Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(7): 15773–15788.
DOI: 10.3934/math.2023805
Received: 03 February 2023
Revised: 31 March 2023
Accepted: 10 April 2023
Published: 28 April 2023

Research article

Cauchy problem for fractional (p, q)-difference equations

Abdelatif Boutiara[1], **Mohamed Rhaima**[2], **Lassaad Mchiri**[3] **and Abdellatif Ben Makhlouf**[4,*]

[1] Department of Mathematics and Computer Science, University of Ghardaia, BP 455 Ghardaia 47000, Algeria

[2] Department of Statistics and Operations Research, College of Sciences, King Saud University, Riyadh 11451, Saudi Arabia

[3] ENSIIE, University of Evry-Val-d'Essonne, 1 square de la Résistance 91025 Évry-Courcouronnes cedex, France

[4] Department of Mathematics, Faculty of Sciences of Sfax, University of Sfax, Tunisia

* **Correspondence:** Email: abdellatif.benmakhlouf@fss.usf.tn.

Abstract: In this research article, we deal with the global convergence of successive approximations (s.a) as well as the existence of solutions to a fractional (p, q)-difference equation. Then, we discuss the existence result of the solutions of Caputo-type (p, q)-difference fractional vector-order equations in a Banach space. Also, we prove a theorem on the global convergence of successive approximations to the unique solution of our problem. Finally, the application of the main results is demonstrated by presenting numerical examples.

Keywords: fractional (p, q)-calculus; global convergence; successive approximations; measure of non-compactness; Meir-Keeler condensing operators
Mathematics Subject Classification: 34A08

1. Introduction

In the literature, we often find the concept of non-integer (fractional) order derivation, which is the generalization of the classical derivation. The concept of fractional derivation and fractional integration are often associated with the names of Riemann-Liouville, while the question of the generalization of these concepts is older. With particular concentration of physicists and engineers, remarkable research work has been devoted to the theory of fractional calculus. They assured that the use of the operators of fractional derivations and fractional integrations is desirable for the description of memory and hereditary properties of various materials and processes. Indeed, it has been found that several theoretical and experimental studies show that certain thermal (heat diffusion) (see [29]), physical (electricity) (see [22, 26]) and rheological (viscoelasticity) phenomena (see [30, 32]) are subject to fractional equations.

On the other hand, the concept of the stability of a fractional differential equation appears when we replaces this equation with an inequality that acts as a perturbation of the equation. Thus, the question for the stability of fractional differential equations is how do the solutions of the inequality differ from those of the given fractional differential equation. Considerable work has been given to the study of the qualitative theory of all kinds of fractional differential equations, and for more details on this information, see [2, 3, 14, 17, 23, 24, 26, 32, 33].

Quantum calculus is the notion of calculus without limits. Quantum calculus was first defined by Jackson (see [19, 20]) in 1910. A generalization of quantum calculus, the (p, q)-calculus or post-quantum calculus, was defined by Chakrabarti and Jagannathan (see [12]). (p, q)-calculus is an extension of q calculus including two independent quantum parameters p and q, equal to q-calculus for the case $p = 1$ and to the classical q-calculus when q goes to 1. Moreover, (p, q)-calculus has many real world applications, such as mechanics, surfaces, physical sciences, etc. (see [5–11] and [13, 15, 16, 21, 27, 28, 31]). In the last decades, the (p, q)-calculus has attracted the attention of many researchers (see [5–11]).

Motivated by the above works in the literature, the aim of the current paper is investigating the solutions of a fractional vector-order difference equation with Caputo fractional (p, q)-difference operator in the Banach space, namely,

$$\begin{cases} {}^{c}D^{\iota}_{p,q}u(\xi) = \mathcal{F}\left(p^{\iota}\xi, u\left(p^{\iota}\xi\right)\right), & \xi \in I^{T}_{p,q}, \\ u(0) = u_0, & u_0 \in \mathbb{E}^{\nu}, \end{cases} \tag{1.1}$$

where $I^{T}_{p,q} := \left\{ \left(\frac{q}{p}\right)^{k} \frac{T}{p} : k \in \mathbb{N}_0 \right\} \cup \{0\}$, $u_a \in \mathbb{E}^{\nu}$, $u_0 = (u_{0,1}, u_{0,2}, \ldots, u_{0,\nu})^{T}$, $u_{0,j} \in \mathbb{E}$, $j = 1, 2, \ldots, \nu$, the parameter $p, q \in (0, 1]$, ${}^{c}D^{\iota}_{p,q}$ denotes the Caputo-type fractional (p, q)-difference of the vector-order $\iota = (\iota_1, \ldots, \iota_\nu)^{T}$, $0 < \iota_j < 1$, $\mathcal{F} : I^{T}_{p,q} \times \mathbb{E}^{\nu} \to \mathbb{E}^{\nu}$ is a given vector-valued function (v.v.f), and the unknown v.v.f $u : I^{T}_{p,q} \to \mathbb{E}^{\nu}$ is continuous on $I^{T}_{p,q}$, where $u(\xi) = (u_1(\xi), \ldots, u_\nu(\xi))^{T}$.

In this paper, let $\mathbb{E}$ be a Banach space with the norm $|\cdot|$ and let $\mathbb{E}^{\nu}$ be ν-dimensional Banach space with the norm $\|u\| = \max_{j=1,2,\ldots,\nu} |u_j|, \quad \nu \in \mathbb{N}$, for every $u \in \mathbb{E}^{\nu}$, $u = (u_1, u_2, \ldots, u_\nu)^{T}$, $u_j \in \mathbb{E}$, $j = 1, 2, \ldots, \nu$. Denote by $C(I^{T}_{p,q}, \mathbb{E}^{\nu})$ the Banach space of continuous functions from $I^{T}_{p,q}$ to $\mathbb{E}^{\nu}$ with the norm $\|u\|_\infty = \sup_{\xi \in I^{T}_{p,q}} \|u(\xi)\|$. By $L^1(I^{T}_{p,q}, \mathbb{E}^{\nu})$, we indicate the space of Bochner integrable functions from $I^{T}_{p,q}$ to $\mathbb{E}^{\nu}$ with the norm $\|u\|_{L^1} = \int_{I^{T}_{p,q}} \|u(\xi)\| \, d\xi$.

The paper is organized as follows: Section 2 is devoted to some preliminary notions. In Section 3, we present our main results about the global convergence of s.a and the existence and the uniqueness of solutions for a fractional (p, q)-difference equation. In Section 4, two examples are given to demonstrate the theoretical results. Finally, in Section 5, we wrap up this paper by a concluding remark.

2. Preliminaries

Definition 2.1. *[34]* For $\iota > 0, 0 < q < p \leq 1$ and f defined on $I_{p,q}^{T}$ the fractional (p,q)-integral is defined by

$$\begin{aligned}\mathcal{I}_{p,q}^{\iota} f(\xi) &:= \frac{1}{p^{\binom{\iota}{2}}\Gamma_{p,q}(\iota)} \int_0^{\xi} (\xi - qs)_{p,q}^{\iota-1} f\left(\frac{s}{p^{\iota-1}}\right) d_{p,q}s \\ &= \frac{(p-q)\xi}{p^{\binom{\iota}{2}}\Gamma_{p,q}(\iota)} \sum_{k=0}^{\infty} \frac{q^k}{p^{k+1}} \left(\xi - \left(\frac{q}{p}\right)^{k+1} \xi\right)_{p,q}^{\iota-1} f\left(\frac{q^k}{p^{k+\iota}}\xi\right),\end{aligned} \tag{2.1}$$

and $\left(\mathcal{I}_{p,q}^{0} f\right)(\xi) = f(\xi)$, where $\binom{\iota}{2}$ is a combination.

Remark 2.1. Note that all integrals are taken in the Bochner sense.

Definition 2.2. *[34]* Let the continuous function f be defined on $I_{p,q}^{T}$. Thus, the Riemann-Liouville fractional (p,q)-difference is defined by

$$\left(D_{p,q}^{\iota} f\right)(\xi) = \left(D_{p,q}^{\lceil\iota\rceil} \mathcal{I}_{p,q}^{\lceil\iota\rceil-\iota} f\right)(\xi), \text{ for } \iota > 0, \tag{2.2}$$

with the smallest integer greater than or equal to ι given by $\lceil\iota\rceil$. Note that if $\iota = 0$, then $\left(D_{p,q}^{0} f\right)(\xi) = f(\xi)$.

Definition 2.3. *[34]* Let f be a continuous function defined on $I_{p,q}^{T}$. If $\iota > 0$, then the Caputo fractional (p,q)-difference is stated by

$$\left({}^{c}D_{p,q}^{a} f\right)(\xi) = \left(\mathcal{I}_{p,q}^{\lceil\iota\rceil-\iota} D_{p,q}^{\lceil\iota\rceil} f\right)(\xi). \tag{2.3}$$

Note that if $\iota = 0$, then $\left({}^{c}D_{p,q}^{0} f\right)(\xi) = f(\xi)$.

Lemma 2.1. *[34]* Let $p, q \in (0,1]$, $0 < \iota < 1$ and $\gamma > 0$. Thus, we have:

$$\left(\mathcal{I}_{p,q}^{\iota}\, \mathcal{I}_{p,q}^{\gamma} u\right)(\xi) = \left(\mathcal{I}_{p,q}^{\gamma}\, \mathcal{I}_{p,q}^{\iota} u\right)(\xi) = \left(\mathcal{I}_{p,q}^{\iota+\gamma,\rho} u\right)(\xi) \text{ with } u \in L^1(I_{p,q}^{T}, \mathbb{E}^{\nu}); \tag{2.4}$$

$$\left({}^{C}\mathcal{D}_{p,q}^{\iota}\, \mathcal{I}_{p,q}^{\iota} u\right)(\xi) = u(\xi) \text{ with } u \in L^1(I_{p,q}^{T}, \mathbb{E}^{\nu}); \tag{2.5}$$

$$\left(\mathcal{I}_{p,q}^{\iota}\, {}^{C}\mathcal{D}_{p,q}^{\iota} u\right)(\xi) = u(\xi) - c, \qquad \text{with } u \in C^1(I_{p,q}^{T}, \mathbb{E}^{\nu}). \tag{2.6}$$

Remark 2.2. Throughout this paper, all operators related to ι or $u \in \mathbb{E}^{\nu}$ are element-wise.

Definition 2.4. Let $u = (u_1, \cdots, u_\nu)^T$, $u \in L^1(I_{p,q}^{T}, \mathbb{E}^{\nu})$, be a v.v.f and $\iota = (\iota_1, \cdots, \iota_\nu)^T$, with $\nu \in \mathbb{N}$, $\iota_j \in \mathbb{R}$ for $j = 1, \cdots, \nu$. The (p,q)-difference Riemann-Liouville vector-order fractional (v.o.f) (p,q)-integral of vector order ι is given by

$$\mathcal{I}_{p,q}^{\iota} u(\xi) = \left(\mathcal{I}_{p,q}^{\iota_1} u_1(\xi), \cdots, \mathcal{I}_{p,q}^{\iota_\nu} u_\nu(\xi)\right)^T.$$

Definition 2.5. Let $u = (u_1, \cdots, u_\nu)^T$, $u \in C^1(I_{p,q}^{T}, \mathbb{E}^{\nu})$, be a vector-valued function and $\iota = (\iota_1, \cdots, \iota_\nu)^T$, with $\nu \in \mathbb{N}$, $\iota_j \in \mathbb{R}$ for $j = 1, \cdots, \nu$. The (p,q)-difference Caputo v.o.f (p,q)-difference of vector order ι is given by

$${}^{c}D_{p,q}^{\iota} u(\xi) = \left({}^{c}D_{p,q}^{\iota_1} u_1(\xi), \cdots, {}^{c}D_{p,q}^{\iota_\nu} u_\nu(\xi)\right)^T.$$

2.1. Measure of non-compactness

In this section, we will exhibit some important properties of the Hausdorff measure of noncompactness (HMNC).

Definition 2.6. *[4]* Let V be a bounded subset of a Banach space $\mathbb{E}$. We define the Hausdorff measure of non-compactness of V as:

$$\kappa(V) = \inf\{\epsilon > 0 : V \text{ can be covered by finitely many balls with radius} < \epsilon\}.$$

Lemma 2.2. *[4]* Let $Y, W \subset \mathbb{E}$ be bounded, and the HMNC possesses the following properties:

i. $\kappa(Y) = 0 \Leftrightarrow Y$ is relatively compact;

ii. $Y \subset W \Rightarrow \kappa(Y) \leq \kappa(W)$;

iii. $\kappa(Y \cup W) = \max\{\kappa(Y), \kappa(W)\}$;

iv. $\kappa(Y) = \kappa(\overline{Y}) = \kappa(conv(Y))$, where $\overline{Y}$ and $conv(Y)$ are the closure and the convex hull of Y, respectively;

v. $\kappa(Y + W) \leq \kappa(Y) + \kappa(W)$, where $Y + W = \{u + v : u \in Y, \ v \in W\}$;

vi. $\kappa(\mu Y) \leq |\mu|\kappa(Y)$, for any $\mu \in \mathbb{R}$.

Definition 2.7. *[25]* Let $d(X, d)$ be a metric space. A mapping $\mathcal{J}$ on X is called a Meir-Keeler contraction if, for any $\epsilon > 0$, there exists $\sigma > 0$ in a way that $\epsilon \leq d(u, v) < \epsilon + \sigma$ implies that $d(\mathcal{J}u, \mathcal{J}v) < \epsilon, \ \ \forall u, v \in X$.

Definition 2.8. *[1]* Let U be a non-empty subset of a Banach space $\mathbb{E}$ and let κ be a measure of non-compactness on $\mathbb{E}$. An operator $\mathcal{J} : U \to U$ is called a Meir-Keeler condensing operator if, for any $\epsilon > 0$, there exists $\sigma > 0$ in a way that $\epsilon \leq \kappa(V) < \epsilon + \sigma$ implies that $\kappa(\mathcal{J}(V)) < \epsilon$, for any bounded V of U.

Theorem 2.1. *[1]* Let U_θ be a non-empty, convex, bounded and closed subset of a Banach space $\mathbb{E}$ and let κ be a measure of non-compactness on $\mathbb{E}$. If $\mathcal{J} : U_\theta \to U_\theta$ is a continuous Meir-Keeler condensing operator, then $\mathcal{J}$ has at least one fixed point, and the set of all fixed points of $\mathcal{J}$ in U_θ is compact.

To end this essential part of the paper, we recall the following lemmas.

Lemma 2.3. *[2]* Let $\mathbb{E}$ be a Banach space, and $V \subset C(I_{p,q}^T, \mathbb{E})$ be bounded and equicontinuous. Then, $\kappa(V(\xi))$ is continuous on $I_{p,q}^T$, and $\kappa_C(V) = \max_{\xi \in I_{p,q}^T} \kappa(V(\xi))$.

Lemma 2.4. *[18]* Let $\mathbb{E}$ be a Banach space and $V \subset \mathbb{E}$ be bounded. Then, for each $\epsilon > 0$, there is a sequence $\{u_n\}_{n=1}^{\infty} \subset V$ in a way that $\kappa(V) \leq 2\kappa\left(\{u_n\}_{n=1}^{\infty}\right) + \epsilon$.

Remark 2.3. We call $V \subset L^1(I_{p,q}^T, \mathbb{E})$ uniformly integrable if there exists $f \in L^1(I_{p,q}^T, \mathbb{R}^+)$ in a way that $\|u(s)\| \leq f(s), \ \ \forall u \in V,$ and $a.e. \ s \in I_{p,q}^T$.

Lemma 2.5. *[18]* If $\{u_n\}_{n=1}^{\infty} \subset L^1(I_{p,q}^T, \mathbb{E})$ is uniformly integrable, then $\xi \mapsto \kappa\left(\left\{u_n(\xi)\right\}_{n=1}^{\infty}\right)$ is measurable, and

$$\kappa\left(\left\{\int_a^\xi u_n(\tau)\, d\tau\right\}_{n=1}^{\infty}\right) \leq 2\int_a^\xi \kappa(\{u_n(\tau)\}_{n=1}^{\infty})\, d\tau.$$

3. Main results

Definition 3.1. A function $u \in C(I^T_{p,q}, \mathbb{E}^\nu)$ is called a solution of the Cauchy problem (1.1) if u satisfies ${}^cD^\iota_{p,q}u(\xi) = \mathcal{F}(p^\iota\xi, u(p^\iota\xi))$ for a.e. $\xi \in I^T_{p,q}$ and the initial condition $u(0) = u_0$.

The next lemma is crucial to the forthcoming discussions.

Lemma 3.1. For any $\varphi \in C(I^T_{p,q}, \mathbb{R})$ the solution u of the linear fractional (p,q)-difference equation

$$\begin{cases} {}^cD^\iota_{p,q}u(\xi) = \varphi(\xi), \text{ for a.e. } \xi \in I^T_{p,q}, \ 0 < \iota < 1, \\ u(0) = u_0 \in \mathbb{R}, \end{cases} \tag{3.1}$$

is defined by the next equation:

$$u(\xi) = u_a + \frac{1}{p^{\binom{\iota}{2}}\Gamma_{p,q}(\iota)} \int_0^\xi (\xi - qs)^{(\iota-1)} \varphi(ps) d_{p,q}s, \quad \xi \in I^T_{p,q}. \tag{3.2}$$

Taking into account Definition 2.5 for the (p,q)-difference Caputo vector-order fractional (p,q)-difference of vector order ι, from Lemma 3.1 we obtain the following result:

Lemma 3.2. For each $\phi \in C(I^T_{p,q}, \mathbb{E}^\nu)$ the solution $u : I^T_{p,q} \to \mathbb{E}^\nu$ of the linear (p,q)-difference vector-order fractional equation

$$\begin{cases} {}^cD^\iota_{p,q}u(\xi) = \phi(\xi), \text{ for a.e. } \xi \in I^T_{p,q}, \\ u(0) = \vartheta, \quad \vartheta \in \mathbb{E}^\nu, \end{cases} \tag{3.3}$$

with $\vartheta = (\vartheta_1, \vartheta_2, \ldots, \vartheta_\nu)^T$, is defined by the next equations:

$$u_i(\xi) = \vartheta_i + \frac{1}{p^{\binom{\iota_i}{2}}\Gamma_{p,q}(\iota_i)} \int_0^\xi (\xi - qs)^{(\iota_i-1)} \phi_i(ps) d_{p,q}s, \quad \xi \in I^T_{p,q}, \ i = 1, 2, \ldots, \nu. \tag{3.4}$$

Alternatively, it can be written by the following form:

$$u(\xi) = \vartheta + \frac{1}{p^{\binom{\iota}{2}}\Gamma_{p,q}(\iota)} \int_0^\xi (\xi - qs)^{(\iota-1)} \phi(ps) d_{p,q}s, \quad \xi \in I^T_{p,q}, \tag{3.5}$$

where

$$\frac{1}{p^{\binom{\iota}{2}}\Gamma_{p,q}(\iota)} \int_0^\xi (\xi - qs)^{(\iota-1)} \phi(ps) d_{p,q}s = \Bigg(\frac{1}{p^{\binom{\iota_1}{2}}\Gamma_{p,q}(\iota_1)} \int_0^\xi (\xi - qs)^{(\iota_1-1)} \phi_1(ps) d_{p,q}s,$$
$$\frac{1}{p^{\binom{\iota_2}{2}}\Gamma_{p,q}(\iota_2)} \int_0^\xi (\xi - qs)^{(\iota_2-1)} \phi_2(ps) d_{p,q}s, \ \ldots, \ \frac{1}{p^{\binom{\iota_\nu}{2}}\Gamma_{p,q}(\iota_\nu)} \int_0^\xi (\xi - qs)^{(\iota_\nu-1)} \phi_\nu(ps) d_{p,q}s \Bigg)^T.$$

Definition 3.2. A function $u \in C(I^T_{p,q}, \mathbb{E}^\nu)$ forms a solution of the Cauchy problem (1.1) if and only if u fulfills the integral equation

$$u(\xi) = \vartheta + \frac{1}{p^{\binom{\iota}{2}}\Gamma_{p,q}(\iota)} \int_0^\xi (\xi - qs)^{(\iota-1)} \mathcal{F}(ps, u(ps)) d_{p,q}s, \quad \xi \in I^T_{p,q}. \tag{3.6}$$

Let us now present our main results.

Theorem 3.1. Suppose the following:

(A1) The function $\mathcal{F} : I_{p,q}^{T} \times \mathbb{E}^{\nu} \to \mathbb{E}^{\nu}$ satisfies Carathodory conditions.

(A2) There exists a continuous function $\omega : I_{p,q}^{T} \to \mathbb{R}^{+}$ and a nondecreasing continuous function $\varphi : \mathbb{R}^{+} \to \mathbb{R}^{+}$ such that

$$\|\mathcal{F}(\xi, u)\| \leq \omega(\xi)\varphi(\|u\|), \quad \xi \in I_{p,q}^{T},\ u \in \mathbb{E}^{\nu}.$$

(A3) For every $V \subset \mathbb{E}^{\nu}$ and each $\xi \in I_{p,q}^{T}$, we have

$$\kappa(\mathcal{F}(\xi, V)) \leq \omega(\xi)\kappa(V).$$

Then, the Cauchy problem (1.1) possesses at least one solution on $I_{p,q}^{T}$, provided that

$$\Gamma_{p,q}(\iota_j + 1) > 4\left(\frac{T}{p}\right)^{\iota_j}\omega^{*}, \tag{3.7}$$

where $\omega^{*} := \sup_{\xi \in I_{p,q}^{T}} \omega(\xi)$.

Proof. Let $\theta > 0$ be any number such that

$$\theta \geq \|u_0\| + \Theta_{\iota}\omega^{*}\varphi(\theta),$$

with $\Theta_{\iota} = \max_{j=1,2,\ldots,\nu}\left\{\frac{\left(\frac{T}{p}\right)^{\iota_j}}{\Gamma_{p,q}(\iota_j+1)}\right\}$.

Define the operator $\mathcal{J} : C(I_{p,q}^{T}, \mathbb{E}^{\nu}) \to C(I_{p,q}^{T}, \mathbb{E}^{\nu})$ by

$$(\mathcal{J}u)(\xi) = u_0 + \frac{1}{p^{\binom{\iota}{2}}\Gamma_{p,q}(\iota)}\int_0^{\xi}(\xi - qs)^{(\iota-1)}\mathcal{F}(ps, u(ps))\, d_{p,q}s, \tag{3.8}$$

with $(\mathcal{J}u)(\xi) = ((\mathcal{J}u_1)(\xi), (\mathcal{J}u_2)(\xi), \ldots, (\mathcal{J}u_{\nu})(\xi))^{T}$ where

$$(\mathcal{J}u_j)(\xi) = u_{0,j} + \frac{1}{p^{\binom{\iota_j}{2}}\Gamma_{p,q}(\iota_j)}\int_0^{\xi}(\xi - qs)^{(\iota_j-1)}\mathcal{F}\left(ps, u_j(ps)\right) d_{p,q}s. \tag{3.9}$$

According to (A1) and (A2), $\mathcal{J}$ is well-defined. Then, the existence of a mild solution of system (1.1) is equivalent to the fixed point problem of $u = \mathcal{J}u$.

Consider the set $U_{\theta} := \{u \in C(I_{p,q}^{T}, \mathbb{E}^{\nu}) : \|u\|_{\infty} \leq \theta\}$. Clearly, the set U_{θ} is a closed, convex and bounded subset of the Banach space $C(I_{p,q}^{T}, \mathbb{E}^{\nu})$. In order to prove that the assumptions of Theorem are satisfied, we will divide the proof into four claims.

Claim 1. $\mathcal{J}(U_{\theta}) \subseteq U_{\theta}$.

For each $\xi \in I_{p,q}^{T}$ and any $u \in U_{\theta}$, by (A2), for any j, one has

$$|(\mathcal{J}u_j)(\xi)| \leq |u_{0,j}| + \frac{1}{p^{\binom{\iota_j}{2}}\Gamma_{p,q}(\iota_j)}\int_0^{\xi}(\xi - qs)^{(\iota_j-1)}|\mathcal{F}\left(ps, u_j(ps)\right)|d_{p,q}s.$$

Taking the maximum over j, we get

$$\begin{aligned}
\|\mathcal{J}u(\xi)\| &\le \|u_0\| + \max_{j=1,2,\ldots,\nu}\left\{\frac{1}{p^{\binom{\iota_j}{2}}\Gamma_{p,q}(\iota_j)}\int_0^{\xi}(\xi-qs)^{(\iota_j-1)}|\mathcal{F}\left(ps,u_j(ps)\right)|d_{p,q}s\right\}\\
&\le \|u_0\| + \max_{j=1,2,\ldots,\nu}\left\{\frac{1}{p^{\binom{\iota_j}{2}}\Gamma_{p,q}(\iota_j)}\int_0^{\xi}(\xi-qs)^{(\iota_j-1)}\omega(\xi)\varphi(\|u\|)d_{p,q}s\right\}\\
&\le \|u_0\| + \max_{j=1,2,\ldots,\nu}\left\{\frac{(\frac{T}{p})^{\iota_j}}{\Gamma_{p,q}(\iota_j+1)}\right\}\omega^*\varphi(\theta)\\
&:= \|u_0\| + \Theta_\iota\omega^*\varphi(\theta)\\
&\le \theta,
\end{aligned}$$

Hence, $\|\mathcal{J}u\|_\infty \le \theta$, which implies that $\mathcal{J}(U_\theta) \subseteq U_\theta$.

Claim 2. *$\mathcal{J}$ is continuous.*

For $\xi \in I^T_{p,q}$, let $\{u_n\}$ be a sequence that converges to u in U_θ. For any j, we have

$$|(\mathcal{J}u_{n,j})(\xi) - (\mathcal{J}u_j)(\xi)| \le \frac{1}{p^{\binom{\iota_j}{2}}\Gamma_{p,q}(\iota_j)}\int_0^{\xi}(\xi-qs)^{(\iota_j-1)}|\mathcal{F}\left(ps,u_{n,j}(ps)\right)-\mathcal{F}\left(ps,u_j(ps)\right)|d_{p,q}s.$$

As in the previous claim, taking the maximum over j gives

$$\begin{aligned}
\|(\mathcal{J}u_n)(\xi) - (\mathcal{J}u)(\xi)\| &\le \max_{j=1,2,\ldots,\nu}\left\{\int_0^{\xi}\frac{(\xi-qs)^{(\iota_j-1)}}{p^{\binom{\iota_j}{2}}\Gamma_{p,q}(\iota_j)}|\mathcal{F}\left(ps,u_{n,j}(ps)\right)-\mathcal{F}\left(ps,u_j(ps)\right)|d_{p,q}s\right\}\\
&\le \Theta_\iota\|\mathcal{F}(\cdot,u_n(\cdot)) - \mathcal{F}(\cdot,u(\cdot))\|.
\end{aligned}$$

In view of (A1), using the Lebesgue dominated convergence theorem, we infer that $\|\mathcal{J}u_n - \mathcal{J}u\|_\infty \to 0$ when $n \to \infty$. Then, $\mathcal{J}$ is continuous.

Claim 3. *$\mathcal{J}$ is bounded and equicontinuous.*

From Claim 1, $\mathcal{J}(U_\theta) = \{\mathcal{J}(u) : u \in U_\theta\} \subset U_\theta$, and consequently, for $u \in U_\theta$, we have $\|\mathcal{J}u\|_\infty \le \theta$, which shows that $\mathcal{J}$ is bounded. Now, we will prove that $\mathcal{J}$ is equicontinuous. For $\xi_1 < \xi_2$, $\xi_1, \xi_2 \in I^T_{p,q}$ and $u \in U_\theta$, for any j, we get

$|(\mathcal{J}u_j)(\xi_2) - (\mathcal{J}u_j)(\xi_1)|$

$$\begin{aligned}
&\le \left|\frac{1}{p^{\binom{\iota_j}{2}}\Gamma_{p,q}(\iota_j)}\int_0^{\xi_1}(\xi_2-qs)^{(\iota_j-1)}\mathcal{F}\left(ps,u_j(ps)\right)d_{p,q}s\right.\\
&\left.+\frac{1}{p^{\binom{\iota_j}{2}}\Gamma_{p,q}(\iota_j)}\int_{\xi_1}^{\xi_2}(\xi_1-qs)^{(\iota_j-1)}\mathcal{F}\left(ps,u_j(ps)\right)|d_{p,q}s\right|\\
&\le \int_0^{\xi_1}\frac{\left((\xi_2-qs)^{(\iota_j-1)}-(\xi_1-qs)^{(\iota_j-1)}\right)}{p^{\binom{\iota_j}{2}}\Gamma_{p,q}(\iota_j)}|\mathcal{F}\left(ps,u_j(ps)\right)|d_{p,q}s\\
&+\frac{1}{p^{\binom{\iota_j}{2}}\Gamma_{p,q}(\iota_j)}\int_{\xi_1}^{\xi_2}(\xi_1-qs)^{(\iota_j-1)}|\mathcal{F}\left(ps,u_j(ps)\right)|d_{p,q}s. \qquad (3.10)
\end{aligned}$$

Thus, applying the maximum over j on (3.10), we derive that

$$\begin{aligned}
&\|(\mathcal{J}u)(\xi_2)-(\mathcal{J}u)(\xi_1)\| \\
&\le \max_{j=1,2,\dots,\nu}\left\{\int_0^{\xi_1}\left[\frac{(\xi_2-qs)^{(\iota_j-1)}}{p^{\binom{\iota_j}{2}}\Gamma_{p,q}(\iota_j)}-\frac{(\xi_1-qs)^{(\iota_j-1)}}{p^{\binom{\iota_j}{2}}\Gamma_{p,q}(\iota_j)}\right]\omega(\xi)\varphi(\|u\|)d_{p,q}s\right\} \\
&+\max_{j=1,2,\dots,\nu}\left\{\frac{1}{p^{\binom{\iota_j}{2}}\Gamma_{p,q}(\iota_j)}\int_{\xi_1}^{\xi_2}(\xi_1-qs)^{(\iota_j-1)}\omega(\xi)\varphi(\|u\|)d_{p,q}s\right\} \\
&\le \max_{j=1,2,\dots,\nu}\left\{\int_0^{\xi_1}\left[\frac{(\xi_2-qs)^{(\iota_j-1)}}{p^{\binom{\iota_j}{2}}\Gamma_{p,q}(\iota_j)}-\frac{(\xi_1-qs)^{(\iota_j-1)}}{p^{\binom{\iota_j}{2}}\Gamma_{p,q}(\iota_j)}\right]d_{p,q}s\right\}\omega^*\varphi(\theta) \\
&+\max_{j=1,2,\dots,\nu}\left\{\frac{1}{p^{\binom{\iota_j}{2}}\Gamma_{p,q}(\iota_j)}\int_{\xi_1}^{\xi_2}(\xi_1-qs)^{(\iota_j-1)}d_{p,q}s\right\}\omega^*\varphi(\theta).
\end{aligned}$$

As $|\xi_2-\xi_1|\to 0$, $\|(\mathcal{J}u)(\xi_2)-(\mathcal{J}u)(\xi_1)\|_\infty\to 0$. Hence, we deduce that $\mathcal{J}(U_\theta)$ is equicontinuous.

Claim 4. *$\mathcal{J}$ is a Meir-Keeler condensing operator.*

We show that for given $\epsilon>0$, $\exists\ \iota>0$ in a way that

$$\epsilon\le\kappa_C(V)<\epsilon+\iota\Longrightarrow\ \kappa_C(\mathcal{J}V)<\epsilon,\quad V\subset U_\theta.$$

In view of Lemma 2.4 and the properties of κ, for every bounded $V\subset U_\theta$ and $\epsilon'>0$, there exists a sequence $\{u_n\}_{n=1}^\infty\subset V$ in a way that

$$\kappa(\mathcal{J}V(\xi))\le 2\kappa\left\{\frac{1}{p^{\binom{\iota}{2}}\Gamma_{p,q}(\iota)}\int_0^\xi(\xi-qs)^{(\iota-1)}\mathcal{F}\left(ps,\{u_n(ps)\}_{n=1}^\infty\right)d_{p,q}s\right\}+\epsilon'.$$

By virtue of Lemma 2.5 and (A3), we get

$$\begin{aligned}
\kappa(\mathcal{J}V(\xi))&\le\frac{4}{p^{\binom{\iota}{2}}\Gamma_{p,q}(\iota)}\int_0^\xi(\xi-qs)^{(\iota-1)}\kappa(\mathcal{F}\left(ps,\{u_n(ps)\}_{n=1}^\infty\right))d_{p,q}s+\epsilon' \\
&\le\frac{4}{p^{\binom{\iota}{2}}\Gamma_{p,q}(\iota)}\int_0^\xi(\xi-qs)^{(\iota-1)}\omega(\xi)\kappa(\{u_n(ps)\}_{n=1}^\infty)d_{p,q}s+\epsilon' \\
&\le\frac{4(\frac{T}{p})^{\iota_j}\omega^*}{\Gamma_{p,q}(\iota_j+1)}\kappa_C(V)+\epsilon'.
\end{aligned}$$

Note that the above inequality holds true for every $\epsilon'>0$. Thus, we get $\kappa(\mathcal{J}V(\xi))\le\frac{4\left(\frac{T}{p}\right)^{\iota_j}\omega^*}{\Gamma_{p,q}(\iota_j+1)}\kappa_C(V)$. Also, since $\mathcal{J}(V)\subset U_\theta$ is bounded and equicontinuous and from Lemma 2.3, we have $\kappa_C(\mathcal{J}(V))=\max_{\xi\in I_{p,q}^T}\kappa(\mathcal{J}(V))$. Hence, $\kappa_C(\mathcal{J}(V))\le\frac{4\left(\frac{T}{p}\right)^{\iota_j}\omega^*}{\Gamma_{p,q}(\iota_j+1)}\kappa_C(V)$.

Note that:

$$\kappa_C(\mathcal{J}(V))\le\frac{4(\frac{T}{p})^{\iota_j}\omega^*}{\Gamma_{p,q}(\iota_j+1)}\kappa_C(V)<\epsilon,$$

and this implies that $\kappa_C(V)<\frac{\Gamma_{p,q}(\iota_j+1)}{4\left(\frac{T}{p}\right)^{\iota_j}\omega^*}\epsilon$.

Hence, for given $\epsilon > 0$ and taking $\sigma = \frac{\Gamma_{p,q}(\iota_j+1)-4\left(\frac{T}{p}\right)^{\iota_j}\omega^*}{4\left(\frac{T}{p}\right)^{\iota_j}\omega^*}\epsilon$, we obtain that $\epsilon \le \kappa_C(V) < \epsilon + \sigma$, which gives $\kappa_C(V) < \epsilon$, for any $V \subset U_\theta$. Consequently, we infer that the operator $\mathcal{J} : U_\theta \to U_\theta$ is a Meir-Keeler operator. So, Theorem 2.1 ensures that $\mathcal{J}$ has at least one fixed point $u \in U_\theta$ which is the solution of the Cauchy problem (1.1), as desired.

Let us now exhibit the main result about the global convergence of successive approximations (s.a).
Set ${}^{\varrho}I^T_{p,q} := \varrho I^T_{p,q}$, for any $\varrho \in [0,1]$. In order to to prove our main results, we need the following assumptions.

(H1) The function $\mathcal{F} : {}^{\varrho}I^T_{p,q} \times \mathbb{E}^\nu \to \mathbb{E}^\nu$ satisfies Carathodory conditions.

(H2) There exist a constant $\varsigma > 0$ and a continuous function $h : I^T_{p,q} \times [0,\varsigma] \to \mathbb{R}_+$ such that $h(\xi,\cdot)$ is nondecreasing for all $\xi \in I^T_{p,q}$, and

$$|\phi(\xi,\rho) - \phi(\xi,\bar{\rho})| \le h(\xi, |\rho - \bar{\rho}|) \tag{3.11}$$

satisfies for all $\xi \in I^T_{p,q}$ and $\rho, \bar{\rho} \in \mathbb{R}$ such that $|\rho - \bar{\rho}| \le \varsigma$.

(H3) There exists a continuous function $\omega : I^T_{p,q} \to \mathbb{R}^+$ and a nondecreasing continuous function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|\mathcal{F}(\xi,u)\| \le \omega(\xi)\varphi(\|u\|), \quad \xi \in I^T_{p,q},\ u \in \mathbb{E}^\nu.$$

(H4) $R \equiv 0$ is the only function in $C\left({}^{\delta}I^T_{p,q}, [0,\varsigma]\right)$ satisfying the integral inequality

$$R(\xi) \le \max_{j=1,2,\dots,\nu}\left\{\int_0^{\xi\delta} \frac{(\xi - qs)^{(\iota_j-1)}}{p^{\binom{\iota_j}{2}}\Gamma_{p,q}(\iota)} h\left(ps, R\left(ps\right)\right) d_{p,q}s\right\} \tag{3.12}$$

with $\varrho \le \delta \le 1$.

Definition 3.3. We present the s.a of the problem (1.1) as

$$\begin{aligned} u_0(\xi) &= u_0, \quad \xi \in I^T_{p,q},\\ u_{n+1}(\xi) &= u_0 + \frac{1}{p^{\binom{\iota}{2}}\Gamma_{p,q}(\iota)}\int_0^{\xi}(\xi - qs)^{(\iota-1)}\mathcal{F}\left(ps, u_n\left(ps\right)\right) d_{p,q}s, \quad \xi \in I^T_{p,q}. \end{aligned}$$

Theorem 3.2. Suppose that $(H1)$–$(H4)$ hold. Thus, the s.a $u_n, n \in \mathbb{N}$ are defined, and we have a convergence towards the unique solution of problem (1.1) uniformly on $I^T_{p,q}$.

Proof. Since u_n is in $C(I^T_{p,q}, \mathbb{E}^\nu)$, there exists $\nu > 0$ satisfying

$$\|u_n\| \le \nu.$$

By $(H1)$, the s.a are well-defined. For every $\xi_1, \xi_2 \in I^T_{p,q}$ with $\xi_1 < \xi_2$, and for all $\xi \in I^T_{p,q}$,

$$|(\mathcal{J}u_{j,n})(\xi_2) - (\mathcal{J}u_{j,n})(\xi_1)|$$

$$\le \left|\frac{1}{p^{\binom{\iota_j}{2}}\Gamma_{p,q}(\iota_j)}\int_0^{\xi_1}(\xi_2 - qs)^{(\iota_j-1)}\mathcal{F}\left(ps, u_{j,n-1}\left(ps\right)\right) d_{p,q}s\right.$$

$$+\frac{1}{p^{\binom{\iota_j}{2}}\Gamma_{p,q}(\iota_j)}\int_{\xi_1}^{\xi_2}(\xi_1-qs)^{(\iota_j-1)}\mathcal{F}\left(ps,u_{j,n-1}(ps)\right)|d_{p,q}s\Bigg|$$

$$\leq\int_0^{\xi_1}\frac{\left((\xi_2-qs)^{(\iota_j-1)}-(\xi_1-qs)^{(\iota_j-1)}\right)}{p^{\binom{\iota_j}{2}}\Gamma_{p,q}(\iota_j)}|\mathcal{F}\left(ps,u_{j,n-1}(ps)\right)|d_{p,q}s$$

$$+\frac{1}{p^{\binom{\iota_j}{2}}\Gamma_{p,q}(\iota_j)}\int_{\xi_1}^{\xi_2}(\xi_1-qs)^{(\iota_j-1)}|\mathcal{F}\left(ps,u_{j,n-1}(ps)\right)|d_{p,q}s. \tag{3.13}$$

Thus, applying the maximum over j on (3.13), we derive that
$\|(\mathcal{J}u)(\xi_2)-(\mathcal{J}u)(\xi_1)\|$

$$\leq\max_{j=1,2,\ldots,\nu}\left\{\int_0^{\xi_1}\left[\frac{(\xi_2-qs)^{(\iota_j-1)}}{p^{\binom{\iota_j}{2}}\Gamma_{p,q}(\iota_j)}-\frac{(\xi_1-qs)^{(\iota_j-1)}}{p^{\binom{\iota_j}{2}}\Gamma_{p,q}(\iota_j)}\right]\omega(\xi)\varphi(\|u\|)d_{p,q}s\right\}$$

$$+\max_{j=1,2,\ldots,\nu}\left\{\frac{1}{p^{\binom{\iota_j}{2}}\Gamma_{p,q}(\iota_j)}\int_{\xi_1}^{\xi_2}(\xi_1-qs)^{(\iota_j-1)}\omega(\xi)\varphi(\|u\|)d_{p,q}s\right\}$$

$$\leq\max_{j=1,2,\ldots,\nu}\left\{\int_0^{\xi_1}\left[\frac{(\xi_2-qs)^{(\iota_j-1)}}{p^{\binom{\iota_j}{2}}\Gamma_{p,q}(\iota_j)}-\frac{(\xi_1-qs)^{(\iota_j-1)}}{p^{\binom{\iota_j}{2}}\Gamma_{p,q}(\iota_j)}\right]d_{p,q}s\right\}\omega^*\varphi(\theta)$$

$$+\max_{j=1,2,\ldots,\nu}\left\{\frac{1}{p^{\binom{\iota_j}{2}}\Gamma_{p,q}(\iota_j)}\int_{\xi_1}^{\xi_2}(\xi_1-qs)^{(\iota_j-1)}d_{p,q}s\right\}\omega^*\varphi(\theta).$$

As $|\xi_2-\xi_1|\to 0$, $\|(\mathcal{J}u)(\xi_2)-(\mathcal{J}u)(\xi_1)\|_\infty\to 0$. Hence, we deduce the equicontinuity on $I_{p,q}^T$. of the sequence $\{u_n, n\in\mathbb{N}\}$.

Set $\quad \mu := \sup\left\{\varrho\in[0,1] : \{u_n(\xi)\} \text{ converges uniformly on } {}^{\varrho}I_{p,q}^T\right\}$.

If $\mu = 1$, we have the global convergence of s.a. We will assume that $\mu < 1$, and $\{u_n(\xi)\}$ is equicontinuous on ${}^{\mu}I_{p,q}^T$, so it converges uniformly towards a function $\tilde{u}(\xi)$. If we prove it, there is $\delta\in(\mu,1]$ satisfying $\{u_n(\xi)\}$ converges uniformly on ${}^{\delta}I_{p,q}^T$, so we find a contradiction.

Set $\quad u(\xi)=\tilde{u}(\xi) \quad$ for all $\quad \xi\in{}^{\mu}I_{p,q}^T$.

By $(H2)$, there are a positive constant ζ and a continuous function $h : I_{p,q}^T\times[0,\ \zeta]\to\mathbb{R}_+$ satisfying (3.11). Then, there are

$$\delta\in[\mu,1] \quad \text{and} \quad n_0\in\mathbb{N},$$

satisfying, for all $\xi\in{}^{\delta}I_{p,q}^T$ and $n,m>n_0$, we get

$$|u_n(\xi)-u_m(\xi)|\leq\zeta.$$

For all $\xi\in{}^{\delta}I_{p,q}^T$, set

$$R^{(n,m)}(\xi)=|u_n(\xi)-u_m(\xi)|,$$
$$R_k(\xi)=\sup_{n,m\geq k}R^{(n,m)}(\xi).$$

$R_k(\xi)$ is a non-increasing sequence, so it converges to a function $R(\xi)$, for all $\xi\in{}^{\delta}I_{p,q}^T$. By the equicontinuity of $\{R_k(\xi)\}$ we derive

$$\lim_{k\to\infty}R_k(\xi)=R(\xi) \text{ uniformly on } {}^{\delta}I_{p,q}^T.$$

Moreover, for each $\xi \in {}^{\delta}I^{T}_{p,q}$ and $n, m \geq k$, we get

$$\begin{aligned}
R^{(n,m)}(\xi) &= \left|u_{j,n}(\xi) - u_{j,m}(\xi)\right| \\
&\leq \left|\frac{1}{p^{\binom{\iota_j}{2}}\Gamma_{p,q}(\iota)} \int_0^{\xi} (\xi - qs)^{(\iota_j - 1)} \mathcal{F}\left(ps, u_{j,n-1}(ps)\right) d_{p,q}s \right. \\
&\left. - \frac{1}{p^{\binom{\iota}{2}}\Gamma_{p,q}(\iota_j)} \int_0^{\xi} (\xi - qs)^{(\iota_j - 1)} \mathcal{F}\left(ps, u_{j,m-1}(ps)\right) d_{p,q}s \right| \\
&\leq \int_0^{\xi} \frac{(\xi - qs)^{(\iota_j - 1)}}{p^{\binom{\iota_j}{2}}\Gamma_{p,q}(\iota)} \left|\mathcal{F}\left(ps, u_{j,n-1}(ps)\right) - \mathcal{F}\left(ps, u_{j,m-1}(ps)\right)\right| d_{p,q}s \\
&\leq \int_0^{\xi\delta} \frac{(\xi - qs)^{(\iota_j - 1)}}{p^{\binom{\iota_j}{2}}\Gamma_{p,q}(\iota)} \left|\mathcal{F}\left(ps, u_{j,n-1}(ps)\right) - \mathcal{F}\left(ps, u_{j,m-1}(ps)\right)\right| d_{p,q}s.
\end{aligned}$$

Taking the maximum over j and using (3.11), we have

$$\begin{aligned}
R^{(n,m)}(\xi) &= |u_n(\xi) - u_m(\xi)| \\
&\leq \max_{j=1,2,\ldots,\nu} \left\{ \int_0^{\xi\delta} \frac{(\xi - qs)^{(\iota_j - 1)}}{p^{\binom{\iota_j}{2}}\Gamma_{p,q}(\iota)} h\left(ps, \left|u_{j,n-1}(ps) - u_{j,m-1}(ps)\right|\right) d_{p,q}s \right\} \\
&\leq \max_{j=1,2,\ldots,\nu} \left\{ \int_0^{\xi\delta} \frac{(\xi - qs)^{(\iota_j - 1)}}{p^{\binom{\iota_j}{2}}\Gamma_{p,q}(\iota)} h\left(ps, R^{(n-1,m-1)}(ps)\right) d_{p,q}s \right\}.
\end{aligned}$$

Thus

$$R_k(\xi) \leq \max_{j=1,2,\ldots,\nu} \left\{ \int_0^{\xi\delta} \frac{(\xi - qs)^{(\iota_j - 1)}}{p^{\binom{\iota_j}{2}}\Gamma_{p,q}(\iota)} h\left(ps, R_{k-1}(ps)\right) d_{p,q}s \right\}.$$

According to the Lebesgue-dominated convergence theorem, we can obtain

$$R(\xi) \leq \max_{j=1,2,\ldots,\nu} \left\{ \int_0^{\xi\delta} \frac{(\xi - qs)^{(\iota_j - 1)}}{p^{\binom{\iota_j}{2}}\Gamma_{p,q}(\iota)} h\left(ps, R(ps)\right) d_{p,q}s \right\}.$$

By $(H1)$ and $(H4)$ we have $R \equiv 0$ on ${}^{\delta}I^{T}_{p,q}$, which gives that $\lim_{k\to\infty} R_k(\xi) = 0$ uniformly on ${}^{\delta}I^{T}_{p,q}$. Thus, $\{u_k(\xi)\}_{k=1}^{\infty}$ is a Cauchy sequence on ${}^{\delta}I^{T}_{p,q}$. So, $\{u_k(\xi)\}_{k=1}^{\infty}$ is uniformly convergent on ${}^{\delta}I^{T}_{p,q}$ which leads to a contradiction.

Then, $\{u_k(\xi)\}_{k=1}^{\infty}$ converges uniformly on $I^{T}_{p,q}$ to a continuous function $u_*(\xi)$.

According to the Lebesgue-dominated convergence theorem, we have

$$\begin{aligned}
&\lim_{k\to\infty} u_0 + \frac{1}{p^{\binom{\iota_j}{2}}\Gamma_{p,q}(\iota)} \int_0^{\xi} (\xi - qs)^{(\iota_j - 1)} \mathcal{F}\left(ps, u_k(ps)\right) d_{p,q}s \\
&= u_0 + \frac{1}{p^{\binom{\iota_j}{2}}\Gamma_{p,q}(\iota)} \int_0^{\xi} (\xi - qs)^{(\iota_j - 1)} \mathcal{F}\left(ps, u_*(ps)\right) d_{p,q}s,
\end{aligned}$$

$\forall \xi \in I^{T}_{p,q}$. This implies that u_* is a solution to the system (1.1).

Now, we will show the uniqueness of the solutions of system (1.1). Set u and w as the two solutions of (1.1). Thus, set

$$\mu := \sup\left\{\varrho \in [0,1] : u_1(\xi) = u_2(\xi) \quad \text{for } \xi \in {}^{\varrho}I^T_{p,q}\right\},$$

and assume that $\mu < 1$. There exist a positive constant ζ and a function $h : {}^{\mu}I^T_{p,q} \times [0,\zeta] \to \mathbb{R}_+$ satisfying (3.11). Choose $\delta \in (\varrho, 1)$ such that

$$|u(\xi) - w(\xi)| \leq \zeta \quad \text{for } \xi \in {}^{\delta}I^T_{p,q}.$$

Then, for any $\xi \in {}^{\delta}I^T_{p,q}$, we get

$$\begin{aligned}
&\left|u_j(\xi) - w_j(\xi)\right| \\
&\leq \left|\frac{1}{p^{\binom{\iota_j}{2}}\Gamma_{p,q}(\iota)} \int_0^{\xi} (\xi - qs)^{(\iota_j - 1)} \mathcal{F}\left(ps, u_j(ps)\right) d_{p,q}s \right.\\
&\left. - \frac{1}{p^{\binom{\iota}{2}}\Gamma_{p,q}(\iota_j)} \int_0^{\xi} (\xi - qs)^{(\iota_j - 1)} \mathcal{F}\left(ps, w_j(ps)\right) d_{p,q}s\right| \\
&\leq \int_0^{\xi} \frac{(\xi - qs)^{(\iota_j - 1)}}{p^{\binom{\iota_j}{2}}\Gamma_{p,q}(\iota)} \left|\mathcal{F}\left(ps, u_j(ps)\right) - \mathcal{F}\left(ps, w_j(ps)\right)\right| d_{p,q}s \\
&\leq \int_0^{\xi\delta} \frac{(\xi - qs)^{(\iota_j - 1)}}{p^{\binom{\iota_j}{2}}\Gamma_{p,q}(\iota)} \left|\mathcal{F}\left(p^{\iota_j}t, u_j(ps)\right) - \mathcal{F}\left(ps, w_j(ps)\right)\right| d_{p,q}s.
\end{aligned}$$

Taking the maximum over j and using (3.11), we get

$$\begin{aligned}
&|u(\xi) - w(\xi)| \\
&\leq \max_{j=1,2,\ldots,\nu} \left\{\int_0^{\xi\delta} \frac{(\xi - qs)^{(\iota_j - 1)}}{p^{\binom{\iota_j}{2}}\Gamma_{p,q}(\iota)} h\left(ps, \left|u_j(ps) - w_j(ps)\right|\right) d_{p,q}s\right\}.
\end{aligned}$$

Again, by $(H1)$ and $(H4)$ we get $u - w \equiv 0$ on ${}^{\delta}I^T_{p,q}$. This gives $u = w$ on ${}^{\delta}I^T_{p,q}$, which is a contradiction. Therefore, $\mu = 1$, and system (1.1) has a unique solution on $I^T_{p,q}$.

4. Applications

In this section we give two examples to illustrate our main result. Let

$$\mathbb{E}^{\nu} = c_0 = \left\{u = (u_1, u_2, \ldots, u_n, \ldots) : u_n \to 0 (n \to \infty)\right\},$$

be the Banach space of real sequences converging to zero, endowed with its usual norm

$$\|u\|_{\infty} = \sup_{n \geq 1} |u_n|.$$

Example 4.1. Consider the following fractional difference equation posed in c_0 :

$$\begin{cases} {}^{c}D^{\iota}_{p,q}u(\xi) = \mathcal{F}(\xi, u(\xi)), \quad \xi \in I^T_{p,q} := [0,1], \\ u(0) = (0, 0, \ldots, 0, \ldots). \end{cases} \tag{4.1}$$

Note that, this problem is a particular case with the following data:

$$I_{p,q}^{T} := [0,1], \iota = 1/2, p = 1/4, q = 1/5.$$

and $\mathcal{F} : I_{p,q}^{T} \times c_0 \longrightarrow c_0$ is given by

$$\mathcal{F}(\xi, u) = \left\{ \frac{1}{e^{\xi} + 3} \left(\frac{1}{n^2} + \arctan(|u_n|) \right) \right\}_{n \geq 1}, \quad \text{for } \xi \in I_{p,q}^{T}, u = \{u_n\}_{n \geq 1} \in c_0.$$

It is clear that condition (A1) holds, and

$$\|\mathcal{F}(\xi, u)\| \leq \frac{1}{e^{\xi} + 3}(1 + \|u\|) = \omega(\xi)\varphi(\|u\|).$$

Therefore, the assumption (A2) of Theorem 3.1 is satisfied with $\omega(\xi) = \frac{1}{e^{\xi}+3}, \xi \in I_{p,q}^{T}$ and $\varphi(x) = 1 + x, x \in [0, \infty)$. On the other hand, for any bounded set $V \subset c_0$, we have

$$\kappa(\mathcal{F}(\xi, V)) \leq \omega(\xi)\kappa(V), \text{ a.e. } t \in I_{p,q}^{T}.$$

Hence, (A3) is satisfied. Now, we check that condition (3.7) is satisfied. Indeed, $\Delta = \frac{4\omega^*}{\Gamma_{p,q}(\iota+1)} = \frac{1}{\Gamma_{p,q}(1/2+1)} < 1$, and $(1 + \theta)\Delta \leq \theta$. Thus,

$$\theta \geq \frac{\Delta}{1 - \Delta} = 2.8143$$

Then, θ can be chosen as $\theta = 3$. Consequently, all the hypotheses of Theorem 3.1 are satisfied, and we conclude that the problem (4.1) has at least one solution $u \in C\left(I_{p,q}^{T}, c_0\right)$.

Example 4.2. We consider the following Caputo fractional (p, q)-difference Cauchy problem:

$$\begin{cases} {}^{c}D_{p,q}^{\iota} u(\xi) = \mathcal{F}(\xi, u(\xi)), \quad \xi \in I_{p,q}^{T} := [0,1], \iota \in (0,1), \\ u(0) = 1, \end{cases} \tag{4.2}$$

where

$$\mathcal{F}(\xi, u(\xi)) = \left(e^{\xi-1} + |u(\xi)|\right) \frac{\xi}{(1 + \xi^2)(1 + |u(\xi)|)}.$$

For each $u, \bar{u} \in \mathbb{R}$ and $\xi \in I_{p,q}^{T}$ we have

$$|\mathcal{F}(\xi, u) - \mathcal{F}(\xi, \bar{u})| \leq \xi \left(1 + e^{\xi-1}\right)|u - \bar{u}|.$$

This leads to the condition (3.11), which holds for each $\xi \in I_{p,q}^{T}$, $\zeta > 0$, and the function

$$h : [0,1] \times [0, \zeta] \to [0, \infty),$$

such that

$$h(\xi, u) = \xi \left(1 + e^{\xi-1}\right)|u|.$$

Then, Theorem 3.2 leads us to the successive approximations $u_n, n \in \mathbb{N}$, defined by

$$u_0(\xi) = 1, \quad \xi \in I_{p,q}^{T},$$
$$u_{n+1}(\xi) = 1 + \frac{1}{p^{\binom{\iota}{2}}\Gamma_{p,q}(\iota)} \int_0^{\xi} (\xi - qs)^{(\iota-1)} \mathcal{F}(ps, u_n(ps))\, d_{p,q}s, \quad t \in I_{p,q}^{T},$$

which converges uniformly on $I_{p,q}^{T}$ to the unique solution of the problem (4.2).

5. Conclusions

In this paper, we investigate the global convergence of s.a and the existence and the uniqueness of solutions to a fractional (p, q)-difference equation by using the measure of non-compactness method with Meir-Keeler fixed point theorem of condensing operators. Two examples are also provided to demonstrate the main results presented in this paper.

Acknowledgments

This research is funded by Researchers Supporting Project number (RSPD2023R683), King Saud University, Riyadh, Saudi Arabia.

Conflict of interest

The authors declare that there are no conflicts of interest.

References

1. A. Aghajani, M. Mursaleen, A. S. Haghighi, Fixed point theorems for Meir-Keeler condensing operators via measure of noncompactness, *Acta Math. Sci.*, **35** (2015), 552–566. https://doi.org/10.1016/S0252-9602(15)30003-5
2. J. P. Aubin, I. Ekeland, *Applied nonlinear analysis*, New York: John Wiley & Sons, 1984.
3. D. Baleanu, K. Diethelm, E. Scalas, J. J. Trujillo, *Fractional calculus models and numerical methods*, Singapore: World Scientific, 2012.
4. J. Banaš, On measures of noncompactness in Banach spaces, *Commentationes Mathematicae Universitatis Carolinae*, **21** (1980), 131–143.
5. A. Boutiara, J. Alzabut, M. Ghaderi, S. Rezapour, On a coupled system of fractional (p, q)-differential equation with Lipschitzian matrix in generalized metric space, *AIMS Math.*, **8** (2023), 1566–1591. https://doi.org/10.3934/math.2023079
6. A. Boutiara, M. Benbachir, K. Guerbati, Measure of noncompactness for nonlinear Hilfer fractional differential equation in Banach spaces, *Ikonion J. Math.*, **1** (2019), 55–67.
7. A. Boutiara, Mixed fractional differential equation with nonlocal conditions in Banach spaces, *J. Math. Model.*, **9** (2021), 451–463. https://doi.org/10.22124/jmm.2021.18439.1582
8. A. Boutiara, S. Etemad, J. Alzabut, A. Hussain, M. Subramanian, S. Rezapour, On a nonlinear sequential four-point fractional q-difference equation involving q-integral operators in boundary conditions along with stability criteria, *Adv. Differ. Equ.*, **2021** (2021), 367. https://doi.org/10.1186/s13662-021-03525-3
9. A. Boutiara, M. Benbachir, M. K. Kaabar, F. Martnez, M. E. Samei, M. Kaplan, Explicit iteration and unbounded solutions for fractional qdifference equations with boundary conditions on an infinite interval, *J. Inequal. Appl.*, **2022** (2022), 29. https://doi.org/10.1186/s13660-022-02764-6

10. A. Boutiara, M. Benbachir, Existence and uniqueness results to a fractional q-difference coupled system with integral boundary conditions via topological degree theory, *Int. J. Nonlinear Anal.*, **13** (2022), 3197–3211. https://doi.org/10.22075/ijnaa.2021.21951.2306

11. A. Boutiara, Multi-term fractional q-difference equations with q-integral boundary conditions via topological degree theory, *Commun. Optim. Theory*, **2021** (2021), 1. https://doi.org/10.23952/cot.2021.1

12. R. Chakrabarti, R. A. Jagannathan, A (p, q)-oscillator realization of two-parameter quantum algebras, *J. Phys. A Math. Gen.*, **24** (1991), L711–L718. https://doi.org/10.1088/0305-4470/24/13/002

13. W. T. Cheng, W. H. Zhang, Q. B. Cai, (p, q)-gamma operators which preserve x^2, *J. Inequal. Appl.*, **2019** (2019), 108. https://doi.org/10.1186/s13660-019-2053-3

14. K. Deimling, *Multivalued differential equations*, New York: De Gruyter, 1992. https://doi.org/10.1515/9783110874228

15. T. Dumrongpokaphan, S. K. Ntouyas, T. Sitthiwirattham, Separate fractional (p, q)-integrodifference equations via nonlocal fractional (p, q)-integral boundary conditions, *Symmetry*, **13** (2021), 2212. https://doi.org/10.3390/sym13112212

16. U. Duran, *Post quantum calculus*, University of Gaziantep, 2016.

17. A. Fernandez, C. Ustaoğlu, On some analytic properties of tempered fractional calculus, *J. Comput. Appl. Math.*, **336** (2020), 112400. https://doi.org/10.1016/j.cam.2019.112400

18. H. P. Heinz, On the behavior of measure of noncompactness with respect to differentiation and integration of vector-valued functions, *Nonlinear Anal. Theor.*, **7** (1983), 1351–1371. https://doi.org/10.1016/0362-546X(83)90006-8

19. F. H. Jackson, On q-difference equations, *Am. J. Math.*, **32** (1910), 305–314. https://doi.org/10.2307/2370183

20. F. H. Jackson, On q-difference integrals, *Q. J. Pure Appl. Math.*, **41** (1910), 193–203.

21. N. Kamsrisuk, C. Promsakon, S. K. Ntouyas, J. Tariboon, Nonlocal boundary value problems for (p, q)-difference equations, *Differ. Equ. Appl.*, **10** (2018), 183–195. https://doi.org/10.7153/dea-2018-10-11

22. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier, 2006.

23. A. Lasota, Z. Opial, An application of the Kakutani-Ky Fan theorem in the theory of ordinary differential equations, *Bull. Acad. Pol. Sci. Ser. Sci. Math. Astron. Phys.*, **13** (1965), 781–786.

24. F. Mainardi, *Fractional calculus and waves in linear viscoelasticity: An introduction to mathematical models*, World Scientific, 2022. https://doi.org/10.1142/p614

25. A. Meir, E. Keeler, A theorem on contraction mappings, *J. Math. Anal. Appl.*, **28** (1969), 326–329. https://doi.org/10.1016/0022-247X(69)90031-6

26. K. S. Miller, B. Ross, *An introduction to the fractional calculus and fractional differential equations*, Wiley, 1993.

27. G. V. Milovanovic, V. Gupta, N. Malik, (p, q)-Beta functions and applications in approximation, *Bol. Soc. Mat. Mex.*, **24** (2018), 219–237. https://doi.org/10.1007/s40590-016-0139-1

28. P. Neang, K. Nonlaopon, J. Tariboon, S. K. Ntouyas, B. Ahmad, Existence and uniqueness results for fractional (p, q)-difference equations with separated boundary conditions, *Mathematics*, **10** (2022), 767. https://doi.org/10.3390/math10050767

29. K. B. Oldham, J. Spanier, *The fractional calculus: Theory and applications of differentiation and integration to arbitrary order*, New York: Academic Press, 1974.

30. I. Podlubny, *Fractional differential equations*, San Diego: Academic Press, 1999.

31. P. N. Sadjang, On the fundamental theorem of (p, q)-calculus and some (p, q)-taylor formulas, *Results Math.*, **73** (2018), 39. https://doi.org/10.1007/s00025-018-0783-z

32. S. Samko, A. Kilbas, O. Marichev, *Fractional integrals and derivatives*, Switzerland: Gordon and Breach Science Publishers, 1993.

33. W. Shatanawi, A. Boutiara, M. S. Abdo, M. B. Jeelani, K. Abodayeh, Nonlocal and multiple-point fractional boundary value problem in the frame of a generalized Hilfer derivative, *Adv. Differ. Equ.*, **2021** (2021), 294. https://doi.org/10.1186/s13662-021-03450-5

34. J. Soontharanon, T. Sitthiwirattham, On fractional (p, q)-calculus, *Adv. Differ. Equ.*, **2020** (2020), 35. https://doi.org/10.1186/s13662-020-2512-7