**AIMS** *Mathematics*

*Research article*

# One new blow-up criterion for the two-dimensional full compressible magnetohydrodynamic equations

**Li Lu**[1,2,*]

[1] School of Mathematics, East China University of Science and Technology, Shanghai 200237, China

[2] College of Mathematics and Information Science, Nanchang Hangkong University, Nanchang 330063, Jiangxi, China

* **Correspondence:** Email: lulid113@163.com.

**Abstract:** This paper concerns the blow-up criterion for two-dimensional (2D) viscous, compressible, and heat conducting magnetohydrodynamic(MHD) flows. When the magnetic field $H$ satisfies the perfect conducting boundary condition $H \cdot n = \text{curl} H = 0$, we prove that for the initial boundary value problem of the two-dimensional full compressible MHD flows with initial density allowed to vanish, the strong solution exists globally provided $\|H\|_{L^\infty(0,T; L^b)} + \|\text{div} u\|_{L^1(0,T; L^\infty)} < \infty$ for any $b > 2$.

## 1. Introduction and main results

In this paper, we consider the two-dimensional viscous, compressible and heat conducting magnetohydrodynamic equations in the Eulerian coordinates (see [1])

$$\begin{cases} \rho_t + \text{div}(\rho u) = 0, \\ (\rho u)_t + \text{div}(\rho u \otimes u) + \nabla P = \mu \triangle u + (\mu + \lambda)\nabla(\text{div} u) + H \cdot \nabla H - \frac{1}{2}\nabla|H|^2, \\ c_v((\rho\theta)_t + \text{div}(\rho u\theta)) + P\text{div} u = \kappa\Delta\theta + \lambda(\text{div} u)^2 + \nu|\text{curl} H|^2 + 2\mu|\mathfrak{D}(u)|^2, \\ H_t + u \cdot \nabla H - H \cdot \nabla u + H\text{div} u = \nu\Delta H, \quad \text{div} H = 0. \end{cases} \tag{1.1}$$

Here $x = (x_1, x_2) \in \Omega$ is the spatial coordinate, $\Omega$ is a bounded smooth domain in $\mathbb{R}^2$, $t \geq 0$ is the time, and the unknown functions $\rho = \rho(x,t)$, $\theta = \theta(x,t)$, $u = (u^1, u^2)(x,t)$ and $H = (H^1, H^2)(x,t)$ denote, respectively, the fluid density, absolute temperature, velocity and magnetic field. In addition, the pressure $P$ is given by

$$P(\rho) = R\theta\rho, \quad (R > 0),$$

where $R$ is a generic gas constant. The deformation tensor $\mathfrak{D}(u)$ is defined by

$$\mathfrak{D}(u) = \frac{1}{2}(\nabla u + (\nabla u)^{tr}).$$

The shear viscosity $\mu$ and the bulk one $\lambda$ satisfy the hypotheses as follows

$$\mu > 0, \quad \mu + \lambda \geq 0.$$

Positive constants $c_v$, $\kappa$ and $\nu$ represent, respectively, the heat capacity, heat conductivity and magnetic diffusivity coefficient.

The initial condition and boundary conditions for Eq (1.1) are given as follows

$$(\rho, \theta, u, H)(x, t = 0) = (\rho_0, \theta_0, u_0, H_0), \tag{1.2}$$

$$\frac{\partial \theta}{\partial n} = 0, \quad u = 0, \quad H \cdot n = 0, \quad \mathrm{curl} H = 0, \quad \text{on } \partial\Omega, \tag{1.3}$$

where $n$ denotes the unit outward normal vector of $\partial\Omega$.

**Remark 1.1.** *The boundary condition imposed on $H$ (1.3) is physical and means that the container is perfectly conducting, see [1–4].*

In the absence of electromagnetic effect, namely, in the case of $H \equiv 0$, the MHD system reduces to the Navier-Stokes equations. Due to the strong coupling and interplay interaction between the fluid motion and the magnetic field, it is rather complicated to investigate the well-posedness and dynamical behaviors of MHD system. There are a huge amount of literature on the existence and large time behavior of solutions to the Navier-Stokes system and MHD one due to the physical importance, complexity, rich phenomena and mathematical challenges, see [1, 5–26] and the reference therein. However, many physically important and mathematically fundamental problems are still open due to the lack of smoothing mechanism and the strong nonlinearity. When the initial density contain vacuum states, the local large strong solutions to Cauchy problem of 3D full MHD equations and 2D isentropic MHD system have been obtained, respectively, by Fan-Yu [5] and Lü-Huang [6]. For the global well-posedness of strong solutions, Li-Xu-Zhang [7] and Lü-Shi-Xu [8] established the global existence and uniqueness of strong solutions to the 3D and 2D MHD equations, respectively, provided the smooth initial data are of small total energy. In particular, the initial density can have compact support in [7, 8]. Furthermore, Hu-Wang [9, 10] and Fan-Yu [11] proved the global existence of renormalized solutions to the compressible MHD equations for general large initial data. However, it is an outstanding challenging open problem to establish the global well-posedness for general large strong solutions with vacuum.

Therefore, it is important to study the mechanism of blow-up and structure of possible singularities of strong (or smooth) solutions to the compressible MHD system (1.1). The pioneering work can be traced to Serrin's criterion [12] on the Leray-Hopf weak solutions to the 3D incompressible Navier-Stokes equations, that is

$$\lim_{t \to T^*} \|u\|_{L^s(0,t;\, L^r)} = \infty, \quad \text{for } \frac{3}{r} + \frac{2}{s} = 1, \ 3 < r \leq \infty, \tag{1.4}$$

where $T^*$ is the finite blow up time. Later, He-Xin [13] established the same Serrin's criterion (1.4) for strong solutions to the incompressible MHD equations.

First of all, we recall several known blow-up criteria for the compressible Navier-Stokes equations. In the isentropic case, Huang-Li-Xin [14] established the Serrin type criterion as follows

$$\lim_{t \to T^*} \left( \|u\|_{L^s(0,t;\ L^r)} + \|\mathrm{div} u\|_{L^1(0,t;\ L^\infty)} \right) = \infty, \quad \text{for } \frac{3}{r} + \frac{2}{s} = 1,\ 3 < r \le \infty. \tag{1.5}$$

For the full compressible Navier-Stokes equations, Fan-Jiang-Ou [15] obtained that

$$\lim_{t \to T^*} \left( \|\theta\|_{L^\infty(0,t;\ L^\infty)} + \|\nabla u\|_{L^1(0,t;\ L^\infty)} \right) = \infty, \tag{1.6}$$

under the condition

$$7\mu > \lambda. \tag{1.7}$$

Later, the restriction (1.7) was removed in Huang-Li-Xin [16]. Recently, Wang [17] established a blow-up criterion for the initial boundary value problem (IBVP) on a smooth bounded domain in $\mathbb{R}^2$, namely,

$$\lim_{t \to T^*} \|\mathrm{div} u\|_{L^1(0,t;\ L^\infty)} = \infty. \tag{1.8}$$

Then, let's return to the compressible MHD system (1.1). Under the three-dimensional isentropic condition, Xu-Zhang [18] founded the same criterion (1.5) as [14]. For the three-dimensional full compressible MHD system, the criterion (1.6) is also established by Lu-Du-Yao [19] under the condition

$$\mu > 4\lambda. \tag{1.9}$$

Soon, the restriction (1.9) was removed by Chen-Liu [20]. Later, for the Cauchy problem and the IBVP of three-dimensional full compressible MHD system, Huang-Li [21] proved that

$$\lim_{t \to T^*} \left( \|u\|_{L^s(0,t;\ L^r)} + \|\rho\|_{L^\infty(0,t;\ L^\infty)} \right) = \infty, \quad \text{for } \frac{3}{r} + \frac{2}{s} \le 1,\ 3 < r \le \infty. \tag{1.10}$$

Recently, Fan-Li-Nakamura [22] extended the results of [17] to the MHD system and established a blow-up criterion which depend only on $H$ and $\mathrm{div} u$ as follows

$$\lim_{t \to T^*} \left( \|H\|_{L^\infty(0,t;\ L^\infty)} + \|\mathrm{div} u\|_{L^1(0,t;\ L^\infty)} \right) = \infty. \tag{1.11}$$

In fact, if $H \equiv 0$ in (1.11), the criterion (1.11) becomes (1.8).

The purpose of this paper is to loosen and weaken the regularity of $H$ required in the blow-up criterion (1.11) for strong solutions of the IBVP (1.1)–(1.3).

In this paper, we denote

$$\int \cdot dx \triangleq \int_\Omega \cdot dx.$$

Furthermore, for $s \ge 0$ and $1 \le r \le \infty$, we define the standard Lebesgue and Sobolev spaces as follows

$$\begin{cases} L^r = L^r(\Omega), \quad W^{s,r} = W^{s,r}(\Omega), \quad H^s = W^{s,2}, \\ W_0^{s,r} = \{f \in W^{s,r} | f = 0 \text{ on } \partial\Omega\}, \quad H_0^s = W_0^{s,2}. \end{cases}$$

To present our results, we first recall the local existence theorem of the strong solution. Fan-Yu [5] attained the local existence and uniqueness of strong solution with full compressible MHD system in $\mathbb{R}^3$. In fact, when $\Omega$ is a bounded domain in $\mathbb{R}^2$, the method applied in [5, 23] can also be used to the case here. The corresponding result can be expressed as follows.

**Theorem 1.1.** *(Local existence theorem) For $q > 2$, assume that the initial data $(\rho_0, \theta_0, u_0, H_0)$ satisfies*

$$\begin{cases} 0 \le \rho_0 \in W^{1,q}, \quad 0 \le \theta_0 \in H^2, \quad u_0 \in H_0^1 \cap H^2, \quad H_0 \in H^2, \quad \mathrm{div}H_0 = 0, \\ \frac{\partial \theta_0}{\partial n}|_{\partial\Omega} = 0, \quad u_0|_{\partial\Omega} = 0, \quad H_0 \cdot n|_{\partial\Omega} = curlH_0|_{\partial\Omega} = 0, \end{cases} \tag{1.12}$$

*and the compatibility conditions as follows*

$$-\mu\triangle u_0 - (\mu + \lambda)\nabla\mathrm{div}u_0 + R\nabla(\rho_0\theta_0) - H_0 \cdot \nabla H_0 + \frac{1}{2}\nabla|H_0|^2 = \rho_0^{1/2}g_1, \tag{1.13}$$

$$-\kappa\triangle\theta_0 - 2\mu|\mathfrak{D}(u_0)|^2 - \lambda(\mathrm{div}u_0)^2 - \nu(curlH_0)^2 = \rho_0^{1/2}g_2, \tag{1.14}$$

*for some $g_1, g_2 \in L^2$. Then there exists a time $T_0 > 0$ such that the IBVP (1.1)–(1.3) has a unique strong solution $(\rho, \theta, u, H)$ on $\Omega \times (0, T_0]$ satisfying that*

$$\begin{cases} 0 \le \rho \in C([0, T_0]; W^{1,q}), \quad \rho_t \in C([0, T_0]; L^q), \\ (u, \theta, H) \in C([0, T_0]; H^2) \cap L^2(0, T_0; W^{2,q}), \quad \theta \ge 0, \\ (u_t, \theta_t, H_t) \in L^2(0, T_0; H^1), \quad (\sqrt{\rho}u_t, \sqrt{\rho}\theta_t, H_t) \in L^\infty(0, T_0; L^2). \end{cases} \tag{1.15}$$

Then, our main result is stated as follows.

**Theorem 1.2.** *Under the assumption of Theorem 1.1 , suppose $(\rho, \theta, u, H)$ is the strong solution of the IBVP (1.1)–(1.3) obtained in Theorem 1.1. If $T^* < \infty$ is the maximum existence time of the strong solution, then*

$$\lim_{t \to T^*} \left( \|H\|_{L^\infty(0,t;L^b)} + \|\mathrm{div}u\|_{L^1(0,t;L^\infty)} \right) = \infty, \tag{1.16}$$

*for any $b > 2$.*

**Remark 1.2.** *Compared to the blow-up criterion (1.11) attained in [22], Theorem 1.2 demonstrates some new message about the blow-up mechanism of the MHD system (1.1)–(1.3). Particularly, beside the same regularity on $\|\mathrm{div}u\|_{L^1(0,t;L^\infty)}$ as (1.11) in [22], our result (1.16) improves the regularity on $\|H\|_{L^\infty(0,t;L^\infty)}$ by relaxing it to $\|H\|_{L^\infty(0,t;L^b)}$ for any $b > 2$.*

The rest of the paper is arranged as follows. We state several basic facts and key inequalities which are helpful for later analysis in Section 2. Sections 3 is devoted to a priori estimate which is required to prove Theorem 1.2, while we give its proof in Section 4.

## 2. Preliminaries

In this section, we will recall several important inequalities and well-known facts. First of all, Gagliardo-Nirenberg inequality (see [27]) is described as follows.

**Lemma 2.1.** *(Gagliardo-Nirenberg) For $q \in (1, \infty), r \in (2, \infty)$ and $s \in [2, \infty)$, there exists some generic constant $C > 0$ which may depend only on $q, r$ and $s$ such that for $f \in C_0^\infty(\Omega)$, we have*

$$\|f\|_{L^s(\Omega)}^s \leq C\|f\|_{L^2(\Omega)}^2 \|\nabla f\|_{L^2(\Omega)}^{s-2}, \tag{2.1}$$

$$\|g\|_{L^\infty(\Omega)} \leq C\|g\|_{L^q(\Omega)}^{q(r-2)/(2r+q(r-2))} \|\nabla g\|_{L^r(\Omega)}^{2r/(2r+q(r-2))}. \tag{2.2}$$

Then, we give several regularity results for the following Lamé system with Dirichlet boundary condition (see [24])

$$\begin{cases} \mathcal{L}U \triangleq \mu\Delta U + (\mu + \lambda)\nabla\text{div}U = F, & x \in \Omega, \\ U = 0, & x \in \partial\Omega. \end{cases} \tag{2.3}$$

We assume that $U \in H_0^1$ is a weak solution of the Lamé system, due to the uniqueness of weak solution, it could be denoted by $U = \mathcal{L}^{-1}F$.

**Lemma 2.2.** *Let $r \in (1, \infty)$, then there exists some generic constant $C > 0$ depending only on $\mu, \lambda, r$ and $\Omega$ such that*

- *If $F \in L^r$, then*

$$\|U\|_{W^{2,r}(\Omega)} \leq C\|F\|_{L^r(\Omega)}. \tag{2.4}$$

- *If $F \in W^{-1,r}$ (i.e., $F = \text{div}f$ with $f = (f_{ij})_{2\times2}, f_{ij} \in L^r$), then*

$$\|U\|_{W^{1,r}(\Omega)} \leq C\|f\|_{L^r(\Omega)}. \tag{2.5}$$

*Furthermore, for the endpoint case, if $f_{ij} \in L^2 \cap L^\infty$, then $\nabla U \in BMO(\Omega)$ and*

$$\|\nabla U\|_{BMO(\Omega)} \leq C\|f\|_{L^\infty(\Omega)} + C\|f\|_{L^2(\Omega)}. \tag{2.6}$$

The following $L^p$-bound for elliptic systems, whose proof is similar to that of [28, Lemma 12], is a direct consequence of the combination of a well-known elliptic theory due to Agmon-Douglis-Nirenberg [29, 30] with a standard scaling procedure.

**Lemma 2.3.** *For $k \geq 0$ and $p > 1$, there exists a constant $C > 0$ depending only on $k$ and $p$ such that*

$$\|\nabla^{k+2}v\|_{L^p(\Omega)} \leq C\|\Delta v\|_{W^{k,p}(\Omega)}, \tag{2.7}$$

*for every $v \in W^{k+2,p}(\Omega)$ satisfying either*

$$v \cdot n = 0, \quad \text{rot}v = 0, \quad \text{on } \partial\Omega,$$

*or*

$$v = 0, \quad \text{on } \partial\Omega.$$

Finally, we give two critical Sobolev inequalities of logarithmic type, which are originally due to Brezis-Gallouet [31] and Brezis-Wainger [32].

**Lemma 2.4.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain and $f \in W^{1,q}$ with $q > 2$, then it holds that*

$$\|f\|_{L^\infty(\Omega)} \leq C\|f\|_{BMO(\Omega)} \ln\left(e + \|f\|_{W^{1,q}(\Omega)}\right) + C, \tag{2.8}$$

*with a constant $C$ depending only on $q$.*

**Lemma 2.5.** *Let $\Omega \subset \mathbb{R}^2$ be a smooth domain and $f \in L^2(s, t; H_0^1 \cap W^{1,q})$ with $q > 2$, then it holds that*

$$\|f\|_{L^2(s,t;L^\infty)}^2 \leq C\|f\|_{L^2(s,t;H^1)}^2 \ln\left(e + \|f\|_{L^2(s,t;W^{1,q})}\right) + C, \tag{2.9}$$

*with a constant $C$ depending only on $q$.*

## 3. A priori estimate

Let $(\rho, \theta, u, H)$ be the strong solution of the IBVP (1.1)–(1.3) obtained in Theorem 1.1. Assume that (1.16) is false, namely, there exists a constant $M > 0$ such that

$$\lim_{t \to T^*} \left( \|H\|_{L^\infty(0,t;L^b)} + \|\mathrm{div}u\|_{L^1(0,t;L^\infty)} \right) \le M < \infty, \quad \text{for any } b > 2. \tag{3.1}$$

First of all, the upper bound of the density can be deduced from $(1.1)_1$ and (3.1), see [14, Lemma 3.4].

**Lemma 3.1.** *Under the assumptions of Theorem 1.2 and* (3.1)*, it holds that for any $t \in [0, T^*)$,*

$$\sup_{0 \le s \le t} \|\rho\|_{L^1 \cap L^\infty} \le C, \tag{3.2}$$

*where (and in what follows) $C$ represents a generic positive constant depending only on $\mu$, $\lambda$, $c_v$, $\kappa$, $\nu$, $q$, $b$, $M$, $T^*$ and the initial data.*

Then, we give the following estimates, which are similar to the energy estimates.

**Lemma 3.2.** *Under the assumptions of Theorem 1.2 and* (3.1)*, it holds that for any $t \in [0, T^*)$,*

$$\sup_{0 \le s \le t} \left( \|\rho\theta\|_{L^1} + \|\rho^{1/2}u\|_{L^2}^2 + \|H\|_{L^2}^2 \right) + \int_0^t \left( \|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2 \right) ds \le C. \tag{3.3}$$

*Proof.* First, using the standard maximum principle to $(1.1)_3$ together with $\theta_0 \ge 0$ (see [15, 25]) gives

$$\inf_{\Omega \times [0,t]} \theta(x, t) \ge 0. \tag{3.4}$$

Then, utilizing the standard energy estimates to (1.1) shows

$$\sup_{0 \le s \le t} \left( \|\rho\theta\|_{L^1} + \|\rho^{1/2}u\|_{L^2}^2 + \|H\|_{L^2}^2 \right) \le C. \tag{3.5}$$

Next, adding $(1.1)_2$ multiplied by $u$ to $(1.1)_4$ multiplied by $H$, and integrating the summation by parts, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left( \|\rho^{1/2}u\|_{L^2}^2 + \|H\|_{L^2}^2 \right) \\ &+ \mu\|\nabla u\|_{L^2}^2 + \nu\|\nabla H\|_{L^2}^2 + (\mu + \lambda)\|\mathrm{div}u\|_{L^2}^2 \le C\|\rho\theta\|_{L^1}\|\mathrm{div}u\|_{L^\infty}, \end{aligned} \tag{3.6}$$

where one has used the following well-known fact

$$\|\nabla H\|_{L^2} \le C\|\mathrm{curl}H\|_{L^2}, \tag{3.7}$$

due to $\mathrm{div}H = 0$ and $H \cdot n|_{\partial\Omega} = 0$.

Hence, the combination of (3.6) with (3.1), (3.4) and (3.5) yields (3.3). This completes the proof of Lemma 3.2. □

The following lemma shows the estimates on the spatial gradients of both the velocity and the magnetic, which are crucial for obtaining the higher order estimates of the solution.

**Lemma 3.3.** *Under the assumptions of Theorem 1.2 and* (3.1)*, it holds that for any $t \in [0, T^*)$,*

$$
\begin{aligned}
\sup_{0 \le s \le t} & \left( \|\rho^{1/2}\theta\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|curlH\|_{L^2}^2 \right) \\
& + \int_0^t \left( \|\rho^{1/2}\dot u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2 + \|H_t\|_{L^2}^2 + \|\Delta H\|_{L^2}^2 \right) ds \le C,
\end{aligned}
\tag{3.8}
$$

*where $\dot f \triangleq u \cdot \nabla f + f_t$ represents the material derivative of $f$.*

*Proof.* Above all, multiplying the equation $(1.1)_3$ by $\theta$ and integrating by parts yield

$$
\frac{c_v}{2}\frac{d}{dt}\|\rho^{1/2}\theta\|_{L^2}^2 + \kappa\|\nabla\theta\|_{L^2}^2 \le \nu \int \theta|curlH|^2 dx + C \int \theta|\nabla u|^2 dx + C\|\rho^{1/2}\theta\|_{L^2}^2\|\mathrm{div}u\|_{L^\infty}.
\tag{3.9}
$$

Firstly, integration by parts together with (3.1) and Gagliardo-Nirenberg inequality implies that

$$
\begin{aligned}
\nu \int \theta|curlH|^2 dx \le & C\|\nabla\theta\|_{L^2}\|H\|_{L^b}\|\nabla H\|_{L^{\tilde b}} + C\|\theta\|_{L^{\tilde b}}\|H\|_{L^b}\|\nabla^2 H\|_{L^2} \\
\le & C\|\nabla\theta\|_{L^2}\|\nabla H\|_{L^{\tilde b}} + C\|\nabla^2 H\|_{L^2}(\|\nabla\theta\|_{L^2} + 1) \\
\le & \varepsilon\|\nabla\theta\|_{L^2}^2 + C\|\nabla^2 H\|_{L^2}^2 + C(\|\nabla H\|_{L^2}^2 + 1),
\end{aligned}
\tag{3.10}
$$

where $\tilde b \triangleq \frac{2b}{b-2} > 2$ satisfies $1/b + 1/\tilde b = 1/2$, and in the second inequality where one has applied the estimate as follows

$$
\|\theta\|_{L^r} \le C(\|\nabla\theta\|_{L^2} + 1), \quad \text{for any } r \ge 1.
\tag{3.11}
$$

Indeed, denote the average of $\theta$ by $\bar\theta = \frac{1}{|\Omega|}\int \theta dx$, it follows from (3.2) and (3.3) that

$$
\bar\theta \int \rho dx \le \int \rho\theta dx + \int \rho|\theta - \bar\theta|dx \le C + C\|\nabla\theta\|_{L^2},
\tag{3.12}
$$

which together with Poincaré inequality yields

$$
\|\theta\|_{L^2} \le C(1 + \|\nabla\theta\|_{L^2}).
\tag{3.13}
$$

And consequently, (3.11) holds.

Secondly, according to [17, 21, 33], Multiplying equations $(1.1)_2$ by $u\theta$ and integrating by parts yield

$$
\begin{aligned}
\mu \int \theta|\nabla u|^2 dx & + (\mu + \lambda) \int \theta|\mathrm{div}u|^2 dx \\
= & - \int \rho\dot u \cdot \theta u dx - \mu \int u \cdot \nabla\theta \cdot \nabla u dx - (\mu + \lambda) \int \mathrm{div}u u \cdot \nabla\theta dx \\
& - \int \nabla P \cdot \theta u dx + \int H \cdot \nabla H \cdot \theta u dx - \frac{1}{2} \int \nabla|H|^2 \cdot \theta u dx \\
\triangleq & \sum_{i=1}^6 I_i.
\end{aligned}
\tag{3.14}
$$

Using the same arguments in [17, 33], we have

$$\sum_{i=1}^{4} I_i \leq \eta \|\rho^{1/2}\dot{u}\|_{L^2}^2 + \varepsilon \|\nabla\theta\|_{L^2}^2 + C\|\rho^{1/2}\theta\|_{L^2}^2 \|\mathrm{div}u\|_{L^\infty}$$
$$+ C\left(\|\rho^{1/2}\theta\|_{L^2}^2 + \|\nabla u\|_{L^2}^2\right)\|u\|_{L^\infty}^2. \tag{3.15}$$

Besides, according to (3.1) and (3.11) yields

$$\sum_{i=5}^{6} I_i \leq C\|\theta\|_{L^{\tilde{b}}}\|H\|_{L^b}\|\nabla H\|_{L^2}\|u\|_{L^\infty} \leq \quad \varepsilon\|\nabla\theta\|_{L^2}^2 + C\|\nabla H\|_{L^2}^2\|u\|_{L^\infty}^2 + C. \tag{3.16}$$

Substituting (3.10), (3.15) and (3.16) into (3.9), and choosing $\varepsilon$ suitably small, we have

$$c_v \frac{d}{dt}\|\rho^{1/2}\theta\|_{L^2}^2 + \kappa\|\nabla\theta\|_{L^2}^2$$
$$\leq 2\eta\|\rho^{1/2}\dot{u}\|_{L^2}^2 + C_1\|\Delta H\|_{L^2}^2 \tag{3.17}$$
$$+ C\left(\|\rho^{1/2}\theta\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2 + 1\right)\left(\|\mathrm{div}u\|_{L^\infty} + \|u\|_{L^\infty}^2 + 1\right),$$

where one has applied the key fact as follows

$$\|\nabla^2 H\|_{L^2} \leq C\|\Delta H\|_{L^2}. \tag{3.18}$$

Furthermore, it follows from (3.1) and $(1.1)_4$ that

$$\|H_t\|_{L^2}^2 + \nu^2\|\Delta H\|_{L^2}^2 + \nu\frac{d}{dt}\|\mathrm{curl}H\|_{L^2}^2$$
$$\leq C\|\nabla u\|_{L^2}\|\nabla u\|_{L^{2\tilde{b}}}\|H\|_{L^b}\|H\|_{L^{2\tilde{b}}} + C\|\nabla H\|_{L^2}^2\|u\|_{L^\infty}^2 \tag{3.19}$$
$$\leq C\|\nabla u\|_{L^2}\|\nabla u\|_{L^{2\tilde{b}}}(\|\nabla H\|_{L^2} + 1) + C\|\nabla H\|_{L^2}^2\|u\|_{L^\infty}^2.$$

In order to estimate $\|\nabla u\|_{L^{2\tilde{b}}}$, according to [24, 26], we divide $u$ into $v$ and $w$. More precisely, let

$$u = v + w, \quad \text{and} \quad v = \mathcal{L}^{-1}\nabla P, \tag{3.20}$$

then we get

$$\mathcal{L}w = \rho\dot{u} - H\cdot\nabla H + \frac{1}{2}\nabla|H|^2. \tag{3.21}$$

And hence, Lemma 2.2 implies that for any $r > 1$,

$$\|\nabla v\|_{L^r} \leq C\|\theta\rho\|_{L^r}, \tag{3.22}$$

and

$$\|\nabla^2 w\|_{L^r} \leq C\|\rho\dot{u}\|_{L^r} + C\|\,|H||\nabla H|\,\|_{L^r}. \tag{3.23}$$

Consequently, it follows from Gagliardo-Nirenberg inequality, (3.2), (3.11), (3.20), (3.22) and (3.23) that for any $s \geq 2$,

$$\|\nabla u\|_{L^s} \leq C\|\nabla v\|_{L^s} + C\|\nabla w\|_{L^s}$$
$$\leq C\|\rho\theta\|_{L^s} + C\|\nabla w\|_{L^2} + C\|\nabla w\|_{L^2}^{2/s}\|\nabla^2 w\|_{L^2}^{1-2/s}$$
$$\leq C\|\rho\theta\|_{L^s} + C\|\nabla w\|_{L^2} + C\|\nabla w\|_{L^2}^{2/s}\left(\|\rho\dot{u}\|_{L^2} + \|\,|H||\nabla H|\,\|_{L^2}\right)^{1-2/s} \tag{3.24}$$
$$\leq \eta\|\rho^{1/2}\dot{u}\|_{L^2} + C\|\rho\theta\|_{L^s} + C\|\nabla w\|_{L^2} + C\|\,|H||\nabla H|\,\|_{L^2}$$
$$\leq \eta\|\rho^{1/2}\dot{u}\|_{L^2} + C\|\nabla u\|_{L^2} + C\|\nabla\theta\|_{L^2} + C\|\nabla H\|_{L^2} + C\|\Delta H\|_{L^2} + C.$$

Putting (3.24) into (3.19) and utilizing Young inequality lead to

$$
\begin{aligned}
&\|H_t\|_{L^2}^2 + \frac{\nu^2}{2}\|\Delta H\|_{L^2}^2 + \nu\frac{d}{dt}\|\mathrm{curl}H\|_{L^2}^2 \\
&\leq \varepsilon\|\nabla\theta\|_{L^2}^2 + \eta\|\rho^{1/2}\dot{u}\|_{L^2}^2 + C\left(\|\nabla u\|_{L^2}^2 + \|u\|_{L^\infty}^2 + 1\right)\left(\|\nabla H\|_{L^2}^2 + 1\right).
\end{aligned}
\tag{3.25}
$$

Adding (3.25) multiplied by $2\nu^{-2}(C_1 + 1)$ to (3.17) and choosing $\varepsilon$ suitably small, we have

$$
\begin{aligned}
&\frac{\kappa}{2}\|\nabla\theta\|_{L^2}^2 + 2\nu^{-2}(C_1 + 1)\|H_t\|_{L^2}^2 + \|\Delta H\|_{L^2}^2 \\
&\quad + \frac{d}{dt}\left(c_v\|\rho^{1/2}\theta\|_{L^2}^2 + 2\nu^{-1}(C_1 + 1)\|\mathrm{curl}H\|_{L^2}^2\right) \\
&\leq C\left(\|\rho^{1/2}\theta\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2 + 1\right)\left(\|\nabla u\|_{L^2}^2 + \|u\|_{L^\infty}^2 + \|\mathrm{div}u\|_{L^\infty} + 1\right) \\
&\quad + C\eta\|\rho^{1/2}\dot{u}\|_{L^2}^2.
\end{aligned}
\tag{3.26}
$$

Then, multiplying $(1.1)_2$ by $u_t$ and integrating by parts, we get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\mu\|\nabla u\|_{L^2}^2 + (\mu + \lambda)\|\mathrm{div}u\|_{L^2}^2\right) + \|\rho^{1/2}\dot{u}\|_{L^2}^2 \\
&\leq \eta\|\rho^{1/2}\dot{u}\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2\|u\|_{L^\infty}^2 \\
&\quad + \frac{d}{dt}\left(\int P\mathrm{div}u\,dx + \frac{1}{2}\int|H|^2\mathrm{div}u\,dx - \int H\cdot\nabla u\cdot H\,dx\right) \\
&\quad - \int P_t\mathrm{div}u\,dx - \int H\cdot H_t\mathrm{div}u\,dx + \int H_t\cdot\nabla u\cdot H\,dx + \int H\cdot\nabla u\cdot H_t\,dx.
\end{aligned}
\tag{3.27}
$$

Notice that

$$
\int P_t\mathrm{div}u\,dx = \int P_t\mathrm{div}v\,dx + \int P_t\mathrm{div}w\,dx,
\tag{3.28}
$$

integration by parts together with (3.20) leads to

$$
\int P_t\mathrm{div}v\,dx = \frac{1}{2}\frac{d}{dt}\left((\mu + \lambda)\|\mathrm{div}v\|_{L^2}^2 + \mu\|\nabla v\|_{L^2}^2\right).
\tag{3.29}
$$

Moreover, define

$$
E \triangleq c_v\theta + \frac{1}{2}|u|^2,
$$

according to (1.1) that $E$ satisfies

$$
\begin{aligned}
(\rho E)_t + \mathrm{div}(\rho u E + Pu) &= \Delta\left(\kappa\theta + \frac{1}{2}\mu|u|^2\right) + \mu\mathrm{div}(u\cdot\nabla u) + \lambda\mathrm{div}(u\mathrm{div}u) \\
&\quad + H\cdot\nabla H\cdot u - \frac{1}{2}u\cdot\nabla|H|^2 + \nu|\mathrm{curl}H|^2.
\end{aligned}
\tag{3.30}
$$

Motivated by [17, 21], it can be deduced from (3.30) that

$$
\begin{aligned}
&- \int P_t \mathrm{div} w dx \\
&= -\frac{R}{c_v} \left( \int (\rho E)_t \mathrm{div} w dx - \int \frac{1}{2} (\rho |u|^2)_t \mathrm{div} w dx \right) \\
&= -\frac{R}{c_v} \Bigg\{ \int \left( (c_v + R) \rho \theta u + \frac{1}{2} \rho |u|^2 u - \kappa \nabla \theta - \mu \nabla u \cdot u - \mu u \cdot \nabla u - \lambda u \mathrm{div} u \right) \cdot \nabla \mathrm{div} w dx \\
&\quad - \frac{1}{2} \int \rho |u|^2 u \cdot \nabla \mathrm{div} w dx - \int \rho \dot{u} \cdot u \mathrm{div} w dx \\
&\quad - \int \mathrm{div} H H \cdot u \mathrm{div} w dx - \int H \cdot \nabla u \cdot H \mathrm{div} w dx - \int (H \cdot u) H \cdot \nabla \mathrm{div} w dx \\
&\quad + \frac{1}{2} \int \mathrm{div} u |H|^2 \mathrm{div} w dx + \frac{1}{2} \int |H|^2 u \cdot \nabla \mathrm{div} w dx \\
&\quad - \nu \int \nabla \mathrm{div} w \times \mathrm{curl} H \cdot H dx - \nu \int \mathrm{curl}(\mathrm{curl} H) \cdot H \mathrm{div} w dx \Bigg\} \\
&\leq C \eta \|\rho^{1/2} \dot{u}\|_{L^2}^2 + C \|\nabla \theta\|_{L^2}^2 + C \|\Delta H\|_{L^2}^2 \\
&\quad + C \left( \|\nabla u\|_{L^2}^2 + \|u\|_{L^\infty}^2 + 1 \right) \left( \|\rho^{1/2} \theta\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2 + 1 \right).
\end{aligned}
\tag{3.31}
$$

Additionally, combining (3.1) and (3.24) yields

$$
\begin{aligned}
&\int H_t \cdot \nabla u \cdot H dx + \int H \cdot \nabla u \cdot H_t dx - \int H \cdot H_t \mathrm{div} u dx \\
&\leq C \|H_t\|_{L^2}^2 + C \|\nabla u\|_{L^{\tilde{b}}}^2 \|H\|_{L^b}^2 \\
&\leq C \eta \|\rho^{1/2} \dot{u}\|_{L^2}^2 + C \left( \|H_t\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2 + \|\Delta H\|_{L^2}^2 + 1 \right).
\end{aligned}
\tag{3.32}
$$

Substituting (3.28), (3.29), (3.31) and (3.32) into (3.27) yields

$$
\begin{aligned}
&\|\rho^{1/2} \dot{u}\|_{L^2}^2 + \frac{d}{dt} \left( \frac{\mu}{2} \left( \|\nabla u\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 \right) + \frac{\mu + \lambda}{2} \left( \|\mathrm{div} u\|_{L^2}^2 + \|\mathrm{div} v\|_{L^2}^2 \right) - A(t) \right) \\
&\leq C_2 \left( \|\nabla \theta\|_{L^2}^2 + \|H_t\|_{L^2}^2 + \|\Delta H\|_{L^2}^2 \right) + C \eta \|\rho^{1/2} \dot{u}\|_{L^2}^2 \\
&\quad + C \left( \|\nabla u\|_{L^2}^2 + \|u\|_{L^\infty}^2 + 1 \right) \left( \|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2 + \|\rho^{1/2} \theta\|_{L^2}^2 + 1 \right),
\end{aligned}
\tag{3.33}
$$

where

$$
A(t) \triangleq \frac{1}{2} \int |H|^2 \mathrm{div} u dx + \int P \mathrm{div} u dx - \int H \cdot \nabla u \cdot H dx,
\tag{3.34}
$$

satisfies

$$
A(t) \leq \frac{\mu}{4} \|\nabla u\|_{L^2}^2 + C_3 \left( \|\rho^{1/2} \theta\|_{L^2}^2 + \|\mathrm{curl} H\|_{L^2}^2 + 1 \right).
\tag{3.35}
$$

Recalling the inequality (3.26), let

$$
C_4 = \min\{2\nu^{-2}(C_1 + 1), \frac{\kappa}{2}, 1\}, \quad C_5 = \min\{2\nu^{-1}(C_1 + 1), c_v\},
\tag{3.36}
$$

adding (3.26) multiplied by $C_6 = \max\{C_4^{-1}(C_2 + 1), C_5^{-1}(C_3 + 1)\}$ into (3.33) and choosing $\eta$ suitably small, we have

$$
\begin{aligned}
&\frac{d}{dt}\tilde{A}(t) + \frac{1}{2}\|\rho^{1/2}\dot{u}\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2 + \|H_t\|_{L^2}^2 + \|\Delta H\|_{L^2}^2 \\
&\leq C\left(\|\rho^{1/2}\theta\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2 + 1\right)\left(\|\nabla u\|_{L^2}^2 + \|u\|_{L^\infty}^2 + \|\operatorname{div}u\|_{L^\infty} + 1\right),
\end{aligned}
\tag{3.37}
$$

where

$$
\begin{aligned}
\tilde{A}(t) &\triangleq C_6\left(c_v\|\rho^{1/2}\theta\|_{L^2}^2 + 2\nu^{-1}(C_1 + 1)\|\operatorname{curl}H\|_{L^2}^2\right) \\
&\quad + \frac{\mu}{2}(\|\nabla u\|_{L^2}^2 + \|\nabla v\|_{L^2}^2) + \frac{\mu + \lambda}{2}(\|\operatorname{div}u\|_{L^2}^2 + \|\operatorname{div}v\|_{L^2}^2) - A(t),
\end{aligned}
\tag{3.38}
$$

satisfies

$$
\begin{aligned}
&\|\rho^{1/2}\theta\|_{L^2}^2 + \frac{\mu}{4}\|\nabla u\|_{L^2}^2 + \|\operatorname{curl}H\|_{L^2}^2 - C \\
&\leq \tilde{A}(t) \leq C\|\rho^{1/2}\theta\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2 + C\|\operatorname{curl}H\|_{L^2}^2 + C.
\end{aligned}
\tag{3.39}
$$

Finally, integrating (3.37) over $(\tau, t)$, along with (3.39) yields

$$
\psi(t) \leq C\int_\tau^t \left(\|\nabla u\|_{L^2}^2 + \|u\|_{L^\infty}^2 + \|\operatorname{div}u\|_{L^\infty} + 1\right)\psi(s)ds + C\psi(\tau),
\tag{3.40}
$$

where

$$
\begin{aligned}
\psi(t) &\triangleq \int_0^t \left(\|\rho^{1/2}\dot{u}\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2 + \|H_t\|_{L^2}^2 + \|\Delta H\|_{L^2}^2\right)ds \\
&\quad + \|\rho^{1/2}\theta\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\operatorname{curl}H\|_{L^2}^2 + 1.
\end{aligned}
\tag{3.41}
$$

Combined with (3.1), (3.3) and Gronwall inequality implies that for any $0 < \tau \leq t < T^*$,

$$
\begin{aligned}
\psi(t) &\leq C\psi(\tau)\exp\left\{\int_\tau^t \left(\|\nabla u\|_{L^2}^2 + \|u\|_{L^\infty}^2 + \|\operatorname{div}u\|_{L^\infty} + 1\right)ds\right\} \\
&\leq C\psi(\tau)\exp\left\{\int_\tau^t \|u\|_{L^\infty}^2 ds\right\}.
\end{aligned}
\tag{3.42}
$$

Utilizing Lemma 2.5, we have

$$
\|u\|_{L^2(\tau,t;L^\infty)}^2 \leq C\|u\|_{L^2(\tau,t;H^1)}^2 \ln\left(e + \|u\|_{L^2(\tau,t;W^{1,b})}\right) + C.
\tag{3.43}
$$

Combining (3.1), (3.2), (3.11), (3.22), (3.23) and Sobolev inequality leads to

$$
\begin{aligned}
\|u\|_{W^{1,b}} &\leq \|v\|_{W^{1,b}} + C\|w\|_{W^{2,2b/(b+2)}} \\
&\leq C\|\rho\dot{u}\|_{L^{2b/(b+2)}} + C\|\rho\theta\|_{L^b} + C\|u\|_{L^2} + C\|H\|\nabla H\|\|_{L^{2b/(b+2)}} \\
&\leq C\|\rho^{1/2}\|_{L^b}\|\rho^{1/2}\dot{u}\|_{L^2} + C\|\nabla\theta\|_{L^2} + C\|\nabla u\|_{L^2} + C\|H\|_{L^b}\|\nabla H\|_{L^2} + C \\
&\leq C\|\rho^{1/2}\dot{u}\|_{L^2} + C\|\nabla\theta\|_{L^2} + C\|\nabla u\|_{L^2} + C\|\nabla H\|_{L^2} + C,
\end{aligned}
\tag{3.44}
$$

this implies that

$$
\|u\|_{L^2(\tau,t;W^{1,b})} \leq C\psi^{1/2}(t).
\tag{3.45}
$$

Substituting (3.45) into (3.43) indicates

$$\|u\|^2_{L^2(\tau,t;L^\infty)} \le C + C\|u\|^2_{L^2(\tau,t;H^1)} \ln(C\psi(t)) \le C + \ln(C\psi(t))^{C_7\|u\|^2_{L^2(\tau,t;H^1)}}. \tag{3.46}$$

Using (3.3), one can choose some $\tau$ which is close enough to $t$ such that

$$C_7\|u\|^2_{L^2(\tau,t;H^1)} \le \frac{1}{2}, \tag{3.47}$$

which together with (3.42) and (3.46) yields

$$\psi(t) \le C\psi^2(\tau) \le C. \tag{3.48}$$

Noticing the definition of $\psi$ in (3.41), we immediately have (3.8). The proof of Lemma 3.3 is completed. □

Now, we show some higher order estimates of the solutions which are needed to guarantee the extension of local solution to be a global one under the conditions (1.12)–(1.14) and (3.1).

**Lemma 3.4.** *Under the assumptions of Theorem 1.2 and* (3.1)*, it holds that for any* $t \in [0, T^*)$,

$$\sup_{0 \le s \le t} (\|\rho\|_{W^{1,q}} + \|\theta\|_{H^2} + \|u\|_{H^2} + \|H\|_{H^2}) \le C. \tag{3.49}$$

*Proof.* First , it follows from (3.8), Gagliardo-Nirenberg and Poincaré inequalities that for $2 \le q < \infty$,

$$\|u\|_{L^q} + \|H\|_{L^q} \le C. \tag{3.50}$$

Combining $(1.1)_4$, (3.3), (3.8) and (3.18) yields

$$\|H\|_{H^2} + \|\nabla H\|^2_{L^4} \le C\|\nabla u\|_{L^4} + C\|H_t\|_{L^2} + C. \tag{3.51}$$

Furthermore, it can be deduced from (3.8), (3.24), (3.50) and (3.51) that

$$\|\nabla u\|_{L^4} \le C\|\rho^{1/2}\dot{u}\|_{L^2} + C\|\nabla\theta\|_{L^2} + C\|H_t\|_{L^2} + C. \tag{3.52}$$

Then, according to (3.11) and Sobolev inequality, we get

$$\|\theta\|^2_{L^\infty} \le \varepsilon\|\nabla^2\theta\|^2_{L^2} + C\|\nabla\theta\|^2_{L^2} + C, \tag{3.53}$$

which combined with $(1.1)_3$ , (3.8), and choosing $\varepsilon$ suitably small yield

$$\|\theta\|^2_{H^2} \le C\|\rho^{1/2}\dot{\theta}\|^2_{L^2} + C\|\nabla\theta\|^2_{L^2} + C\|\nabla u\|^4_{L^4} + C\|\nabla H\|^4_{L^4} + C. \tag{3.54}$$

Therefore, the combination of (3.51) and (3.52) yields

$$\sup_{0 \le s \le t} (\|\theta\|_{L^r} + \|\nabla\theta\|_{L^2} + \|\nabla u\|_{L^4} + \|H\|_{H^2} + \|\nabla H\|_{L^4}) \le C, \quad \forall\, r \ge 1. \tag{3.55}$$

Together with (3.53) and (3.54) gives

$$\sup_{0 \le s \le t} (\|\theta\|_{H^2} + \|\theta\|_{L^\infty}) \le C. \tag{3.56}$$

Now, we bound $\|\nabla\rho\|_{W^{1,q}}$ and $\|u\|_{H^2}$. For $r \in [2, q]$, it holds that

$$
\begin{aligned}
\frac{d}{dt}\|\nabla\rho\|_{L^r} &\leq C\|\nabla\rho\|_{L^r}(\|\nabla u\|_{L^\infty} + 1) + C\|\nabla^2 u\|_{L^r} \\
&\leq C\|\nabla\rho\|_{L^r}(\|\nabla v\|_{L^\infty} + \|\nabla w\|_{L^\infty} + 1) + C\|\nabla^2 v\|_{L^r} + C\|\nabla^2 w\|_{L^r} \\
&\leq C\|\nabla\rho\|_{L^r}(\|\nabla v\|_{L^\infty} + \|\nabla w\|_{L^\infty} + 1) + C\|\nabla^2 w\|_{L^r} + C,
\end{aligned}
\tag{3.57}
$$

where in the last inequality one has applied the following fact

$$
\|\nabla^2 v\|_{L^r} \leq C\|\nabla\rho\|_{L^r} + C.
\tag{3.58}
$$

Taking (3.2), (3.56), (3.58) and Lemmas 2.2–2.4, we get

$$
\|\nabla v\|_{L^\infty} \leq C \ln(e + \|\nabla\rho\|_{L^r}) + C.
\tag{3.59}
$$

Putting (3.59) into (3.57), it can be deduced from Gronwall inequality that

$$
\|\nabla\rho\|_{L^r} \leq C.
\tag{3.60}
$$

Finally, let $r = 2$ in (3.60), according to Lemma 2.2, (3.50), (3.55) and (3.58) yields

$$
\|u\|_{H^2} \leq C.
\tag{3.61}
$$

Therefore, together with (3.55), (3.56), (3.60) and (3.61), we get (3.49). The proof of Lemma 3.4 is completed. $\qquad\square$

## 4. Proof of Theorem 1.2

With the priori estimates in Lemmas 3.1–3.4, we can prove Theorem 1.2.

*Proof of Theorem 1.2.* Assume that (1.16) is false, namely, (3.1) holds. Notice that the general constant $C$ in Lemmas 3.1–3.4 is independent of $t$, that is, all the priori estimates attained in Lemmas 3.1–3.4 are uniformly bounded for any $t \leq T^*$. Therefore, the function

$$
(\rho, \theta, u, H)(x, T^*) \triangleq \lim_{t\to T^*}(\rho, \theta, u, H)(x, t)
$$

satisfies the initial conditions (1.12) at $t = T^*$.

Due to

$$
(\rho\dot{u}, \rho\dot{\theta})(x, T^*) = \lim_{t\to T^*}(\rho\dot{u}, \rho\dot{\theta}) \in L^2,
$$

therefore

$$
-\mu\triangle u - (\mu + \lambda)\nabla\mathrm{div}u + R\nabla(\rho\theta) - H \cdot \nabla H + \frac{1}{2}\nabla|H|^2|_{t=T^*} = \rho^{1/2}(x, T^*)g_1(x),
$$

$$
-\kappa\triangle\theta - 2\mu|\mathfrak{D}(u)|^2 - \lambda(\mathrm{div}u)^2 - \nu(\mathrm{curl}H)^2|_{t=T^*} = \rho^{1/2}(x, T^*)g_2(x),
$$

with

$$
g_1(x) \triangleq \begin{cases} \rho^{-1/2}(x, T^*)(\rho\dot{u})(x, T^*), & \text{for } x \in \{x|\rho(x, T^*) > 0\}, \\ 0, & \text{for } x \in \{x|\rho(x, T^*) = 0\}, \end{cases}
$$

and

$$g_2(x) \triangleq \begin{cases} \rho^{-1/2}(x, T^*)(c_v\rho\dot{\theta} + R\theta\rho\text{div}u)(x, T^*), & \text{for } x \in \{x|\rho(x, T^*) > 0\}, \\ 0, & \text{for } x \in \{x|\rho(x, T^*) = 0\}, \end{cases}$$

satisfying $g_1, \ g_2 \in L^2$. Thus, $(\rho, \theta, u, H)(x, T^*)$ also satisfies (1.13) and (1.14).

Hence, Theorem 1.1 shows that we could extend the local strong solutions beyond $T^*$, while taking $(\rho, \theta, u, H)(x, T^*)$ as the initial data. This contradicts the hypothesis of Theorem 1.2 that $T^*$ is the maximum existence time of the strong solution. This completes the proof of theorem 1.2. □

## 5. Conclusions

This paper concerns the blow-up criterion for the initial boundary value problem of the two-dimensional full compressible magnetohydrodynamic equations in the Eulerian coordinates. When the initial density allowed to vanish, and the magnetic field $H$ satisfies the perfect conducting boundary condition $H \cdot n = \text{curl}H = 0$, we prove the blow-up criterion $\lim_{t \to T^*} (\|H\|_{L^\infty(0,t;L^b)} + \|\text{div}u\|_{L^1(0,t;L^\infty)}) = \infty$ for any $b > 2$, which depending on both $H$ and div$u$.

## Acknowledgments

## Conflict of interest

The author declares no conflict of interest in this paper.

## References

1. P. A. Davidson, *An introduction to magnetohydrodynamics*, Cambridge: Cambridge University Press, 2001.

2. B. Ducomet, E. Feireisl, The equations of magnetohydrodynamics: on the interaction between matter and radiation in the evolution of gaseous stars, *Commun. Math. Phys.*, **266** (2006), 595–629. https://doi.org/10.1007/s00220-006-0052-y

3. C. S. Dou, S. Jiang, Q. C. Ju, Global existence and the low Mach number limit for the compressible magnetohydrodynamic equations in a bounded domain with perfectly conducting boundary, *Z. Angew. Math. Phys.*, **64** (2013), 1661–1678. https://doi.org/10.1007/s00033-013-0311-7

4. J. F. Gerbeau, C. Le Bris, T. Leliévre, *Mathematical methods for the magnetohydrodynamics of Liquid metals, numerical mathematics and scientific computation*, Oxford: Oxford University Press, 2006.

5. J. S. Fan, W. H. Yu, Strong solution to the compressible magnetohydrodynamic equations with vacuum, *Nonlinear Anal. Real World Appl.*, **10** (2009), 392–409. https://doi.org/10.1016/j.nonrwa.2007.10.001

6. B. Q. Lü, B. Huang, On strong solutions to the Cauchy problem of the two-dimensional compressible magnetohydrodynamic equations with vacuum, *Nonlinearity*, **28** (2015), 509–530. https://doi.org/10.1088/0951-7715/28/2/509

7. H. L. Li, X. Y. Xu, J. W. Zhang, Global classical solutions to 3D compressible magnetohydrodynamic equations with large oscillations and vacuum, *SIAM J. Math. Anal.*, **45** (2013), 1356–1387. https://doi.org/10.1137/120893355

8. B. Q. Lü, X. D. Shi, X. Y. Xu, Global well-posedness and large time asymptotic behavior of strong solutions to the compressible magnetohydrodynamic equations with vacuum, *Indiana Univ. Math. J.*, **65** (2016), 725–975.

9. X. P. Hu, D. H. Wang, Global solutions to the three-dimensional full compressible magnetohydrodynamic flows, *Commun. Math. Phys.*, **283** (2008), 255–284. https://doi.org/10.1007/s00220-008-0497-2

10. X. P. Hu, D. H. Wang, Global existence and large-time behavior of solutions to the threedimensional equations of compressible magnetohydrodynamic flows, *Arch. Ration. Mech. Anal.*, **197** (2010), 203–238. https://doi.org/10.1007/s00205-010-0295-9

11. J. S. Fan, W. H. Yu, Global variational solutions to the compressible magnetohydrodynamic equations, *Nonlinear Anal.*, **69** (2008), 3637–3660. https://doi.org/10.1016/j.na.2007.10.005

12. J. Serrin, On the interior regularity of weak solutions of the Navier-Stokes equations, *Arch. Rational Mech. Anal.*, **9** (1962), 187–195.

13. C. He, Z. P. Xin , On the regularity of solutions to the magnetohydrodynamic equations, *J. Differ. Equ.*, **213** (2005), 235–254. https://doi.org/10.1016/j.jde.2004.07.002

14. X. D. Huang, J. Li, Z. P. Xin, Serrin type criterion for the three-dimensional compressible flows, *SIAM J. Math. Anal.*, **43** (2011), 1872–1886. https://doi.org/10.1137/100814639

15. J. S. Fan, S. Jiang, Y. B. Ou, A blow-up criterion for compressible viscous heat-conductive flows, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **27** (2010), 337–350. https://doi.org/10.1016/j.anihpc.2009.09.012

16. X. D. Huang, J. Li, Z. P. Xin, Blowup criterion for viscous barotropic flows with vacuum states, *Comm. Math. Phys.*, **301** (2011), 23–35. https://doi.org/10.1007/s00220-010-1148-y

17. Y. Wang, One new blowup criterion for the 2D full compressible Navier-Stokes system, *Nonlinear Anal. Real World Appl.*, **16** (2014), 214–226. https://doi.org/10.1016/j.nonrwa.2013.09.020

18. X. Y. Xu, J. W. Zhang, A blow-up criterion for 3D compressible magnetohydrodynamic equations with vaccum, *Math. Models Method. Appl. Sci.*, **22** (2012), 1150010. https://doi.org/10.1142/S0218202511500102

19. M. Lu, Y. Du, Z. A. Yao, Blow-up criterion for compressible MHD equations, *J. Math. Anal. Appl.*, **379** (2011), 425–438. https://doi.org/10.1016/j.jmaa.2011.01.043

20. M. T. Chen, S. Q. Liu, Blow-up criterion for 3D viscous-resistive compressible magnetohydrodynamic equations, *Math. Method. Appl. Sci.*, **36** (2013), 1145–1156. https://doi.org/10.1002/mma.2674

21. X. D. Huang, J. Li, Serrin-type blowup criterion for viscous, compressible, and heat conducting Navier-Stokes and magnetohydrodynamic flows, *Comm. Math. Phys.*, **324** (2013), 147–171. https://doi.org/10.1007/s00220-013-1791-1

22. J. S. Fan, F. C. Li , G. Nakamura, A blow-up criterion to the 2D full compressible magnetohydrodynamic equations, *Math. Meth. Appl. Sci.*, **38** (2015), 2073–2080.

23. Y. Cho, H. Kim, Existence results for viscous polytropic fluids with vacuum, *J. Differ. Equ.*, **228** (2006), 377–411. https://doi.org/10.1016/j.jde.2006.05.001

24. Y. Z. Sun, C. Wang, Z. F. Zhang, A Beale-Kato-Majda Blow-up criterion for the 3-D compressible Navier-Stokes equations, *J. Math. Pures Appl.*, **95** (2011), 36–47. https://doi.org/10.1016/j.matpur.2010.08.001

25. E. Feireisl, *Dynamics of viscous compressible fluids*, Oxford: Oxford Science Publication, 2004.

26. H. Y. Wen, C. J. Zhu, Blow-up criterions of strong solutions to 3D compressible Navier-Stokes equations with vacuum, *Adv. Math.*, **248** (2013), 534–572. https://doi.org/10.1016/j.aim.2013.07.018

27. L. Nirenberg, On elliptic partial differential equations, *Ann. Scuola Norm. Sup. Pisa*, **13** (1959), 115–162.

28. Y. Cho, H. J. Choe, H. Kim, Unique solvability of the initial boundary value problems for compressible viscous fluids, *J. Math. Pures Appl.*, **83** (2004), 243–275. https://doi.org/10.1016/j.matpur.2003.11.004

29. S. Agmon, A. Douglis, L. Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions I, *Comm. Pure Appl. Math.*, **12** (1959), 623–727.

30. S. Agmon, A. Douglis, L. Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions II, *Comm. Pure Appl. Math.*, **17** (1964), 35–92. https://doi.org/10.1002/cpa.3160170104

31. H. Brezis, T. Gallouet, Nonlinear Schrödinger evolution equations, *Nonlinear Anal. TMA*, **4** (1980), 677–681.

32. H. Brezis , S. Wainger, A note on limiting cases of Sobolev embedding and convolution inequalities, *Commun. Part. Diff. Eq.*, **5** (1980), 773–789.

33. X. D. Huang, J. Li, Y. Wang, Serrin-type blowup criterion for full compressible Navier-Stokes system, *Arch. Ration. Mech. Anal.*, **207** (2013), 303–316. https://doi.org/10.1007/s00205-012-0577-5