})^d})$ is a complete normed quasilinear space. With the topology induced by norm $\|\cdot\|_{\mathcal{I}(\mathbb{R})^d}$ on $\mathcal{I}(\mathbb{R})^d$, it follows that $int\mathcal{I}(\mathbb{R}_+)^d = \{\boldsymbol{A} = (A_1, \ldots, A_d) \mid A_i \in int\mathcal{I}(\mathbb{R}_+), i = 1, \ldots, d\}$.

Lemma 2.2. *(See [28]) For two convergent sequences $\{\boldsymbol{A}^m\}$ and $\{\boldsymbol{B}^m\}$ in $\mathcal{I}(\mathbb{R})^d$,*

$$\lim_{m \to \infty} (\boldsymbol{A}^m \ominus \boldsymbol{B}^m) = \lim_{m \to \infty} \boldsymbol{A}^m \ominus \lim_{m \to \infty} \boldsymbol{B}^m.$$

It is noteworthy that although $+$ is associative in $\mathcal{I}(\mathbb{R})$, for $A, B \in \mathcal{I}(\mathbb{R})$, the interval $(A \ominus B) + B$ is not always equal to A. Ghosh et el. [21] verified that

$$\|A \ominus B\|_{\mathcal{I}(\mathbb{R})} \leq \|A \ominus C\|_{\mathcal{I}(\mathbb{R})} + \|C \ominus B\|_{\mathcal{I}(\mathbb{R})}, \text{ for all } A, B, C \in \mathcal{I}(\mathbb{R}). \tag{2.1}$$

Lemma 2.3. *For all $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in \mathcal{I}(\mathbb{R})^d$, $\|\boldsymbol{A} \ominus \boldsymbol{B}\|_{\mathcal{I}(\mathbb{R})^d} \leq \|\boldsymbol{A} \ominus \boldsymbol{C}\|_{\mathcal{I}(\mathbb{R})^d} + \|\boldsymbol{C} \ominus \boldsymbol{B}\|_{\mathcal{I}(\mathbb{R})^d}$.*

Proof. See Appendix A. □

Definition 2.4. (See [28]) Let $\boldsymbol{A} = (A_1, \ldots, A_d)$ and $\boldsymbol{B} = (B_1, \ldots, B_d)$ be two elements of $\mathcal{I}(\mathbb{R})^d$.

(1) $\boldsymbol{A}$ is said to be dominated by $\boldsymbol{B}$ from below if $\boldsymbol{A} \ominus \boldsymbol{B} \in \mathcal{I}(\mathbb{R}_+)^d$, and then we write $\boldsymbol{B} \preceq \boldsymbol{A}$. Otherwise, we write $\boldsymbol{B} \npreceq \boldsymbol{A}$.

(2) $\boldsymbol{A}$ is said to be strictly dominated by $\boldsymbol{B}$ from below if $\boldsymbol{A} \ominus \boldsymbol{B} \in int\mathcal{I}(\mathbb{R}_+)^d$, and then we write $\boldsymbol{B} \prec \boldsymbol{A}$. Otherwise, we write $\boldsymbol{B} \nprec \boldsymbol{A}$.

2.1.2. Interval-vector-valued functions

Let $\mathcal{X}$ be a Hausdorff topological space, and let $F = \{f_1, \ldots, f_d\} : \mathcal{X} \to \mathcal{I}(\mathbb{R})^d$ be an interval-vector-valued function. F is *upper semi-continuous* (resp., *lower semi-continuous*) provided that for each $x \in \mathcal{X}$ and for each $\epsilon > 0$, there exists an open neighborhood $o(x)$ of x such that for each $x' \in o(x)$, $F(x') \ominus F(x) \in V_\epsilon^d - \mathcal{I}(\mathbb{R}_+)^d$ (resp., $F(x') \ominus F(x) \in V_\epsilon^d + \mathcal{I}(\mathbb{R}_+)^d$), where $V_\epsilon^d = \{(A_1, \ldots, A_d) \mid A_i \subset (-\epsilon, \epsilon), i = 1, \ldots, d\}$. Furthermore, F is continuous if it is both lower and upper semi-continuous.

Definition 2.5. (See [28]) Let $\mathcal{K}$ be a nonempty convex subset of a Hausdorff topological space $\mathcal{Y}$, and let $F = \{f_1, \ldots, f_d\} : \mathcal{K} \rightarrow \mathcal{I}(\mathbb{R})^d$ be an interval-vector-valued function. Then, F is said to be *generalized* $\mathcal{I}(\mathbb{R}_+)^d$-*quasi-concave* on $\mathcal{K}$ provided that for $x_1, x_2 \in \mathcal{K}$, $A \in \mathcal{I}(\mathbb{R})^d$, and for $\lambda \in [0, 1]$, if $A \ominus F(x_i) \notin int\mathcal{I}(\mathbb{R}_+)^d$ with $i = 1, 2$, we have $A \ominus F(\lambda x_1 + (1 - \lambda)x_2) \notin int\mathcal{I}(\mathbb{R}_+)^d$.

Lemma 2.6. *(See [28]) Let C be a nonempty compact subset of a Hausdorff topological space X and $F = \{f_1, \ldots, f_d\} : C \rightarrow \mathcal{I}(\mathbb{R})^d$ be an interval-vector-valued function. If there exists $i \in \{1, \ldots, d\}$ such that f_i is upper semi-continuous on C, then there is $x^* \in C$ such that $F(x^*) \nprec F(x)$ for all $x \in C$.*

Definition 2.7. (See [28]) Let $F = \{f_1, \ldots, f_d\} : \mathcal{K} \subset \mathbb{R} \rightarrow \mathcal{I}(\mathbb{R})^d$ be an interval-vector-valued function. Then, F is said to be non-decreasing (resp., non-increasing) on $\mathcal{K}$ if, for $x_1, x_2 \in \mathcal{K}$ and $x_1 < x_2$, we have $F(x_1) \preceq F(x_2)$ (resp., $F(x_2) \preceq F(x_1)$).

Lemma 2.8. *(See [28]) An interval-vector-valued function $F = \{f_1, \ldots, f_d\} : \mathcal{K} \subset \mathbb{R} \rightarrow \mathcal{I}(\mathbb{R})^d$ is non-decreasing (resp., non-increasing) if and only if $\underline{f}_i$ and $\overline{f}_i$ are non-decreasing (resp., non-increasing) for each i.*

2.2. Fuzzy sets and fuzzy mapping

Let X denote a universal set. A fuzzy subset $\widetilde{a}$ of X is defined by its membership function $\mu_{\widetilde{a}} : X \rightarrow [0, 1]$, which assigns to each element $x \in X$ a real number $\mu_{\widetilde{a}}(x)$ in the interval $[0, 1]$. Especially, $\mu_{\widetilde{a}}(x)$ is the grade of membership of x in the set $\widetilde{a}$. The α-cut of $\widetilde{a}$, denoted by $\widetilde{a}^\alpha$, is a set defined by $\widetilde{a}^\alpha = \{x \in X \mid \mu_{\widetilde{a}}(x) \geq \alpha\}$, when $\alpha \in (0, 1]$, and $\widetilde{a}^0 = cl\{x \in X \mid \mu_{\widetilde{a}}(x) > 0\}$, where cl represents the closure of sets. For more detail, see [17, 55].

Definition 2.9. (See [17]) A fuzzy number $\widetilde{I}$ is a fuzzy set on $\mathbb{R}$, whose membership function $\mu_{\widetilde{I}}(\cdot) : \mathbb{R} \rightarrow [0, 1]$ satisfies the following conditions:

(1) $\mu_{\widetilde{I}}(x) = 0$ for all $x \in (-\infty, c]$,

(2) $\mu_{\widetilde{I}}(\cdot)$ is strictly increasing and continuous on $[c, a]$,

(3) $\mu_{\widetilde{I}}(x) = 1$ for all $x \in [a, b]$,

(4) $\mu_{\widetilde{I}}(\cdot)$ is strictly decreasing and continuous on $[b, d]$,

(5) $\mu_{\widetilde{I}}(x) = 0$ for all $x \in [d, +\infty)$,

where $-\infty < c \leq a \leq b \leq d < +\infty$.

As usual, write $\mathcal{F}(\mathbb{R})$ for the family of all fuzzy numbers, and $\mathcal{F}(\mathbb{R})^d$ for the family of all d-dimensional fuzzy vectors. We use $\widetilde{A}, \widetilde{B}, \widetilde{C}, \cdots$ to denote the elements of $\mathcal{F}(\mathbb{R})^d$. Given $\widetilde{A} = (\widetilde{A}_1, \ldots, \widetilde{A}_d) \in \mathcal{F}(\mathbb{R})^d$ and $\alpha \in [0, 1]$, the α-cut of $\widetilde{A}$ is an interval vector $\widetilde{A}^\alpha = (\widetilde{A}_1^\alpha, \ldots, \widetilde{A}_d^\alpha) \in \mathcal{I}(\mathbb{R})^d$, where $\widetilde{A}_i^\alpha$ is the α-cut of $\widetilde{A}_i$ for $i = 1, ..., d$.

Let $\mathcal{X}$ and $\mathcal{Y}$ be two Hausdorff topological vector spaces and $X \subset \mathcal{X}$, $Y \subset \mathcal{Y}$ be two nonempty convex subsets. A mapping $\widetilde{S} : X \rightarrow \mathcal{F}(Y)$ is called a fuzzy mapping where $\mathcal{F}(Y)$ denotes the collection of fuzzy sets on Y. If $\widetilde{S}$ is a fuzzy mapping, $\widetilde{S}(x)$ (denoted by $\widetilde{S}_x$) is a fuzzy set, and $\widetilde{S}_x(y)$ is the membership function of y in $\widetilde{S}_x$.

Definition 2.10. (See [11]) A fuzzy mapping $\widetilde{S} : X \to \mathcal{F}(Y)$ is convex if, for every $x \in X$, $y, z \in Y$ and $\lambda \in [0, 1]$, we have $\widetilde{S}_x(\lambda y + (1 - \lambda)z) \geq \min\{\widetilde{S}_x(y), \widetilde{S}_x(z)\}$.

Lemma 2.11. *(See [11]) Let $s : X \to [0, 1]$ be a function. If $\widetilde{S} : X \to \mathcal{F}(Y)$ is convex, and $S : X \rightrightarrows Y$ is defined by $S(x) = (\widetilde{S}_x)^{s(x)}$ for $x \in X$, then S is a correspondence with convex values.*

2.3. Set valued analysis

Let X and Y be two Hausdorff topological spaces and $F : X \rightrightarrows Y$ be a correspondence. F is *upper semi-continuous* (resp., *lower semi-continuous*) if, for each $x \in X$ and for each open set $G \subset Y$ with $F(x) \subset G$ (resp., $F(x) \cap G \neq \emptyset$), there exists a neighborhood $o(x)$ of x such that $F(x') \subset G$ (resp., $F(x') \cap G \neq \emptyset$) for each $x' \in o(x)$. F is *continuous* if it is both lower and upper semi-continuous. The graph of F is $Graph(F) = \{(x, y) \in X \times Y \mid y \in F(x)\}$, and F is closed if $Graph(F)$ is closed on $X \times Y$. For more details, see [3, 18, 22, 50].

Lemma 2.12. *(See [50]) Let X and Y be two Hausdorff topological spaces, and Y be compact. Then, $F : X \rightrightarrows Y$ is upper semi-continuous with nonempty and compact values if and only if it is closed.*

Lemma 2.13. *(See [50]) Let $F, G : X \rightrightarrows Y$ be two correspondences with $F(x) \cap G(x) \neq \emptyset$ for all $x \in X$. If F is closed, and G is upper semi-continuous with nonempty and compact values, then $F \cap G$ is upper semi-continuous.*

Theorem 2.14. *(Fan-Glicksberg [18, 22]) Let $\mathcal{V}$ be a locally convex Hausdorff topological vector space, and let X be a nonempty compact subset of $\mathcal{V}$. If $\Phi : X \rightrightarrows X$ is upper semi-continuous with nonempty, convex and compact values, then there exists $x^* \in X$ such that $x^* \in \Phi(x^*)$.*

2.4. Structural stability and robustness

Let us recall the structural stability and robustness of the bounded rationality model according to [2, 48, 49].

Definition 2.15. (See [2]) Let (Γ, ρ) be a metric space.

(1) The model $\mathcal{M}$ is said to be structurally stable at $\gamma \in \Gamma$ if the equilibrium correspondence $E : \Gamma \times \mathbb{R}_+ \rightrightarrows X$ is continuous at $\gamma \in \Gamma$.

(2) The model $\mathcal{M}$ is said to be robust to ϵ-equilibria at $\gamma \in \Gamma$ if for any $\delta > 0$, there exists $\hat{\epsilon} > 0$ such that $H(E(\gamma', \epsilon), E(\gamma')) < \delta$, for any $\gamma' \in \Gamma$ with $\epsilon < \hat{\epsilon}$ and $\rho(\gamma, \gamma') < \hat{\epsilon}$, where H is the Hausdorff distance defined on X.

Definition 2.16. (See [34]) Let (Γ, ρ) be a metric space.

(1) The model $\mathcal{M}$ is said to be (γ, ϵ)-stable if the equilibrium correspondence $E : \Gamma \times \mathbb{R}_+ \rightrightarrows X$ is continuous at $(\gamma, \epsilon) \in \Gamma \times \mathbb{R}_+$.

(2) The model $\mathcal{M}$ is said to be (γ, ϵ)-robust if for any $\delta > 0$, there exists $\hat{\epsilon} > 0$ such that $H(E(\gamma', \epsilon), E(\gamma', \epsilon')) < \delta$, for any $(\gamma', \epsilon') \in \Gamma \times \mathbb{R}_+$ with $|\epsilon - \epsilon'| < \hat{\epsilon}$ and $\rho(\gamma, \gamma') < \hat{\epsilon}$, where H is the Hausdorff distance defined on X.

In particular, taking $\epsilon' = 0$ in Definition 2.16 (2), we can get the definition of robustness to ϵ-equilibria. A model that is $(\gamma, 0)$-stable is structurally stable, that is, $E : \Gamma \rightrightarrows X$ is continuous at $\gamma \in \Gamma$.

Lemma 2.17. *(See [34, 48, 49, 53]) Let Γ be a complete metric space and X be a compact metric space. Let $k : \Gamma \rightrightarrows X$ be an upper semi-continuous correspondence, and let $R : Graph(k) \to \mathbb{R}_+$ be a lower semi-continuous function. For any $\gamma \in \Gamma$, $E(\gamma) \neq \emptyset$,*

(1) the correspondence $E : \Gamma \rightrightarrows X$ is upper semi-continuous;

(2) there exists a dense G_δ subset Q of Γ such that $\mathcal{M}$ is structurally stable at every $\gamma \in Q$;

(3) if the model $\mathcal{M}$ is structurally stable at $\gamma \in \Gamma$, then $\mathcal{M}$ is robust to ϵ-equilibria at $\gamma \in \Gamma$, and $\mathcal{M}$ is robust to ϵ-equilibria at $\gamma \in Q$;

(4) if $\gamma \in Q$, $\gamma_m \to \gamma$, and $\epsilon_m \to 0$, then $H(E(\gamma_m, \epsilon_m), E(\gamma)) \to 0$;

(5) if $E(\gamma)$ is a singleton set, then $\mathcal{M}$ is structurally stable and robust to ϵ-equilibria at $\gamma \in \Gamma$;

(6) if the model $\mathcal{M}$ is (γ, ϵ)-stable, then $\mathcal{M}$ is (γ, ϵ)-robust.

3. Existence of fuzzy Pareto-Nash equilibria

In this section, we focus on the fuzzy Pareto-Nash equilibria of the FCMGFP. We first discuss the semi-continuity and generalized quasi-concavity of fuzzy-vector-valued functions. Then, we obtain Berge's maximum theorem for fuzzy-vector-valued functions. Based on this theorem, we establish two existence theorems of fuzzy Pareto-Nash equilibria and fuzzy α-Pareto-Nash equilibria for FCMGFP. Throughout this section, we write $\mathcal{X}$ for a Hausdorff topological space, $\mathcal{Y}$ for a Hausdorff topological vector space and $\mathcal{V}_i$ for a locally convex Hausdorff topological vector space.

3.1. Fuzzy-vector-valued functions

Let $\widetilde{F} = \{\widetilde{f_1}, ..., \widetilde{f_d}\} : \mathcal{X} \to \mathcal{F}(\mathbb{R})^d$ be a fuzzy-vector-valued function. By the α-cuts of fuzzy vectors, we can obtain the interval-vector-valued function $\widetilde{F}^\alpha = \{\widetilde{f_1^\alpha}, ..., \widetilde{f_d^\alpha}\} : \mathcal{X} \times [0, 1] \to \mathcal{I}(\mathbb{R})^d$, defined for $(x, \alpha) \in \mathcal{X} \times [0, 1]$ by $\widetilde{F}^\alpha(x) = (\widetilde{f_1^\alpha}(x), ..., \widetilde{f_d^\alpha}(x))$.

We use the related properties of interval-vector-valued functions to establish the semi-continuity and generalized quasi-concavity of fuzzy-vector-valued functions.

Definition 3.1. A fuzzy-vector-valued function $\widetilde{F} = \{\widetilde{f_1}, ..., \widetilde{f_d}\} : \mathcal{X} \to \mathcal{F}(\mathbb{R})^d$ is *upper semi-continuous* (*resp., lower semi-continuous, continuous*) if the corresponding interval-vector-valued function $\widetilde{F}^\alpha = \{\widetilde{f_1^\alpha}, ..., \widetilde{f_d^\alpha}\} : \mathcal{X} \times [0, 1] \to \mathcal{I}(\mathbb{R})^d$ is *upper semi-continuous* (*resp., lower semi-continuous, continuous*).

Remark 3.2. A fuzzy-vector-valued function $\widetilde{F} = \{\widetilde{f_1}, ..., \widetilde{f_d}\} : \mathcal{X} \to \mathcal{F}(\mathbb{R})^d$ is upper semi-continuous (resp., lower semi-continuous, continuous) if and only if each $\widetilde{f_i} : \mathcal{X} \to \mathcal{F}(\mathbb{R})$ is upper semi-continuous (resp., lower semi-continuous, continuous).

Assume that $\mathcal{F}_0(\mathbb{R})$ is the family of all triangular fuzzy numbers. Let $\widetilde{A} = (a, b, c) \in \mathcal{F}_0(\mathbb{R})$ and $\alpha \in [0, 1]$. Then, the α-cut of $\widetilde{A}$ is an interval $\widetilde{A}^\alpha = [a + \alpha(b - a), c - \alpha(c - b)]$.

Example 3.3. Consider the fuzzy-vector-function $\widetilde{F}(x) = (\widetilde{f_1}(x), \widetilde{f_2}(x))$, where $\widetilde{f_1}(x) = (2, 3, 4)x + (3, 4, 5)$, and $\widetilde{f_2}(x) = (1, 2, 3)x^2 + (2, 3, 4)$ for $x \in \mathbb{R}$. Since $\widetilde{f_1}^{\alpha}(x) = [2 + \alpha, 4 - \alpha]x + [3 + \alpha, 5 - \alpha]$ and $\widetilde{f_2}^{\alpha}(x) = [1 + \alpha, 3 - \alpha]x^2 + [2 + \alpha, 4 - \alpha]$ are continuous on $\mathbb{R} \times [0, 1]$, $\widetilde{F}$ is a continuous fuzzy-vector-valued function.

Based on the partial order of interval vectors, we use α-cuts to establish the following partial order of fuzzy vectors.

Definition 3.4. Let $\widetilde{A}$ and $\widetilde{B}$ be two elements of $\mathcal{F}(\mathbb{R})^d$.

(1) $\widetilde{A}$ is said to be dominated by $\widetilde{B}$ from below if $\widetilde{B}^{\alpha} \preceq \widetilde{A}^{\alpha}$ for all $\alpha \in [0, 1]$, and we rewrite this property as $\widetilde{B} \preceq \widetilde{A}$. Otherwise, we write $\widetilde{B} \npreceq \widetilde{A}$.

(2) $\widetilde{A}$ is said to be strictly dominated by $\widetilde{B}$ from below if $\widetilde{B}^{\alpha} \prec \widetilde{A}^{\alpha}$ for all $\alpha \in [0, 1]$, and we rewrite this property as $\widetilde{B} \prec \widetilde{A}$. Otherwise, we write $\widetilde{B} \nprec \widetilde{A}$.

In particular, when $d = 1$, the partial order $\preceq$ in Definition 3.4 is equivalent to the fuzzy maximum order in [32, 39]. According to Definitions 2.4 and 3.4, the following lemma is obvious.

Lemma 3.5. *For two elements $\widetilde{A}$ and $\widetilde{B}$ of $\mathcal{F}(\mathbb{R})^d$, the following assertions hold:*

(1) $\widetilde{B} \preceq \widetilde{A}$ if and only if $\widetilde{A}^{\alpha} \ominus \widetilde{B}^{\alpha} \in \mathcal{I}(\mathbb{R}_+)^d$ for all $\alpha \in [0, 1]$.

(2) $\widetilde{B} \nprec \widetilde{A}$ if and only if there is $\alpha \in [0, 1]$ such that $\widetilde{A}^{\alpha} \ominus \widetilde{B}^{\alpha} \notin int\mathcal{I}(\mathbb{R}_+)^d$.

Definition 3.6. Let $\mathcal{K}$ be a nonempty convex subset of $\mathcal{Y}$. A fuzzy-vector-valued function $\widetilde{F} = \{\widetilde{f_1}, ..., \widetilde{f_d}\} : \mathcal{K} \to \mathcal{F}(\mathbb{R})^d$ is said to be *generalized quasi-concave* on $\mathcal{K}$ provided that for $x_1, x_2 \in \mathcal{K}$ and $\lambda \in [0, 1]$, and for $\widetilde{A} \in \mathcal{F}(\mathbb{R})^d$, if $\widetilde{F}(x_i) \nprec \widetilde{A}$ with $i = 1, 2$, we have

$$\widetilde{F}(\lambda x_1 + (1 - \lambda)x_2) \nprec \widetilde{A}.$$

Lemma 3.7. *Let $\widetilde{F} = \{\widetilde{f_1}, ..., \widetilde{f_d}\} : \mathcal{K} \subset \mathbb{R} \to \mathcal{F}(\mathbb{R})^d$ be a fuzzy-vector-valued function. If $\widetilde{F}^{\alpha} : \mathcal{K} \to \mathcal{I}(\mathbb{R})^d$ is non-decreasing (or non-increasing) for each $\alpha \in [0, 1]$, then $\widetilde{F}$ is generalized quasi-concave.*

Proof. See Appendix B. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

In Example 3.3, for each $\alpha \in [0, 1]$, since $\underline{\widetilde{f_1}^{\alpha}}(x) = (2 + \alpha)x + (3 + \alpha)$, $\overline{\widetilde{f_1}^{\alpha}}(x) = (4 - \alpha)x + (5 - \alpha)$, $\underline{\widetilde{f_2}^{\alpha}}(x) = (1 + \alpha)x^2 + (2 + \alpha)$ and $\overline{\widetilde{f_2}^{\alpha}}(x) = (3 - \alpha)x^2 + (4 - \alpha)$ are non-decreasing on $[0, +\infty)$, from Lemma 3.7, it follows that $\widetilde{F}$ is generalized quasi-concave on $[0, +\infty)$.

3.2. Existence of fuzzy Pareto-Nash equilibria

We now consider a fuzzy constrained multi-objective game with fuzzy payoffs (FCMGFP) as follows:

$$\gamma = (X_i, \widetilde{S}_i, s_i, \widetilde{F}_i)_{i \in N}$$

which has the following assumptions:

(1) $N = \{1, 2, ..., n\}$ is a set of players;

(2) X_i is player i's strategy space;

(3) $\widetilde{S}_i : X_{-i} \to \mathcal{F}(X_i)$ is player i's fuzzy constraint mapping;

(4) $s_i : X_{-i} \to [0, 1]$ is player i's feasible function, where $s_i(x_{-i})$ represent the feasible degree on strategy profile $x_{-i} \in X_{-i}$ for player i;

(5) $\widetilde{F}_i = \{\widetilde{f}_1^i, \ldots, \widetilde{f}_d^i\} : X = \prod_{i \in \mathrm{N}} X_i \to \mathcal{F}(\mathbb{R})^d$ is player i's payoff function.

As usual, for each player $i \in N$, write

$$-i = N \setminus \{i\} \quad \text{and} \quad X_{-i} = \prod_{j \neq i} X_j,$$

where the latter is endowed with the product topology.

Definition 3.8. A point $x^* = (x_i^*, x_{-i}^*) \in X$ is called a fuzzy Pareto-Nash equilibrium of γ if for each $i \in N$, $\widetilde{S}_{ix_{-i}^*}(x_i^*) \geq s_i(x_{-i}^*)$, and for each u_i with $\widetilde{S}_{ix_{-i}^*}(u_i) \geq s_i(x_{-i}^*)$, we have

$$\widetilde{F}_i(x_i^*, x_{-i}^*) \nprec \widetilde{F}_i(u_i, x_{-i}^*). \tag{3.1}$$

Definition 3.9. Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in [0, 1]^n$. A point $x^* = (x_i^*, x_{-i}^*) \in X$ is called a fuzzy α-Pareto-Nash equilibrium of γ if for each $i \in N$, $\widetilde{S}_{ix_{-i}^*}(x_i^*) \geq s_i(x_{-i}^*)$, and for each u_i with $\widetilde{S}_{ix_{-i}^*}(u_i) \geq s_i(x_{-i}^*)$, we have

$$\widetilde{F}_i^{\alpha_i}(u_i, x_{-i}^*) \ominus \widetilde{F}_i^{\alpha_i}(x_i^*, x_{-i}^*) \notin int\mathcal{I}(\mathbb{R}_+)^d, \tag{3.2}$$

where $[0, 1]^n = [0, 1] \times \cdots \times [0, 1]$.

For $\alpha = (\alpha_1, \ldots, \alpha_n) \in [0, 1]^n$, we observed that a fuzzy α-Pareto-Nash equilibrium of γ must be a fuzzy Pareto-Nash equilibrium of γ, but a fuzzy Pareto-Nash equilibrium of γ is not necessarily a fuzzy α-Pareto-Nash equilibrium of γ.

Using the fuzzy mapping $\widetilde{S}_i$ and the corresponding feasibility function s_i, we consider the correspondence $S_i : X_{-i} \rightrightarrows X_i$ as follows: For all $x_{-i} \in X_{-i}$,

$$S_i(x_{-i}) = (\widetilde{S}_{ix_{-i}})^{s_i(x_{-i})} = \{x_i \in X_i \mid \widetilde{S}_{ix_{-i}}(x_i) \geq s_i(x_{-i})\}.$$

In what follows, S_i is called a correspondence induced by the fuzzy mapping $\widetilde{S}_i$. Then, a strategy profile $x^* = (x_i^*, x_{-i}^*) \in X$ is called a fuzzy Pareto-Nash equilibrium of γ if for each $i \in N$, $x_i^* \in S_i(x_{-i}^*)$, and for each $u_i \in S_i(x_{-i}^*)$, $\widetilde{F}_i(x_i^*, x_{-i}^*) \nprec \widetilde{F}_i(u_i, x_{-i}^*)$. A strategy profile $x^* = (x_i^*, x_{-i}^*) \in X$ is called a fuzzy α-Pareto-Nash equilibrium of γ if for each $i \in N$, $x_i^* \in S_i(x_{-i}^*)$, and for each $u_i \in S_i(x_{-i}^*)$,

$$\widetilde{F}_i^{\alpha_i}(u_i, x_{-i}^*) \ominus \widetilde{F}_i^{\alpha_i}(x_i^*, x_{-i}^*) \notin int\mathcal{I}(\mathbb{R}_+)^d. \tag{3.3}$$

In order to prove the existence of fuzzy Pareto-Nash equilibria, we need the following lemma.

Theorem 3.10. *Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be two Hausdorff topological spaces, let $\widetilde{F} = \{\widetilde{f}_1, \ldots, \widetilde{f}_d\} : \mathcal{X}_1 \times \mathcal{X}_2 \to \mathcal{F}(\mathbb{R})^d$ be a continuous fuzzy-vector-valued function, and let $S : \mathcal{X}_2 \rightrightarrows \mathcal{X}_1$ be a continuous correspondence with nonempty and compact values. Moreover, for all $y \in \mathcal{X}_2$, let*

$$\Phi(y) = \{x \in S(y) \mid \widetilde{F}(x, y) \nprec \widetilde{F}(u, y), \text{ for all } u \in S(y)\}.$$

Then $\Phi : \mathcal{X}_2 \rightrightarrows \mathcal{X}_1$ is an upper semi-continuous correspondence with nonempty and compact values.

Proof. Let $y \in \mathcal{X}_2$, and then

$$\Phi(y) = \bigcup_{\alpha \in [0,1]} \Phi_\alpha(y), \tag{3.4}$$

where $\Phi_\alpha(y) = \{x \in S(y) \mid \widetilde{F}^\alpha(u, y) \ominus \widetilde{F}^\alpha(x, y) \notin int\mathcal{I}(\mathbb{R}_+)^d, \text{ for all } u \in S(y)\}$.

Since $S(y)$ is nonempty and compact. Meanwhile, the continuity of $\widetilde{F}$ yields that of each $\widetilde{f}_i^\alpha$. So, from Definition 2.4 and Lemma 2.6, it follows that $\Phi_\alpha(y) \neq \emptyset$ for each $\alpha \in [0, 1]$. Consequently, $\Phi(y) \neq \emptyset$.

Next, we verify the compactness of $\Phi(y)$. Since $\Phi(y)$ and $S(y)$ are compact, and $\Phi(y) \subset S(y)$, we need only to show that $\Phi(y)$ is closed. For this aim, consider a net $\{x^\tau\}_{\tau \in D} \subset \Phi(y)$ having the properties $x^\tau \to x$ and $x \notin \Phi(y)$. Then, there exists $u_0 \in S(y)$ such that

$$\widetilde{F}(x, y) \prec \widetilde{F}(u_0, y). \tag{3.5}$$

Since $x^\tau \in \Phi(y)$, from (3.4) there is $\alpha_\tau \in [0, 1]$ such that for all $u \in S(y)$,

$$\widetilde{F}^{\alpha_\tau}(u, y) \ominus \widetilde{F}^{\alpha_\tau}(x^\tau, y) \notin int\mathcal{I}(\mathbb{R}_+)^d. \tag{3.6}$$

Furthermore, there is a subnet $\{\alpha_{\tau'}\}$ of $\{\alpha_\tau\}$ such that $\alpha_{\tau'} \to \alpha \in [0, 1]$. From the compactness of $S(y)$ and $u_0 \in S(y)$, there is a net $\{u^\tau\}_{\tau \in D}$ such that $u^\tau \in S(y)$ and $u^\tau \to u_0$. Then, from the continuity of $\widetilde{F}$, (3.6) and Lemma 2.2, it follows that

$$\widetilde{F}^\alpha(u_0, y) \ominus \widetilde{F}^\alpha(x, y) \notin int\mathcal{I}(\mathbb{R}_+)^d,$$

which contradicts the formula (3.5). Therefore, $\Phi(y)$ is compact.

Finally, we prove that Φ is upper semi-continuous. Note that

$$\Phi(y) = S(y) \bigcap \left\{ x \in \mathcal{X}_1 \mid \widetilde{F}(x, y) \nprec \widetilde{F}(u, y), \text{for all } u \in S(y) \right\},$$

and by Lemma 2.13, we turn to verify the closeness of the correspondence $T : \mathcal{X}_2 \rightrightarrows \mathcal{X}_1$, which is defined for $y \in \mathcal{X}_2$ by

$$T(y) = \left\{ x \in \mathcal{X}_1 \mid \widetilde{F}(x, y) \nprec \widetilde{F}(u, y), \text{for all } u \in S(y) \right\}.$$

For this aim, assume that $\{y^\tau\}_{\tau \in D} \subset \mathcal{X}_2$ with $y^\tau \to y$, and $x^\tau \in T(y^\tau)$ with $x^\tau \to x \in \mathcal{X}_1$. It suffices to show that $x \in T(y)$.

We argue by contradiction and assume $x \notin T(y)$. Then, there exists $u_0 \in S(y)$, such that

$$\widetilde{F}(x, y) \prec \widetilde{F}(u_0, y). \tag{3.7}$$

On the other hand, since $x^\tau \in T(y^\tau)$, this implies that $x^\tau \in \mathcal{X}_1$, and for all $u \in S(y^\tau)$, we have $\widetilde{F}(x^\tau, y^\tau) \nprec \widetilde{F}(u, y^\tau)$. In other words, there is $\alpha_\tau \in [0, 1]$, such that

$$\widetilde{F}^{\alpha_\tau}(u, y^\tau) \ominus \widetilde{F}^{\alpha_\tau}(x^\tau, y^\tau) \notin int\mathcal{I}(\mathbb{R}_+)^d. \tag{3.8}$$

Furthermore, there is a subnet $\{\alpha_{\tau'}\}$ of $\{\alpha_\tau\}$ such that $\alpha_{\tau'} \to \alpha \in [0, 1]$. Guaranteed by the lower semi-continuity of S, there is a net $\{u^\tau\}_{\tau \in D}$ such that $u^\tau \in S(y^\tau)$ and $u^\tau \to u_0$. Then, from the continuity of $\widetilde{F}$, (3.8) and Lemma 2.2, it follows that

$$\widetilde{F}^\alpha(u_0, y) \ominus \widetilde{F}^\alpha(x, y) \notin int\mathcal{I}(\mathbb{R}_+)^d,$$

which contradicts the formula (3.7). Hence, Φ is closed. The proof is completed. $\square$

A special result of Theorem 3.10 is the following corollary, which was first proved by Li et al. [28].

Corollary 3.11. *Let X_1 and X_2 be two Hausdorff topological spaces, let $F = \{f_1, \ldots, f_d\} : X_1 \times X_2 \to I(\mathbb{R})^d$ be a continuous interval-vector-valued function, and let $S : X_2 \rightrightarrows X_1$ be a continuous correspondence with nonempty and compact values. Moreover, for each $y \in X_2$, let*

$$\Phi(y) = \{x \in S(y) \mid F(u, y) \ominus F(x, y) \notin int I(\mathbb{R})_+^d, for\ all\ u \in S(y)\}.$$

Then, $\Phi : X_2 \rightrightarrows X_1$ is an upper semi-continuous correspondence with nonempty compact values.

It is noted that for the case $d = 1$ in Corollary 3.11, if $F : X_1 \times X_2 \to \mathbb{R}$ is continuous, and $S : X_2 \rightrightarrows X_1$ is a continuous correspondence with nonempty and compact values, then Corollary 3.11 becomes Berge's maximum theorem (see [5]).

Theorem 3.12. *Let $\gamma = (X_i, \widetilde{S}_i, s_i, \widetilde{F}_i)_{i \in N}$ be an FCMGFP, where X_i is a nonempty, convex and compact subset of $\mathcal{V}_i$. Then, γ has at least one fuzzy Pareto-Nash equilibrium if the following conditions are fulfilled: For each $i \in N$,*

(1) $\widetilde{S}_i$ is a convex fuzzy mapping, and S_i is continuous with nonempty and compact values;

(2) there is $k_i \in \{1, \ldots, d\}$ such that $\widetilde{f}^i_{k_i} : X \to \mathcal{F}(\mathbb{R})$ is continuous;

(3) for each $x_{-i} \in X_{-i}$, $x_i \mapsto \widetilde{f}^i_{k_i}(x_i, x_{-i})$ is generalized quasi-concave.

Proof. Let $\Phi : X \rightrightarrows X$ be the correspondence defined for $x = (x_1, \ldots, x_n) \in X$ by

$$\Phi(x) = \prod_{i \in N} \Phi_i(x_{-i}).$$

Here, for $i \in N$ and $x_{-i} \in X_{-i}$, we have

$$\Phi_i(x_{-i}) = \left\{ a_i \in S_i(x_{-i}) \mid \widetilde{f}^i_{k_i}(a_i, x_{-i}) \not\prec \widetilde{f}^i_{k_i}(u_i, x_{-i}),\ \text{for all}\ u_i \in S_i(x_{-i}) \right\}.$$

According to Theorem 3.10, each $\Phi_i : X_{-i} \rightrightarrows X_i$ is upper semi-continuous with nonempty compact values.

Next, we verify the convexity of $\Phi_i(x_{-i})$. For this aim, let $a_1, a_2 \in \Phi_i(x_{-i})$ and $\lambda \in [0, 1]$. That is, for $a_1, a_2 \in S_i(x_{-i})$ and for all $u_i \in S_i(x_{-i})$, it follows that

$$\widetilde{f}^i_{k_i}(a_1, x_{-i}) \not\prec \widetilde{f}^i_{k_i}(u_i, x_{-i}) \quad \text{and} \quad \widetilde{f}^i_{k_i}(a_2, x_{-i}) \not\prec \widetilde{f}^i_{k_i}(u_i, x_{-i}).$$

Since the fuzzy mapping $\widetilde{S}_i$ is convex, from Lemma 2.11 it follows that $S_i(x_{-i})$ is convex. So, $\lambda a_1 + (1 - \lambda)a_2 \in S_i(x_{-i})$. Since $x_i \mapsto \widetilde{f}^i_{k_i}(x_i, x_{-i})$ is generalized quasi-concave, we have

$$\widetilde{f}^i_{k_i}(\lambda a_1 + (1 - \lambda)a_2, x_{-i}) \not\prec \widetilde{f}^i_{k_i}(u_i, x_{-i}).$$

That is, $\lambda a_1 + (1 - \lambda)a_2 \in \Phi_i(x_{-i})$. Therefore, $\Phi_i(x_{-i})$ is convex.

Recall that each Φ_i is upper semi-continuous with nonempty convex compact values, and so is Φ. According to Theorem 2.14, there exists $x^* \in X$ such that $x^* \in \Phi(x^*)$. In other words, for each $i \in N$, $x_i^* \in S_i(x_{-i}^*)$, and for all $u_i \in S_i(x_{-i}^*)$, $\widetilde{f}^i_{k_i}(x_i^*, x_{-i}^*) \not\prec \widetilde{f}^i_{k_i}(u_i, x_{-i}^*)$, that is, $\widetilde{F}_i(x_i^*, x_{-i}^*) \not\prec \widetilde{F}_i(u_i, x_{-i}^*)$. This completes the proof. $\qquad\square$

It should be noted that if, for each $i \in N$, $\widetilde{F}_i : X \to \mathbb{R}^d$ and $s_i(x_{-i}) = 1$ $(x_{-i} \in X_{-i})$ in Theorem 3.12, then the FCMGFP becomes a constrained multi-objective game (see [48]). When each player has only one objective in an FCMGFP, the FCMGFP degenerates into a fuzzy constrained game with fuzzy payoffs (FCGFP). Letting $d = 1$ within Theorem 3.12, we obtain the following corollary.

Corollary 3.13. *Let $\gamma' = (X_i, \widetilde{S}_i, s_i, \widetilde{f}_i)_{i \in N}$ be an FCGFP, where X_i is a nonempty, convex and compact subset of $\mathcal{V}_i$. Then, γ' has at least one fuzzy Pareto-Nash equilibrium if the following conditions are fulfilled: For each $i \in N$,*

(1) $\widetilde{S}_i$ is a convex fuzzy mapping, and S_i is continuous with nonempty and compact values;

(2) $\widetilde{f}_i : X \to \mathcal{F}(\mathbb{R})$ is a continuous fuzzy-valued function;

(3) for each $x_{-i} \in X_{-i}$, $x_i \mapsto \widetilde{f}_i(x_i, x_{-i})$ is generalized quasi-concave.

Through Theorem 3.12 and Corollary 3.13, we can get the following conclusion that a fuzzy Pareto-Nash equilibrium of FCMGFP $(X_i, \widetilde{S}_i, s_i, \widetilde{F}_i)_{i \in N}$ must be a fuzzy Pareto-Nash equilibrium of a certain FCGFP $(X_i, \widetilde{S}_i, s_i, \widetilde{f}_{k_i}^i)_{i \in N}$ $(k_i \in \{1, \ldots, d\})$. Therefore, the fuzzy Pareto-Nash equilibria of FCMGFP can be transformed into the fuzzy Pareto-Nash equilibria of the FCGFP.

Theorem 3.14. *Let $\gamma = (X_i, \widetilde{S}_i, s_i, \widetilde{F}_i)_{i \in N}$ be an FCMGFP, where X_i is a nonempty, convex, and compact subset of $\mathcal{V}_i$, and let $\alpha = (\alpha_1, \ldots, \alpha_n) \in [0, 1]^n$. Then, γ has at least one fuzzy α-Pareto-Nash equilibrium if the following conditions are fulfilled: For each $i \in N$,*

(1) $\widetilde{S}_i$ is a convex fuzzy mapping, and S_i is continuous with nonempty and compact values;

(2) there is $k_i \in \{1, \ldots, d\}$ such that $\widetilde{f}_{k_i}^{i\alpha_i} : X \to \mathcal{I}(\mathbb{R})$ is continuous;

(3) for each $x_{-i} \in X_{-i}$, $x_i \mapsto \widetilde{f}_{k_i}^{i\alpha_i}(x_i, x_{-i})$ is generalized $\mathcal{I}(\mathbb{R}_+)$-quasi-concave.

Proof. By Corollary 3.11 and the proof method of Theorem 3.12, we prove Theorem 3.14. $\square$

In real-world games, decision makers often determine the membership degrees of each player's fuzzy payoffs according to their needs, that is, determine the α-cuts of fuzzy vectors. Given an $\alpha = (\alpha_1, \ldots, \alpha_n) \in [0, 1]^n$, we can obtain a corresponding fuzzy constrained multi-objective game with interval payoffs (FCMGIP) $(X_i, \widetilde{S}_i, s_i, \widetilde{F}_i^{\alpha_i})_{i \in N}$ from $(X_i, \widetilde{S}_i, s_i, \widetilde{F}_i)_{i \in N}$ in Theorem 3.14. When each player has only one objective in an FCMGIP, the FCMGIP degenerates into a fuzzy constrained game with interval payoffs (FCGIP). Letting $d = 1$ within Theorem 3.14, we obtain the following corollary.

Corollary 3.15. *Let $\gamma' = (X_i, \widetilde{S}_i, s_i, \widetilde{f}_i)_{i \in N}$ be an FCGFP, where X_i is a nonempty, convex and compact subset of $\mathcal{V}_i$, and let $\alpha = (\alpha_1, \ldots, \alpha_n) \in [0, 1]^n$. Then, γ' has at least one fuzzy α-Pareto-Nash equilibrium if the following conditions are fulfilled: For each $i \in N$,*

(1) $\widetilde{S}_i$ is a convex fuzzy mapping, and S_i is continuous with nonempty and compact values;

(2) there is $k_i \in \{1, \ldots, d\}$ such that $\widetilde{f}_i^{\alpha_i} : X \to \mathcal{I}(\mathbb{R})$ is continuous;

(3) for each $x_{-i} \in X_{-i}$, $x_i \mapsto \widetilde{f}_i^{\alpha_i}(x_i, x_{-i})$ is generalized $\mathcal{I}(\mathbb{R}_+)$-quasi-concave.

According to Theorems 3.12 and 3.14, Corollaries 3.13 and 3.15, we can get that the fuzzy Pareto-Nash equilibrium and fuzzy α-Pareto-Nash equilibrium of an FCMGFP can be transformed into weak Pareto-Nash equilibrium of an FCGIP. The specific relationship is shown in Figure 1. It should be noted that the membership degrees α and α' of fuzzy payoffs can be the same or different in Figure 1.
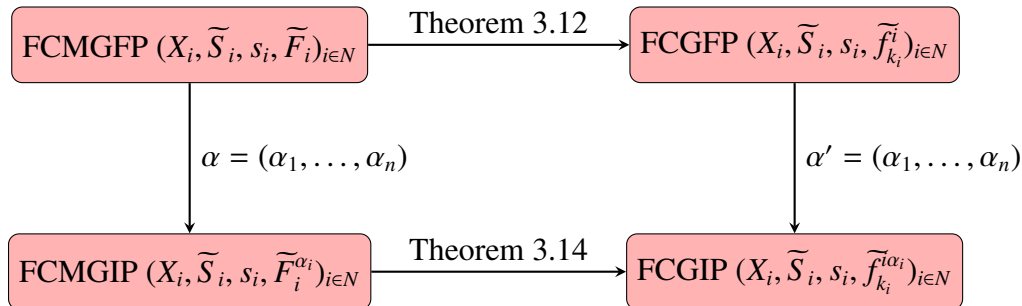
Figure 1. The relationship between fuzzy Pareto-Nash equilibrium and fuzzy α-Pareto-Nash equilibrium for an FCMGFP.

Example 3.16. Consider a two-player fuzzy constrained multi-objective game with fuzzy payoffs $(X_i, \widetilde{S}_i, s_i, \widetilde{F}_i)_{i \in \{1,2\}}$ as follows: For $i = 1, 2$,

(1) $X_i = [0, 3]$,

(2) $\widetilde{S}_i : [0, 3] \to \mathcal{F}([0, 3])$ is player i's fuzzy constraint mapping given by

$$\widetilde{S}_{1x_2}(x_1) = \begin{cases} \frac{3}{5}, & \text{if } x_2 \in [0, 3], x_1 \in [0, 2], \\ 0, & \text{otherwise}, \end{cases}$$

and

$$\widetilde{S}_{2x_1}(x_2) = \begin{cases} \frac{4}{5}, & \text{if } x_1 \in [0, 3], x_2 \in [1, 3], \\ 0, & \text{otherwise}, \end{cases}$$

(3) $s_i : [0, 3] \to [0, 1]$ is defined by $s_1(x_2) = \frac{1}{2}$ and $s_2(x_1) = \frac{3}{4}$, for $x_1, x_2 \in [0, 3]$, that is, the feasible degree on strategy $x_2 \in [0, 3]$ is $\frac{1}{2}$ for player 1, and the feasible degree on strategy $x_1 \in [0, 3]$ is $\frac{3}{4}$ for player 2.

(4) $\widetilde{F}_i : [0, 3] \times [0, 3] \to \mathcal{F}_0(\mathbb{R})^2$ is the triangular fuzzy-vector-valued function given for $(x_1, x_2) \in [0, 3] \times [0, 3]$ by

$$\widetilde{F}_i(x_1, x_2) = (\widetilde{f}_{i1}(x_1, x_2), \widetilde{f}_{i2}(x_1, x_2)),$$

where

$$\widetilde{f}_{11}(x_1, x_2) = (2, 3, 4)x_1 - (3, 4, 5)x_2, \quad \widetilde{f}_{12}(x_1, x_2) = (1, 2, 3)x_2^2 - (2, 3, 4)x_1^2.$$
$$\widetilde{f}_{21}(x_1, x_2) = (3, 4, 5)x_2 - (2, 3, 4)x_1, \quad \widetilde{f}_{22}(x_1, x_2) = (2, 3, 4)x_1^2 - (1, 2, 3)x_2^2.$$

For such a game, the conditions within Theorem 3.12 are satisfied.

First, for $i = 1, 2$, it is not difficult to verify the convexity of $\widetilde{S}_i$. The correspondence $S_i : [0, 3] \rightrightarrows [0, 3]$ induced by the fuzzy mapping $\widetilde{S}_i$ is precisely given by

$$S_1(x_2) = (\widetilde{S}_{1x_2})^{s_1(x_2)} = \{x_1 \in [0, 3] \mid \widetilde{S}_{1x_2}(x_1) \geq s_1(x_2)\} = [0, 2], \text{ for } x_2 \in [0, 3],$$

$$S_2(x_1) = (\widetilde{S}_{2x_1})^{s_2(x_1)} = \{x_2 \in [0,3] \mid \widetilde{S}_{2x_1}(x_2) \geq s_2(x_1)\} = [1,3], \text{ for } x_1 \in [0,3].$$

Obviously, S_i is continuous with nonempty convex compact values.

Second, for all $(x_1, x_2, \alpha) \in [0,2] \times [1,3] \times [0,1]$, observe that

$$\widetilde{f_{11}^\alpha}(x_1, x_2) = [2+\alpha, 4-\alpha]x_1 - [3+\alpha, 5-\alpha]x_2,$$

$$\widetilde{f_{12}^\alpha}(x_1, x_2) = [1+\alpha, 3-\alpha]x_2^2 - [2+\alpha, 4-\alpha]x_1^2,$$

$$\widetilde{f_{21}^\alpha}(x_1, x_2) = [3+\alpha, 5-\alpha]x_2 - [2+\alpha, 4-\alpha]x_1,$$

$$\widetilde{f_{22}^\alpha}(x_1, x_2) = [2+\alpha, 4-\alpha]x_1^2 - [1+\alpha, 3-\alpha]x_2^2.$$

We see that $\underline{\widetilde{f_{11}^\alpha}}(x_1, x_2) = (2+\alpha)x_1 - (3+\alpha)x_2$ and $\overline{\widetilde{f_{11}^\alpha}}(x_1, x_2) = (4-\alpha)x_1 - (5-\alpha)x_2$ are both continuous. So, $\widetilde{f_{11}^\alpha}$ is continuous. Similarly, we can also verify the continuity of $\widetilde{f_{21}^\alpha}$. According to Definition 3.1, the second condition of Theorem 3.12 is fulfilled.

Finally, for $\alpha \in [0,1]$, by Lemma 2.6, $x_1 \mapsto \widetilde{f_{11}^\alpha}(x_1, x_2)$ is non-decreasing for each $x_2 \in [1,3]$, and $x_2 \mapsto \widetilde{f_{21}^\alpha}(x_1, x_2)$ is non-decreasing for each $x_1 \in [0,2]$. From Lemma 3.7, the considered game also fulfills the last condition of Theorem 3.12. By calculation, we can deduce that $(2,3)$ is a fuzzy Pareto-Nash equilibrium of the game $(X_i, \widetilde{S}_i, s_i, \widetilde{f_{i1}})_{i \in \{1,2\}}$.

In fact, for $\alpha \in [0,1]$, $u_1 \in [0,2]$, $u_2 \in [1,3]$, we immediately arrive at the following consequences:

$$\widetilde{f_{11}^\alpha}(u_1, 3) \ominus \widetilde{f_{11}^\alpha}(2, 3) = [(4-\alpha)(u_1 - 2), (2+\alpha)(u_1 - 2)] \notin int\mathcal{I}(\mathbb{R}_+),$$

$$\widetilde{f_{21}^\alpha}(2, u_2) \ominus \widetilde{f_{21}^\alpha}(2, 3) = [(5-\alpha)(u_2 - 3), (3+\alpha)(u_2 - 3)] \notin int\mathcal{I}(\mathbb{R}_+).$$

Furthermore, for all $u_1 \in [0,2]$, $u_2 \in [1,3]$, $\widetilde{f_{11}}(2,3) \not\prec \widetilde{f_{11}}(u_1, 3)$, and $\widetilde{f_{21}}(2,3) \not\prec \widetilde{f_{21}}(2, u_2)$. From Definition 3.8, $(2,3)$ is a fuzzy Pareto-Nash equilibrium of the game $(X_i, \widetilde{S}_i, s_i, \widetilde{f_{i1}})_{i \in \{1,2\}}$.

Similarly, $\widetilde{f_{12}^\alpha}$ and $\widetilde{f_{22}^\alpha}$ are continuous. At the same time, for $\alpha \in [0,1]$, by Lemma 2.8, $x_1 \mapsto \widetilde{f_{12}^\alpha}(x_1, x_2)$ is non-increasing for each $x_2 \in [1,3]$, and $x_2 \mapsto \widetilde{f_{22}^\alpha}(x_1, x_2)$ is non-increasing for each $x_1 \in [0,2]$. For all $u_1 \in [0,2]$, $u_2 \in [1,3]$,

$$\widetilde{f_{12}^\alpha}(u_1, 1) \ominus \widetilde{f_{12}^\alpha}(0, 1) = [(4-\alpha)(0 - u_1^2), (2+\alpha)(0 - u_1^2)] \notin int\mathcal{I}(\mathbb{R}_+),$$

$$\widetilde{f_{22}^\alpha}(0, u_2) \ominus \widetilde{f_{22}^\alpha}(0, 1) = [(3-\alpha)(1 - u_2^2), (1+\alpha)(1 - u_2^2)] \notin int\mathcal{I}(\mathbb{R}_+),$$

that is, $\widetilde{f_{12}}(0,1) \not\prec \widetilde{f_{12}}(u_1, 1)$, and $\widetilde{f_{22}}(0,1) \not\prec \widetilde{f_{22}}(1, u_2)$. Hence, $(0,1)$ is a fuzzy Pareto-Nash equilibrium of the game $(X_i, \widetilde{S}_i, s_i, \widetilde{f_{i2}})_{i \in \{1,2\}}$. In the same way, we can verify that $(2,1)$ and $(0,3)$ are two fuzzy Pareto-Nash equilibria of the games $(X_i, \widetilde{S}_i, s_i, \widetilde{f_{ii}})_{i \in \{1,2\}}$ and $(X_i, \widetilde{S}_i, s_i, \widetilde{f_{ij(j \neq i)}})_{i \in \{1,2\}}$. Therefore, $(2,3)$, $(0,1)$, $(2,1)$ and $(0,3)$ are four fuzzy Pareto-Nash equilibria of the game $(X_i, \widetilde{S}_i, s_i, \widetilde{F}_i)_{i \in \{1,2\}}$. $\square$

Remark 3.17. For any given $\alpha = (\alpha_1, \alpha_2) \in [0,1]^2$, from Definition 3.9, $(2,3)$, $(0,1)$, $(2,1)$ and $(0,3)$ are not only four fuzzy Pareto-Nash equilibria of $(X_i, \widetilde{S}_i, s_i, \widetilde{F}_i)_{i \in \{1,2\}}$ but also four fuzzy α-Pareto-Nash equilibria of $(X_i, \widetilde{S}_i, s_i, \widetilde{F}_i)_{i \in \{1,2\}}$ in Example 3.16.

4. Stability of the fuzzy Pareto-Nash equilibria set

The last section presents a comprehensive discussion on a bounded rationality model for FCMGFPs. First, we construct a metric space of all FCMGFPs with fuzzy Pareto-Nash equilibria. Then, we verify the completeness of the metric space. Furthermore, with the help of the nonlinear scalarization function, we introduce a rationality function for FCMGFP. Finally, we prove that the structural stability $((\gamma, \epsilon)$-stability) of this model implies its robustness to ϵ-equilibria $((\gamma, \epsilon)$-robustness).

4.1. The metric space of all FCMGFPs with fuzzy Pareto-Nash equilibria

For each $i \in N$, let (X_i, d_i) be a compact metric space, and let Γ be the set of all FCMGFPs, where each $\gamma = (X_i, \widetilde{S}_i, s_i, \widetilde{F}_i)_{i \in N}$ satisfies the following conditions:

(1) $S_i : X_{-i} \rightrightarrows X_i$ is a continuous correspondence with nonempty and compact values induced by $\widetilde{S}_i$;

(2) $\widetilde{F}_i : X \to \mathcal{F}(\mathbb{R})^d$ is continuous on X with

$$\sup_{(x,\alpha)\in X\times[0,1]} \sum_{i=1}^{n} \|\widetilde{F}_i^{\alpha}(x)\|_{\mathcal{I}(\mathbb{R})^d} < +\infty;$$

(3) there exists $x = (x_1, \ldots, x_n) \in X$, for each $i \in N$, $x_i \in S_i(x_{-i})$, and for each $u_i \in S_i(x_{-i}^*)$, $\widetilde{F}(x_i^*, x_{-i}^*) \nprec \widetilde{F}_i(u_i, x_{-i}^*)$.

Denote by $E(\gamma)$ all fuzzy Pareto-Nash equilibria of $\gamma \in \Gamma$. Clearly, $E : \Gamma \rightrightarrows X$ is a correspondence, and $E(\gamma) \neq \emptyset$.

For $\gamma_j = (X_i, S_{ij}, \widetilde{F}_{ij})_{i\in N} \in \Gamma$ with $j = 1, 2$, define

$$\rho(\gamma_1, \gamma_2) = \sup_{(x,\alpha)\in X\times[0,1]} \sum_{i=1}^{n} \|\widetilde{F}_{i1}^{\alpha}(x) \ominus \widetilde{F}_{i2}^{\alpha}(x)\|_{\mathcal{I}(\mathbb{R})^d} + \sup_{x\in X} \sum_{i=1}^{n} H_i(S_{i1}(x_{-i}), S_{i2}(x_{-i}))$$

where H_i denotes the Hausdorff distance on $K_0(X_i)$, the set of all nonempty compact subsets in X_i. It is easy to show that (Γ, ρ) is a metric space.

Lemma 4.1. *The metric space (Γ, ρ) is complete.*

Proof. Let $\{\gamma_m\}_m$ be a Cauchy sequence in Γ, where $\gamma_m = (X_i, S_{im}, \widetilde{F}_{im})_{i\in N}$ for each $m \in \mathbb{N}$. Then, for $\epsilon > 0$, there is $Z \in \mathbb{N}$ such that for all $m, p > Z$,

$$\rho(\gamma_m, \gamma_p) = \sup_{(x,\alpha)\in X\times[0,1]} \sum_{i=1}^{n} \|\widetilde{F}_{im}^{\alpha}(x) \ominus \widetilde{F}_{ip}^{\alpha}(x)\|_{\mathcal{I}(\mathbb{R})^d} + \sup_{x\in X} \sum_{i=1}^{n} H_i(S_{im}(x_{-i}), S_{ip}(x_{-i})) < \epsilon.$$

Associated with each $i \in N$ are the following consequences. First, for each $(x, \alpha) \in X \times [0, 1]$, the sequence $\{\widetilde{F}_{im}^{\alpha}(x)\}_m$ is a Cauchy sequence in $\mathcal{I}(\mathbb{R})^d$. Second, for each $x_{-i} \in X_{-i}$, the sequence $\{S_{im}(x_{-i})\}_m$ is a Cauchy sequence in X_i.

In order to prove the existence of a limit $\gamma \in \Gamma$ of $\{\gamma_m\}_m$, we divide our proof into three steps in the following.

Step I. We show that there is $S_i : X_{-i} \rightrightarrows X_i$ such that $S_{im} \to S_i$, and S_i is continuous with convex compact values on X_{-i}.

From the completeness of $(K_0(X_i), H_i)$, there is $S_i : X_{-i} \rightrightarrows X_i$ such that $S_{im}(x_{-i}) \to S_i(x_{-i})$ for all $x_{-i} \in X_{-i}$. Recall that each S_{im} is continuous with convex compact values on X_{-i}, and so is S_i.

Step II. We prove that there is $\widetilde{F}_i : X \to \mathcal{F}(\mathbb{R})^d$ such that $\widetilde{F}_{im} \to \widetilde{F}_i$, and $\widetilde{F}_i$ is continuous on X with

$$\sup_{(x,\alpha) \in X \times [0,1]} \sum_{i=1}^{n} \|\widetilde{F}_i^\alpha(x)\|_{\mathcal{I}(\mathbb{R})^d} < +\infty. \tag{4.1}$$

Since $(\mathcal{I}(\mathbb{R})^d, \|\cdot\|_{\mathcal{I}(\mathbb{R})^d})$ is complete, for each $\alpha \in [0,1]$, there is $\widetilde{F}_i^\alpha : X \to \mathcal{I}(\mathbb{R})^d$ such that $\widetilde{F}_{im}^\alpha(x) \to \widetilde{F}_i^\alpha(x)$ for all $x \in X$.

As for the continuity of $\widetilde{F}_i$ on X, notice that for any $\{(x^m, \alpha^m)\}_m \subset X \times [0,1]$, we have

$$(x^m, \alpha^m) \to (x, \alpha), \quad \text{as } m \to \infty.$$

Since $\widetilde{F}_{im}^\alpha(x)$ is continuous on $X \times [0,1]$, by Lemma 2.3, we have

$$
\begin{aligned}
\|\widetilde{F}_i^{\alpha_m}(x^m) \ominus \widetilde{F}_i^\alpha(x)\|_{\mathcal{I}(\mathbb{R})^d} &\le \|\widetilde{F}_i^{\alpha_m}(x^m) \ominus \widetilde{F}_{im}^{\alpha_m}(x^m)\|_{\mathcal{I}(\mathbb{R})^d} + \|\widetilde{F}_{im}^{\alpha_m}(x^m) \ominus \widetilde{F}_i^\alpha(x)\|_{\mathcal{I}(\mathbb{R})^d} \\
&\le \|\widetilde{F}_i^{\alpha_m}(x^m) \ominus \widetilde{F}_{im}^{\alpha_m}(x^m)\|_{\mathcal{I}(\mathbb{R})^d} + \|\widetilde{F}_{im}^{\alpha_m}(x^m) \ominus \widetilde{F}_{im}^\alpha(x)\|_{\mathcal{I}(\mathbb{R})^d} \\
&\quad + \|\widetilde{F}_{im}^\alpha(x) \ominus \widetilde{F}_i^\alpha(x)\|_{\mathcal{I}(\mathbb{R})^d} \\
&\le 2 \sup_{(x,\alpha) \in X \times [0,1]} \|\widetilde{F}_{im}^\alpha(x) \ominus \widetilde{F}_i^\alpha(x)\|_{\mathcal{I}(\mathbb{R})^d} + \|\widetilde{F}_{im}^{\alpha_m}(x^m) \ominus \widetilde{F}_{im}^\alpha(x)\|_{\mathcal{I}(\mathbb{R})^d} \\
&\to 0 \text{ as } m \to \infty.
\end{aligned}
$$

That is, $\widetilde{F}_i$ is continuous for all $i \in N$.

Finally, (4.1) follows from the estimate

$$\sup_{(x,\alpha) \in X \times [0,1]} \sum_{i=1}^{n} \|\widetilde{F}_{im}^\alpha(x)\|_{\mathcal{I}(\mathbb{R})^d} < +\infty$$

together with the convergence $\widetilde{F}_{im}^\alpha(x) \to \widetilde{F}_i^\alpha(x)$ for all $(x, \alpha) \in X \times [0,1]$.

Step III. We now verify that there exists $x^* \in X$, for each $i \in N$, $x_i^* \in S_i(x_{-i}^*)$, and for each $u_i \in S_i(x_{-i}^*)$, $\widetilde{F}(x_i^*, x_{-i}^*) \not\prec \widetilde{F}_i(u_i, x_{-i}^*)$.

In fact, from $\{\gamma_m\}_m \subset \Gamma$ it follows that there exists a sequence $\{x^m\}_m \subset X$ such that each x^m is a fuzzy Pareto-Nash equilibrium of γ_m. In other words, for each $i \in N$, $x_i^m \in S_{im}(x_{-i}^m)$, and for each $u_i \in S_{im}(x_{-i}^m)$, $\widetilde{F}_{im}(x_i^m, x_{-i}^m) \not\prec \widetilde{F}_{im}(u_i, x_{-i}^m)$. That is, there is $\alpha_{im} \in [0,1]$ such that for each $u_i \in S_{im}(x_{-i}^m)$,

$$\widetilde{F}_{im}^{\alpha_{im}}(u_i, x_{-i}^m) \ominus \widetilde{F}_{im}^{\alpha_{im}}(x_i^m, x_{-i}^m) \notin int\mathcal{I}(\mathbb{R}_+)^d. \tag{4.2}$$

Because of the compactness of X, without loss of generality, we may assume $x^m \to x^* \in X$. Since

$$H_i(S_{im}(x_{-i}^m), S_i(x_{-i}^*)) \le H_i(S_{im}(x_{-i}^m), S_i(x_{-i}^m)) + H_i(S_i(x_{-i}^m), S_i(x_{-i}^*)),$$

from the continuity of S_i, it follows that $H_i(S_{im}(x_{-i}^m), S_i(x_{-i}^*)) \to 0$ as $m \to \infty$. Thus, we have

$$
\begin{aligned}
d_i(x_i^*, S_i(x_{-i}^*)) &\le d_i(x_i^*, x_i^m) + d_i(x_i^m, S_{im}(x_{-i}^m)) + H_i(S_{im}(x_{-i}^m), S_i(x_{-i}^*)) \\
&= d_i(x_i^*, x_i^m) + H_i(S_{im}(x_{-i}^m), S_i(x_{-i}^*)) \to 0,
\end{aligned}
\tag{4.3}
$$

which implies that $x_i^* \in S_i(x_{-i}^*)$.

For all $u_i \in S_i(x_{-i}^*)$, we note that the sets $S_i(x_{-i}^*)$ and $S_{im}(x_{-i}^*)$ are both compact. Thus, from the convergence $S_{im}(x_{-i}^m) \to S_i(x_{-i}^*)$, there is $u_i^m \in S_{im}(x_{-i}^m)$ such that $u_i^m \to u_i$ as $m \to \infty$. Then, by (4.2), we obtain

$$\widetilde{F}_{im}^{\alpha_{im}}(u_i^m, x_{-i}^m) \ominus \widetilde{F}_{im}^{\alpha_{im}}(x_i^m, x_{-i}^m) \notin int\mathcal{I}(\mathbb{R}_+)^d. \tag{4.4}$$

Moreover, there is a subsequence of $\{\alpha_{im}\}$ such that $\alpha_{im} \to \alpha_i$ (for the sake of simplicity, $\{\alpha_{im}\}$ also denotes its subsequence). Since $\widetilde{F}_{im}$ is continuous, from Lemma 2.3, it follows that

$$
\begin{aligned}
\|(\widetilde{F}_{im}^{\alpha_{im}}(x_i^m, x_{-i}^m)) \ominus \widetilde{F}_i^{\alpha_i}(x_i^*, x_{-i}^*)\|_{\mathcal{I}(\mathbb{R})^d} &\leq \|(\widetilde{F}_{im}^{\alpha_{im}}(x_i^m, x_{-i}^m)) \ominus \widetilde{F}_i^{\alpha_{im}}(x_i^m, x_{-i}^m)\|_{\mathcal{I}(\mathbb{R})^d} \\
&\quad + \|\widetilde{F}_i^{\alpha_{im}}(x_i^m, x_{-i}^m) \ominus \widetilde{F}_i^{\alpha_i}(x_i^*, x_{-i}^*)\|_{\mathcal{I}(\mathbb{R})^d} \\
&\leq \sup_{(x,\alpha) \in X \times [0,1]} \|\widetilde{F}_{im}^{\alpha}(x) \ominus \widetilde{F}_i^{\alpha}(x)\|_{\mathcal{I}(\mathbb{R})^d} \\
&\quad + \|\widetilde{F}_i^{\alpha_{im}}(x_i^m, x_{-i}^m) \ominus \widetilde{F}_i^{\alpha_i}(x_i^*, x_{-i}^*)\|_{\mathcal{I}(\mathbb{R})^d} \\
&\to 0 \text{ as } m \to \infty.
\end{aligned}
\tag{4.5}
$$

Thus, as $m \to \infty$,

$$\widetilde{F}_{im}^{\alpha_{im}}(x_i^m, x_{-i}^m) \to \widetilde{F}_i^{\alpha_i}(x_i^*, x_{-i}^*). \tag{4.6}$$

Similarly, it can be verified that as $m \to \infty$,

$$\widetilde{F}_{im}^{\alpha_{im}}(u_i^m, x_{-i}^m) \to \widetilde{F}_i^{\alpha_i}(u_i, x_{-i}^*). \tag{4.7}$$

Then, combining (4.4), (4.6) and (4.7), finish Step III. The proof is completed. $\qquad\square$

4.2. *The bounded rationality model for FCMGFPs*

In order to construct the bounded rationality model for FCMGFPs, we first establish a nonlinear scalarization function for an interval vector by the Minkowski functional. This nonlinear scalarization function yields a principle of constructing bounded rationality functions.

Definition 4.2. For any fixed $\boldsymbol{e} = (e_1, \dots, e_d) \in int\mathcal{I}(\mathbb{R}_+)^d$, the nonlinear scalarization function $\xi_{\boldsymbol{e}} : \mathcal{I}(\mathbb{R})^d \to \mathbb{R}_+$ is defined for $\boldsymbol{A} = (A_1, \dots, A_d) \in \mathcal{I}(\mathbb{R})^d$ by

$$\xi_{\boldsymbol{e}}(\boldsymbol{A}) = \inf\{ r \in \mathbb{R}_+ \mid \boldsymbol{A} \notin r \cdot \boldsymbol{e} + int\mathcal{I}(\mathbb{R}_+)^d \}.$$

Clearly, $\xi_{\boldsymbol{e}}(\boldsymbol{A}) = \min\{\xi_{e_1}(A_1), \dots, \xi_{e_d}(A_d)\}$.

Lemma 4.3. *The nonlinear scalarization function $\xi_{\boldsymbol{e}} : \mathcal{I}(\mathbb{R})^d \to \mathbb{R}_+$ has the following properties:*

(1) $\xi_{\boldsymbol{e}}$ is positively homogeneous and continuous.

(2) $\xi_{\boldsymbol{e}}(\boldsymbol{A}) \leq r$ if and only if $\boldsymbol{A} \notin r \cdot \boldsymbol{e} + int\mathcal{I}(\mathbb{R}_+)^d$.

Proof. See Appendix C. $\qquad\square$

Next, we consider the bounded rationality model for FCMGFPs $\mathcal{M} = \{\Gamma, X, K, R\}$ as follows:

(1) (Γ, ρ) is a complete metric space;

(2) X is a compact metric space;

(3) $K : \Gamma \times X \rightrightarrows X$ is defined by

$$K(\gamma, x) = \prod_{i \in N} S_i(x_{-i}),$$

for all $(\gamma, x) \in \Gamma \times X$;

(4) $k : \Gamma \rightrightarrows X$ is defined by

$$k(\gamma) = \{x \in X \mid x \in K(\gamma, x)\}$$

for all $\gamma \in \Gamma$;

(5) $R : Graph(k) \to \mathbb{R}_+$ is a rationality function, given by choosing $e \in int\mathcal{I}(\mathbb{R}_+)^d$ and then for each $x \in k(\gamma)$ defining

$$R(\gamma, x) = \max_{i \in N} \min_{\alpha \in [0,1]} \max_{u_i \in S_i(x_i)} \xi_e(\widetilde{F}_i^\alpha(u_i, x_{-i}) \ominus \widetilde{F}_i^\alpha(x_i, x_{-i})).$$

It is clear that $R(\gamma, x) \geq 0$ for each $\gamma \in \Gamma$ and $x \in k(\gamma)$. From Lemma 4.3, $R(\gamma, x) = 0$ if and only if x is a fuzzy Pareto-Nash equilibrium of γ.

For any $\gamma \in \Gamma$ and $\epsilon \geq 0$, the set of all ϵ-equilibria of γ is defined as

$$E(\gamma, \epsilon) = \{x \in k(\gamma) \mid R(\gamma, x) \leq \epsilon\}.$$

Lemma 4.4. *Let $\gamma \in \Gamma$ and $\epsilon \geq 0$ with $x \in k(\gamma)$. Then, $x \in E(\gamma, \epsilon)$ if and only if for each $i \in N$, there is $\alpha_i \in [0, 1]$ such that for all $u_i \in S_i(x_{-i})$,*

$$\widetilde{F}_i^{\alpha_i}(u_i, x_{-i}) \ominus \widetilde{F}_i^{\alpha_i}(x_i, x_{-i}) \notin \epsilon \cdot e + int\mathcal{I}(\mathbb{R}_+)^d. \tag{4.8}$$

Consequently, $E(\gamma, 0) = \{x \in k(\gamma) \mid R(\gamma, x) = 0\} \neq \emptyset$ is the set of fuzzy Pareto-Nash equilibria of γ.

Proof. Assume that $x \in E(\gamma, \epsilon)$, i.e., $R(\gamma, x) \leq \epsilon$. Then,

$$\max_{i \in N} \min_{\alpha \in [0,1]} \max_{u_i \in S_i(x_{-i})} \xi_e(\widetilde{F}_i^\alpha(u_i, x_{-i}) \ominus \widetilde{F}_i^\alpha(x_i, x_{-i})) \leq \epsilon.$$

So, for each $i \in N$, there is $\alpha_i \in [0, 1]$ such that $\xi_e(\widetilde{F}_i^{\alpha_i}(u_i, x_{-i}) \ominus \widetilde{F}_i^{\alpha_i}(x_i, x_{-i})) \leq \epsilon$ for all $u_i \in S_i(x_{-i})$. Using Lemma 4.3, we obtain (4.8).

Conversely, if, for each $i \in N$, there is $\alpha_i \in [0, 1]$ such that (4.8) holds for all $u_i \in S_i(x_{-i})$, then from Lemma 4.3, $\xi_e(\widetilde{F}_i^{\alpha_i}(u_i, x_{-i}) \ominus \widetilde{F}_i^{\alpha_i}(x_i, x_{-i})) \leq \epsilon$ for all $u_i \in S_i(x_{-i})$, and thus, $R(\gamma, x) \leq \epsilon$. $\square$

Theorem 4.5. *The bounded rationality model for FCMGFPs $\mathcal{M} = \{\Gamma, X, K, R\}$ has the following properties:*

(1) The correspondence $E : \Gamma \rightrightarrows X$ is upper semi-continuous;

(2) there exists a dense G_δ subset Q of Γ such that $\mathcal{M}$ is structurally stable at every $\gamma \in Q$;

(3) if the model $\mathcal{M}$ is structurally stable at $\gamma \in \Gamma$, then $\mathcal{M}$ is robust to ϵ-equilibria at $\gamma \in \Gamma$, and $\mathcal{M}$ is robust to ϵ-equilibria at $\gamma \in Q$;

(4) if $\gamma \in Q$, $\gamma_n \to \gamma$ and $\epsilon_n \to 0$, then $H(E(\gamma_n, \epsilon_n), E(\gamma)) \to 0$;

(5) if $E(\gamma)$ is a singleton set, then $\mathcal{M}$ is structurally stable and robust to ϵ-equilibria at $\gamma \in \Gamma$;

(6) if the model $\mathcal{M}$ is (γ, ϵ)-stable, then $\mathcal{M}$ is (γ, ϵ)-robust.

To prove Theorem 4.5, we need the following lemmas.

Lemma 4.6. *The correspondence $k : \Gamma \rightrightarrows X$ is upper semi-continuous.*

Proof. According to Lemma 2.12, we only need to show that k is closed on Γ. That is, the set Graph$(k) = \{(\gamma, x) \in \Gamma \times X \mid x \in k(\gamma)\}$ is closed on $\Gamma \times X$.

Assume that $\{(\gamma_m, x^m)\}_m \subset$ Graph(k) with $(\gamma_m, x^m) \to (\gamma, x)$. We turn to show that $(\gamma, x) \in$ Graph(k). From the completeness of (Γ, ρ), it follows that $\gamma \in \Gamma$. Meanwhile, by $x^m \in k(\gamma_m)$, we obtain $x_i^m \in S_{im}(x_{-i}^m)$ for all $i \in N$. Moreover, in the same way as the proof of (4.3), we get $x_i \in S_i(x_{-i})$ for all $i \in N$. Thus, $x \in k(\gamma)$ and $(\gamma, x) \in$ Graph(k). $\qquad\square$

Lemma 4.7. *The rationality function R is lower semi-continuous at (γ, x) for any $(\gamma, x) \in \Gamma \times X$.*

Proof. Let $r \in \mathbb{R}$, and we aim to prove the closeness of the lower level set

$$\{(\gamma, x) \in \Gamma \times X \mid R(\gamma, x) \le r\}.$$

Suppose that $\{(\gamma_m, x^m)\}_m \subset \Gamma \times X$ with $R(\gamma_m, x^m) \le r$ for all m, and $(\gamma_m, x^m) \to (\gamma, x)$. We turn to show that $R(\gamma, x) \le r$.

Since $R(\gamma_m, x^m) \le r$, for each $i \in N$, there is $\alpha_i \in [0, 1]$ such that

$$\max_{u_i \in S_{im}(x_{-i}^m)} \xi_e(\widetilde{F}_{im}^{\alpha_i}(u_i, x_{-i}^m) \ominus \widetilde{F}_{im}^{\alpha_i}(x_i^m, x_{-i}^m)) \le r. \tag{4.9}$$

Meanwhile, note that the sets $S_i(x_{-i})$ and $S_{im}(x_{-i}^m)$ are both compact. For any $u_i \in S_i(x_{-i})$, from $S_{im}(x_{-i}^m) \to S_i(x_{-i})$, there exists $u_i^m \in S_{im}(x_{-i}^m)$ such that $u_i^m \to u_i$. Combining this with (4.9), we obtain

$$\xi_e(\widetilde{F}_{im}^{\alpha_i}(u_i^m, x_{-i}^m) \ominus \widetilde{F}_{im}^{\alpha_i}(x_i^m, x_{-i}^m)) \le r. \tag{4.10}$$

By an argument analogous to the proof of (4.5), we can derive that

$$\widetilde{F}_{im}^{\alpha_i}(x_i^m, x_{-i}^m) \to \widetilde{F}_i^{\alpha_i}(x_i, x_{-i}) \tag{4.11}$$

and

$$\widetilde{F}_{im}^{\alpha_i}(u_i^m, x_{-i}^m) \to \widetilde{F}_i^{\alpha_i}(u_i, x_{-i}). \tag{4.12}$$

Using the continuity of ξ_e, (4.11) and (4.12), let $m \to \infty$ within (4.10), and we get

$$\xi_e(\widetilde{F}_i^{\alpha_i}(u_i, x_{-i}) \ominus \widetilde{F}_i^{\alpha_i}(x_i, x_{-i})) \le r.$$

By the definition of R, it follows that

$$R(\gamma, x) = \max_{i \in N} \min_{\alpha \in [0,1]} \max_{u_i \in S_i(x_{-i})} \xi_e(\widetilde{F}_i^{\alpha}(u_i, x_{-i}) \ominus \widetilde{F}_i^{\alpha}(x_i, x_{-i})) \le r.$$

The proof is completed. $\qquad\square$

Proof of Theorem 4.5. Combining Lemma 2.17 with Lemmas 4.1, 4.6 and 4.7, we can directly obtain Theorem 4.5.

5. Conclusions

In this paper, we investigate a new class of fuzzy constrained multi-objective games with fuzzy payoffs. First, by α-cuts of fuzzy vectors, we establish the partial order of fuzzy vectors as well as the continuity and generalized quasi-concavity of fuzzy vector-valued functions. Second, we prove Berge's maximum theorem for fuzzy vector-valued functions (Theorem 3.10). Combining our result with the Fan-Glicksberg fixed point theorem leads to two existence theorems of fuzzy Pareto-Nash equilibria and fuzzy α-Pareto-Nash equilibria for FCMGFPs (Theorems 3.12 and 3.14). Finally, through a nonlinear scalar method, we construct the bounded rationality model $\mathcal{M}$, a class of FCMGFPs with bounded rational functions. After verifying the lower semi-continuity of the bounded rational function, a series of relevant conclusions about the structural stability $((\gamma, \epsilon)$-stability) and the robustness to ϵ-equilibria $((\gamma, \epsilon)$-robustness) of the model are proved (Theorem 4.5). The results obtained in this paper are generalizations and extensions of relevant conclusions in some of the literature, such as in [24, 28, 44, 48, 52]. In the future, we will focus on the application of FCMGFPs in practical problems. In addition, when considering mixed strategies, the multi-objective bi-matrix game with fuzzy payoffs is a special case of the two-player multi-objective game with fuzzy payoffs. The conditions for the existence and the specific calculation method of fuzzy Pareto-Nash equilibrium for this game are the issues we will discuss in the future. This paper discussed the stability of the FCMGFP in the framework of bounded rationality. Next, we intend to study the general stability of fuzzy Pare-Nash equilibrium for the FCMGFP from the perspective of essential stability.

Appendix A. Proof of Lemma 2.3.

Proof. Let $\boldsymbol{A} = (A_1, \ldots, A_d)$, $\boldsymbol{B} = (B_1, \ldots, B_d)$, $\boldsymbol{C} = (C_1, \ldots, C_d)$, by (2.1) and the Minkowski inequality, we get

$$
\begin{aligned}
\|\boldsymbol{A} \ominus \boldsymbol{B}\|_{\mathcal{I}(\mathbb{R})^d} &= \left(\sum_{i=1}^{d} \|A_i \ominus B_i\|_{\mathcal{I}(\mathbb{R})}^2 \right)^{1/2} \\
&\leq \left(\sum_{i=1}^{k} [\|A_i \ominus C_i\|_{\mathcal{I}(\mathbb{R})} + \|C_i \ominus B_i\|_{\mathcal{I}(\mathbb{R})}]^2 \right)^{1/2} \\
&\leq \left(\sum_{i=1}^{k} \|A_i \ominus C_i\|_{\mathcal{I}(\mathbb{R})}^2 \right)^{1/2} + \left(\sum_{i=1}^{k} \|C_i \ominus B_i\|_{\mathcal{I}(\mathbb{R})}^2 \right)^{1/2} \\
&= \|\boldsymbol{A} \ominus \boldsymbol{C}\|_{\mathcal{I}(\mathbb{R})^d} + \|\boldsymbol{C} \ominus \boldsymbol{B}\|_{\mathcal{I}(\mathbb{R})^d}.
\end{aligned}
$$

$\square$

Appendix B. Proof of Lemma 3.7.

Proof. Let $\widetilde{A} \in \mathcal{F}(\mathbb{R})^d$, let $x_1, x_2 \in X$, and let $\lambda \in [0, 1]$. Suppose that

$$\widetilde{F}(x_1) \not\prec \widetilde{A} \text{ and } \widetilde{F}(x_2) \not\prec \widetilde{A}. \tag{5.1}$$

We only need to prove that

$$\widetilde{F}(\lambda x_1 + (1 - \lambda)x_2) \not\prec \widetilde{A}. \tag{5.2}$$

From (5.1), there are $\alpha_1, \alpha_2 \in [0, 1]$ such that

$$\widetilde{F}^{\alpha_1}(x_1) \not\prec \widetilde{A}^{\alpha_1} \quad \text{and} \quad \widetilde{F}^{\alpha_2}(x_2) \not\prec \widetilde{A}^{\alpha_2}.$$

We will discuss the following two cases.

Case 1: $x_1 \leq x_2$. Since $\widetilde{F}^{\alpha_1}$ is non-decreasing, we have $\widetilde{F}^{\alpha_1}(x_1) \preceq \widetilde{F}^{\alpha_1}(\lambda x_1 + (1 - \lambda)x_2)$. From $\widetilde{F}^{\alpha_1}(x_1) \not\prec \widetilde{A}^{\alpha_1}$, it follows that $\widetilde{F}^{\alpha_1}(\lambda x_1 + (1 - \lambda)x_2) \not\prec \widetilde{A}^{\alpha_1}$.

Case 2: $x_1 > x_2$. Since $\widetilde{F}^{\alpha_2}$ is non-decreasing, we have $\widetilde{F}^{\alpha_2}(x_2) \preceq \widetilde{F}^{\alpha_2}(\lambda x_1 + (1 - \lambda)x_2)$. From $\widetilde{F}^{\alpha_2}(x_2) \not\prec \widetilde{A}^{\alpha_2}$, it follows that $\widetilde{F}^{\alpha_2}(\lambda x_1 + (1 - \lambda)x_2) \not\prec \widetilde{A}^{\alpha_2}$.

As mentioned for the above two cases, (5.2) holds. If $\widetilde{F}^{\alpha}$ is non-increasing for each $\alpha \in [0, 1]$, (5.2) holds by similar argument. The proof is completed. $\qquad \square$

Appendix C. Proof of Lemma 4.3.

Proof. (1) Let $\boldsymbol{e} = ([\underline{e}_1, \overline{e}_1], \ldots, [\underline{e}_d, \overline{e}_d])$ and $\boldsymbol{A} = ([\underline{a}_1, \overline{a}_1], \ldots, [\underline{a}_d, \overline{a}_d])$. For each $t > 0$,

$$\xi_e(t \cdot \boldsymbol{A}) = \inf\{r \in \mathbb{R}_+ \mid t \cdot \boldsymbol{A} \notin r \cdot \boldsymbol{e} + int\mathcal{I}(\mathbb{R}_+)^d\}$$

$$= \min\{\max\{\frac{t\underline{a}_1}{\underline{e}_1}, 0\}, \ldots, \max\{\frac{t\underline{a}_d}{\underline{e}_d}, 0\}\}$$

$$= t \cdot \min\{\max\{\frac{\underline{a}_1}{\underline{e}_1}, 0\}, \ldots, \max\{\frac{\underline{a}_d}{\underline{e}_d}, 0\}\}$$

$$= t \cdot \xi_e(\boldsymbol{A}),$$

which shows the positive homogeneity of ξ_e.

In what follows, we verify the continuity of ξ_e. If

$$\boldsymbol{A} = ([\underline{a}_1, \overline{a}_1], \ldots, [\underline{a}_d, \overline{a}_d]) \notin int\mathcal{I}(\mathbb{R}_+)^d,$$

then there is i such that $[\underline{a}_i, \overline{a}_i] \notin int\mathcal{I}(\mathbb{R}_+)$, and thus the definition of ξ_e implies that $\xi_e(\boldsymbol{A}) = 0$. Consequently, for $\epsilon > 0$, taking $\delta = \min\{\epsilon \cdot \underline{e}_1, \ldots, \epsilon \cdot \underline{e}_d\}$, we see that $\xi_e(\boldsymbol{A}') < \epsilon$ for all $\boldsymbol{A}' \in o(\boldsymbol{A}, \delta)$. As for the case

$$\boldsymbol{A} = ([\underline{a}_1, \overline{a}_1], \ldots, [\underline{a}_d, \overline{a}_d]) \in int\mathcal{I}(\mathbb{R}_+)^d,$$

from the fact that $[\underline{a}_i, \overline{a}_i] \in int\mathcal{I}(\mathbb{R}_+)$ for all i, together with the definition of ξ_e, it follows that

$$\xi_e(\boldsymbol{A}) = \min\left\{\frac{\underline{a}_1}{\underline{e}_1}, \ldots, \frac{\underline{a}_d}{\underline{e}_d}\right\}.$$

Then, for $\epsilon > 0$, taking $\delta = \min\{\epsilon \cdot \underline{e}_1, \ldots, \epsilon \cdot \underline{e}_d\}$, we see that $|\xi_e(\boldsymbol{A}') - \xi_e(\boldsymbol{A})| < \epsilon$ for all $\boldsymbol{A}' \in o(\boldsymbol{A}, \delta)$.

(2) Let $\xi_e(\boldsymbol{A}) = r'$. If $r' \leq r$, then the definition of ξ_e implies that $\boldsymbol{A} \notin r' \cdot \boldsymbol{e} + int\mathcal{I}(\mathbb{R}_+)^d$, and thus, $\boldsymbol{A} \notin r \cdot \boldsymbol{e} + int\mathcal{I}(\mathbb{R}_+)^d$ for all $r' < r$. Conversely, if $\boldsymbol{A} \notin r \cdot \boldsymbol{e} + int\mathcal{I}(\mathbb{R}_+)^d$, then the definition of ξ_e implies that $\xi_e(\boldsymbol{A}) \leq r$. $\qquad \square$

Acknowledgments

Conflict of interest

The authors declared that they have no conflicts of interest to this work.

References

1. G. Alefeld, G. Mayer, Interval analysis: Theory and applications, *J. Comput. Appl. Math.,* **121** (2000), 421–464.

2. L. Anderlini, D. Canning, Structural stability implies robustness to bound rationality, *J. Econ. Theory,* **101** (2001), 395–322.

3. J. P. Aubin, I. Ekeland, *Applied nonlinear analysis*, New York: John Wiley and Sons Inc., 1984.

4. R. E. Bellman, L. A. Zadeh, Decision making in a fuzzy environment, *Manage. Sci.,* **17** (1970), 141–164.

5. C. Berge, *Topological spaces*, New York: Dover Publications, 1997.

6. H. Bigdeli, H. Hassanpour, A satisfactory strategy of multiobjective two person matrix games with fuzzy payoffs, *Iran. J. Fuzzy Syst.,* **13** (2016), 17–33.

7. J. J. Buckley, Multiple goals non cooperative conflict under uncertainty: A fuzzy set approach, *Fuzzy Set. Syst.,* **13** (1984), 107–124.

8. D. Butnariu, Fuzzy games: A description of the concept, *Fuzzy Set. Syst.,* **1** (1978), 181–192.

9. L. Campos, Fuzzy linear programming models to solve fuzzy matrix games, *Fuzzy Set. Syst.,* **32** (1989), 275–289.

10. Y. Chalco-Cano, A. Rufián-Lizana, H. Román-Flores, M. D. Jiménez-Gamero, Calculus for interval-valued functions using generalized Hukuhara derivative and applications, *Fuzzy Set. Syst.,* **219** (2013), 49–67. http://dx.doi.org/10.1016/j.fss.2012.12.004

11. S. S. Chang, Y. G. Zhu, On variational inequalities for fuzzy mappings, *Fuzzy Set. Syst.,* **32** (1989), 359–367.

12. M. Clemente, F. R. Fernández, J. Puerto, Pareto-optimal security strategies in matrix games with fuzzy payoffs, *Fuzzy Set. Syst.,* **176** (2011), 36–45. http://dx.doi.org/10.1016/j.fss.2011.03.006

13. W. D. Collins, C. Hu, Studying interval valued matrix games with fuzzy logic, *Soft Comput.,* **12** (2008), 147–155. http://dx.doi.org/10.1007/s00500-007-0207-6

14. X. P. Ding, Constrained multiobjective games in general topological spaces, *Comput. Math. Appl.,* **39** (2000), 23–30. https://doi.org/10.1016/S0898-1221(99)00330-2

15. X. P. Ding, Existence of Pareto equilibria for constrained multiobjective games in *H*-space, *Comput. Math. Appl.,* **39** (2000), 125–134.

16. X. P. Ding, Pareto equilibria of multicriteria games without compactness, continuity and concavity, *Appl. Math. Mech.,* **17** (1996), 847–854. https://doi.org/10.1007/BF00127184

17. D. Dubois, H. Prade, Operations on fuzzy numbers, *Int. J. Syst. Sci.,* **9** (1978), 613–626. https://doi.org/10.1080/00207727808941724

18. K. Fan, Fixed point and minimax theorems in locally convex topological linear spaces, *Proc. Natl. Acad. Sci. USA,* **38** (1952), 121–126. http://dx.doi.org/10.1073/pnas.38.2.121

19. D. Garagic, J. B. Cruz, An approach to fuzzy non-cooperative Nash games, *J. Optim. Theory Appl.,* **118** (2003), 475–491.

20. D. Ghosh, Newton method to obtain efficient solutions of the optimization problems with interval-valued objective functions, *J. Appl. Math. Comput.,* **53** (2017), 709–731. http://dx.doi.org/10.1007/s12190-016-0990-2

21. D. Ghosh, R. S. Chauhan, R. Mesiar, A. K. Debnath, Generalized Hukuhara Gâteaux and Fréchet derivatives of interval-valued functions and their application in optimization with interval-valued functions, *Inform. Sci.,* **510** (2020), 317–340. http://dx.doi.org/10.1016/j.ins.2019.09.023

22. I. L. Glicksberg, A further generalization of the Kakutani fixed point theorem with applications to Nash equilibrium points, *Proc. Amer. Math. Soc.,* **3** (1952), 170–174.

23. A. González, M. A. Vila, A discrete method for studying indifference and order relations between fuzzy numbers, *Inform. Sci.,* **56** (1991), 245–258.

24. N. V. Hung, V. M. Tam, D. O'Regan, Y. J. Cho, A new class of generalized multiobjective games in bounded rationality with fuzzy mappings: Structural (λ, ϵ)-stability and (λ, ϵ)-robustness to ϵ-equilibria, *J. Comput. Appl. Math.,* **372** (2020), 112735. http://dx.doi.org/10.1016/j.cam.2020.112735

25. W. K. Kim, K. H. Lee, Generalized fuzzy games and fuzzy equilibria, *Fuzzy Set. Syst.,* **122** (2001), 293–301.

26. M. Larbani, Solving bi-matrix games with fuzzy payoffs by introducing nature as a third player, *Fuzzy Set. Syst.,* **160** (2009), 657–666.

27. D. F. Li, Linear programming approach to solve interval-valued matrix games, *Omega,* **39** (2011), 655–666. http://dx.doi.org/10.1016/j.omega.2011.01.007

28. W. Li, D. Y. Li, Y. Q. Feng, D. Zou, Weak Pareto-Nash equilibria of generalized interval-valued multiobjective games with fuzzy mappings, unpublished work.

29. Z. Lin, Essential components of the set of weakly Pareto-Nash equilibrium points for multiobjective generalized games in two different topological spaces, *J. Optim. Theory Appl.,* **124** (2005), 387–405.

30. S. T. Liu, C. Kao, Matrix games with interval data, *Comput. Ind. Eng.,* **56** (2009), 1697–1700. http://dx.doi.org/10.1016/j.cie.2008.06.002

31. V. Lupulescu, Hukuhara differentiability of interval-valued functions and interval differential equations on time scales, *Inform. Sci.,* **248** (2013), 50–67. http://dx.doi.org/10.1016/j.ins.2013.06.004

32. T. Maeda, On characterization of equilibrium strategy of two-person zero-sum games with fuzzy payoffs, *Fuzzy Set. Syst.,* **139** (2003), 283–296.

33. Y. Miyazaki, A remark on topological robustness to bounded rationality in semialgebraic models, *J. Math. Econom.,* **55** (2014), 33–35. http://dx.doi.org/10.1016/j.jmateco.2014.09.008

34. Y. Miyazaki, H. Azuma, (λ, ϵ)-stable model and essential equilibria, *Math. Soc. Sci.,* **65** (2013), 85–91. http://dx.doi.org/10.1016/j.mathsocsci.2012.08.002

35. R. E. Moore, *Interval analysis*, Englewood Cliffs: Prentice-Hall, 1966.

36. R. E. Moore, *Method and applications of interval analysis*, Philadelphia: Society for Industrial and Applied Mathematics, 1987.

37. I. Nishizaki, M. Sakawa, Equilibrium solutions in multiobjective bimatrix games with fuzzy payoffs and fuzzy goals, *Fuzzy Set. Syst.,* **111** (2000), 99–116.

38. R. Osuna-Gómez, Y. Chalco, B. Hernández-Jiménez, G. Ruiz-Garzón, New efficiency conditions for multiobjective interval-valued programming problems, *Inform. Sci.,* **420** (2017), 235–248. http://dx.doi.org/10.1016/j.ins.2017.08.022

39. J. Ramík, J. Rímánek, Inequality relation between fuzzy numbers and its use in fuzzy optimization, *Fuzzy Set. Syst.,* **16** (1985), 123–150.

40. V. Scalzo, On the existence of maximal elements, fixed points and equilibria of generalized games in a fuzzy environment, *Fuzzy Set. Syst.,* **272** (2015), 126–133. http://dx.doi.org/10.1016/j.fss.2015.02.006

41. Q. Q. Song, L. S. Wang, On the stability of the solution for multiobjective generalized games with the payoffs perturbed, *Nonlinear Anal.,* **73** (2010), 2680–2685. http://dx.doi.org/10.1016/j.na.2010.06.048

42. L. Stefanini, B. Bede, Generalized Hukuhara differentiability of interval-valued functions and interval differential equations, *Nonlinear Anal.,* **71** (2009), 1311–1328. http://dx.doi.org/10.1016/j.na.2008.12.005

43. J. Tao, Z. Zhang, Properties of interval-valued function space under the gH-difference and their application to semi-linear interval differential equations, *Adv. Differ. Equ.,* **2016** (2016), 45–72. http://dx.doi.org/10.1186/s13662-016-0759-9

44. S. Y. Wang, Existence of a Pareto equilibrium, *J. Optim. Theory Appl.,* **79** (1993), 373–384.

45. C. Wang, R. P. Agarwal, D. O'Regan, Calculus of fuzzy vector-valued functions and almost periodic fuzzy vector-valued functions on time scales, *Fuzzy Set. Syst.,* **375** (2019), 1–52. http://dx.doi.org/10.1016/j.fss.2018.12.008

46. H. C. Wu, The Karush-Kuhn-Tucker optimality conditions in an optimization problem with interval-valued objective function, *Eur. J. Oper. Res.,* **176** (2007), 46–59. http://dx.doi.org/10.1016/j.ejor.2005.09.007

47. Z. Yang, A coalitional extension of generalized fuzzy games, *Fuzzy Set. Syst.,* **383** (2019), 68–79. http://dx.doi.org/10.1016/j.fss.2019.06.010

48. C. Yu, J. Yu, Bounded rationality in multiobjective games, *Nonlinear Anal.-Theor.,* **67** (2007), 930–937. http://dx.doi.org/10.1016/j.na.2006.06.050

49. C. Yu, J. Yu, On structural stability and robustness to bounded rationality, *Nonlinear Anal.-Theor.,* **65** (2006), 583–529. http://dx.doi.org/10.1016/j.na.2005.09.039

50. J. Yu, *Game theory and nonlinear analysis*, Beijing: Science Press, 2011.

51. J. Yu, H. Yang, C. Yu, Structural stability and robustness to bounded rationality for non-compact cases, *J. Global Optim.,* **44** (2009), 149–157. http://dx.doi.org/10.1007/s10898-008-9316-8

52. J. Yu, X. Z. Yuan, The study of Pareto equilibria for multiobjective games by fixed point and ky fan minimax inequality methods, *Comput. Math. Appl.,* **35** (1998), 17–24.

53. J. Yu, Z. Yang, N. F. Wang, Further results on structural stability and robustness to bounded rationality, *J. Math. Econ.,* **67** (2016), 49–53. http://dx.doi.org/10.1016/j.jmateco.2016.09.009

54. X. Z. Yuan, E. Tarafdar, Non-compact Pareto equilibria for multiobjective games, *J. Math. Anal. Appl.,* **204** (1996), 156–163.

55. L. A. Zadeh, Fuzzy sets, *Inform. Control,* **8** (1965), 338–353.

56. M. Zeleny, Game with multiple payoffs, *Int. J. Game Theory,* **4** (1976), 179–191.