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Research article

Certain differential subordination results for univalent functions associated with q-Salagean operators

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Abstract: In this paper, we employ the concept of the q-derivative to derive certain differential and integral operators, $D^n_{q,\lambda}$ and $I^n_{q,\lambda}$, resp., to generalize the class of Salagean operators over the set of univalent functions. By means of the new operators, we establish the subclasses $M^n_{q,\lambda}$ and $D^n_{q,\lambda}$ of analytic functions on an open unit disc. Further, we study coefficient inequalities for each function in the given classes. Over and above, we derive some properties and characteristics of the set of differential subordinations by following specific techniques. In addition, we study the general properties of $D^n_{q,\lambda}$ and $I^n_{q,\lambda}$ and obtain some interesting differential subordination results. Several results are also derived in some details.

Keywords: q-derivative operator; univalent function; Salagean operator; geometric function theory
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1. Introduction

The theory of the q-calculus operators has been included in diverse areas of science including fractional q-calculus, optimal control, q-difference and q-integral equations. An application to the existed q-calculus operators is given by [1]. Meanwhile, the geometric function theory of the area of complex analysis is described by Srivastava [2]. In [3], the authors present the q-Salagean and Ruscheweyh differential operators as a special case of analytic functions. The Jackson q-derivative of conformable bi-univalent functions is discussed in [4]. Authors in [5] discuss q-calculus and symmetric

Salagean differential operators. Arif et al. [6] investigate the multivalent functions by using a q-derivative operator. Ismail et al. [7] obtain some properties of starlike functions by using q-derivative operators. Srivastava et al. [8] derive some properties of analytic functions based on a q-Noor integral operator. However, some important properties of the q-calculus theory in the geometric class of analytic functions are studied by various authors, see, e.g., [9–20] and [21–24]. See also [25–27] for further integral transforms and applications.

Let f be a complex valued function and $0 < q < 1$. Then, the Jackson q-difference operator is defined by [28]

$$D_q f(z) = \frac{f(z) - f(qz)}{z - qz}, \; z \in D \quad (D = \{z : |z| < 1\}).$$

Let $\mathcal{A}$ consist of analytic functions on the unite disc D normalized by $f(0) = 0$ and $f'(0) = 1$. Then, the expansion of the function $f \in \mathcal{A}$ has the following form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

The class of univalent functions in $\mathcal{A}$ is denoted by S. Whereas, the class of starlike functions and the class of convex functions are respectively denoted by S^* and K [29]. Many important properties of the aforementioned subclasses of univalent functions are given by [30–32].

Here, we denote by $\mathcal{P}$ the class of analytic functions p which are analytic in D such that $p(0) = 0$ and $Re\{p\} > 0$ [29]. Therefore, the function p can be written in the form

$$p(z) = 1 + \sum_{k=1}^{\infty} p_n z^n, \quad z \in D. \tag{1.2}$$

Assume that the function f is given by (1.1) and the function g is given by the following form

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \;\; z \in D.$$

Then, the convolution (or Hadamard) product of two functions f and g is presented in [33] as

$$f(z) * g(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \;\; z \in D.$$

Let $f \in \mathcal{A}$ be given by (1.1) and g be analytic function on the open unit disk D with $g(0) = 0$. We say that the function f is subordinate to a function g written as $f \prec g$ if

$$f(z) = g(w(z)) \quad (z \in D),$$

where w is a Schwartz function with $w(0) = 0$ and $|w(z)| \leq |z|$. Note that the function g need not be univalent [34] (see also [35]).

Let $f \in \mathcal{A}$ be a univalent convex function given by (1.1). A sequence $\{b_k\}_{k=1}^{\infty}$ of complex numbers is said to be a subordinating factor sequence if it satisfies the following differential subordination ([36, 37])

$$\sum_{k=1}^{\infty} a_k b_k z^k \prec f(z), \;\; z \in D, \tag{1.3}$$

where $a_1 = 1$. The univalent function $h(z)$ is called a dominant of the solution of the differential subordination or, more simply, a dominant if

$$f(z) \prec h(z), \qquad (z \in D),$$

for all $f(z)$ satisfying (1.3). A dominant $\tilde{h}(z)$ that satisfies

$$\tilde{h}(z) \prec h(z), \qquad (z \in D),$$

for all of $h(z)$ satisfying (1.3) is said to be the best dominant [34].

Let $f \in \mathcal{A}$ be given by (1.1), then the Salagean differential operator, introduced in [38], is denoted by $D^n f$, where

$$D^n f(z) = z + \sum_{n=2}^{\infty} k^n a_k z^k. \tag{1.4}$$

The Salagean differential operator conciliated many researchers to generalize it; see, for example, [39–41]. In this paper, benefited from the idea of Salagean and the q-derivative operator, we introduce a q-analogue of Salagean differential and integral operators. We also define a new subclass of univalent functions and establish coefficient bounds for functions in these subclasses. Finally, we obtain some differential subordination results.

Lemma 1. *[29] Let the function $p \in \mathcal{P}$ be given by* (1.2)*. Then, the coefficients are bounded by*

$$|p_n| \leq 2, \ \ \textit{for all} \ \ n \in \mathbb{N},$$

where the bound of coefficients is sharp.

Lemma 2. *[36] The sequence $\{b_k\}_{k=1}^{\infty}$ is a subordinating factor if and only if*

$$Re\left(1 + 2\sum_{k=1}^{\infty} b_k z^k\right) > 0, \ \ \textit{for all} \ \ z \in D.$$

1.1. Generalized Salagean differential operator

Several differential operators have been recently introduced to generalize (1.4) [42]. Here, we define a differential operator as follows:

$$\begin{aligned} D_q^0 f(z) &= f(z); \\ D_{q,\lambda}^1 f(z) &= (1-\lambda) f(z) + \lambda D_q f(z); \\ &\vdots \\ D_{q,\lambda}^n f(z) &= D_{q,\lambda}(D_{q,\lambda}^{n-1} f(z)), \end{aligned}$$

where $n \in \mathbb{N}_0 = \mathbb{N} \bigcup \{0\}$. Let f be defined by (1.1), then we have

$$D_{q,\lambda}^n f(z) = z + \sum_{k=2}^{\infty} \left[(1-\lambda) + \lambda [k]_q\right]^n a_k z^k. \tag{1.5}$$

From the definition of the operator (1.5), we, for $q \to 1^-$, obtain

$$\begin{aligned}\lim_{q\to 1^-} D^n_{q,\lambda} f(z) &= \lim_{q\to 1^-}\left[z+\sum_{k=2}^{\infty}\left[(1-\lambda)+\lambda[k]_q\right]^n a_k z^k\right]\\ &= z+\sum_{k=2}^{\infty}[1+(k-1)\lambda]^n a_k z^k = D^n_\lambda f(z),\end{aligned}$$

where $D_\lambda f$ is the generalized Salagean differential operator defined in [42].

1.2. Generalized Salagean integral operator

Suppose that

$$\psi(z)=\frac{1-\lambda}{1-z}+\frac{\lambda z}{(1-z)(1-qz)},$$

and

$$\Psi(z)=\underbrace{\psi(z)*...\psi(z)}_{k\text{–times}}=z+\sum_{k=2}^{\infty}\left[1-\lambda+\lambda[k]_q\right]^n z^k.$$

Then, for every univalent function $f \in \mathcal{A}$ we define the integral operator $I^n_{q,\lambda} f$ such that

$$I^n_{q,\lambda} f(z) := [\Psi(z)]^{-1} * f(z),$$

where

$$[\Psi(z)]^{-1} * \Psi(z) = \frac{z}{1-z} = z+\sum_{k=2}^{\infty} z^k \quad (z\in D).$$

This, indeed, implies that

$$[\Psi(z)]^{-1} = z+\sum_{k=2}^{\infty}\frac{1}{\left[(1-\lambda)+\lambda[k]_q\right]^n} z^k \quad (z\in D).$$

Therefore, we have

$$I^n_{q,\lambda} f(z) = z+\sum_{k=2}^{\infty}\frac{a_k}{\left[(1-\lambda)+\lambda[k]_q\right]^n} z^k \quad (z\in D). \tag{1.6}$$

Remark 1. *Note that for $\lambda = 1$, the integral operator* (1.6) *reduces to the following integral operator*

$$I^n_{q,1} f(z) = z+\sum_{k=2}^{\infty}\frac{a_k}{[k]^n_q} z^k, \quad (z\in D).$$

Lemma 3. *If $f \in \mathcal{A}$, then we have*

- $I^0_{q,\lambda} f(z) = f(z)$,
- $I^1_{q,1} f(z) = \int_0^z \frac{f(t)}{t} d_q t$.

2. Coefficient estimate

Definition 1. *Let $M^n_{q,\lambda}(\mu)$ be a subclass of $\mathcal{A}$ consisting of functions f such that the following inequality holds*

$$Re\left\{\frac{zD_q(D^n_{q,\lambda}f(z))}{D^n_{q,\lambda}f(z)}\right\} < \mu,$$

for $z \in D$ and some μ ($\mu > 1$). For $n = 1$, we define $M^1_{q,\lambda}(\mu) = M_{q,\lambda}(\mu)$.

Definition 2. *Let $N^n_{q,\lambda}(\mu)$ be a subclass of $\mathcal{A}$ consisting of functions f such that the following inequality holds*

$$Re\left\{\frac{zD_q(I^n_{q,\lambda}f(z))}{I^n_{q,\lambda}f(z)}\right\} < \mu,$$

for $z \in D$ and some μ ($\mu > 1$). For $n = 1$, we define $N^1_{q,\lambda}(\mu) = N_{q,\lambda}(\mu)$.

In the following, we derive a sufficient condition so that the function f belongs to the classes $M^n_{q,\lambda}$ and $N^n_{q,\lambda}$. We also derive theorems and discuss conditions so that the coefficient inequalities hold.

Theorem 1. *Let $\mu < [k]_q$, $\beta < [k]_q$ and $k \in \mathbb{N}$. If $f \in \mathcal{A}$ satisfies the inequality*

$$\sum_{n=2}^{\infty} |[k]_q - \mu|[1 - \lambda + \lambda[k]_q]^n |a_k| \leq \mu - 1, \tag{2.1}$$

for some $\mu > 1$, then f belongs to $M^n_{q,\lambda}(\mu)$.

Proof. Assume that the inequality (2.1) holds. Then, it suffices to show that

$$\left|\frac{\frac{zD_q(D^n_{q,\lambda}f(z))}{D^n_{q,\lambda}f(z))} - \beta}{\frac{zD_q(D^n_{q,\lambda}f(z)))}{D^n_{q,\lambda}f(z))} - (2\mu - \beta)}\right| < 1.$$

For, we derive

$$\begin{aligned}
&\left|\frac{\frac{zD_q(D^n_{q,\lambda}f(z))}{D^n_{q,\lambda}f(z))} - \beta}{\frac{zD_q(D^n_{q,\lambda}f(z)))}{D^n_{q,\lambda}f(z))} - (2\mu - \beta)}\right| \\
&= \left|\frac{z + \sum_{k=2}^{\infty}[1 - \lambda + \lambda[k]_q]^n[k]_q a_k z^k - \beta z + \sum_{k=2}^{\infty}\beta[1 - \lambda + \lambda[k]_q]^n a_k z^k}{z + \sum_{k=2}^{\infty}[1 - \lambda + \lambda[k]_q]^n[k]_q a_k z^k - (2\mu - \beta)z - \sum_{k=2}^{\infty}(2\mu - \beta)[1 - \lambda + \lambda[k]_q]^n a_k z^k}\right| \\
&= \left|\frac{1 - \beta + \sum_{k=2}^{\infty}([k]_q - \beta)[1 - \lambda + \lambda[k]_q]^n a_k z^{k-1}}{2\mu - 1 - \beta - \sum_{k=2}^{\infty}([k]_q + \beta - 2\mu)[1 - \lambda + \lambda[k]_q]^n a_k z^{k-1}}\right|.
\end{aligned}$$

This, indeed, yields

$$\left|\frac{\frac{zD_q(D^n_{q,\lambda}f(z))}{D^n_{q,\lambda}f(z))} - \beta}{\frac{zD_q(D^n_{q,\lambda}f(z)))}{D^n_{q,\lambda}f(z))} - (2\mu - \beta)}\right| \leq \frac{1 - \beta + \sum_{k=2}^{\infty}([k]_q - \beta)[1 - \lambda + \lambda[k]_q]^n |a_k||z|^{k-1}}{2\mu - 1 - \beta - \sum_{k=2}^{\infty}|[k]_q + \beta - 2\mu|[1 - \lambda + \lambda[k]_q]^n |a_k||z|^{k-1}} = \Lambda^{\mu}_{q,\lambda}(z).$$

If $\Lambda^{\mu}_{q,\lambda}(z) \leq 1$, then we get

$$1-\beta+\sum_{k=2}^{\infty}([k]_q-\beta)[1-\lambda+\lambda[k]_q]^n|a_k| \leq (2\mu-1-\beta)-\sum_{k=2}^{\infty}|[k]_q+\beta-2\mu|[1-\lambda+\lambda[k]_q]^n|a_k|.$$

Therefore, we have

$$\sum_{k=2}^{\infty}(|[k]_q-\beta|+|[k]_q+\beta-2\mu|)[1-\lambda+\lambda[k]_q]^n|a_k| \leq 2\mu-2,$$

which is equivalent to assertion (2.1). Thus, the proof of theorem is completed. □

In the special case, putting $n = 1$ in Theorem 1, we derive the following corollary.

Corollary 1. *Let $\mu < [k]_q$, $\beta < [k]_q$, $k \in \mathbb{N}$. If for some $\mu > 1$ the function $f \in \mathcal{A}$ satisfies the inequality*

$$\sum_{n=2}^{\infty}([k]_q-\mu)(1-\lambda+\lambda[k]_q)|a_k| \leq \mu-1,$$

then $f(z)$ belongs to $M_{q,\lambda}(\mu)$.

Similar Theorem 1, we state the following theorem.

Theorem 2. *Let $\mu < [k]_q$, $\beta < [k]_q$ and $k \in \mathbb{N}$. If $f \in \mathcal{A}$ satisfies the inequality*

$$\sum_{n=2}^{\infty}\frac{[k]_q-\mu}{[1-\lambda+\lambda[k]_q]^n}|a_k| \leq \mu-1, \tag{2.2}$$

for some $\mu > 1$, then f belongs to $N^n_{q,\lambda}(\mu)$.

As a special case, by putting $n = 1$ in Theorem 2, we arrive at the following corollary.

Corollary 2. *Let $\mu < [k]_q$, $\beta < [k]_q$, $k \in \mathbb{N}$. If for some $\mu > 1$ the function $f \in \mathcal{A}$ satisfies the inequality*

$$\sum_{n=2}^{\infty}\frac{[k]_q-\mu}{1-\lambda+\lambda[k]_q}|a_k| \leq \mu-1,$$

then $f \in N_{q,\lambda}(\mu)$.

Lemma 4. *Let $\mu > 1$. If the sequence $\{\Lambda_j\}_{k=1}^{\infty}$ is defined by*

$$\Lambda_1 = 1,$$

$$\Lambda_k = \frac{2(\mu-1)}{|[k]_q-1|}\sum_{j=1}^{k-1}\Lambda_j, \tag{2.3}$$

then

$$\Lambda_2 = \frac{2(\mu-1)}{[2]_q-1}, \tag{2.4}$$

and

$$\Lambda_k = \frac{2(\mu-1)}{[k]_q-1}\prod_{j=2}^{k-1}\left(1+\frac{2(\mu-1)}{[j]_q-1}\right), \quad (k \geq 3). \tag{2.5}$$

Proof. We can easily prove assertion (2.4). To prove assertion (2.5), we use the induction on k. Indeed, from (2.3), we get

$$\Lambda_3 = \frac{2(\mu-1)}{[3]_q-1}\left(1+\frac{2(\mu-1)}{[2]_q-1}\right),$$

which implies that (2.4) holds for $k = 3$. Suppose that assertion (2.5) holds for $k = m$, then we have

$$\begin{aligned}
\Lambda_{m+1} &= \frac{2(\mu-1)}{[m+1]_q-1}\sum_{j=1}^{m}\Lambda_j = \frac{2(\mu-1)}{[m+1]_q-1}\left(\sum_{j=1}^{m-1}\Lambda_j+\Lambda_m\right)\\
&= \frac{2(\mu-1)}{[m+1]_q-1}\left(\frac{[m]_q-1}{2(\mu-1)}+1\right)\Lambda_m\\
&= \frac{2(\mu-1)}{[m+1]_q-1}\left(\frac{[m]_q-1}{2(\mu-1)}+1\right)\frac{2(\mu-1)}{[m]_q-1}\prod_{j=2}^{m-1}\left(1+\frac{2(\mu-1)}{[j]_q-1}\right)\\
&= \frac{2(\mu-1)}{[m+1]_q-1}\prod_{j=2}^{m}\left(1+\frac{2(\mu-1)}{[j]_q-1}\right).
\end{aligned}$$

This implies that (2.5) holds for $k = m + 1$. This completes the proof of the Lemma. □

Theorem 3. *Let $\mu > 1$ and $f \in M^n_{q,\lambda}(\mu)$, then we have*

$$|a_2| \leq \frac{2(\mu-1)}{([2]_q-1)[1-\lambda+\lambda[2]_q]^n}, \tag{2.6}$$

and

$$|a_k| \leq \frac{2(\mu-1)}{([k]_q-1)[(1-\lambda+\lambda[k]_q]^n}\prod_{j=2}^{k-1}\left(1+\frac{2(\mu-1)}{[j]_q-1}\right) \;\; for \;\; k \geq 3. \tag{2.7}$$

Proof. In view of $D^n_{q,\lambda}f$ presented in (1.5), we write

$$D^n_{q,\lambda}f(z) = z + \sum_{k=2}^{\infty}\Delta_k z^k,$$

where

$$\Delta_k = [(1-\lambda)+\lambda[k]_q]^n a_k. \tag{2.8}$$

Now, we consider

$$p(z)=\frac{\mu-1-\left(\frac{zD_q(D^n_{q,\lambda}f(z))}{D^n_{q,\lambda}f(z)}-1\right)}{\mu-1}=1+p_1z+p_2z^2+... \tag{2.9}$$

Then it is easy to show that $p\in\mathcal{P}$. In view of (1.5), we derive

$$zD_q(D^n_{q,\lambda}f(z))=\mu D^n_{q,\lambda}f(z)-(\mu-1)p(z)D^n_{q,\lambda}f(z). \tag{2.10}$$

From Eqs (2.1), (2.9) and (2.10), we establish

$$\begin{aligned}z+[2]_q\Delta_2z^2+...+[k]_q\Delta_kz^k+... &= \mu[z+\Delta_2z^2+...+\Delta_kz^k+...]\\ &-(\mu-1)[1+p_1z+p_2z^2+...+p_kz^k+...][z+\Delta_2z^2+...+\Delta_kz^k+...].\end{aligned} \tag{2.11}$$

By evaluating the coefficients of z^k, in both sides of (2.11), we infer

$$[k]_q\Delta_k=\mu\Delta_k-(\mu-1)[p_k-p_{k-1}\Delta_2+p_{k-1}\Delta_3+...+p_1\Delta_{k-1}+\Delta_k].$$

As $p\in\mathcal{P}$, we apply Lemma 1 to yield

$$|\Delta_k|\le\frac{2(\mu-1)}{[k]_q-1}\sum_{j=1}^{k-1}|\Delta_j|,\quad(\Delta_1=1,\ j\in\mathbb{N}-\{1\}).$$

Next, we find the sequence $\{\Lambda\}_{k=1}^{\infty}$ such that

$$\Lambda_1 = 1,$$

and

$$\Lambda_k = \frac{2(\mu-1)}{[k]_q-1}\sum_{j=1}^{k-1}\Lambda_j.$$

Let us show that

$$|\Delta_k|\le\Lambda_k,\quad(k\in\mathbb{N}-\{1\}). \tag{2.12}$$

For $k=2$, we have

$$|\Delta_2|\le\frac{2(\mu-1)}{[2]_q-1}.$$

Assume that

$$|\Delta_m|\le|\Lambda_m|,\quad(m\in\{2,3,...,k\}).$$

Then, we have

$$|\Delta_{k+1}|\le\frac{2(\mu-1)}{[k+1]_q-1}\sum_{j=1}^{k}|\Delta_j|\le\frac{2(\mu-1)}{[k+1]_q-1}\sum_{j=1}^{k}\Lambda_j=\Lambda_{k+1}.$$

Therefore, by applying Lemma 4, we reach to the assertions (2.4) and (2.5). From Eqs (2.8), (2.12), (2.4) and (2.5), we establish the coefficient estimates in (2.6) and (2.7). Thus, the proof of Theorem 3 is completed. □

Similar to Theorem 3, we state without proof the following theorem.

Theorem 4. *Let* $\mu > 1$ *and* $f \in N^n_{q,\lambda}(\mu)$*, then we have*

$$|a_2| \leq \frac{2(\mu - 1)[1 - \lambda + \lambda[2]_q]^n}{[2]_q - 1},$$

and

$$|a_k| \leq \frac{2(\mu - 1)[(1 - \lambda + \lambda[k]_q]^n}{[k]_q - 1} \prod_{j=2}^{k-1} \left(1 + \frac{2(\mu - 1)}{[j]_q - 1}\right), \quad (k \geq 3).$$

3. Properties of differential subordination

In this section, by applying the inequalities (2.1) and (2.2), we introduce the subclasses $\overline{M}^n_{q,\lambda}(\mu)$ and $\overline{N}^n_{q,\lambda}(\mu)$. Further, we construct differential subordination results for these subclasses.

Definition 3. *The function* $f \in \mathcal{A}$ *belongs to the class* $\overline{M}^n_{q,\lambda}(\mu)$ *if the Taylor-Maclaurin coefficient satisfies the inequality* (2.1)*. When* $q \to 1^-$*, the class* $\overline{M}^n_{q,\lambda}(\mu)$ *is denoted by* $\overline{M}^n_{\lambda}(\mu)$*.*

Theorem 5. *Let* $f \in \mathcal{A}$ *be given by* (1.1)*,* $0 < q < 1$*,* $\mu < [k]_q$ *and* $0 \leq \lambda \leq 1$*. If* $f \in \overline{M}^n_{q,\lambda}(\mu)$ *and* g *is a convex function, then we have*

$$\Lambda_{q,\lambda}(n, \mu)(f * g)(z) \prec g(z), \tag{3.1}$$

and

$$Re\,(f\,(z)) > -\frac{\mu - 1 + ([2]_q - \mu)[1 - \lambda + \lambda[2]_q]^n}{([2]_q - \mu)[1 - \lambda + \lambda[2]_q]^n}, \tag{3.2}$$

where

$$\Lambda_{q,\lambda}(n, \mu) = \frac{([2]_q - \mu)[1 - \lambda + \lambda[2]_q]^{n-1}}{\mu - 1 + ([2]_q - \mu)[1 - \lambda + \lambda[2]_q]^n}. \tag{3.3}$$

Proof. Suppose that $f \in \overline{M}^n_{q,\lambda}(\mu)$ and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$, then we conclude

$$\Lambda_{q,\lambda}(n, \mu)(f * g)(z) = \Lambda_{q,\lambda}(n, \mu)\left(z + \sum_{k=2}^{\infty} a_k b_k z^k\right), \tag{3.4}$$

where $\Lambda_{q,\lambda}(n, \mu)$ has the significance of (3.3). If $\{\Lambda_{q,\lambda}(n, \mu)a_k\}_{k=1}^{\infty}$ is a subordinate factor sequence with $a_1 = 1$, then the subordination (3.1) holds.
By applying Lemma 2, the inequality (3.4) is equivalent to

$$Re\left\{1 + \sum_{k=1}^{\infty} \frac{2([2]_q - \mu)[1 - \lambda + \lambda[2]_q]^{n-1}}{\mu - 1 + ([2]_q - \mu)[1 - \lambda + \lambda[2]_q]^n} a_k z^k\right\} > 0, \quad \text{for } z \in D. \tag{3.5}$$

Note that $\{|[k]_q - \mu|[1 - \lambda + \lambda[k]_q]^n\}_{k=1}^{\infty}$ is an increasing sequence. Now, by applying Theorem 1, we infer

$$\begin{aligned}
&Re\left\{1 + \sum_{k=1}^{\infty} \frac{2([2]_q - \mu)[1 - \lambda + \lambda[2]_q]^{n-1}}{\mu - 1 + ([2]_q - \mu)[1 - \lambda + \lambda[2]_q]^n} a_k z^k\right\} \\
&\geq Re\left\{1 + \sum_{k=1}^{\infty} \frac{([2]_q - \mu)[1 - \lambda + \lambda[2]_q]^n}{\mu - 1 + ([2]_q - \mu)[1 - \lambda + \lambda[2]_q]^n} a_k z^k\right\} \\
&= Re\left\{1 + \frac{2([2]_q - \mu)[1 - \lambda + \lambda[2]_q]^n}{\mu - 1 + ([2]_q - \mu)[1 - \lambda + \lambda[2]_q]^n} a_1 z\right. \\
&\qquad \left. + \frac{1}{\mu - 1 + ([2]_q - \mu)[1 - \lambda + \lambda[2]_q]^n} \sum_{k=2}^{\infty} ([2]_q - \mu)[1 - \lambda + \lambda[2]_q]^n a_k z^k\right\} \\
&\geq 1 - \frac{2([2]_q - \mu)[1 - \lambda + \lambda[2]_q]^n}{\mu - 1 + ([2]_q - \mu)[1 - \lambda + \lambda[2]_q]^n} r \\
&\qquad - \frac{1}{\mu - 1 + ([2]_q - \mu)[1 - \lambda + \lambda[2]_q]^n} \sum_{k=2}^{\infty} ([2]_q - \mu)[1 - \lambda + \lambda[2]_q]^n |a_k| r^k \\
&> 1 - \frac{2([2]_q - \mu)[1 - \lambda + \lambda[2]_q]^n}{\mu - 1 + ([2]_q - \mu)[1 - \lambda + \lambda[2]_q]^n} r - \frac{\mu - 1}{\mu - 1 + ([2]_q - \mu)[1 - \lambda + \lambda[2]_q]^n} r \\
&= 1 - r,
\end{aligned}$$

for $|z| = r < 1$. This proves inequality (3.5). Hence, the differential subordination (3.1) is established. We can verify inequality (3.2) by setting $g(z) = \frac{z}{1-z}$. This completes the proof of Theorem 5. □

Remark 2. *Suppose that the function f_2 is defined as follows*

$$f_2(z) = z - \frac{\mu - 1}{([2]_q - \mu)[1 - \lambda + \lambda[2]_q]^n} z^2, \quad (n \in \mathbb{N}, 1 < \mu < [2]_q, 0 \leq \lambda \leq 1).$$

It can be easily shown that $f_2(z) \in \overline{M}^n_{q,\lambda}(\mu)$. Thus, from the differential subordination (3.1)*, we find that*

$$\Lambda_{q,\lambda}(n, \mu) f_2(z) \prec \frac{z}{1 - z}, \qquad (z \in D), \tag{3.6}$$

where $\Lambda_{q,\lambda}(n, \mu)$ has the meaning of (3.3). *This implies that*

$$min_{z \in D} \left\{\Lambda_{q,\lambda}(n, \mu) f_2(z)\right\} = -\frac{1}{2}.$$

This shows that, the constant $\Lambda_{q,\lambda}(n, \mu)$ in (3.1) *is the best unique dominant. Indeed, we can't replace this constant by a large one.*

By setting $q \to 1^-$ in Theorem 5, one may derive the following corollary.

Corollary 3. *Let $f \in \mathcal{A}$ be given by* (1.1), $1 < \mu < 2$ *and* $0 \leq \lambda \leq 1$. *If $f \in \overline{M}^n_{\lambda}(\mu)$ and g is a convex function, then*

$$\Lambda_{\lambda}(n, \mu)(f * g)(z) \prec g(z), \tag{3.7}$$

and

$$Re\,(f\,(z)) > -\frac{\mu - 1 + (2-\mu)(1+\lambda)^n}{(2-\mu)(1+\lambda)^n},$$

where

$$\Lambda_\lambda(n,\mu) = \frac{(2-\mu)(1+\lambda)^{n-1}}{\mu - 1 + (2-\mu)(1+\lambda)^n}.$$

Remark 3. *Similar to Remark 2, we infer that the constant $\Lambda_\lambda(n,\mu)$ in* (3.7) *is the best unique dominant. Indeed, we can't replace this constant by a large one.*

Definition 4. *The function $f \in \mathcal{A}$ belongs to the class $\overline{N}^{n}_{q,\lambda}(\mu)$ if the Taylor-Maclaurin coefficient satisfies the inequality* (2.2).
For $q \to 1^-$, the class $\overline{N}^{n}_{q,\lambda}(\mu)$ is denoted by $\overline{N}^{n}_{\lambda}(\mu)$.

Similar to Theorem 5, we can derive the following theorem.

Theorem 6. *Let $f \in \mathcal{A}$ be given by* (1.1), $0 < q < 1$, $\mu < [k]_q$ *and* $0 \le \lambda \le 1$. *If $f \in \overline{N}^{n}_{q,\lambda}(\mu)$ and g is a convex function, then*

$$\Delta_{q,\lambda}(n,\mu)(f * g)(z) \prec g(z), \tag{3.8}$$

and

$$Re\,(f\,(z)) > -\frac{(\mu-1)[1-\lambda+\lambda[2]_q]^n + ([2]_q - \mu)}{[2]_q - \mu},$$

where

$$\Delta_{q,\lambda}(n,\mu) = \frac{([2]_q - \mu)(1-\lambda+\lambda[2]_q)}{(\mu-1)[1-\lambda+\lambda[2]_q]^n + [2]_q - \mu}.$$

Remark 4. *Similar to Remark 2, we claim that the constant $\Delta_{q,\lambda}(n,\mu)$ in* (3.8) *is the best unique dominant. Indeed, we can't replace this constant by a large one.*

By allowing $q \to 1^-$ in Theorem 6, one may state- without proof the following corollary.

Corollary 4. *Let $f \in \mathcal{A}$ be given by* (1.1), $1 < \mu < 2$ *and* $0 \le \lambda \le 1$. *If $f \in \overline{N}^{n}_{\lambda}(\mu)$ and g is a convex function, then*

$$\Delta_\lambda(n,\mu)(f * g)(z) \prec g(z), \tag{3.9}$$

and

$$Re\,(f\,(z)) > -\frac{(\mu-1)(1+\lambda)^n + 2 - \mu}{2-\mu},$$

where

$$\Delta_\lambda(n,\mu) = \frac{(2-\mu)(1+\lambda)}{(\mu-1)(1+\lambda)^n + 2 - \mu}.$$

We cannot replace the constant $\Delta_\lambda(n,\mu)$ in (3.9) *by a larger one.*

Remark 5. *Similar to Remark 2, we establish that the constant $\Delta_\lambda(n,\mu)$ in* (3.9) *is the best unique dominant. Indeed, we can't replace this constant by a large one.*

4. Conclusions

In this paper, new subclasses of analytic functions and q-analogues of Salagean differential operators were studied by virtue of an idea of Salagean operators. Several subclasses of univalent functions associated with q-Salagean differential operators are obtained. Further, coefficient bounds for functions in the aforementioned subclasses are discussed. Some reliable results for differential subordinations of the analytic functions are also investigated. However, our results may be used in generalizing several Salagean differential operators, which in turn extend different types of q-analogues of univalent functions. Moreover, by applying different types of fractional integral operators, some subclasses of univalent functions can be introduced and various subordination chains with applications to classes of univalent functions may be established by using the Loewner chain.

Conflict of interest

Authors declare no competing interests regarding the publication of the article are found.

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