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Research article

Existence and stability results for nonlinear coupled singular fractional-order differential equations with time delay

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Abstract: The objective of the manuscript is to build coupled singular fractional-order differential equations with time delay. To study the underline problem, an integral representation is initially discussed and the operator form of the solution is investigated using various supplementary hypotheses. Also, the existence and uniqueness of the considered problem are investigated by using the Lebesgue-dominated convergence theorem and some analysis results. Moreover, the stability analysis to determine the nature of the proposed model's solution is examined. Finally, two supportive examples are provided to demonstrate our analysis as applications.

Keywords: time delay; coupled fractional order differential equation; Lebesgue-dominated convergence theorem; Schaefer theorem; stability result
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1. Introduction and position of the model

In the past, it was believed that scientific disciplines were completely separate; but now, after the tremendous development and modern theories in basic science techniques, they have become completely connected. For example, mathematics in which the level of development in different disciplines has varied dramatically in contemporary times. Not knowing mathematics in a mathematically-driven world is like walking around a museum without looking at its walls. Learning and appreciating mathematics can help you appreciate certain things you would not otherwise focus on in your surrounding world. Mathematics is everywhere in nature. A typical example is the celebrated Fibonacci sequence of numbers, which is present in the reproduction of species in nature. Mathematics is also useful to formulate epidemic models via differential or difference systems of equations that describe the couplings of the dynamics among sub-populations like susceptible, exposed, infectious, or recovered with immunity.

Fractional differential equations appear naturally in diverse fields of science and engineering. They constitute an important field of research. It should be noted that most papers dealing with the existence of solutions of nonlinear initial value problems of fractional differential equations mainly use the techniques of nonlinear analysis, such as fixed point techniques, stability, the Leray–Schauder result, etc. Relatively, fractional calculus and fractional differential/integral equations are very fresh topics for the researchers. Fractional differential equations (FDEs) are more useful and have a higher degree of freedom. Consequently, academics have employed fractional order derivatives and integrals to describe the behavior of numerous real-world problems. For more details; see [1–14].

Stability analysis of FDEs with different types of initial and boundary conditions have attracted many researchers who discussed the analysis of stability in the setting of Ulam-Hyers. It should be noted that most papers dealing with the existence of solutions of nonlinear initial value problems of fractional differential equations mainly use the techniques of nonlinear analysis, such as fixed point techniques, stability and the Leray–Schauder result. By using fixed point techniques, the existence and uniqueness of solutions to differential/integral equations involving fractional operators were studied by a large number of researchers. For further related results, see for example [15–28].

Another important class of this theory is fractional delay differential equations (FDDES), which constitute a large class of equations including discrete, proportional and continuous type delay terms. FDDEs plays a significant role in molding various physical process and phenomenon. FDDEs have various applications in different fields, such as probability theory of structures, growth cell, quantum mechanics, dynamics of both linear and nonlinear systems, astrophysics, and electrodynamics, see [29–34].

There are several approaches for noninteger order derivatives, such as Weyl, Hadamard, Riemann–Liouville, Grunwald–Letnikov and Caputo etc. Caputo fractional order derivative (CFOD) is very suitable for describing the behavior of many real world problems because of its good physical interpretation of initial and boundary conditions. That is why the FDEs are extensively studied by many researchers using the Caputo fractional order derivative with several types of boundary and initial conditions.

On the other hand, delay differential equations (DDEs) constitute a large class of differential equations. The mentioned area has numerous applications in modeling real world problems from the past to the present. It is worth mentioning that theoretical aspects like existence, uniqueness, and stability of solution and analytical aspects like analytical and numerical methods for finding solutions of the aforesaid problems need bit more mathematical maturity as compared to ordinary differential equations [35, 36]. The classical form of the aforesaid equations has been studied very well. On the other hand, FDDEs are equations in which fractional derivatives involve time delays. As compared to ordinary derivatives, fractional derivatives are nonlocal in nature and have the ability to model memory effects more comprehensively, whereas time delays represent the history of a past state. Therefore, researchers have given attention to investigate FDDEs from various aspects, including existence theory and stability analysis. Plenty of research studies have been established addressing the theory and applications of FDDEs. For detail we refer [37–40].

Further, stability results play a significant role in the analysis of dynamical problems. The aforementioned theory has many applications. There are numerous kinds of stability analysis, including Lyapunov and Mittag-Leffler as well as exponential stability. The mentioned forms have been investigated for many FDDEs problems, see [41, 42]. Recently, Hyers–Ulam (HU) type stability has received proper attention. The concerned stability has been established for large numbers of classical as well as fractional order problems (see [43–45]). So far, we know the said stability has also been investigated for proportional type delay problems, we refer [46, 47].

Continuing on the same path, we investigate the following CFDDEs subject to conditions:

$$\begin{cases} {}^{c}D^{\ell}\varpi(t) = \Im\left(t, \varpi(t-\rho), \varpi(t), \varphi(t)\right), \ t \in D, \ \ell \in (1,2], \\ {}^{c}D^{\tau}\varphi(t) = \Im\left(t, \varphi(t-\rho), \varphi(t), \varpi(t)\right), \ t \in D, \ \tau \in (1,2], \\ \varpi(t) = \varrho(t), \ \varpi(1) = \Lambda(\varpi, \varphi), \ t \in D_0, \\ \varphi(t) = \varkappa(t), \ \varphi(1) = \Lambda(\varphi, \varpi), \ t \in D_0, \end{cases} \tag{1.1}$$

where $\varrho, \varkappa : D_0 \to \mathbb{R}$, $\Lambda : \mho \times \mho \to \mho$ and $\Im : D \times \mho^3 \to \mho$ are continuous functions. Also, we will derive a sufficient condition for at least one solution via CFOD for the considered model (1.1). Moreover, the Banach contraction principle and the Schaefer fixed point (FP) theorem are used to discuss the existence of FPs for the supposed problem. In addition to, with the tools of analysis, we will investigate different types of HU stability, such as generalized HU stability, HU Rassias (HUR) stability and generalized HUR (GHUR, for short) type stability for our considered problem (1.1). Finally, two illustrative examples are investigated to support and strengthen theoretical results.

For the benefit of our readers, the remainder of the article is organized as follows: In Section 2, some useful definitions and theorems are provided. In Section 3, we present results for existence and uniqueness of the concerned model (1.1). Section 4 is committed to stability results for the aforementioned model (1.1). Section 5 is devoted to presenting illustrative examples. Conclusion of the work is provided at the end of the paper.

2. Preliminaries

This component of the paper is devoted to the fundamental findings, definitions of FP theory, and nonlinear analysis that are required for the exploration of the main results, see [1, 4, 9–12]. Assume that $\mho_1$, $\mho_2$ and $\mho_3$ are spaces of all continuous function from $D = [0,1]$ and $D_0 = [-\rho, 0]$ to $\mathbb{R}$, respectively, equipped with the norms below

$$\begin{aligned} \|\varpi\|_{\mho_1} &= \sup_{t\in D} |\varpi(t)|, \ \varpi \in D, \\ \|\varphi\|_{\mho_2} &= \sup_{t\in D} |\varphi(t)|, \ \varphi \in D, \\ \|\widetilde{\varpi}\|_{\mho_3} &= \sup_{t\in D_0} |\widetilde{\varpi}(t)|, \ \widetilde{\varpi} \in D_0. \end{aligned}$$

Clearly, the product $\mho = \mho_1 \times \mho_2$ is the space of all continuous functions with the norm $\|\varpi + \varphi\|_{\mho} = \|\varpi\|_{\mho_1} + \|\varphi\|_{\mho_2}$, for all $\varpi, \varphi \in D$.

Definition 2.1. [48] For the function $\eta \in L^1[0,r]$, the integral of fractional order ℓ is denoted by $I^\ell\eta$ and described as

$$I^\ell\eta(t) = \int_0^t \frac{\eta(\upsilon)}{\Gamma(\ell)(t-\upsilon)^{1-\ell}}d\upsilon.$$

Definition 2.2. [48] For the function $\eta(t) \in L^1([0,r],\mathbb{R}_+)$, the fractional order Caputo derivative on the interval $[0,r]$ is denoted by ${}^cD^\ell\eta$ and described as

$${}^cD^\ell\eta(t) = \int_0^t \frac{\eta^n(\upsilon)}{\Gamma(n-\ell)(t-\upsilon)^{1-n+\ell}}d\upsilon,$$

where n is the smallest integer equal or greater than ℓ, i.e., $n = [\ell]$.

Theorem 2.3. *[48] The FDE*

$${}^cD^\ell\eta(t) = 0,\ \ell \in (n-1,n],$$

has the following solution

$$\eta(t) = Q_1 + Q_2t + Q_3t^2 + ...Q_nt^{n-1},$$

where $Q_j \in \mathbb{R}$ *for* $j = 1,2,...,n$.

Lemma 2.4. *[48] The relation between a fractional order integral and its derivative is provided as*

$$I^\ell\left[{}^cD^\ell\varkappa(t)\right] = C_1 + C_2t + C_3t^2 + ... + C_nt^{n-1} + \varkappa(t),$$

where $C_i \in \mathbb{R}$ *for* $i = 1,2,...,n$.

Definition 2.5. [49] The mapping $\mathfrak{R} : \mho \to \mho^*$ on normed linear spaces is continuous and complete, if for each bounded $Z \in \mho$, $\overline{\mathfrak{R}(Z)} \in \mho^*$ is compact.

Definition 2.6. [50] Let $(\mho,d)$ be a metric space. If there is $\vartheta \geq 0$ so that

$$d(\mathfrak{R}\varpi,\mathfrak{R}\varphi) \leq \vartheta d(\varpi,\varphi),\ \text{for all } \varpi,\varphi \in \mho.$$

Then $\mathfrak{R} : \mho \to \mho$ is called Lipschitz mapping and ϑ is the Lipschitz constant. If $\vartheta \in (0,1)$, then $\mathfrak{R}$ is called a contraction.

Theorem 2.7. *[50] Every self contraction mapping* $\mathfrak{R}$ *in a complete metric space* $(\mho,d)$ *has a unique FP.*

Theorem 2.8. *[49] Assume that* $\mho$ *is a Banach space and* $\mathfrak{R}$ *is equi-continuous on* $\mho$. *Then* $\mathfrak{R}$ *have either a FP or the set* $\Pi_0 = \{\varpi \in \mho : \varpi = \theta\mathfrak{R}(\varpi), \text{for some } \theta \in (0,1)\}$ *is unbounded.*

For the benefit of our readers, the remainder of the article is organized as follows: The results demonstrating the existence and uniqueness of the relevant model (1.1) are presented in Section 3. Results for stability for the aforementioned mode (1.1) are the focus of Section 4. The examples in Section 5 are meant to serve as applications to our study. The paper's conclusion is offered at the end.

3. Integral representation for the suggested model

The existence of the solutions for the proposed class of CFDDEs has been affirmed in this section. We define the Green functions that corresponds to the created integral equations and offer the integral representation of the problem under consideration. Also, we are able to provide the necessary conditions for solving the underlying problem of CFDDEs by using the findings of FP theory and analysis.

Theorem 3.1. *Let $\ell, \tau \in (1,2]$. Then the system of CFDDEs (1.1) has the following solution*

$$\varpi(t) = \begin{cases} s(t) + \int_0^1 \aleph(t,\upsilon)\,\Im(\upsilon, \varpi(\upsilon-\rho), \varpi(\upsilon), \varphi(\upsilon))\,d\upsilon, \ t \in D, \\ \varrho(t), \ t \in D_0, \end{cases} \tag{3.1}$$

and

$$\varphi(t) = \begin{cases} r(t) + \int_0^1 \Xi(t,\upsilon)\,\Im(\upsilon, \varphi(\upsilon-\rho), \varphi(\upsilon), \varpi(\upsilon))\,d\upsilon, \ t \in D, \\ \varkappa(t), \ t \in D_0, \end{cases} \tag{3.2}$$

where $\aleph, \Xi : D \times [0,1] \to \mathbb{R}$ are green's functions defined in the proof below.

Proof. Suppose that $\begin{cases} \Im(t, \varpi(t-\rho), \varpi(t), \varphi(t)) = \eta(t), \\ \Im(t, \varphi(t-\rho), \varphi(t), \varpi(t)) = \eta^*(t) \end{cases}$ for $t \in D$, then the considered problem (1.1) takes the form

$$\begin{cases} {}^cD^\ell \varpi(t) - \eta(t) = 0, \ t \in D, \ 1 < \ell \le 2, \\ {}^cD^\tau \varphi(t) - \eta^*(t) = 0, t \in D, \ 1 < \tau \le 2, \\ \varpi(t) = \varrho(t), \ \varpi(1) = \Lambda(\varpi, \varphi), \ t \in D_0, \\ \varphi(t) = \varkappa(t)\ \varphi(1) = \Lambda(\varphi, \varpi), \ t \in D_0. \end{cases} \tag{3.3}$$

In the light of Lemma 2.4, problem (3.3) can be written as

$$\begin{cases} \varpi(t) = Q_1 + Q_2 t + I^\ell \eta(t), \\ \varphi(t) = P_1 + P_2 t + I^\tau \eta^*(t). \end{cases} \tag{3.4}$$

Applying the conditions $\varpi(0) = \varrho(0)$ and $\varphi(0) = \varkappa(0)$, we have

$$Q_1 = \varrho(0) \text{ and } P_1 = \varkappa(0). \tag{3.5}$$

From (3.5) in (3.4), we get

$$\begin{cases} \varpi(t) = \varrho(0) + Q_2 t + I^\ell \eta(t), \\ \varphi(t) = \varkappa(0) + P_2 t + I^\tau \eta^*(t). \end{cases} \tag{3.6}$$

Substituting $\varpi(1) = \Lambda(\varpi, \varphi)$ in the first equation of (3.6), one can obtain

$$\Lambda(\varpi, \varphi) = \varrho(0) + Q_2 + \frac{1}{\Gamma(\ell)} \int_0^1 (1-\upsilon)^{\ell-1} \eta(\upsilon)\,d\upsilon,$$

which yields that

$$Q_2 = \Lambda(\varpi,\varphi) - \varrho(0) - \frac{1}{\Gamma(\ell)}\int_0^1 (1-\upsilon)^{\ell-1}\eta(\upsilon)\,d\upsilon. \tag{3.7}$$

Similarly, from the condition $\varphi(1) = \Lambda(\varphi,\varpi)$ in the second equation of (3.6), we get

$$P_2 = \Lambda(\varphi,\varpi) - \varkappa(0) - \frac{1}{\Gamma(\tau)}\int_0^1 (1-\upsilon)^{\tau-1}\eta(\upsilon)\,d\upsilon. \tag{3.8}$$

Applying (3.7) and (3.8) in (3.6), we get

$$\begin{aligned}
\varpi(t) &= \varrho(0) - t\varrho(0) + t\Lambda(\varpi,\varphi) - \int_0^1 \frac{\eta(\upsilon)}{\Gamma(\ell)(1-\upsilon)^{1-\ell}}d\upsilon + \int_0^t \frac{\eta(\upsilon)}{\Gamma(\ell)(1-\upsilon)^{1-\ell}}d\upsilon \\
&= (1-t)\varrho(0) + t\Lambda(\varpi,\varphi) \\
&\quad - \int_0^t \frac{\eta(\upsilon)}{\Gamma(\ell)(1-\upsilon)^{1-\ell}}d\upsilon - \int_t^1 \frac{\eta(1)}{\Gamma(\ell)(1-\upsilon)^{1-\ell}}d\upsilon + \int_0^t \frac{\eta(\upsilon)}{\Gamma(\ell)(1-\upsilon)^{1-\ell}}d\upsilon,
\end{aligned}$$

and

$$\begin{aligned}
\varphi(t) &= (1-t)\varkappa(0) + t\Lambda(\varphi,\varpi) \\
&\quad - \int_0^t \frac{\eta^*(\upsilon)}{\Gamma(\tau)(1-\upsilon)^{1-\tau}}d\upsilon - \int_t^1 \frac{\eta^*(1)}{\Gamma(\tau)(1-\upsilon)^{1-\tau}}d\upsilon + \int_0^t \frac{\eta^*(\upsilon)}{\Gamma(\tau)(1-\upsilon)^{1-\tau}}d\upsilon.
\end{aligned}$$

In the light of the proven linear BVP results (3.1) and (3.2), we have

$$\varpi(t) = s(t) + \int_0^1 \aleph(t,\upsilon)\,\Im(\upsilon,\varpi(\upsilon-\rho),\varpi(\upsilon),\varphi(\upsilon))\,d\upsilon, \tag{3.9}$$

and

$$\varphi(t) = r(t) + \int_0^1 \Xi(t,\upsilon)\,\Im(\upsilon,\varphi(\upsilon-\rho),\varphi(\upsilon),\varpi(\upsilon))\,d\upsilon, \tag{3.10}$$

where

$$\varpi(t) = \begin{cases} s(t) + \int_0^1 \aleph(t,\upsilon)\,\Im(\upsilon,\varpi(\upsilon-\rho),\varpi(\upsilon),\varphi(\upsilon))\,d\upsilon, & t\in D, \\ \varrho(t), & t\in D_0. \end{cases}$$

and

$$\varphi(t) = \begin{cases} r(t) + \int_0^1 \Xi(t,\upsilon)\,\Im(\upsilon,\varphi(\upsilon-\rho),\varphi(\upsilon),\varpi(\upsilon))\,d\upsilon, & t\in D, \\ \varkappa(t), & t\in D_0. \end{cases}$$

Moreover, the Green functions are represented as

$$\aleph(t,\upsilon)=\frac{1}{\Gamma(\ell)}\begin{cases}(t-\upsilon)^{\ell-1}-(1-\upsilon)^{\ell-1}, & 0\le\upsilon\le t\le 1,\\ -(1-\upsilon)^{\ell-1}, & 0\le t\le\upsilon\le 1,\end{cases}$$

and

$$\Xi(t,\upsilon)=\frac{1}{\Gamma(\tau)}\begin{cases}(t-\upsilon)^{\tau-1}-(1-\upsilon)^{\tau-1}, & 0\le\upsilon\le t\le 1,\\ -(1-\upsilon)^{\tau-1}, & 0\le t\le\upsilon\le 1.\end{cases}$$

Therefore, the integral representation of the considered problem (1.1) is the equations (3.9) and (3.10).

□

4. Existence results

This part is devoted to present the desired solution of the CFDDEs as an operator equation and investigated some hypotheses to obtain the existence results for the stated problem.

Let us consider the operator $\mathfrak{K}:\mho\to\mho$ such that

$$\mathfrak{K}(\varpi,\varphi)(t)=\begin{cases}\mathfrak{K}_1(\varpi,\varphi)(t)+\mathfrak{K}_2(\varpi,\varphi)(t), \ t\in D,\\ \varrho(t), \ t\in D_0,\end{cases}$$

and

$$\mathfrak{K}(\varphi,\varpi)(t)=\begin{cases}\mathfrak{K}_1(\varphi,\varpi)(t)+\mathfrak{K}_2(\varphi,\varpi)(t), \ t\in D,\\ \varkappa(t), \ t\in D_0,\end{cases}$$

where

$$\begin{aligned}\mathfrak{K}_1(\varpi,\varphi)(t) &= (1-t)\varrho(0)+t\Lambda(\varpi,\varphi),\\ \mathfrak{K}_1(\varphi,\varpi)(t) &= (1-t)\varkappa(0)+t\Lambda(\varphi,\varpi),\end{aligned}$$

$$\begin{aligned}\mathfrak{K}_2(\varpi,\varphi)(t) &= \int_0^1\aleph(t,\upsilon)\,\Im(\upsilon,\varpi(\upsilon-\rho),\varpi(\upsilon),\varphi(\upsilon))\,d\upsilon,\\ \mathfrak{K}_2(\varphi,\varpi)(t) &= \int_0^1\Xi(t,\upsilon)\,\Im(\upsilon,\varphi(\upsilon-\rho),\varphi(\upsilon),\varpi(\upsilon))\,d\upsilon,\end{aligned}$$

$$\Omega=\sup_{t\in D}\int_0^1\aleph(t,\upsilon)\,d\upsilon, \text{ and } \Omega^*=\sup_{t\in D}\int_0^1\Xi(t,\upsilon)\,d\upsilon.$$

The following are some essential assumptions that will be applied in this study.

(p_i) For $\varpi_1,\varpi_2,\varphi_1,\varphi_2\in\mho$, there is $B_{\Im}\ge 0$ so that

$$\left|\Im(t,\varpi_1(t-\rho),\varpi_1(t),\varphi_1(t))-\Im(t,\varpi_2(t-\rho),\varpi_2(t),\varphi_2(t))\right|\le B_{\Im}\left(\|\varpi_1-\varpi_2\|+\|\varphi_1-\varphi_2\|\right).$$

(p_{ii}) For $\varpi_1, \varpi_2, \varphi_1, \varphi_2 \in \mho$, there is $B_\Lambda, B_\Lambda^* \in [0,1)$ so that

$$|\Lambda(\varpi_1, \varphi_1) - \Lambda(\varpi_2, \varphi_2)| \leq B_\Lambda (\|\varpi_1 - \varpi_2\| + \|\varphi_1 - \varphi_2\|).$$

(p_{iii}) For any $\varpi, \varphi \in \mho$, there are $R_\Lambda, T_\Lambda, T_\Lambda^* \geq 0$ and $u \in [0,1)$ so that

$$|\Lambda(\varpi, \varphi)| \leq R_\Lambda (\|\varpi\|^u + \|\varphi\|^u) + T_\Lambda \text{ and } |\Lambda(\varphi, \varpi)| \leq R_\Lambda (\|\varpi\|^u + \|\varphi\|^u) + T_\Lambda^*.$$

(p_{iv}) For any $u_0 \in [0,1)$, $\varpi, \varphi \in \mho$, there are $R_\mathfrak{J}, T_\mathfrak{J}, T_\mathfrak{J}^* \geq 0$ so that

$$\begin{aligned} \left|\mathfrak{J}(t, \varpi(t-\rho), \varpi(t), \varphi(t))\right| &\leq R_\mathfrak{J} (\|\varpi\|^{u_0} + \|\varphi\|^{u_0}) + T_\mathfrak{J}, \\ \text{and } \left|\mathfrak{J}(t, \varphi(t-\rho), \varphi(t), \varpi(t))\right| &\leq R_\mathfrak{J} (\|\varpi\|^{u_0} + \|\varphi\|^{u_0}) + T_\mathfrak{J}^*. \end{aligned}$$

Now, our first main result is as follows:

Theorem 4.1. *In view of the assumptions (p_i) and (p_{ii}), the operator $\mathfrak{R}$ has at most one FP provided that $B_\Lambda + \Omega B_\mathfrak{J} = B < 1$ and $B_\Lambda + \Omega^* B_\mathfrak{J} = B^* < 1$.*

Proof. Let $\varpi^*, \varphi^* \in \mho$, if $t \in D_0$, then

$$\left\|\mathfrak{R}(\varpi, \varphi)(t) - \mathfrak{R}(\varpi^*, \varphi^*)(t)\right\| \geq 0.$$

If $t \in D$, we have

$$\begin{aligned} & \left|\mathfrak{R}(\varpi, \varphi)(t) - \mathfrak{R}(\varpi^*, \varphi^*)(t)\right| \\ = & \left|\mathfrak{R}_1(\varpi, \varphi)(t) + \mathfrak{R}_2(\varpi, \varphi)(t) - \mathfrak{R}_1(\varpi^*, \varphi^*)(t) - \mathfrak{R}_2(\varpi^*, \varphi^*)(t)\right| \\ \leq & \left|\mathfrak{R}_1(\varpi, \varphi)(t) - \mathfrak{R}_1(\varpi^*, \varphi^*)(t)\right| + \left|\mathfrak{R}_2(\varpi, \varphi)(t) - \mathfrak{R}_2(\varpi^*, \varphi^*)(t)\right| \\ \leq & |(1-t)\varrho(0) + t\Lambda(\varpi, \varphi) - (1-t)\varrho(0) - t\Lambda(\varpi^*, \varphi^*)| \\ & + \left|\int_0^1 \aleph(t, \upsilon)\, \mathfrak{J}(\upsilon, \varpi(\upsilon-\rho), \varpi(\upsilon), \varphi(\upsilon))\, d\upsilon - \int_0^1 \aleph(t, \upsilon)\, \mathfrak{J}(\upsilon, \varpi^*(\upsilon-\rho), \varpi^*(\upsilon), \varphi^*(\upsilon))\, d\upsilon\right| \\ \leq & |\Lambda(\varpi, \varphi) - \Lambda(\varpi^*, \varphi^*)| \\ & + \int_0^1 |\aleph(t, \upsilon)| \left|\mathfrak{J}(\upsilon, \varpi(\upsilon-\rho), \varpi(\upsilon), \varphi(\upsilon)) - \mathfrak{J}(\upsilon, \varpi^*(\upsilon-\rho), \varpi^*(\upsilon), \varphi^*(\upsilon))\right| d\upsilon. \end{aligned}$$

Using assumptions (p_i) and (p_{ii}), we can write

$$\begin{aligned} \left\|\mathfrak{R}(\varpi, \varphi) - \mathfrak{R}(\varpi^*, \varphi^*)\right\|_\mho \leq & B_\Lambda \left(\|\varpi - \varpi^*\|_{\mho_1} + \|\varphi - \varphi^*\|_{\mho_2}\right) \\ & + \int_0^1 |\aleph(t, \upsilon)|\, B_\mathfrak{J} \left(\|\varpi - \varpi^*\|_{\mho_1} + \|\varphi - \varphi^*\|_{\mho_2}\right) d\upsilon \\ \leq & \left(B_\Lambda + B_\mathfrak{J} \int_0^1 |\aleph(t, \upsilon)|\, d\upsilon\right) \left(\|\varpi - \varpi^*\|_{\mho_1} + \|\varphi - \varphi^*\|_{\mho_2}\right) \end{aligned}$$

$$\begin{aligned} &\leq (B_\Lambda + \Omega B_{\mathfrak{J}})\left(\|\varpi - \varpi^*\|_{\mho_1} + \|\varphi - \varphi^*\|_{\mho_2}\right) \\ &= B\left(\|\varpi - \varpi^*\|_{\mho_1} + \|\varphi - \varphi^*\|_{\mho_2}\right). \end{aligned} \tag{4.1}$$

Similarly, one can obtain

$$\left\|\mathfrak{R}(\varphi, \varpi) - \mathfrak{R}(\varphi^*, \varpi^*)\right\|_{\mho} \leq B^*\left(\|\varphi - \varphi^*\|_{\mho_2} + \|\varpi - \varpi^*\|_{\mho_1}\right), \tag{4.2}$$

where $B = B_\Lambda + \Omega B_{\mathfrak{J}} < 1$, and $B^* = B_\Lambda + \Omega^* B_{\mathfrak{J}} < 1$.

It follows from (4.1), (4.2) and the Banach contraction principle that there is a unique FP of the operator $\mathfrak{R}$. Thus, there is a unique solution to the problem (1.1) under consideration. □

Theorem 4.2. *Under the assumptions (p_{iii}), the operator $\mathfrak{R}_1$ is equi-continuious and satisfies the following growth conditions:*

$$\left\|\mathfrak{R}_1(\varpi, \varphi)(t)\right\|_{\mho_1} = R_\Lambda(\|\varpi\|^u + \|\varphi\|^u) + T \text{ and } \left\|\mathfrak{R}_1(\varphi, \varpi)(t)\right\|_{\mho_1} = R_\Lambda(\|\varpi\|^u + \|\varphi\|^u) + T^*.$$

Proof. Assume that $\varpi_n, \varphi_n \in \phi \subseteq \mho$, which converge to ϖ and φ respectively. For the continuity of $\mathfrak{R}_1$, we have to prove $\mathfrak{R}_1(\varpi_n, \varphi_n) \to \mathfrak{R}_1(\varpi, \varphi)$ as $n \to \infty$, where $\phi = \{(\varpi, \varphi) \in \mho^2 : \|(\varpi, \varphi)\| \leq l \in \mathbb{R}\}$. For this, consider

$$\begin{aligned} \left\|\mathfrak{R}_1(\varpi_n, \varphi_n)(t) - \mathfrak{R}_1(\varpi, \varphi)(t)\right\|_{\mho_1} &= \|(1-t)\varrho(0) + t\Lambda(\varpi_n, \varphi_n) - (1-t)\varrho(0) - t\Lambda(\varpi, \varphi)\| \\ &= \|t\Lambda(\varpi_n, \varphi_n) - t\Lambda(\varpi, \varphi)\| \\ &= \|t\|\,\|\Lambda(\varpi_n, \varphi_n) - \Lambda(\varpi, \varphi)\|. \end{aligned}$$

Since Λ is continuous, we get

$$\left\|\mathfrak{R}_1(\varpi_n, \varphi_n)(t) - \mathfrak{R}_1(\varpi, \varphi)(t)\right\|_{\mho_1} \to 0, \text{ as } n \to \infty.$$

Thus, we conclude that $\mathfrak{R}_1$ is continuous. For the growth stipulation, consider

$$\begin{aligned} \left\|\mathfrak{R}_1(\varpi, \varphi)(t)\right\|_{\mho_1} &= \|(1-t)\varrho(0) - t\Lambda(\varpi, \varphi)\| \\ &\leq \|(1-t)\varrho(0)\| + \|t\Lambda(\varpi, \varphi)\| \\ &\leq \|(1-t)\|\,\|\varrho(0)\| + \|t\|\,\|\Lambda(\varpi, \varphi)\| \\ &\leq |\varrho(0)| + |\Lambda(\varpi, \varphi)| \\ &= |\varrho(0)| + R_\Lambda(\|\varpi\|^u + \|\varphi\|^u) + T_\Lambda \\ &= T + R_\Lambda(\|\varpi\|^u + \|\varphi\|^u). \end{aligned}$$

In the same scenario, one can get

$$\left\|\mathfrak{R}_1(\varphi, \varpi)(t)\right\|_{\mho_1} \leq T^* + R_\Lambda(\|\varpi\|^u + \|\varphi\|^u),$$

where $T = |\varrho(0)| + T_\Lambda$ and $T^* = |\varkappa(0)| + T_\Lambda$. Hence, $\mathfrak{R}_1$ fulfills the above growth stipulations. To complete the prove, assume that ϕ^* is any bounded subsetof ϕ, then using the growth stipulations of $\mathfrak{R}_1$, we have $\mathfrak{R}_1(\phi^*)$ is bounded. Suppose that $\mathfrak{R}_1$ is a mapping from bounded set into equi-continuous set, then, we get

$$\left\|\mathfrak{R}_1(\varpi, \varphi)(t_1) - \mathfrak{R}_1(\varpi, \varphi)(t_2)\right\|_{\mho_1} = \|(1-t_1)\varrho(0) - t_1\Lambda(\varpi, \varphi) - (1-t_2)\varrho(0) - t_2\Lambda(\varpi, \varphi)\|$$

$$= \|(t_2 - t_1)\varrho(0) - (t_1 - t_2)\,\Lambda(\varpi, \varphi)\| \to 0, \text{ as } t_1 \to t_2.$$

Similarly,

$$\left\|\Re_1(\varphi, \varpi)(t_1) - \Re_1(\varphi, \varpi)(t_2)\right\|_{\mho_1} = \|(t_2 - t_1)\varkappa(0) - (t_1 - t_2)\,\Lambda(\varphi, \varpi)\| \to 0, \text{ as } t_1 \to t_2.$$

Therefore, $\Re_1$ is equi-continuious. □

Theorem 4.3. *Via the assumption* (p_{iv})*, the operator* $\Re_2$ *is completely continuous and fulfills the growth conditions below:*

$$\left\|\Re_2(\varpi, \varphi)(t)\right\|_{\mho_2} = Z\left(\|\varpi\|^{u_0} + \|\varphi\|^{u_0}\right) + W \text{ and } \left\|\Re_2(\varphi, \varpi)(t)\right\|_{\mho_2} = Z\left(\|\varpi\|^{u} + \|\varphi\|^{u}\right) + W^*.$$

Proof. Similar to the proof of Theorem 4.2, suppose that $\varpi_n, \varphi_n \in \phi \subseteq \mho$, which converge to ϖ and φ respectively. For the continuity of $\Re_2$, we have to show $\Re_2(\varpi_n, \varphi_n) \to \Re_2(\varpi, \varphi)$ as $n \to \infty$. For this, consider

$$\begin{aligned}
&\left|\Re_2(\varpi_n, \varphi_n)(t) - \Re_2(\varpi, \varphi)(t)\right| \\
= &\left|\int_0^1 \aleph(t, \upsilon)\,\Im(\upsilon, \varpi_n(\upsilon - \rho), \varpi_n(\upsilon), \varphi_n(\upsilon))\,d\upsilon - \int_0^1 \aleph(t, \upsilon)\,\Im(\upsilon, \varpi(\upsilon - \rho), \varpi(\upsilon), \varphi(\upsilon))\,d\upsilon\right| \\
\leq &\left| t\int_0^1 \left[\Im(\upsilon, \varpi_n(\upsilon - \rho), \varpi_n(\upsilon), \varphi_n(\upsilon)) - \Im(\upsilon, \varpi(\upsilon - \rho), \varpi(\upsilon), \varphi(\upsilon))\right]\frac{(1-\upsilon)^{\ell-1}}{\Gamma(\ell)}d\upsilon\right. \\
&\left. + \int_0^t \left[\Im(\upsilon, \varpi_n(\upsilon - \rho), \varpi_n(\upsilon), \varphi_n(\upsilon)) - \Im(\upsilon, \varpi(\upsilon - \rho), \varpi(\upsilon), \varphi(\upsilon))\right]\frac{(t-\upsilon)^{\ell-1}}{\Gamma(\ell)}d\upsilon\right| \\
\leq &\ |t|\int_0^1 \left|\Im(\upsilon, \varpi_n(\upsilon - \rho), \varpi_n(\upsilon), \varphi_n(\upsilon)) - \Im(\upsilon, \varpi(\upsilon - \rho), \varpi(\upsilon), \varphi(\upsilon))\right|\frac{(1-\upsilon)^{\ell-1}}{\Gamma(\ell)}d\upsilon \\
&\left. + \int_0^t \left|\Im(\upsilon, \varpi_n(\upsilon - \rho), \varpi_n(\upsilon), \varphi_n(\upsilon)) - \Im(\upsilon, \varpi(\upsilon - \rho), \varpi(\upsilon), \varphi(\upsilon))\right|\frac{(t-\upsilon)^{\ell-1}}{\Gamma(\ell)}d\upsilon\right|,
\end{aligned} \tag{4.3}$$

and

$$\begin{aligned}
&\left|\Re_2(\varphi_n, \varpi_n)(t) - \Re_2(\varphi, \varpi)(t)\right| \\
= &\left|\int_0^1 \Xi(t, \upsilon)\,\Im(\upsilon, \varphi_n(\upsilon - \rho), \varphi_n(\upsilon), \varpi_n(\upsilon))\,d\upsilon - \int_0^1 \Xi(t, \upsilon)\,\Im(\upsilon, \varphi(\upsilon - \rho), \varphi(\upsilon), \varpi(\upsilon))\,d\upsilon\right| \\
\leq &\ |t|\int_0^1 \left|\Im(\upsilon, \varpi_n(\upsilon - \rho), \varpi_n(\upsilon), \varphi_n(\upsilon)) - \Im(\upsilon, \varpi(\upsilon - \rho), \varpi(\upsilon), \varphi(\upsilon))\right|\frac{(1-\upsilon)^{\tau-1}}{\Gamma(\tau)}d\upsilon \\
&\left. + \int_0^t \left|\Im(\upsilon, \varphi_n(\upsilon - \rho), \varphi_n(\upsilon), \varpi_n(\upsilon)) - \Im(\upsilon, \varphi(\upsilon - \rho), \varphi(\upsilon), \varpi(\upsilon))\right|\frac{(t-\upsilon)^{\tau-1}}{\Gamma(\tau)}d\upsilon\right|.
\end{aligned} \tag{4.4}$$

Taking $n \to \infty$ in (4.3) and (4.4), we conclude that

$$\left\|\mathfrak{R}_2(\varpi_n,\varphi_n)(t)-\mathfrak{R}_2(\varpi,\varphi)(t)\right\|_{\mho_2}\to 0 \text{ and } \left\|\mathfrak{R}_2(\varphi_n,\varpi_n)(t)-\mathfrak{R}_2(\varphi,\varpi)(t)\right\|_{\mho_2}\to 0.$$

For the growth stipulations, we proceed as

$$\begin{aligned}
\left\|\mathfrak{R}_2(\varphi,\varpi)(t)\right\|_{\mho_2} &= \left\|\int_0^1 \mathfrak{I}(\upsilon,\varphi(\upsilon-\rho),\varphi(\upsilon),\varpi(\upsilon))\,\Xi(t,\upsilon)\,d\upsilon\right\| \\
&\leq \left\|t\int_0^1 \mathfrak{I}(\upsilon,\varphi(\upsilon-\rho),\varphi(\upsilon),\varpi(\upsilon))\frac{(1-\upsilon)^{\ell-1}}{\Gamma(\ell)}d\upsilon\right\| \\
&\quad+\left\|\int_0^t \mathfrak{I}(\upsilon,\varphi(\upsilon-\rho),\varphi_n(\upsilon),\varpi(\upsilon))\frac{(t-\upsilon)^{\ell-1}}{\Gamma(\ell)}d\upsilon\right\| \\
&\leq \sup_{t\in D}\left\{|t|\int_0^1 \left|\mathfrak{I}(\upsilon,\varphi(\upsilon-\rho),\varphi(\upsilon),\varpi(\upsilon))\right|\frac{(1-\upsilon)^{\ell-1}}{\Gamma(\ell)}d\upsilon\right\} \\
&\quad+\sup_{t\in D}\left\{\int_0^t \left|\mathfrak{I}(\upsilon,\varphi(\upsilon-\rho),\varphi(\upsilon),\varpi(\upsilon))\right|\frac{(t-\upsilon)^{\ell-1}}{\Gamma(\ell)}d\upsilon\right\} \\
&\leq \sup_{t\in D}\left\{\int_0^1 \left|\mathfrak{I}(\upsilon,\varphi(\upsilon-\rho),\varphi(\upsilon),\varpi(\upsilon))\right|\frac{(1-\upsilon)^{\ell-1}}{\Gamma(\ell)}d\upsilon\right\} \\
&\quad+\sup_{t\in D}\left\{\int_0^t \left|\mathfrak{I}(\upsilon,\varphi(\upsilon-\rho),\varphi(\upsilon),\varpi(\upsilon))\right|\frac{(t-\upsilon)^{\ell-1}}{\Gamma(\ell)}d\upsilon\right\},
\end{aligned}$$

which yields that

$$\begin{aligned}
\left\|\mathfrak{R}_2(\varphi,\varpi)(t)\right\|_{\mho_2} &\leq \frac{1}{\Gamma(\ell)}\sup_{t\in D}\left\{\int_0^1 (1-\upsilon)^{\ell-1}\left[R_{\mathfrak{I}}\left(\|\varpi\|^{u_0}+\|\varphi\|^{u_0}\right)+T_{\mathfrak{I}}\right]d\upsilon\right\} \\
&\quad+\frac{1}{\Gamma(\ell)}\sup_{t\in D}\left\{\int_0^t (t-\upsilon)^{\ell-1}\left[R_{\mathfrak{I}}\left(\|\varpi\|^{u_0}+\|\varphi\|^{u_0}\right)+T_{\mathfrak{I}}\right]d\upsilon\right\} \\
&\leq \frac{\left(R_{\mathfrak{I}}\left(\|\varpi\|^{u_0}+\|\varphi\|^{u_0}\right)+T_{\mathfrak{I}}\right)}{\Gamma(\ell)}\left(\sup_{t\in D}\int_0^1 (1-\upsilon)^{\ell-1}d\upsilon\right) \\
&\quad+\frac{\left(R_{\mathfrak{I}}\left(\|\varpi\|^{u_0}+\|\varphi\|^{u_0}\right)+T_{\mathfrak{I}}\right)}{\Gamma(\ell)}\left(\sup_{t\in D}\int_0^t (t-\upsilon)^{\ell-1}d\upsilon\right) \\
&= \frac{\left(R_{\mathfrak{I}}\left(\|\varpi\|^{u_0}+\|\varphi\|^{u_0}\right)+T_{\mathfrak{I}}\right)}{\ell\Gamma(\ell)}+\frac{\left(R_{\mathfrak{I}}\left(\|\varpi\|^{u_0}+\|\varphi\|^{u_0}\right)+T_{\mathfrak{I}}\right)}{\Gamma(\ell)}\sup_{t\in D}\left(\frac{t^{\ell}}{\ell}\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{(R_{\mathfrak{I}}(\|\varpi\|^{u_0}+\|\varphi\|^{u_0})+T_{\mathfrak{I}})}{\ell\Gamma(\ell)}+\frac{(R_{\mathfrak{I}}(\|\varpi\|^{u_0}+\|\varphi\|^{u_0})+T_{\mathfrak{I}})}{\ell\Gamma(\ell)}\\
&= \frac{2}{\Gamma(\ell+1)}(R_{\mathfrak{I}}(\|\varpi\|^{u_0}+\|\varphi\|^{u_0})+T_{\mathfrak{I}}) \leq Z(\|\varpi\|^{u_0}+\|\varphi\|^{u_0})+W.
\end{aligned}
$$

Analogously,

$$
\begin{aligned}
\left\|\mathfrak{R}_2(\varphi,\varpi)(t)\right\|_{\mho_2} &\leq \sup_{t\in D}\left\{\int_0^1 \left|\mathfrak{I}(\upsilon,\varpi(\upsilon-\rho),\varpi(\upsilon),\varphi(\upsilon))\right|\frac{(1-\upsilon)^{\tau-1}}{\Gamma(\tau)}d\upsilon\right\}\\
&\quad+\sup_{t\in D}\left\{\int_0^t \left|\mathfrak{I}(\upsilon,\varpi(\upsilon-\rho),\varpi(\upsilon),\varphi(\upsilon))\right|\frac{(t-\upsilon)^{\tau-1}}{\Gamma(\tau)}d\upsilon\right\}\\
&\leq \frac{1}{\Gamma(\ell)}\sup_{t\in D}\left\{\int_0^1 (1-\upsilon)^{\ell-1}\left[R_{\mathfrak{I}}(\|\varpi\|^{u_0}+\|\varphi\|^{u_0})+T^*_{\mathfrak{I}}\right]d\upsilon\right\}\\
&\quad+\frac{1}{\Gamma(\ell)}\sup_{t\in D}\left\{\int_0^t (t-\upsilon)^{\ell-1}\left[R_{\mathfrak{I}}(\|\varpi\|^{u_0}+\|\varphi\|^{u_0})+T^*_{\mathfrak{I}}\right]d\upsilon\right\}\\
&\leq \frac{\left(R_{\mathfrak{I}}(\|\varpi\|^{u_0}+\|\varphi\|^{u_0})+T^*_{\mathfrak{I}}\right)}{\tau\Gamma(\tau)}+\frac{(R_{\mathfrak{I}}(\|\varpi\|^{u_0}+\|\varphi\|^{u_0})+T_{\mathfrak{I}})}{\Gamma(\tau)}\sup_{t\in D}\left(\frac{t^\tau}{\tau}\right)\\
&= \frac{2}{\Gamma(\tau+1)}\left(R_{\mathfrak{I}}(\|\varpi\|^{u_0}+\|\varphi\|^{u_0})+T^*_{\mathfrak{I}}\right) \leq Z(\|\varpi\|^{u_0}+\|\varphi\|^{u_0})+W^*,
\end{aligned}
$$

where

$$
Z=\frac{2R_{\mathfrak{I}}}{\Gamma(\tau+1)},\ W=\frac{2T_{\mathfrak{I}}}{\Gamma(\tau+1)} \text{ and } W^*=\frac{2T^*_{\mathfrak{I}}}{\Gamma(\tau+1)}.
$$

Therefore, $\mathfrak{R}_2$ satisfies the growth stipulations above. Assume that ϕ^* is any bounded subset of ϕ, from the growth stipulations of $\mathfrak{R}_2$, we have $\mathfrak{R}_2(\phi^*)$ is bounded. Now, we claim that $\mathfrak{R}_2$ is a mapping from bounded set into equi-continuous set.For this, we consider $t_2 \leq t_1$. So, we have the following cases:

(1) If $0 \leq t \leq \upsilon \leq 1$, we get

$$
\begin{aligned}
&\left|\mathfrak{R}_2(\varpi,\varphi)(t_1)-\mathfrak{R}_2(\varpi,\varphi)(t_2)\right|\\
&= \left|\int_0^1 \aleph(t_1,\upsilon)\,\mathfrak{I}(\upsilon,\varpi(\upsilon-\rho),\varpi(\upsilon),\varphi(\upsilon))\,d\upsilon-\int_0^1 \aleph(t_2,\upsilon)\,\mathfrak{I}(\upsilon,\varpi(\upsilon-\rho),\varpi(\upsilon),\varphi(\upsilon))\,d\upsilon\right|\\
&= \left|\int_0^1 [\aleph(t_1,\upsilon)-\aleph(t_2,\upsilon)]\,\mathfrak{I}(\upsilon,\varpi(\upsilon-\rho),\varpi(\upsilon),\varphi(\upsilon))\,d\upsilon\right|\\
&= |t_1-t_2|\left|\int_0^1 (\mathfrak{I}(\upsilon,\varpi(\upsilon-\rho),\varpi(\upsilon),\varphi(\upsilon)))\frac{(1-\upsilon)^{\ell-1}}{\Gamma(\ell)}d\upsilon\right| \to 0 \text{ as } t_1\to t_2,
\end{aligned}
$$

and

$$
\begin{aligned}
& \left|\Re_2(\varphi,\varpi)(t_1)-\Re_2(\varphi,\varpi)(t_2)\right| \\
= & \left|\int_0^1 \Xi(t_1,\upsilon)\,\Im(\upsilon,\varphi(\upsilon-\rho),\varphi(\upsilon),\varpi(\upsilon))\,d\upsilon-\int_0^1 \Xi(t_2,\upsilon)(\upsilon,\varphi(\upsilon-\rho),\varphi(\upsilon),\varpi(\upsilon))\,d\upsilon\right| \\
= & \left|\int_0^1 [\Xi(t_1,\upsilon)-\Xi(t_2,\upsilon)]\,\Im(\upsilon,\varphi(\upsilon-\rho),\varphi(\upsilon),\varpi(\upsilon))\,d\upsilon\right| \\
= & |t_1-t_2|\left|\int_0^1 (\Im(\upsilon,\varphi(\upsilon-\rho),\varphi(\upsilon),\varpi(\upsilon)))\frac{(1-\upsilon)^{\tau-1}}{\Gamma(\tau)}d\upsilon\right| \to 0 \text{ as } t_1\to t_2.
\end{aligned}
$$

(2) If $0\le\upsilon\le t\le 1$, we get

$$
\begin{aligned}
& \left|\Re_2(\varpi,\varphi)(t_1)-\Re_2(\varpi,\varphi)(t_2)\right| \\
= & \left|\int_0^1 \aleph(t_1,\upsilon)\,\Im(\upsilon,\varpi(\upsilon-\rho),\varpi(\upsilon),\varphi(\upsilon))\,d\upsilon-\int_0^1 \aleph(t_2,\upsilon)\,\Im(\upsilon,\varpi(\upsilon-\rho),\varpi(\upsilon),\varphi(\upsilon))\,d\upsilon\right| \\
= & \left|\int_0^1 [\aleph(t_1,\upsilon)-\aleph(t_2,\upsilon)]\,\Im(\upsilon,\varpi(\upsilon-\rho),\varpi(\upsilon),\varphi(\upsilon))\,d\upsilon\right| \\
\le & \left|(t_1-t_2)\int_0^1 (\Im(\upsilon,\varpi(\upsilon-\rho),\varpi(\upsilon),\varphi(\upsilon)))\frac{(1-\upsilon)^{\ell-1}}{\Gamma(\ell)}d\upsilon\right| \\
& +\left|\int_0^{t_1} (\Im(\upsilon,\varpi(\upsilon-\rho),\varpi(\upsilon),\varphi(\upsilon)))\frac{(t_1-\upsilon)^{\ell-1}-(t_2-\upsilon)^{\ell-1}}{\Gamma(\ell)}d\upsilon\right| \\
& +\left|\int_{t_1}^{t_2} (\Im(\upsilon,\varpi(\upsilon-\rho),\varpi(\upsilon),\varphi(\upsilon)))\frac{(t_2-\upsilon)^{\ell-1}}{\Gamma(\ell)}d\upsilon\right|
\end{aligned}
$$

which implies that

$$
\begin{aligned}
& \left|\Re_2(\varpi,\varphi)(t_1)-\Re_2(\varpi,\varphi)(t_2)\right| \\
\le & \frac{1}{\Gamma(\ell)}|t_1-t_2|\int_0^1 (1-\upsilon)^{\ell-1}\left(R_{\Im}\left(\|\varpi\|^{u_0}+\|\varphi\|^{u_0}\right)+T_{\Im}\right)d\upsilon \\
& +\frac{1}{\Gamma(\ell)}\int_0^{t_1}\left[(t_1-\upsilon)^{\ell-1}-(t_2-\upsilon)^{\ell-1}\right]\left(R_{\Im}\left(\|\varpi\|^{u_0}+\|\varphi\|^{u_0}\right)+T_{\Im}\right)d\upsilon \\
& +\frac{1}{\Gamma(\ell)}\int_{t_1}^{t_2}(t_2-\upsilon)^{\ell-1}\left(R_{\Im}\left(\|\varpi\|^{u_0}+\|\varphi\|^{u_0}\right)+T_{\Im}\right)d\upsilon
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{(R_{\mathfrak{I}}(\|\varpi\|^{u_0}+\|\varphi\|^{u_0})+T_{\mathfrak{I}})}{\Gamma(\ell)}\left\{|t_1-t_2|\int_0^1(1-\upsilon)^{\ell-1}d\upsilon\right.\\
&\left.+\int_0^{t_1}\left((t_1-\upsilon)^{\ell-1}-(t_2-\upsilon)^{\ell-1}\right)d\upsilon+\int_{t_1}^{t_2}(t_2-\upsilon)^{\ell-1}d\upsilon\right\}\\
&= \frac{(R_{\mathfrak{I}}(\|\varpi\|^{u_0}+\|\varphi\|^{u_0})+T_{\mathfrak{I}})}{\Gamma(\ell)}\left(\frac{|t_1-t_2|}{\ell}+\frac{t_1^{\ell}-t_2^{\ell}+(t_2-t_1)^{\ell}}{\ell}+\frac{(t_2-t_1)^{\ell}}{\ell}\right)\to 0,
\end{aligned}
$$

as $t_1\to t_2$. Similarly,

$$
\begin{aligned}
&\left|\mathfrak{R}_2(\varphi,\varpi)(t_1)-\mathfrak{R}_2(\varphi,\varpi)(t_2)\right|\\
&\leq \frac{1}{\Gamma(\tau)}|t_1-t_2|\int_0^1(1-\upsilon)^{\tau-1}\left(R_{\mathfrak{I}}(\|\varpi\|^{u_0}+\|\varphi\|^{u_0})+T^*_{\mathfrak{I}}\right)d\upsilon\\
&+\frac{1}{\Gamma(\tau)}\int_0^{t_1}\left[(t_1-\upsilon)^{\tau-1}-(t_2-\upsilon)^{\tau-1}\right]\left(R_{\mathfrak{I}}(\|\varpi\|^{u_0}+\|\varphi\|^{u_0})+T^*_{\mathfrak{I}}\right)d\upsilon\\
&+\frac{1}{\Gamma(\tau)}\int_{t_1}^{t_2}(t_2-\upsilon)^{\tau-1}\left(R_{\mathfrak{I}}(\|\varpi\|^{u_0}+\|\varphi\|^{u_0})+T^*_{\mathfrak{I}}\right)d\upsilon\\
&\leq \frac{\left(R_{\mathfrak{I}}(\|\varpi\|^{u_0}+\|\varphi\|^{u_0})+T^*_{\mathfrak{I}}\right)}{\Gamma(\tau)}\left\{|t_1-t_2|\int_0^1(1-\upsilon)^{\tau-1}d\upsilon\right.\\
&\left.+\int_0^{t_1}\left((t_1-\upsilon)^{\tau-1}-(t_2-\upsilon)^{\tau-1}\right)d\upsilon+\int_{t_1}^{t_2}(t_2-\upsilon)^{\tau-1}d\upsilon\right\}\\
&= \frac{\left(R_{\mathfrak{I}}(\|\varpi\|^{u_0}+\|\varphi\|^{u_0})+T^*_{\mathfrak{I}}\right)}{\Gamma(\tau)}\left(\frac{|t_1-t_2|}{\tau}+\frac{t_1^{\tau}-t_2^{\tau}+(t_2-t_1)^{\tau}}{\tau}+\frac{(t_2-t_1)^{\tau}}{\tau}\right)\to 0,
\end{aligned}
$$

as $t_1\to t_2$. Based on the cases (1) and (2), we conclude that $\mathfrak{R}_2$ is completely continuous. □

Theorem 4.4. *Assume that the hypotheses* (p_{iii}) *and* (p_{iv}) *are true. Then the CFDDEs (1.1) have a solution.*

Proof. The completely continuous of $\mathfrak{R}_1$ and $\mathfrak{R}_2$ leads to the equi-continuous of $\mathfrak{R}$. Let us consider the set $\Pi_0=\{(\varpi,\varphi)\in\mho:\varpi=\theta\mathfrak{R}(\varpi,\varphi)$ and $\varphi=\theta\mathfrak{R}(\varphi,\varpi)$, for some $\theta<1\}$. If $t\in D$, we have

$$
\begin{aligned}
\|\varpi\|_{\mho} &= \left\|\theta\mathfrak{R}(\varpi,\varphi)\right\|_{\mho}=|\theta|\left\|\mathfrak{R}(\varpi,\varphi)\right\|_{\mho}=\theta\left\|\mathfrak{R}(\varpi,\varphi)\right\|_{\mho}\\
&\leq \left\|\mathfrak{R}_1(\varpi,\varphi)\right\|_{\mho_1}+\left\|\mathfrak{R}_2(\varpi,\varphi)\right\|_{\mho_2}\\
&\leq T+R_{\Lambda}(\|\varpi\|^{u}+\|\varphi\|^{u})+Z(\|\varpi\|^{u_0}+\|\varphi\|^{u_0})+W, \qquad (4.5)
\end{aligned}
$$

and

$$
\|\varphi\|_{\mho} = \left\|\theta\mathfrak{R}(\varphi,\varpi)\right\|_{\mho}=|\theta|\left\|\mathfrak{R}(\varphi,\varpi)\right\|_{\mho}=\theta\left\|\mathfrak{R}(\varphi,\varpi)\right\|_{\mho}
$$

$$\leq \left\|\mathfrak{R}_1(\varphi,\varpi)\right\|_{\mho_1} + \left\|\mathfrak{R}_2(\varphi,\varpi)\right\|_{\mho_2}$$
$$\leq T^* + R_\Lambda\left(\|\varpi\|^u + \|\varphi\|^u\right) + Z\left(\|\varpi\|^{u_0} + \|\varphi\|^{u_0}\right) + W^*. \tag{4.6}$$

Also, for $t \in D_0$, we get

$$\|\varpi\|_{\mho_3} = |\theta| \left\|\mathfrak{R}(\varpi,\varphi)\right\| = \theta \|\varrho(t)\| \leq \|\varrho(t)\|, \tag{4.7}$$

and

$$\|\varphi\|_{\mho_3} = |\theta| \left\|\mathfrak{R}(\varphi,\varpi)\right\| = \theta \|\varkappa(t)\| \leq \|\varkappa(t)\|. \tag{4.8}$$

It follows from (4.5)-(4.8) that Π_0 is bounded. If Π_0 is unbounded and let say $\Pi_0 = U \to \infty$. Then from(4.5) and (4.6), we obtain that for any $\varpi, \varphi \in \Pi_0$

$$1 \leq \lim_{U\to\infty}\left(\frac{T + R_\Lambda\left(\|\varpi\|^u + \|\varphi\|^u\right)}{U} + \frac{Z\left(\|\varpi\|^{u_0} + \|\varphi\|^{u_0}\right) + W}{U}\right) = 0,$$

and

$$1 \leq \lim_{U\to\infty}\left(\frac{T^* + R_\Lambda\left(\|\varpi\|^u + \|\varphi\|^u\right)}{U} + \frac{Z\left(\|\varpi\|^{u_0} + \|\varphi\|^{u_0}\right) + W^*}{U}\right) = 0,$$

which is a contradiction. Thus, Π_0 is bounded. In view of Theorem 2.8, $\mathfrak{R}$ has a FP which is a solution to the proposed problem (1.1). □

5. Stability analysis

Here, for the stability study of the suggested problem (1.1), the authors give some specific predictions. We begin this part with the following definitions.

Definition 5.1. We say that a solution $(\varpi(t), \varphi(t))$ of the considered system (1.1) is HU stable, if for a unique solution $(\widehat{\varpi}(t), \widehat{\varphi}(t))$, there is a constant $P_{\mathfrak{J}} \geq 0$ so that for each $\varpi, \varphi \in \mho$, $\varepsilon \geq 0$ and $t \in D$, we have

$$\left|{}^cD^\ell\varpi(t) - \mathfrak{J}\left(t, \varpi(t-\rho), \varpi(t), \varphi(t)\right)\right| \leq \varepsilon \text{ and } \left|{}^cD^\tau\varphi(t) - \mathfrak{J}\left(t, \varphi(t-\rho), \varphi(t), \varpi(t)\right)\right| \leq \varepsilon, \tag{5.1}$$

such that

$$\left\|(\varpi, \widehat{\varpi}) - (\varphi - \widehat{\varphi})\right\|_{\mho} \leq \varepsilon P_{\mathfrak{J}}.$$

Moreover, the solution is called GHU stable, if there is a function $M_{\mathfrak{J}} : (0,\infty) \to (0,\infty)$ with $M_{\mathfrak{J}}(0) = 0$ so that

$$\left\|(\varpi, \widehat{\varpi}) - (\varphi - \widehat{\varphi})\right\| \leq M_{\mathfrak{J}}(\varepsilon).$$

Definition 5.2. The solution $(\varpi(t), \varphi(t))$ of the proposed system (1.1) is HUR stable with respect to a continuous function $O \in \mho$, if there is a constant $P_{\mathfrak{J}} \geq 0$ such that for each $\varpi, \varphi \in \mho$, $\varepsilon \geq 0$ and $t \in D$, the differential inequalities

$$\left|{}^cD^\ell\varpi(t) - \mathfrak{J}\left(t, \varpi(t-\rho), \varpi(t), \varphi(t)\right)\right| \leq \varepsilon O(t),$$
$$\left|{}^cD^\tau\varphi(t) - \mathfrak{J}\left(t, \varphi(t-\rho), \varphi(t), \varpi(t)\right)\right| \leq \varepsilon O(t), \tag{5.2}$$

have a unique solution $(\widehat{\varpi}(t), \widehat{\varphi}(t)) \in \mho$ so that

$$\left\|(\varpi, \widehat{\varpi}) - (\varphi - \widehat{\varphi})\right\| \leq O(t) P_{\mathfrak{J}} \varepsilon.$$

Further, we say that the problem (1.1) is GHUR stable with respect to a continuous function $O \in \mho$, if

$$\left\|(\varpi, \widehat{\varpi}) - (\varphi - \widehat{\varphi})\right\| \leq O(t) P_{\Im}.$$

Remark 5.3. The functions $\overline{\varpi}, \overline{\varphi} \in \mho$ are a solution to the differential inequalities (5.1) iff we can find a continuous functions $U, V : D \to \mathbb{R}$ depend on ϖ and φ, respectively such that

(i) $U(t) \leq \varepsilon$ and $V(t) \leq \varepsilon$, $t \in D$;
(ii) ${}^{c}D^{\ell}\varpi(t) - \Im(t, \varpi(t-\rho), \varpi(t), \varphi(t)) + U(t) = 0$;
(iii) ${}^{c}D^{\ell}\varphi(t) - \Im(t, \varphi(t-\rho), \varphi(t), \varpi(t)) + V(t) = 0$.

Remark 5.4. The functions $\overline{\varpi}, \overline{\varphi} \in \mho$ are a solution to the differential inequalities (5.2) iff we can find a continuous functions $U, V : D \to \mathbb{R}$ depend on ϖ and φ, respectively so that for $\upsilon \in \mho$, we have

(r_1) $U(t) \leq \upsilon(t)\varepsilon$ and $V(t) \leq \upsilon(t)\varepsilon$, $t \in D$;
(r_{ii}) ${}^{c}D^{\ell}\varpi(t) - \Im(t, \varpi(t-\rho), \varpi(t), \varphi(t)) + U(t) = 0$;
(r_{iii}) ${}^{c}D^{\ell}\varphi(t) - \Im(t, \varphi(t-\rho), \varphi(t), \varpi(t)) + V(t) = 0$.

Lemma 5.5. *If ϖ and φ are the solution of the CFDDEs (1.1). Then ϖ and φ satisfy the following inequalities:*

$$\left|\varpi(t) - \Re_1(\varpi, \varphi) - \Re_2(\varpi, \varphi)\right| \leq \varepsilon\Omega \text{ and } \left|\varphi(t) - \Re_1(\varphi, \varpi) - \Re_2(\varphi, \varpi)\right| \leq \varepsilon\Omega^*.$$

Proof. Assume that ϖ and φ are the solution of (1.1), then

$$\varpi(t) = \Re_1(\varpi, \varphi) + \int_0^1 \aleph(t, \upsilon)\left[\Im(\upsilon, \varpi(\upsilon-\rho), \varpi(\upsilon), \varphi(\upsilon)) + U(\upsilon)\right] d\upsilon,$$

which implies that

$$\varpi(t) - \Re_1(\varpi, \varphi) - \int_0^1 \Im(\upsilon, \varpi(\upsilon-\rho), \varpi(\upsilon), \varphi(\upsilon))\, d\upsilon = \int_0^1 \aleph(t, \upsilon)\, U(\upsilon)\, d\upsilon.$$

Hence,

$$\begin{aligned}
&\left|\varpi(t) - \Re_1(\varpi, \varphi) - \Re_2(\varpi, \varphi)\right| \\
= &\left|\varpi(t) - \Re_1(\varpi, \varphi) - \int_0^1 \Im(\upsilon, \varpi(\upsilon-\rho), \varpi(\upsilon), \varphi(\upsilon))\, d\upsilon\right| \\
= &\left|\int_0^1 \aleph(t, \upsilon)\, U(\upsilon)\, d\upsilon\right| \leq \int_0^1 |\aleph(t, \upsilon)|\, |U(\upsilon)|\, d\upsilon \leq \varepsilon \int_0^1 |\aleph(t, \upsilon)|\, d\upsilon \leq \varepsilon\Omega.
\end{aligned}$$

Analogously,

$$\left|\varphi(t) - \Re_1(\varphi, \varpi) - \Re_2(\varphi, \varpi)\right|$$

$$
\begin{aligned}
&= \left|\varphi(t) - \mathfrak{R}_1(\varphi, \varpi) - \int_0^1 \mathfrak{I}\left(\upsilon, \varphi(\upsilon - \rho), \varphi(\upsilon), \varpi(\upsilon)\right) d\upsilon\right| \\
&= \left|\int_0^1 \Xi\left(t, \upsilon\right) V\left(\upsilon\right) d\upsilon\right| \leq \int_0^1 \left|\Xi\left(t, \upsilon\right)\right| \left|V\left(\upsilon\right)\right| d\upsilon \leq \varepsilon \int_0^1 \left|\Xi\left(t, \upsilon\right)\right| d\upsilon \leq \varepsilon \Omega^*.
\end{aligned}
$$

This completes the proof. □

Theorem 5.6. *Suppose that the postulates* (p_i) *and* (p_{ii}) *are true. Then the proposed system (1.1) is HU stable and GHU stable if* $W = 1 - \frac{BB^*}{(1-B)(1-B^*)} > 0$ *provided that* $\max\{B, B^*\} < 1$.

Proof. Let $(\widehat{\varpi}(t), \widehat{\varphi}(t)) \in \mho$ be a unique solution of (1.1), so for any solution $\varpi, \varphi \in \mho$, we have

$$
\begin{aligned}
&\left|\varpi(t) - \widehat{\varpi}(t)\right| \\
= &\left|\varpi(t) - \mathfrak{R}_1(\widehat{\varpi}, \widehat{\varphi}) + \int_0^1 \aleph\left(t, \upsilon\right)\left(\mathfrak{I}\left(\upsilon, \widehat{\varpi}(\upsilon - \rho), \widehat{\varpi}(\upsilon), \widehat{\varphi}(\upsilon)\right)\right) d\upsilon\right| \\
= &\Bigg|\varpi(t) - \mathfrak{R}_1(\varpi, \varphi) - \int_0^1 \aleph\left(t, \upsilon\right)\left(\mathfrak{I}\left(\upsilon, \varpi(\upsilon - \rho), \varpi(\upsilon), \varphi(\upsilon)\right)\right) d\upsilon \\
&- \mathfrak{R}_1(\widehat{\varpi}, \widehat{\varphi}) + \mathfrak{R}_1(\varpi, \varphi) + \int_0^1 \aleph\left(t, \upsilon\right)\left(\mathfrak{I}\left(\upsilon, \varpi(\upsilon - \rho), \varpi(\upsilon), \varphi(\upsilon)\right)\right) d\upsilon \\
&- \int_0^1 \aleph\left(t, \upsilon\right)\left(\mathfrak{I}\left(\upsilon, \widehat{\varpi}(\upsilon - \rho), \widehat{\varpi}(\upsilon), \widehat{\varphi}(\upsilon)\right)\right) d\upsilon\Bigg| \\
\leq &\left|\varpi(t) - \mathfrak{R}_1(\varpi, \varphi) - \mathfrak{R}_2(\varpi, \varphi)\right| + \left|\mathfrak{R}_1(\varpi, \varphi) - \mathfrak{R}_1(\widehat{\varpi}, \widehat{\varphi})\right| \\
&+ \left|\int_0^1 \aleph\left(t, \upsilon\right)\left(\mathfrak{I}\left(\upsilon, \varpi(\upsilon - \rho), \varpi(\upsilon), \varphi(\upsilon)\right)\right) d\upsilon - \int_0^1 \aleph\left(t, \upsilon\right)\left(\mathfrak{I}\left(\upsilon, \widehat{\varpi}(\upsilon - \rho), \widehat{\varpi}(\upsilon), \widehat{\varphi}(\upsilon)\right)\right) d\upsilon\right|,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\left\|\varpi - \widehat{\varpi}\right\|_{\mho_1} &\leq \Omega\varepsilon + t\left|\Lambda(\varphi, \varpi) - \Lambda(\widehat{\varphi}, \widehat{\varpi})\right| \\
&\quad + \int_0^1 \left|\aleph\left(t, \upsilon\right)\right| \left|\mathfrak{I}\left(\upsilon, \varpi(\upsilon - \rho), \varpi(\upsilon), \varphi(\upsilon)\right) - \mathfrak{I}\left(\upsilon, \widehat{\varpi}(\upsilon - \rho), \widehat{\varpi}(\upsilon), \widehat{\varphi}(\upsilon)\right)\right| d\upsilon \\
&\leq \Omega\varepsilon + \left|\Lambda(\varphi, \varpi) - \Lambda(\widehat{\varphi}, \widehat{\varpi})\right| + \Omega B_{\mathfrak{I}}\left(\left\|\varpi - \widehat{\varpi}\right\|_{\mho_1} + \left\|\varphi - \widehat{\varphi}\right\|_{\mho_2}\right) \\
&\leq \Omega\varepsilon + B_{\Lambda}\left(\left\|\varpi - \widehat{\varpi}\right\| + \left\|\varphi - \widehat{\varphi}\right\|\right) + \Omega B_{\mathfrak{I}}\left(\left\|\varpi - \widehat{\varpi}\right\|_{\mho_1} + \left\|\varphi - \widehat{\varphi}\right\|_{\mho_2}\right) \\
&= \Omega\varepsilon + \left(B_{\Lambda} + \Omega B_{\mathfrak{I}}\right)\left(\left\|\varpi - \widehat{\varpi}\right\|_{\mho_1} + \left\|\varphi - \widehat{\varphi}\right\|_{\mho_2}\right) \\
&= \Omega\varepsilon + B\left(\left\|\varpi - \widehat{\varpi}\right\|_{\mho_1} + \left\|\varphi - \widehat{\varphi}\right\|_{\mho_2}\right).
\end{aligned}
$$

Hence,

$$\left\|\varpi-\widehat{\varpi}\right\|_{\mho_1}-\frac{B}{1-B}\left\|\varphi-\widehat{\varphi}\right\|_{\mho_2}\leq\frac{\Omega\varepsilon}{1-B}. \tag{5.3}$$

Similarly, one can write

$$\left\|\varphi-\widehat{\varphi}\right\|_{\mho_2}-\frac{B^*}{1-B^*}\left\|\varpi-\widehat{\varpi}\right\|_{\mho_1}\leq\frac{\Omega^*\varepsilon}{1-B^*}. \tag{5.4}$$

The inequalities (5.3) and (5.4) can be written as

$$\begin{bmatrix} 1 & -\frac{B}{1-B} \\ -\frac{B^*}{1-B^*} & 1 \end{bmatrix}\begin{bmatrix} \left\|\varpi-\widehat{\varpi}\right\|_{\mho_1} \\ \left\|\varphi-\widehat{\varphi}\right\|_{\mho_2} \end{bmatrix}\leq\begin{bmatrix} \frac{\Omega\varepsilon}{1-B} \\ \frac{\Omega^*\varepsilon}{1-B^*} \end{bmatrix},$$

which implies that

$$\begin{bmatrix} \left\|\varpi-\widehat{\varpi}\right\|_{\mho_1} \\ \left\|\varphi-\widehat{\varphi}\right\|_{\mho_2} \end{bmatrix}\leq\begin{bmatrix} \frac{1}{W} & \frac{B}{1-B}\frac{1}{W} \\ \frac{B^*}{1-B^*}\frac{1}{W} & \frac{1}{W} \end{bmatrix}\begin{bmatrix} \frac{\Omega\varepsilon}{1-B} \\ \frac{\Omega^*\varepsilon}{1-B^*} \end{bmatrix}, \tag{5.5}$$

where $W=1-\frac{BB^*}{(1-B)(1-B^*)}$. According to the system (5.5), we can write

$$\left\|\varpi-\widehat{\varpi}\right\|_{\mho_1}\leq\frac{\Omega\varepsilon}{W(1-B)}+\frac{B\Omega^*\varepsilon}{W(1-B)(1-B^*)},$$

and

$$\left\|\varphi-\widehat{\varphi}\right\|_{\mho_2}\leq\frac{B^*\Omega\varepsilon}{W(1-B^*)(1-B)}+\frac{\Omega^*\varepsilon}{W(1-B^*)}.$$

Hence,

$$\begin{aligned}\left\|\varpi-\widehat{\varpi}\right\|_{\mho_1}+\left\|\varphi-\widehat{\varphi}\right\|_{\mho_2} \leq\; & \frac{\Omega\varepsilon}{W(1-B)}+\frac{\Omega^*\varepsilon}{W(1-B^*)} \\ & +\frac{B\Omega^*\varepsilon}{W(1-B)(1-B^*)}+\frac{B^*\Omega\varepsilon}{W(1-B^*)(1-B)}.\end{aligned}$$

Put

$$P_{\mathfrak{J}}=\frac{\Omega}{W(1-B)}+\frac{\Omega^*}{W(1-B^*)}+\frac{B\Omega^*}{W(1-B)(1-B^*)}+\frac{B^*\Omega}{W(1-B^*)(1-B)}.$$

Then,

$$\left\|(\varpi,\widehat{\varpi})-(\varphi,\widehat{\varphi})\right\|_{\mho}\leq P_{\mathfrak{J}}\varepsilon.$$

Therefore, the solution of (1.1) is HU stable. Moreover, if we define a function $M_{\mathfrak{J}}(\varepsilon)=\varepsilon$ so that $M_{\mathfrak{J}}(0)=0$, we conclude that

$$\left\|(\varpi,\widehat{\varpi})-(\varphi,\widehat{\varphi})\right\|_{\mho}\leq M_{\mathfrak{J}}(\varepsilon),$$

which guarantees the GHU stability (1.1). □

Lemma 5.7. *If ϖ and φ are the solution of the CFDDEs (1.1). Then ϖ and φ satisfy the following inequalities:*

$$\left|\varpi(t)-\mathfrak{R}_1(\varpi,\varphi)-\mathfrak{R}_2(\varpi,\varphi)\right|\leq\varepsilon\upsilon(t)\Omega \text{ and } \left|\varphi(t)-\mathfrak{R}_1(\varphi,\varpi)-\mathfrak{R}_2(\varphi,\varpi)\right|\leq\varepsilon\upsilon(t)\Omega^*.$$

Proof. Using Remark 5.4, we obtain the desired result in the same manner as Lemma 5.5. □

Theorem 5.8. *Via Lemma 5.7 and the conditions (p_i) and (p_{ii}), the solution of (1.1) is HUR stable and GHUR if $B + B^* \neq 1$.*

Proof. Using Remark 5.4 and Theorem 5.6, we arrive to the desired result. □

6. Supportive examples

Here, we provide some applications to support our findings.

Example 6.1. Consider the following CFDDEs:

$$\begin{cases} {}^cD^{\frac{7}{4}}\varpi(t) = \frac{1}{29+\sin(t)} + \frac{|\varpi(t-5)|}{(1+|\varpi(t-5)|)(37+3\sin(t))} + \frac{|\varpi(t)|}{(1+|\varpi(t)|)(37+3\sin(t))} + \frac{|\varphi(t)|}{(1+|\varphi(t)|)(37+3\sin(t))} \\ {}^cD^{\frac{7}{4}}\varphi(t) = \frac{1}{29+\sin(t)} + \frac{|\varphi(t-5)|}{(1+|\varphi(t-5)|)(37+3\sin(t))} + \frac{|\varphi(t)|}{(1+|\varphi(t)|)(37+3\sin(t))} + \frac{|\varpi(t)|}{(1+|\varpi(t)|)(37+3\sin(t))} \end{cases}, \tag{6.1}$$

for $t \in D$ with boundary conditions

$$\varpi(t) = \varphi(t) = 0,\ \varpi(1) = \varphi(1) = \Lambda(\varpi, \varphi) = \Lambda(\varphi, \varpi) = \frac{\varpi(t) + \varphi(t)}{30} + 35,\ t \in [-5, 0].$$

Now, consider

$$\begin{aligned} & \left|\Im\left(t, \varpi_1(t-5), \varpi_1(t), \varphi_1(t)\right) - \Im\left(t, \varpi_2(t-5), \varpi_2(t), \varphi_2(t)\right)\right| \\ = & \left|\frac{1}{29+\sin(t)} + \frac{|\varpi_1(t-5)|}{(1+|\varpi_1(t-5)|)(37+3\sin(t))} + \frac{|\varpi_1(t)|}{(1+|\varpi_1(t)|)(37+3\sin(t))}\right. \\ & + \frac{|\varphi_1(t)|}{(1+|\varphi_1(t)|)(37+3\sin(t))} - \frac{1}{29+\sin(t)} - \frac{|\varpi_2(t-5)|}{(1+|\varpi_1(t-5)|)(37+3\sin(t))} \\ & \left. - \frac{|\varpi_2(t)|}{(1+|\varpi_1(t)|)(37+3\sin(t))} - \frac{|\varphi_2(t)|}{(1+|\varphi_1(t)|)(37+3\sin(t))}\right| \\ \leq & \frac{\|\varpi_1 - \varpi_2\|}{40} + \frac{\|\varpi_1 - \varpi_2\|}{40} + \frac{\|\varphi_1 - \varphi_2\|}{40} \\ \leq & \frac{1}{20}\left(\|\varpi_1 - \varpi_2\| + \|\varphi_1 - \varphi_2\|\right), \end{aligned}$$

and

$$\begin{aligned} & \left|\Im\left(t, \varphi_1(t-5), \varphi_1(t), \varpi_1(t)\right) - \Im\left(t, \varphi_2(t-5), \varphi_2(t), \varpi_2(t)\right)\right| \\ = & \left|\frac{1}{29+\sin(t)} + \frac{|\varphi_1(t-5)|}{(1+|\varphi_1(t-5)|)(37+3\sin(t))} + \frac{|\varphi_1(t)|}{(1+|\varphi_1(t)|)(37+3\sin(t))}\right. \\ & + \frac{|\varpi_1(t)|}{(1+|\varpi_1(t)|)(37+3\sin(t))} - \frac{1}{29+\sin(t)} - \frac{|\varphi_2(t-5)|}{(1+|\varphi_2(t-5)|)(37+3\sin(t))} \\ & \left. - \frac{|\varphi_2(t)|}{(1+|\varphi_1(t)|)(37+3\sin(t))} - \frac{|\varpi_2(t)|}{(1+|\varpi_1(t)|)(37+3\sin(t))}\right| \\ \leq & \frac{\|\varphi_1 - \varphi_2\|}{40} + \frac{\|\varphi_1 - \varphi_2\|}{40} + \frac{\|\varpi_1 - \varpi_2\|}{40} \end{aligned}$$

$$\leq \frac{1}{20}\left(\|\varphi_1-\varphi_2\|+\|\varpi_1-\varpi_2\|\right).$$

Hence, $B_{\mathfrak{I}} = \frac{1}{20}$, $B_{\Lambda} = \frac{1}{30}$, $\ell = \tau = \frac{7}{4}$ and $\Omega = \Omega^* = \frac{1}{\Gamma(\frac{11}{4})} \approx 0.6218$. Based on these values, we get $B_\Lambda + \Omega B_{\mathfrak{I}} = B_\Lambda + \Omega^* B_{\mathfrak{I}} \approx 0.06442 < 1$. Hence, in view of Theorem 4.1, the problem (6.1) has a unique solution. Additionally, conditions created for Ulam forms of stability are still in place. Therefore, in light of Theorem 5.6, the solution of (6.1) is UH and GUH stable. (note $B + B^* \approx 0.12884 \neq 1$).

Example 6.2. Assume the following CFDDEs:

$$\begin{cases} {}^cD^{\frac{8}{5}}\varpi(t) = \frac{1}{15+e^t} + \frac{\left|\sqrt{\varpi(t-4)}\right|}{(1+|\varpi(t-5)|)(18+2e^t))} + \frac{\sqrt{|\varpi(t)|}}{(1+|\varpi(t)|)(18+2e^t)} + \frac{\sqrt{|\varphi(t)|}}{(1+|\varphi(t)|)(18+2e^t)} \\ {}^cD^{\frac{8}{5}}\varphi(t) = \frac{1}{15+e^t} + \frac{\left|\sqrt{\varphi(t-4)}\right|}{(1+|\varphi(t-5)|)(18+2e^t))} + \frac{\sqrt{|\varphi(t)|}}{(1+|\varphi(t)|)(18+2e^t)} + \frac{\sqrt{|\varpi(t)|}}{(1+|\varpi(t)|)(18+2e^t)} \end{cases}, \tag{6.2}$$

for $t \in D$ with boundary conditions

$$\varpi(t) = \varphi(t) = 0,\ \varpi(1) = \varphi(1) = \Lambda(\varpi,\varphi) = \Lambda(\varphi,\varpi) = \frac{\varpi(t)+\varphi(t)}{8},\ t \in [-4,0].$$

Let

$$\begin{aligned} \left|\mathfrak{I}\left(t,\varpi(t-4),\varpi(t),\varphi(t)\right)\right| &\leq \left|\frac{1}{16}\right| + \left|\frac{(\varpi(t-4))^{\frac{1}{2}}}{20}\right| + \left|\frac{(\varpi(t))^{\frac{1}{2}}}{20}\right| + \left|\frac{(\varphi(t))^{\frac{1}{2}}}{20}\right| \\ &\leq \frac{1}{16} + \frac{|(\varpi(t))|^{\frac{1}{2}}}{20} + \frac{|(\varpi(t))|^{\frac{1}{2}}}{20} + \frac{|(\varphi(t))|^{\frac{1}{2}}}{20} \\ &= \frac{1}{16} + \frac{|(\varpi(t))|^{\frac{1}{2}}}{10} + \frac{|(\varphi(t))|^{\frac{1}{2}}}{20} \\ &\leq \frac{1}{16} + \frac{1}{10}\left(|(\varpi(t))|^{\frac{1}{2}} + |(\varphi(t))|^{\frac{1}{2}}\right) \\ &= T_{\mathfrak{I}} + R_{\mathfrak{I}}\left(\|\varpi\|^{u_0} + \|\varphi\|^{u_0}\right), \end{aligned}$$

and

$$\begin{aligned} \left|\mathfrak{I}\left(t,\varphi(t-4),\varphi(t),\varpi(t)\right)\right| &\leq \left|\frac{1}{16}\right| + \left|\frac{(\varphi(t-4))^{\frac{1}{2}}}{20}\right| + \left|\frac{(\varphi(t))^{\frac{1}{2}}}{20}\right| + \left|\frac{(\varpi(t))^{\frac{1}{2}}}{20}\right| \\ &\leq \frac{1}{16} + \frac{|(\varphi(t))|^{\frac{1}{2}}}{20} + \frac{|(\varphi(t))|^{\frac{1}{2}}}{20} + \frac{|(\varpi(t))|^{\frac{1}{2}}}{20} \\ &= \frac{1}{16} + \frac{|(\varphi(t))|^{\frac{1}{2}}}{10} + \frac{|(\varpi(t))|^{\frac{1}{2}}}{20} \\ &\leq \frac{1}{16} + \frac{1}{10}\left(|(\varpi(t))|^{\frac{1}{2}} + |(\varphi(t))|^{\frac{1}{2}}\right) \\ &= T^*_{\mathfrak{I}} + R_{\mathfrak{I}}\left(\|\varpi\|^{u_0} + \|\varphi\|^{u_0}\right). \end{aligned}$$

Assigning the following values and using boundary conditions, we get $R_{\mathfrak{I}} = \frac{1}{10}$, $T_{\mathfrak{I}} = T^*_{\mathfrak{I}} = \frac{1}{16}$, $R_\Lambda = \frac{1}{8}$, $T_\Lambda = T^*_\Lambda = 0$, $u = u_0 = \frac{1}{2}$ and $\ell = \tau = \frac{8}{5} \in (1,2]$. In light of Theorem 4.4, CFDDEs (6.2) have a unique solution.

7. Conclusion and future works

The theory of delay for differential equations underwent significant development. This had a number of practical implications whose investigation required the solutions from delay equations. Such equations are required to describe processes whose pace relies on their initial conditions. These procedures are frequently referred to as "processes with delays" or "with aftereffects". As a result of the fact that nonlinear differential equations (both ODEs and PDEs) can be used to model a variety of physical processes, the discussion and investigation of different analytical and numerical approaches to solving nonlinear differential equations is crucial for evaluating scientific engineering challenges. For a class of nonlocal FDDEs, the authors have developed a thorough mathematical study using the Caputo derivative framework. In order to construct the needed conditions for existence and stability of the solution, we used the findings of FP theory and nonlinear analysis to explain the dynamics of the suggested CFDDEs. In order to support the main findings of this study, we provide two examples. As future works, we raise the following five points:

- Generalizing the aforementioned findings for CFDDEs with stricter BVPs.
- Applying the results on arbitrary fractional-order differential equations and Hadamard fractional derivatives.
- Applying the results on linear and nonlinear fractional integrodifferential systems.
- Investigating brand-new numerical discoveries relating to the operator with a higher order.
- Replacing the current kernels with Mittag-Leffler kernels.

Availability of data and materials

The data used to support the findings of this study are available from the corresponding author upon request.

Conflict of interest

The authors declare that they have no competing interests.

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