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Research article

Spatiotemporal dynamics of a diffusive predator-prey system incorporating social behavior

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Abstract: This research concerned with a new formulation of a spatial predator-prey model with Leslie-Gower and Holling type II schemes in the presence of prey social behavior. The aim interest here is to distinguish the influence of Leslie-Gower term on the spatiotemporal behavior of the model. Interesting results are obtained as Hopf bifurcation, Turing bifurcation and Turing-Hopf bifurcation. A rigorous mathematical analysis shows that the presence of Leslie-Gower can induce Turing pattern, which shows that this kind of interaction is very important in modeling different natural phenomena. The direction of Turing-Hopf bifurcation is studied with the help of the normal form. The obtained results are tested numerically.

Keywords: Leslie-Gower predator-prey model; social behavior; Hopf bifurcation; turing pattern; T-H

bifurcation
Mathematics Subject Classification: 93A30, 34C23

1. Introduction

The prey predator model (denoted P-P model) is considered one of many powerful tools for predicting the evolution of species in nature [17, 18, 26]. In recent years, a well-known prey behavior dented by 'herd behavior' or 'social behavior' that been successfully modeled and analyzed. When the prey makes a group defense (which know by prey's social behavior), the predators cannot reach the prey located in the center of the herd, so the predator will hunt only the prey located in the frontier of the herd [31]. Therefore, the interaction happens only on the borders of the group. There are numerous approaches to determining the number of prey on the outer line of the group. The simplest one is to consider that the prey makes a group o a square shape, then the number of the prey will be proportional to the square root of the prey density [33]. For modeling this specific behavior we presume that the resources make a group with a square shape. Hence, the density of the resources in the frontier of the group is four times the square root of the number of the resources on the herd frontier. Hence, inspired by the manner of proposing the classical Holling I interaction functional we can deduce that the functional response that describes this conduct is $F(M, N) = \gamma N \sqrt{M}$, where γ is the hunting rate, and M is the density o the resources, N is the density of the consumers. The manner of building this interaction functional is discussed in details through the paper [1]. Further, by changing the group structure or shape the density of the resources will be changed accordingly. As epitome if we consider that the resources make a group with a sphere shape it will influence the number of the resources in the outer bound where it will be $4^{\frac{1}{3}} \times 3^{\frac{2}{3}} \times \pi^{\frac{1}{3}} \times M^{\frac{2}{3}}$. For generalizing the previous results, it is considered that in [3, 37] that the functional response F can be generalized by the one $F_2(M, N) = \gamma N M^{\alpha}$, where the parameter $0 < \alpha < 1$ indicates the structure of the resources group. In the case of the square or circle group shape this rate will become $\alpha = \frac{1}{2}$, and in the case of $\alpha = \frac{2}{3}$ we got the case of the sphere of cube group shape, which means that this functional response generalizes all the previous cases of the prey herd structure. Dealing with the resources that exhibit grouping behavior is not always easy for the consumers, where the from one herd to another, and from predator to another, handling with the prey in the outer bound of the group changes and takes different time (different handling time for the predator to handle with a prey) which is been investigated in the first time by Holling to propose the Holling II interaction functional [17]. This point of view is applied for modeling the intermingling between the resources and the consumers in this case through the paper [6], where a new functional response is obtained $F_3(M, N) = \frac{\gamma N M^{\alpha}}{1+\gamma t_h M^{\alpha}}$, which summarizes all the intermingling functions and cases on interaction. This intermingling function was the subject of an investigation on many occasions for the purpose of modeling many behaviors in nature we mention a few [10, 13]. In fact, using the functional response F_3 as intermingling function next to the logistic increasing of the prey and linear mortality of resources (see as an example the paper [6]), we get a Gause-type model [4], which means that the mathematical investigation is trivial and can be distinguished easily from the paper [4]. Investigating the herd behavior in mathematical models is the subject of the recent activities, we cite the researches [2, 5, 7–10, 12, 13, 32], and for more reading about different mathematical modeling of some natural phenomenon we cite the papers [11, 14, 16, 21–25, 30]. In this research, we will use a different approach where we will incorporate the Leslie-Gower intermingling functional with the functional $F_3(M, N)$ functional responses. Incorporating Leslie-Gower's functional response form with resources social behavior is a recent step and attracts any researchers, we cite for instance the papers [15, 18, 27, 28, 35, 36], hence, it is the subject of interest in this research.

Modeling the intermingling resource-consumer in the case of resources social depends on the spatial positioning, so, it is wise to consider a spatiotemporal model [34]. Our purpose in this research is to study the influence of the Leslie-Gower forme on the evolution of the two species, where we will use a comparative analysis for achieving this goal. In these regards, we consider the first model that models the intermingling resource-consumer in the case of the resources social conduct and no Leslie-Gower form. The investigated model is:

$$\begin{cases} \frac{\partial}{\partial \tau} M(x,\tau) = \beta M(x,\tau)\left(1 - \frac{M(x,\tau)}{L}\right) + \delta \Delta M(x,\tau) - \frac{\gamma M^{\alpha}(x,\tau) N(x,\tau)}{1+\gamma\tau_h M^{\alpha}(x,\tau)}, & x \in (0, l\pi),\ \tau > 0, \\ \frac{\partial}{\partial \tau} N(x,\tau) = -\mu N(x,\tau) + \frac{e\gamma M^{\alpha}(x,\tau) N(x,\tau)}{1+\gamma\tau_h M^{\alpha}(x,\tau)} + \eta \Delta N(x,\tau), & x \in (0, l\pi),\ \tau > 0, \\ \frac{\partial}{\partial \vec{n}} M(x,\tau) = \frac{\partial}{\partial \vec{n}} N(x,\tau) = 0, & x \in (0, l\pi),\ \tau > 0, \\ M(x,0) = M_0(x) \geq 0, \quad N(x,0) = N_0(x) \geq 0, & x \in (0, l\pi), \end{cases} \tag{1.1}$$

with $M(x,\tau)$ (resp. $N(x,\tau)$) is the number of the resources (rep. consumer) at t and position x, $\beta M(x,\tau)\left(1 - \frac{M(x,\tau)}{L}\right)$ is the logistic increasing of the resources with increasing rate β and the crying capacity of the environment L for the resources, μ in the mortality coefficient for the consumer, e is the conversion rate of the resources into consumer, δ (resp. η) is the diffusion rate for the resources (resp. consumer). To mention the Neumann boundary conditions highlights that neither the resources or the consumers cannot move cross the borders. Our main contribution consists to cooperate the Leslie-Gower with the prey social behavior interaction function and determine its effect on the temporal behavior of the solutions. The investigated model is given as:

$$\begin{cases} \frac{\partial}{\partial \tau} M(x,\tau) = \beta M(x,\tau)\left(1 - \frac{M(x,\tau)}{L}\right) + \delta \Delta M(x,\tau) - \frac{\gamma M^{\alpha}(x,\tau) N(x,\tau)}{1+\gamma\tau_h M^{\alpha}(x,\tau)}, & x \in (0, l\pi),\ \tau > 0, \\ \frac{\partial}{\partial \tau} N(x,\tau) = \sigma N(x,\tau)\left(1 - \frac{N(x,\tau)}{M(x,\tau)}\right) + \eta \Delta N(x,\tau), & \\ \frac{\partial}{\partial \vec{n}} M(x,\tau) = \frac{\partial}{\partial \vec{n}} N(x,\tau) = 0, & x \in (0, l\pi),\ \tau > 0, \\ M(x,0) = M_0(x) \geq 0, \quad N(x,0) = N_0(x) \geq 0, & x \in (0, l\pi). \end{cases} \tag{1.2}$$

Our purpose is to investigate with the influence of the new approximation provided in (1.2) in modeling the interaction resources-consumer in the presence of the resources grouping behavior. In fact, we will use a comparative analysis between the two models (1.1) and (1.2), where we will show that the system (1.2) have a very rich dynamics. For achieving these aims we use the sections:

In Sec. 2 we analyze (1.1), where it is proved that it can undergo Hopf bifurcation (H-bifurcation), and cannot have Turing instability (T-instability), which means that it is not possible to have Turing-Hopf bifurcation (T-H bifurcation). The third section is used to analyze the system (1.2), where we will show that the system (1.2) undergoes many types of bifurcation as T-bifurcation, H-bifurcation, and T-H bifurcation. However, the system (1.1) can undergo only Hopf bifurcation (there is no T-bifurcation, and then there is no T-H bifurcation). The normal form of T-H bifurcation is utilized for study of steady states solution near the T-H bifurcation. The obtained mathematical results are confirmed using numerical simulation.

2. Long time behavior of the model (1.1)

We know that the equilibrium states for the system (1.1) are solutions of the following system

$$\begin{cases} \beta M\left(1-\frac{M}{L}\right)-\frac{\gamma M^{\alpha}N}{1+\gamma\tau_h M^{\alpha}}=0, \\ -\mu N+\frac{e\gamma M^{\alpha}N}{1+\gamma\tau_h M^{\alpha}}=0. \end{cases} \tag{2.1}$$

Clearly, (2.1) has three equilibria $E_0(0,0)$, $E_1(L,0)$, and the unique positive equilibrium point $E_*(M_*,N_*)$ where

$$M_*=\left(\frac{\mu}{\gamma(e-\mu t_h)}\right)^{\frac{1}{\alpha}},\quad N_*=\frac{e\beta}{\mu}\left(1-\frac{M_*}{L}\right),$$

which exists if the following conditions hold

$$(H_1): e>\mu t_h \text{ and } M_*<L\ \left(i.e. \gamma>\gamma^*=\frac{\mu}{L^{\alpha}(e-\mu t_h)}\right). \tag{2.2}$$

The linearized system of (1.1) evaluated at (M_*,N_*) is

$$\begin{pmatrix} \frac{\partial M}{\partial\tau} \\ \frac{\partial N}{\partial\tau} \end{pmatrix}=(D\Delta+J_{E_*}(M_*,N_*))\begin{pmatrix} u \\ v \end{pmatrix}, \tag{2.3}$$

where $D\Delta=diag(\delta\frac{\partial^2}{\partial x^2},\eta\frac{\partial^2}{\partial x^2})$ and

$$J_{E_*}(M_*,N_*)=\begin{pmatrix} \beta\left(1-2\frac{M_*}{L}\right)-\frac{\alpha\gamma(M_*)^{\alpha-1}N_*}{(1+\gamma\tau_h(M_*)^{\alpha})^2} & -\frac{\gamma(M_*)^{\alpha}}{1+\gamma\tau_h(M_*)^{\alpha}} \\ \frac{e\gamma\alpha(M_*)^{\alpha-1}N_*}{(1+\gamma\tau_h(M_*)^{\alpha})^2} & 0 \end{pmatrix}. \tag{2.4}$$

Then the characteristic equation of system (2.3) takes the following form

$$\lambda^2-Tr_k\lambda+Det_k=0,\qquad k\in\mathbb{N}_0, \tag{2.5}$$

where

$$Tr_k=\beta\left(1-2\frac{M_*}{L}\right)-\frac{\alpha\gamma(M_*)^{\alpha-1}N_*}{(1+\gamma\tau_h(M_*)^{\alpha})^2}-(\delta+\eta)\left(\frac{k}{l}\right)^2, \tag{2.6}$$

and

$$Det_k=\delta\eta\left(\frac{k}{l}\right)^4-\left(\beta\left(1-2\frac{M_*}{L}\right)-\frac{\alpha\gamma(M_*)^{\alpha-1}N_*}{(1+\gamma\tau_h(M_*)^{\alpha})^2}\right)\eta\left(\frac{k}{l}\right)^2+Det_0, \tag{2.7}$$

with

$$Det_0=\frac{e\alpha\gamma^2(M_*)^{2\alpha-1}N_*}{(1+\gamma\tau_h(M_*)^{\alpha})^3}>0.$$

Using the fact that

$$\beta\left(1-\frac{M_*}{L}\right)=\frac{\alpha\gamma(M_*)^{\alpha-1}N_*}{1+\gamma\tau_h(M_*)^{\alpha}},\ \text{and } 1+\gamma\tau_h(M_*)^{\alpha}=\frac{e\gamma}{\mu}(M_*)^{\alpha},$$

it follows that (2.6) becomes

$$Tr_k = \beta\left(1-\alpha+\alpha\frac{\mu\tau_h}{e}\right) - \frac{\beta}{L}\left(2-\alpha+\alpha\frac{\mu\tau_h}{e}\right)\left(\frac{\mu}{\gamma(e-\mu t_h)}\right)^{\frac{1}{\alpha}} - (\delta+\eta)\left(\frac{k}{l}\right)^2, \tag{2.8}$$

and

$$Det_k = \delta\eta\left(\frac{k}{l}\right)^4 - \left(\beta\left(1-\alpha+\alpha\frac{\mu\tau_h}{e}\right) - \frac{\beta}{L}\left(2-\alpha+\alpha\frac{\mu\tau_h}{e}\right)\left(\frac{\mu}{\gamma(e-\mu t_h)}\right)^{\frac{1}{\alpha}}\right)\eta\left(\frac{k}{l}\right)^2 + Det_0. \tag{2.9}$$

Putting

$$L_H = \left(\frac{2-\alpha+\alpha\frac{\mu\tau_h}{e}}{1-\alpha+\alpha\frac{\mu\tau_h}{e}}\right)\left(\frac{\mu}{\gamma(e-\mu t_h)}\right)^{\frac{1}{\alpha}}.$$

Clearly, for $k = 0$ we have $Det_0 > 0$. Then, if $L > L_H$ we have $Tr_0 < 0$ which means that E^* is locally stable and if $L < L_H$ we get $Tr_0 < 0$, then E^* is unstable. Now, taking L as the bifurcation parameter. Thus, we have

Theorem 2.1. *Presume that* (H_1) *holds and we put*

$$L = L_k := \frac{\left(2-\alpha+\alpha\frac{\mu\tau_h}{e}\right)\left(\frac{\mu}{\gamma(e-\mu t_h)}\right)^{\frac{1}{\alpha}}}{l^2\left(1-\alpha+\alpha\frac{\mu\tau_h}{e}\right) - (\delta+\eta)k^2}. \tag{2.10}$$

Then, there exists an integer k^* *for which the model (1.1) undergoes a H-bifurcation at* E^* *when* $L = L_k$ *for* $0 \le k \le k^*$*. Further, if* $k = 0$ *the periodic solution is homogeneous (spatially), and if* $k = 1, ..., k^*$ *the periodic solution is nonhomogeneous (spatially) when* $k = 1, ..., k^*$*, where* $[.]$ *is the integer part function.*

Proof. Denotes

$$\bar{k} = \left[l\sqrt{\frac{1-\alpha+\alpha\frac{\mu\tau_h}{e}}{\delta+\eta}}\right].$$

Clearly, for an integer $k < \bar{k}$ we have $L_k > 0$. We consider that Det_k defined by (3.10) is a function of k. Clearly, $Det_0 > 0$, therefore there exists $\tilde{k} > 0$ (it can be $+\infty$ if $Det_k > 0$ for all integer $k = 0, 1, 2, ...$) a positive integer which represents the first integer that satisfy $Det_{\tilde{k}} > 0$ and $Det_{\tilde{k}+1} \le 0$. Taking $k^* = \min\{\bar{k}, \tilde{k}\}$. In this case, we guarantees that for $L = L_k$, $k = 0, 1, ..., k^*$ we have $Tr_k = 0$ and $Det_k > 0$ for all $k = 0, 1, ..., k^*$ which implies that (3.5) has purely imaginary roots. Letting

$$\lambda_k(L) = \theta_k(L) \pm i\omega_k(L), \qquad k = 0, 1, ..., k^*$$

be the solution of Eq (3.5) verifying

$$\theta_k(L_k) = 0, \quad \omega_k(L_k) = \sqrt{Det_k(L_k)}.$$

Then, we get

$$\theta_k'(L_k) = \left.\frac{dRe\lambda_k}{dL}\right|_{L=L_k} = \frac{\left[l^2\left(1-\alpha+\alpha\frac{\mu\tau_h}{e}\right) - (\delta+\eta)k^2\right]^2}{\left(2-\alpha+\alpha\frac{\mu\tau_h}{e}\right)\left(\frac{\mu}{\gamma(e-\mu t_h)}\right)^{\frac{1}{\alpha}}} > 0.$$

It follows that the transversal condition is justified at each L_k, $k = 0, 1, ..., k^*$, that is to say that system (1.1) undergoes H-bifurcation at $L = L_k$. □

Now, we focus on proving that the system (1.1) cannot exhibit the existence of Turing instability. This phenomena is know by the diffusion driven instability, where considering the presence of spatial diffusion (with distinct diffusion rates) can destabilize a stable equilibrium, which means that if an equilibrium is stable in the absence of the spatial diffusion can become instable in the presence of the diffusion (with distinct dispersal rates).

Lemma 2.2. *The model (1.1) cannot have T-instability at E^*.*

Proof. Before proceeding to prove Lemma 3.6, it is necessary to assume that the conditions that guarantees the existence and the stability of E^* in the absence of diffusion holds which consists to suppose that (H_1) holds and $L < L_H$. Immediately, we find that

$$\beta\left(1-2\frac{M_*}{L}\right)-\frac{\alpha\gamma(M_*)^{\alpha-1}N_*}{(1+\gamma\tau_h(M_*)^{\alpha})^2}=\beta\left(1-\alpha+\alpha\frac{\mu\tau_h}{e}\right)-\frac{\beta}{L}\left(2-\alpha+\alpha\frac{\mu\tau_h}{e}\right)\left(\frac{\mu}{\gamma(e-\mu t_h)}\right)^{\frac{1}{\alpha}}<0.$$

Consequently, it follows by $Det_k > 0,\ \forall k \in \mathbb{N}$ the non occurrence of T-instability. □

3. Long time behavior of the model (1.2)

In this section, we consider the diffusive predator-prey model with Leslie-Gower term (1.2). We discuss the existence of Hopf bifurcation, after that we derive the condition for Turing pattern which leads to the occurrence of T-H bifurcation. First of all, we prove that system (1.2) has unique positive solution

3.1. Existence and boundedness

Here, we investigate the existence and positivity of solution for (1.2).

Theorem 3.1. *Assume that $\beta, L, \gamma, \tau_h, \delta$ and η are all positive, if $M_0(x,\tau) \geq 0$ and $N_0(x,\tau) \geq 0$ for $(x,\tau) \in [0, l\pi] \times [0, +\infty)$. Hence, (1.2) has a unique positive solution verifying*

$$0 \leq M(x,\tau) \leq M^{**}(\tau), 0 \leq N(x,\tau) \leq N^{**}(\tau) \quad for \quad (x,\tau) \in [0, l\pi] \times [0, +\infty],$$

such that $(M^{**}(\tau), N^{**}(\tau))$ is the unique solution of

$$\begin{cases} M_\tau = \beta M\left(1-\frac{M}{L}\right), \\ N_\tau = N\left(1-\frac{N}{M}\right), \\ M(0) = M_0^{**} = \sup\limits_{x\in[0,l\pi]} M_0(x)\,,\ N(0) = N_0^{**} = \sup\limits_{x\in[0,l\pi]} N_0(x). \end{cases} \tag{3.1}$$

Proof. Putting

$$f(M,N) = \beta M\left(1-\frac{M}{L}\right) - \frac{\gamma M^{\alpha}N}{1+\gamma\tau_h M^{\alpha}}, \quad g(M,N) = \sigma N\left(1-\frac{N}{M}\right).$$

Since, $f_N \leq 0$ and $g_M \geq 0$ for $(M,N) \in \mathbb{R}_+^2 = \{(M,N)|M \geq 0\}$ and from [29] yields that f, g are mixed quasi-monotone functionals in $\mathbb{R}$. Now, letting

$$(\tilde{M}(x,\tau), \tilde{N}(x,\tau)) = (0,0) \text{ and } (\hat{M}(x,\tau), \hat{N}(x,\tau)) = (M^{**}(\tau), N^{**}(\tau)).$$

From

$$\frac{\partial \hat{M}}{\partial \tau} - \delta\Delta\hat{M} - f(\hat{M}, \tilde{N}) \quad = \quad 0 \geq 0 = \frac{\partial \tilde{M}}{\partial \tau} - \delta\Delta\tilde{M} - f(\tilde{M}, \hat{N}),$$

$$\frac{\partial \hat{N}}{\partial \tau} - \eta\Delta\hat{N} - f(\hat{M}, \hat{N}) \quad = \quad 0 \geq 0 = \frac{\partial \tilde{N}}{\partial \tau} - \eta\Delta\tilde{N} - f(\tilde{M}, \tilde{N})$$

and $0 \leq M_0(x,\tau) \leq M_0^{**}, 0 \leq N_0(x,\tau) \leq> N_0^{**}$, for $(x,\tau) \in [0, l\pi] \times [0, +\infty]$, we can conclude that $(\tilde{M}, \tilde{N})$ and $(\hat{M}, \hat{N})$ are the upper and lower solutions of (1.2), respectively. From [29], we get the existence of a unique globally defined solution $(M(x,\tau), N(x,\tau))$ of system (1.2) satisfying $0 \leq M(x,\tau) \leq M^{**}(\tau)$ and $0 \leq N(x,\tau) \leq N^{**}(\tau)$.

Applying the strong maximum principle to (1.2), we obtain that $M(x,\tau) > 0$, $N(x,\tau) > 0$ for $(x,\tau) \in (0, l\pi) \times (0, +\infty)$. □

Now, we mainly focus on checking the existence of feasible steady states. We can easily verify that $E_1 = (L, 0)$ are always equilibrium for (1.2). Next, we investigate the existence of a positive steady state of our proposed system (1.2) which is denoted by $E_2 = (M^*, N^*)$, then we have the following theorem

Theorem 3.2. *Suppose that all parameters are positive, then the system (1.2) has a unique positive steady state $E_2 = (M^*, N^*)$ with $M^* = N^*$ and $0 < M^* < L$ (please see Figure 1).*

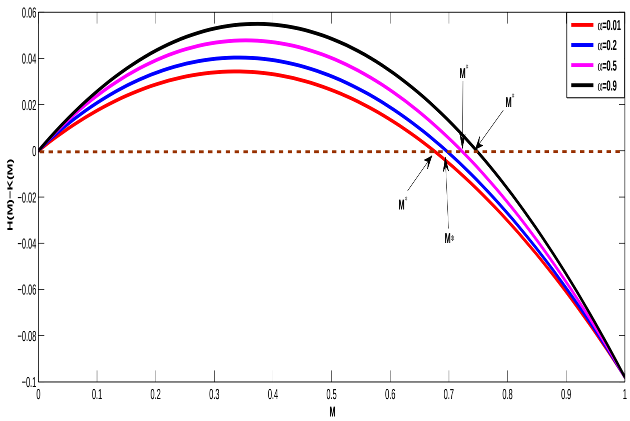

Figure 1. Influence of prey group shape on E_2 for different value of the parameter α. Here we take $L = 1, \beta = 0.3, \gamma = 0.1$ and $\tau_h = 0.2$.

Proof. Obviously, E_2 is the solution of:

$$\begin{cases} \beta M^* \left(1 - \frac{M^*}{L}\right) - \frac{\gamma M^{*\alpha} N^*}{1+\gamma\tau_h M^{*\alpha}} = 0, \\ \sigma \left(1 - \frac{N^*}{M^*}\right) = 0. \end{cases} \tag{3.2}$$

Using the second equation of (3.2), we can notice that $M^* = N^*$. Substituting $M^* = N^*$ into the first equation of (3.2), gives $h(M^*) = 0$, with

$$h(M) = \frac{\gamma M^\alpha}{1 + \gamma\tau_h M^\alpha} + \beta\Big(\frac{M}{L} - 1\Big),$$

and

$$h'(M) = \frac{\alpha\gamma M^{\alpha-1}}{(1 + \gamma\tau_h M^\alpha)^2} + \frac{\beta}{L} > 0, \text{ for } M \in [0, L].$$

Obviously, $h(0) = -\beta < 0$ and $h(L) = \frac{\gamma L^\alpha}{1+\gamma\tau_h L^\alpha} > 0$. Hence, $h(M)$ is an increasing function for $M \in [0, L]$, and bisect the horizontal axis at M^*, where $0 < M^* < L$. The proof of Theorem 3.2 is completed. □

Now, we define the Sobolev space

$$X = \left\{U = (M, N)^T \in H^2(0, l\pi) \times H^2(0, l\pi) | \partial_{\vec{n}} M = \partial_{\vec{n}} N = 0, \quad x = 0, l\pi\right\}, \tag{3.3}$$

and its complexification, $X_{\mathbb{C}} := X \oplus iX = \{x_1 + ix_2 | x_1, x_1 \in X\}$, with the the inner product $\langle ., . \rangle$ as

$$< U_1, U_2 >= \int_0^{l\pi} (\overline{M_1} M_2 + \overline{N_1} N_2)\, \mathrm{d}x, \qquad U_i = (M_i, N_i)^T \in X_{\mathbb{C}}, \quad i = 1, 2.$$

The linearization of (1.2) at (M, N) is

$$\begin{pmatrix} \frac{\partial M}{\partial \tau} \\ \frac{\partial N}{\partial \tau} \end{pmatrix} = (D\Delta + J_k(M, N)) \begin{pmatrix} M \\ N \end{pmatrix}, \tag{3.4}$$

where $D\Delta = diag(\delta\frac{\partial^2}{\partial x^2}, \eta\frac{\partial^2}{\partial x^2})$ and

$$J_k(M, N) = \begin{pmatrix} \beta\Big(1 - 2\frac{M}{L}\Big) - \frac{\alpha\gamma M^{\alpha-1} N}{(1+\gamma\tau_h M^\alpha)^2} & -\frac{\gamma M^\alpha}{1+\gamma\tau_h M^\alpha} \\ \sigma\Big(\frac{N}{M}\Big)^2 & \sigma\Big(1 - 2\frac{N}{M}\Big) \end{pmatrix}. \tag{3.5}$$

Clearly, the problem

$$-\Delta\phi = \mu\phi, \quad x \in (0, l\pi); \quad \phi'(0) = \phi'(l\pi) = 0,$$

has the eigenvalues $\xi_k = \left(\frac{k}{l}\right)^2 (k = 0, 1, 2, ...)$ where the corresponding normalized eigenfunctions defining in the Sobolev space X are

$$\xi_k(x) = \frac{\cos\frac{k}{l}x}{\left\|\cos\frac{k}{l}x\right\|} = \begin{cases} \sqrt{\frac{1}{l\pi}}, & k = 0, \\ \sqrt{\frac{2}{l\pi}}\cos\frac{k}{l}x, & k \geq 0. \end{cases} \tag{3.6}$$

Now, let

$$U(x, \tau) = \sum_{k=0}^{+\infty} \begin{pmatrix} a_k \\ b_k \end{pmatrix} \cos\Big(\frac{k\pi}{l}x\Big) e^{\lambda_k \tau}, \tag{3.7}$$

be the nontrivial solution of the system (3.4) yields the existence of $k \in \mathbb{N}_0$ such that λ satisfy

$$det(\lambda I - C_k - J_k(M,N)) = 0,$$

where, I is 2×2 identity matrix, and $C_k = -\left(\frac{k}{l}\right)^2 diag(\delta, \eta)$. After a straightforward calculation we obtain the characteristic equation of (3.4) as

$$\lambda^2 - Tr_k \lambda + Det_k = 0, \quad k \in \mathbb{N}_0, \tag{3.8}$$

where

$$Tr_k = \beta\left(1 - 2\frac{M}{L}\right) - \frac{\alpha\gamma M^{\alpha-1}N}{(1+\gamma\tau_h M^\alpha)^2} + \sigma\left(1 - 2\frac{N}{M}\right) - (\delta+\eta)\left(\frac{k}{l}\right)^2, \tag{3.9}$$

and

$$\begin{aligned} Det_k = \delta\eta\left(\frac{k}{l}\right)^4 - \left[\eta\beta\left(1 - 2\frac{M}{L}\right) - \eta\frac{\alpha\gamma M^{\alpha-1}N}{(1+\gamma\tau_h M^\alpha)^2} + \delta\sigma\left(1 - 2\frac{N}{M}\right)\right]\left(\frac{k}{l}\right)^2 \\ +\sigma\left[\left(\beta\left(1 - 2\frac{M}{L}\right) - \frac{\alpha\gamma M^{\alpha-1}N}{(1+\gamma\tau_h M^\alpha)^2}\right)\left(1 - 2\frac{N}{M}\right) + \frac{\gamma M^\alpha}{1+\gamma\tau_h M^\alpha}\left(\frac{N}{M}\right)^2\right]. \end{aligned} \tag{3.10}$$

For the semi trivial steady state E_1 we obtain

$$\begin{cases} Tr_k|_{E_1} = -\beta + \sigma - (\delta+\eta)\left(\frac{k}{l}\right)^2, & \text{for } k \geq 0, \\ Det_k|_{E_1} = \delta\eta\left(\frac{k}{l}\right)^4 - (\sigma\delta - \eta\beta)\left(\frac{k}{l}\right)^2 - \sigma\beta, & \text{for } k \geq 0. \end{cases} \tag{3.11}$$

Obviously, we have $Det_0|_{E_1} = -\sigma\beta < 0$ in the absence of the spatial diffusion, which means that E_1 is always unstable.

Now, let's concentrate on analyzing the stability and the bifurcation properties of E_2. From Theorem 3.2, (1.2) has a unique equilibrium denoted by $E_2 = (M^*, N^*)$ where, $M^* = N^*$ and $0 < M^* < L$. Submitting (M^*, N^*) into (3.9) and (3.10) and using the fact that

$$\beta M^*\left(1 - \frac{M^*}{L}\right) = \frac{\gamma(M^*)^\alpha}{1+\gamma\tau_h(M^*)^\alpha} \text{ and } 1 + \gamma\tau_h(M^*)^\alpha = \frac{\gamma(M^*)^\alpha}{\beta\left(1 - \frac{M^*}{L}\right)}, \tag{3.12}$$

then, we obtain

$$\begin{cases} Tr_k(\sigma) = \sigma_0 - \sigma - (\delta+\eta)\left(\frac{k}{l}\right)^2, & \text{for } k \geq 0, \\ Det_k(\sigma) = \delta\eta\left(\frac{k}{l}\right)^4 - (\eta\sigma_0 - \sigma\delta)\left(\frac{k}{l}\right)^2 + \sigma C_*, & \text{for } k \geq 0, \end{cases} \tag{3.13}$$

where

$$C_* = \frac{\alpha\beta^2}{\gamma M^*}\left(1 - \frac{M^*}{L}\right)^2 + \frac{1}{2}\beta M^* > 0, \tag{3.14}$$

and

$$\sigma_0 = \beta\left(1 - 2\frac{M^*}{L}\right) - \frac{\alpha\beta^2}{\gamma(M^*)^\alpha}\left(1 - \frac{M^*}{L}\right)^2. \tag{3.15}$$

Now, putting

$$\beta^* = \frac{\gamma\left(1 - 2\frac{M^*}{L}\right)(M^*)^\alpha}{\alpha\left(1 - \frac{M^*}{L}\right)^2}. \tag{3.16}$$

Remark 3.3. *If $\frac{L}{2} < M^* < L$, i.e. $\beta^* < 0$, we obtain that $Tr_k(\sigma) < 0$ and $Det_k(\sigma) > 0$, so E_2 is always locally asymptotically stable (with or without diffusion). Thus, there is no Hopf bifurcation.*

Now, assume that the following condition holds

$$(\mathbf{H_2}) : 0 < M^* < \frac{L}{2} \text{ and } 0 < \beta < \beta^*.$$

So, we have the following theorem

Theorem 3.4. *If $\eta = \delta = 0$, then:*

- **(i)** *The positive steady state E_2 of system (1.2) is locally asymptotically stable if $\sigma > \sigma_0$ and unstable if $\sigma < \sigma_0$ (see Figure 2).*
- **(ii)** *The local system of the diffusion system (1.2) undergoes Hopf bifurcation for $\sigma = \sigma_0$, where σ_0 is defined in (3.15) (please see Figure 3).*

Proof. Under the condition $(\mathbf{H_2})$ and from (3.13), if $k = 0$ one can immediately get that E_2 is locally asymptotically stable when $\sigma > \sigma_0$ Figure 2 and unstable when $\sigma < \sigma_0$. For $\sigma = \sigma_0$, Eq (3.8) has a pair of purely imaginary roots $\pm i\sqrt{\sigma C_*}$. Let $\lambda(\sigma) = \upsilon(\sigma) \pm i\omega(\sigma)$ be the root of (3.8) satisfying $\upsilon(\sigma_0) = 0,\ \omega(\sigma_0) = \pm i\sqrt{\sigma C_*}$. A simple calculate gives following transversality condition:

$$\frac{d}{d\sigma}\upsilon(\sigma)\Big|_{\sigma=\sigma_0} = -\frac{1}{2} < 0. \tag{3.17}$$

Therefore, we conclude that the non diffusive system associated with (1.2) undergoes H-bifurcation at E_2 when $\sigma = \sigma_0$ (see Figure 3). □

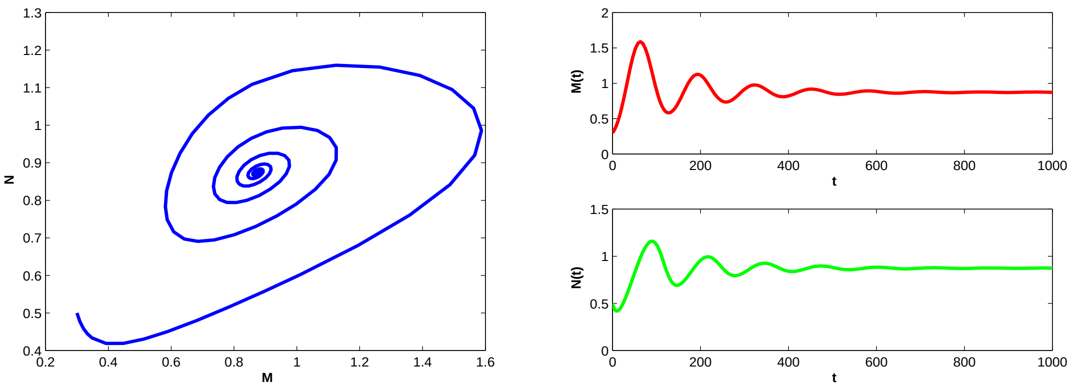

Figure 2. The behavior of (1.2) in the absence of diffusion for $\alpha = 2/3, \beta = 0.1, L = 10, \tau_h = 0.01, \sigma = 0.79 > \sigma_0 = 0.76$ where, $(M^* = N^* = 0.873 < L = 10$, then, E_2 is locally asymptotically stable. Here $(M(0), N(0)) = (0.3, 0.4)$.

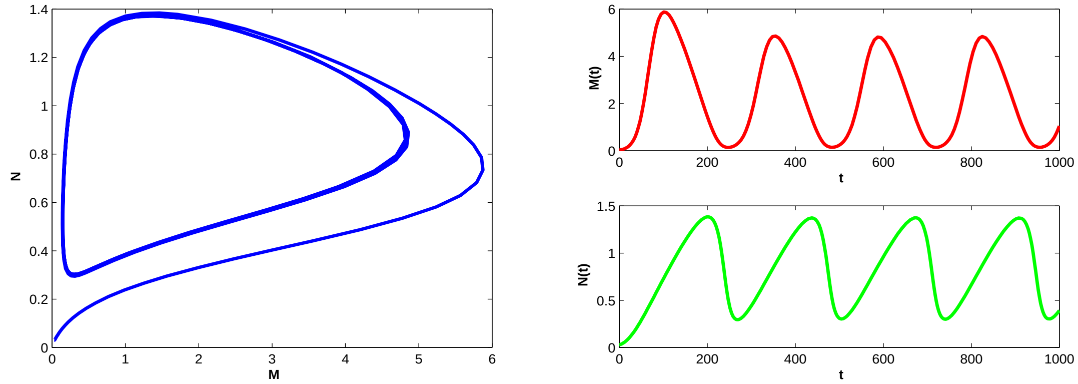

Figure 3. The behavior of (1.2) in the absence of diffusion for $\alpha = 2/3, \beta = 0.1, L = 10, \tau_h = 0.01, \sigma = 0.01 < \sigma_C = 0.76$ where, $(M^* = N^* = 0.873 < L = 10$, then, E_2 loses its stability and a H-bifurcation occurs. Here $(M(0), N(0)) = (0.1, 0.05)$.

3.2. H-bifurcation

Now, concentrating on studying the occurrence of time-periodic solutions for (1.2) generated by H-bifurcation. throughout the rest of this paper, we assume that $(\mathbf{H_2})$ holds i.e. $\sigma_0 > 0$ and taking σ as the bifurcation parameter. Recall that H-bifurcation appears if

$$Tr_k = 0, \quad Det_k > 0 \ \text{ and } \ \frac{\partial}{\partial\sigma}\lambda(\sigma)\Big|_{\sigma=\sigma_0} \neq 0.$$

Obviously

$$\sigma = \sigma_k = \sigma_0 - (\delta + \eta)\left(\frac{k}{l}\right)^2, \quad k \in \mathbb{N}_0 \tag{3.18}$$

where

$$k^* = \max\{n \in \mathbb{N} \mid Det_k(\sigma_k) > 0 \text{ and } \sigma_k > 0 \text{ for } k = 0, 1, ..., n-1\} \tag{3.19}$$

are the critical points for H-bifurcation. The values of the H-bifurcation are highlighted as follows.

Theorem 3.5.

(i) *The diffusive model (1.2) undergoes H-bifurcation at E_2 when $\sigma = \sigma_k$, for $k = 0, 1, ..., k^*-1$, where σ_k and k^* are defined in (3.18) and (3.19), respectively (see Figure 4). Further, for $k = 0$ we get a homogeneous periodic solution and a non homogeneous periodic solution for $k = 1, 2, ..., k^* - 1$.*
(ii) *The eventual H-bifurcation points $(\sigma_k)_{0\leq k\leq k^*-1}$ satisfying the following relationships*

$$\sigma_{k^*-1} < ... < \sigma_{k+1} < \sigma_k < \sigma_{k-1} < ... < \sigma_1 < \sigma_0.$$

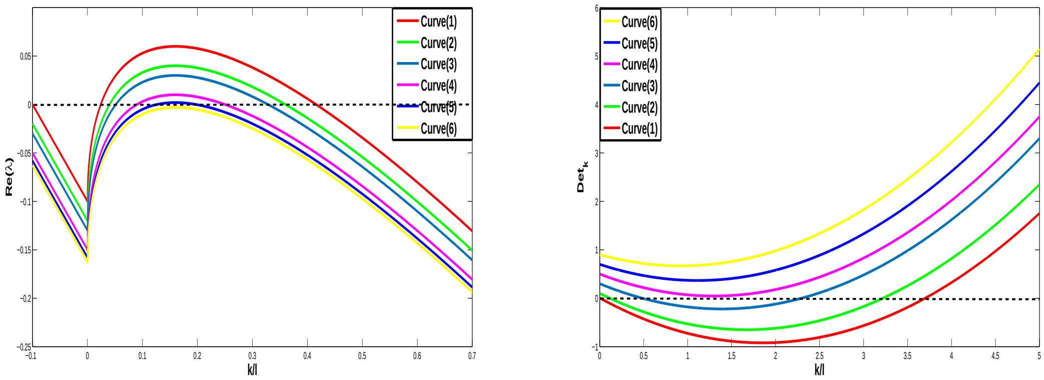

Figure 4. Left: the relation between $Re(\lambda)$ and k/l, where for the yellow curve T-instability does not exist, and for the other curves T-instability exists. Right: plot of Det_k with respect to k/l with the parameter values $\beta = 0.3,\ L = 1, \gamma = 0.5,\ \alpha = 2/3,\ \tau_h = 0.1$ and different value of σ. curve(1): $\sigma_1 = 0.0.21$; curve(2): $\sigma_2 = 0.0.27$; curve(3): $\sigma_3 = 0.0.32$; curve(4): $\sigma_4 = 0.0.36$; curve(5): $\sigma_5 = 0.0.41$; curve(6): $\sigma_6 = 0.0.46$.

Proof.

(i) From the definition of the integer k^*, we can easily affirm that when $\sigma = \sigma_k$, $Tr_k(\sigma_k) = 0$ and $Det_k(\sigma_k) > 0$ for $k = 0, 1, ..., k^* - 1$, which follows purely imaginary roots of Eq (3.8). Letting

$$\lambda_k(\sigma) = A_k(\sigma) \pm iB_k(\sigma), \quad k = 0, 1, ..., k^* - 1$$

be the roots of Eq (3.8) which verifies

$$A_k(\sigma_k) = 0, \quad B_k = \sqrt{Det_k(\sigma_k)}.$$

It follows that if σ is in the neighborhood σ_k, the solutions of the characteristic equation (3.8) take the following form

$$A_k(\sigma) \pm iB_k(\sigma) = \frac{Tr_k(\sigma) \pm \sqrt{Tr_k^2(\sigma) - 4Det_k(\sigma)}}{2}$$

with

$$A_k(\sigma) = \frac{Tr_k(\sigma)}{2}, \quad B_k(\sigma) = \sqrt{Det_k(\sigma) - \frac{Tr_k^2(\sigma)}{4}}$$

and we have

$$A'(\sigma_k) = -\frac{1}{2} < 0.$$

This yields to the verification of the transversality condition for each σ_k where $k = 0, 1, ..., k^* - 1$.

(ii) Now, we are in the position to prove the second affirmation. After a simple calculate we find

$$\sigma_{k+1} - \sigma_k = -\frac{(1 + 2n)(\delta + \eta)}{l^2} < 0.$$

This implies that (σ_k) is strictly decreasing sequence for all $k = 0, 1, ..., k^* - 1$. The proof of Theorem 3.5 is completed.

□

3.3. T-instability

In the following, we mainly prove that under certain sufficient condition, the system (1.2) may exhibits T-instability unlike the situation in the system without Leslie-Gower term (please see Sec. 2). Mentioning that T-instability holds when the equilibrium point is locally stable in the non diffusive system and becomes unstable in the case of diffusive system (i.e. $Det_k < 0$ for some value of integer k). Notice that E_2 is locally stable if the condition ($\mathbf{H_2}$) is satisfied and $\sigma > \sigma_0$. In this case we have

$$Tr_0(\sigma) < 0, \text{ and } Det_0(\sigma) > 0.$$

In order to study the occurrence of the Turing instability, we define the functional Θ as

$$\Theta\left(\left(\frac{k}{l}\right)\right) := Det_k(\sigma) = \delta\eta\left(\left(\frac{k}{l}\right)^2\right)^2 - (\eta\sigma_0 - \sigma\delta)\left(\frac{k}{l}\right)^2 + \sigma C_*,$$

which considered as a quadratic polynomial in $\left(\frac{k}{l}\right)^2$ and σ_0 is defined in (3.15).

Lemma 3.6. *If*

$$\frac{\eta}{\delta} < \frac{\sigma}{\sigma_0}, \tag{3.20}$$

the system (1.2) cannot undergo T-instability.

Proof. Clearly, under the condition (3.20) we have $\Theta\left(\left(\frac{k}{l}\right)^2\right) > 0$ which means that system (1.2) has no diffusion driven instability. □

In the next, we presume that

$$(\mathbf{H_3}) : \frac{\eta}{\delta} > \frac{\sigma}{\sigma_0}.$$

Obviously, if $\Theta\left(\left(\frac{k}{l}\right)^2\right) < 0$, then Eq (3.8) has one of the two roots is positive. If $F(\eta, \delta) = \eta\sigma_0 - \sigma\delta > 0$, $\Theta\left(\left(\frac{k}{l}\right)\right)$ has a minimum at

$$\left(\frac{k}{l}\right)^2_{\min} = \frac{\eta\sigma_c - \sigma\delta}{2\eta\delta} > 0.$$

Evaluating Θ at this minimum, we get

$$\min_{\left(\frac{k}{l}\right)^2} \Theta\left(\left(\frac{k}{l}\right)^2\right) = \sigma C_* - \frac{(\eta\sigma_0 - \sigma\delta)^2}{4\eta\delta}, \tag{3.21}$$

where C^* is defined by (3.14). Next we show that under ($\mathbf{H_3}$), $\min_{\left(\frac{k}{l}\right)^2} \Theta\left(\left(\frac{k}{l}\right)^2\right) < 0$ for some values $\eta/\delta > 0$, which it known by the condition of T-instability. Defining the ratio $\xi = \eta/\delta$ and

$$\Pi(\eta, \delta) = (\eta\sigma_0 - \sigma\delta)^2 - 4\eta\delta\sigma C_* = \sigma_0^2\eta^2 - 2\sigma(\sigma_0 + 2C_*)\eta\delta + \sigma^2\delta^2.$$

Then, $\Pi(\eta, \delta) = 0$ and $F(\eta, \delta) = 0$ are equivalent to

$$\sigma_0^2\xi^2 - 2\sigma(\sigma_0 + 2C_*)\xi + \sigma^2 = 0, \tag{3.22}$$

and

$$\xi = \xi^* = \frac{\sigma}{\sigma_0}. \tag{3.23}$$

Notice that

$$4\sigma^2(\sigma_0 + 2C_*)^2 - 4\sigma_0^2\sigma^2 = 16\sigma^2 C_*(C_* + \sigma_0) > 0,$$

which means that Eq (3.22) has two has two positive real roots

$$\xi_1 = \frac{\sigma(\sigma_0 + 2C_*) + 2\sigma\sqrt{C_*(C_* + \sigma_0)}}{\sigma_0^2}, \quad \xi_2 = \frac{\sigma(\sigma_0 + 2C_*) - 2\sigma\sqrt{C_*(C_* + \sigma_0)}}{\sigma_0^2}. \tag{3.24}$$

Easily, one can see that $0 < \xi_2 < \xi^* < \xi_1$ and if $\eta/\delta > \xi_1$, we have $\min_{(\frac{k}{l})^2} \Theta\left(\left(\frac{k}{l}\right)^2\right) < 0$ and $F(\eta, \delta) > 0$. Here, the positive equilibrium E_2 becomes unstable, which means that T-instability occurs.

Now, defining the set

$$R_T := \{(\eta, \delta) : \delta > 0,\ \eta > 0 \text{ and } \eta/\delta > \xi_1\}.$$

Hence, we get the results

Theorem 3.7. *Presume that* $(\mathbf{H_2})$ *holds and* $\sigma > \sigma_0$ *(for having the stability of the positive equilibrium). Then there exists an unbounded set* R_T *such that for any* $(\eta, \delta) \in R_T$*, the equilibrium* E_2 *becomes unstable, that is , Turing instability (for illustrations we refer Figure 5).*

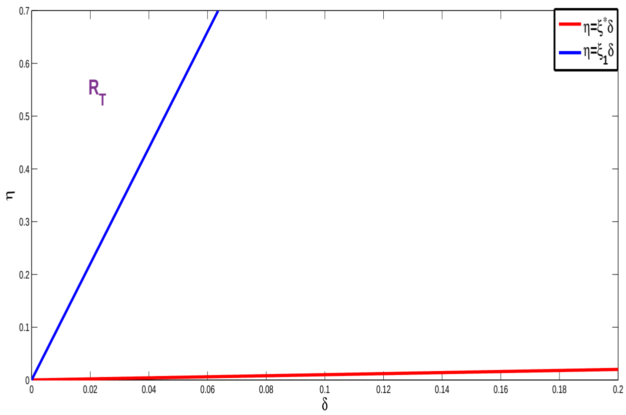

Figure 5. Graph represents Bifurcation diagram for T-instability generated by the diffusive system (1.2) in $\delta - \eta$ plane with the parameter values $\beta = 0.9,\ L = 10,\ \gamma = 0.2,\ \tau_h = 0.5,\ \alpha = 2/3,\ \sigma = 0.5$.

3.4. T-H bifurcation

In this subsection, our aim is to investigate the occurrence of T-H bifurcation. This type of bifurcation happen if there are two integers k_1 and k_2 where for $k = k_1$, (1.2) has H-bifurcation and for $k = k_2$ the system (1.2) undergoes T-bifurcation, this kind of bifurcation is a bi-dimensional bifurcation which means that we need to choose two bifurcation parameters. Hence, we choose σ and

δ as bifurcation parameters. Assuming that $\sigma_0 > 0$, it is well known that $Tr_0(\sigma) = 0$ and $Det_0(\sigma) < 0$ are necessary conditions for Hopf bifurcation to occur. From (3.13) and (3.15), $Tr_0 = 0$ is equivalent to

$$\sigma = \sigma^H(\delta) = \sigma_0 = \beta\left(1 - 2\frac{M^*}{L}\right) - \frac{\alpha\beta^2}{\gamma(M^*)^\alpha}\left(1 - \frac{M^*}{L}\right)^2, \tag{3.25}$$

which is the line of H-bifurcation in $\delta - \sigma$ plan, where the frequency of the oscillations is

$$\omega^H = Im(\lambda) = \sqrt{\sigma C_*}.$$

Besides, T-instability occurs when $\frac{\eta}{\delta} > \xi_1$, where ξ is given in (3.24). It follows that the critical value of the T-bifurcation for the parameter σ is

$$\sigma = \sigma^T(\delta) = \frac{\eta\sigma_0^2}{\delta\left((\sigma_0 + 2C_*) + 2\sqrt{C_*(C_* + \sigma_0)}\right)} = \frac{\eta\sigma_0^2}{\left(\sqrt{C_*} + \sqrt{C_* + \sigma_0}\right)^2}. \tag{3.26}$$

Now, we prove the existence of intersection point between the H-bifurcation curve σ^H and T-instability curve σ^T in $\delta - \sigma$ plane. Defining the following function

$$h(x) = \frac{\eta\sigma_0^2}{x\left((\sigma_0 + 2C_*) + 2\sqrt{C_*(C_* + \sigma_0)}\right)}, \quad x > 0.$$

Clearly, h is monotonously decreases with the increasing of x. In addition, we have $\lim_{x\to 0^+} h(x) = +\infty$. Therefore, we conclude that the H-bifurcation line σ^H cuts the T-bifurcation curve σ^T at $(\delta^{T-H}, \sigma^{T-H}) = (\hat{\delta}, \sigma_0)$ where

$$\hat{\delta} = \frac{\eta\sigma_0}{\left(\sqrt{C_*} + \sqrt{C_* + \sigma_0}\right)^2}. \tag{3.27}$$

Now, we will examine the transversality condition. Fixing δ, and taking σ as parameter, letting $\lambda(\sigma)$ the roots of (3.8), hence:

$$\frac{d}{d\sigma}Re\lambda(\sigma)\Big|_{\sigma^H} = \frac{d}{d\sigma}Re\lambda(\sigma)\Big|_{\sigma^T} = -\frac{1}{2} < 0,$$

then, we get

Theorem 3.8. *Presume that the conditions* ($\mathbf{H_2}$) *and* ($\mathbf{H_3}$) *are verified, then:*

(i) *The H-bifurcation line σ^H cuts the T-bifurcation curve σ^T in $\delta - \sigma$-parameter space at the unique point $(\hat{\delta}, \sigma_0)$, where σ_0 and $\hat{\delta}$ are defined in (3.15) and (3.27) (for illustrations we refer Figure 6).*

(ii) *At $(\delta, \sigma) = (\hat{\delta}, \sigma_0)$ the characteristic equation (3.8) has a simple zero root.*

In order to illustrate numerically the obtained result in Theorem 3.8, T-bifurcation curve and H-bifurcation curves are plotted in $\delta - \sigma$ plane (please see Figure 5). we fix the parameters $\beta = 0.9$, $L = 10$, $\gamma = 0.2$, $\tau_h = 0.5$, $\alpha = 2/3$, $\sigma = 0.5$ and $\eta = 1$. It follows that $M^* = N^* = 4.97 < L = 10$, $\sigma_0 = 0.4478 < \sigma = 0.5$, $\alpha^* = 0.432 < \alpha = 0.667$, $C_* = 5.6316$ and $\hat{\delta} = 0.049$. From Figure 6, we can see that the intersection point (1) divide the $\delta - \sigma$ plan into four regions. In D_1, E_2 is stable. D_2 represents the T-bifurcation region. D_3 is the domain in which the pure H-bifurcation occurs. In

D_4, both T-instability and H-bifurcation occur. In this situation, the diffusive system (1.2) produces a complex spatiotemporal patterns, where the two instabilities T- bifurcation and H-bifurcation coincide.

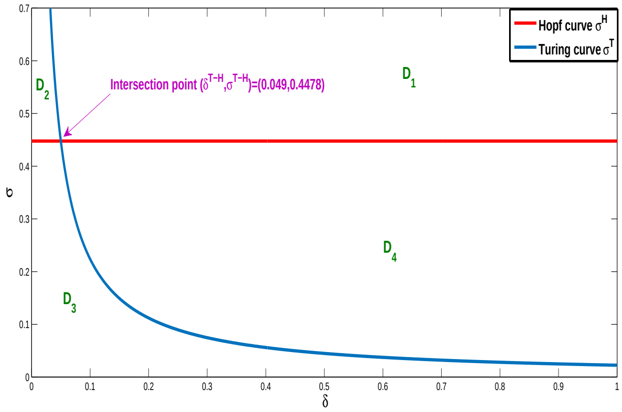

Figure 6. Graph represents the existence of T-H bifurcation point in $\delta-\sigma$ plane near E_2 with the parameter values $\beta = 0.9,\ L = 10,\ \gamma = 0.2,\ \tau_h = 0.5,\ \alpha = 2/3,\ \sigma = 0.5$ and $\eta = 1$.

4. Normal forms for T-H bifurcation

Here, we mainly focus on the calculate of the normal form of T-H bifurcation in order to determine the properties and the spatiotemporal dynamics of (1.2) at $E_2 = (M^*, N^*)$ near the T-H bifurcation point $(\hat{\delta}, \sigma_0)$. At first, we set $\mu = (\mu_1, \mu_2) \in \mathbb{R}^2)$ where $\mu_1 = \delta - \hat{\delta},\ \mu_2 = \sigma - \sigma_0$. Then, we we apply the translation $\overline{M} = M - M^*,\ \overline{N} = N - N^*$ to (1.2) and introduce a new parameter $\mu = (\mu_1, \mu_2) \in \mathbb{R}^2$). we denote $\overline{M}$ by M and $\overline{N}$ by N. Therefore, the diffusive system (1.2) becomes

$$\begin{cases} \frac{\partial}{\partial \tau} M(x,\tau) = \beta(M(x,\tau) + M^*)\left(1 - \frac{(M(x,\tau)+M^*)}{L}\right) + (\hat{\delta} + \mu_1)\Delta M(x,\tau) \\ \qquad - \frac{\gamma(M(x,\tau)+M^*)^\alpha (N(x,\tau)+M^*)}{1+\gamma\tau_h (M(x,\tau)+M^*)^\alpha}, \quad x \in (0, l\pi),\ \tau > 0, \\ \frac{\partial}{\partial \tau} N(x,\tau) = (\sigma_0 + \mu_2)(N(x,\tau) + M^*)\left(1 - \frac{N(x,\tau)+M^*}{M(x,\tau)+M^*}\right) + \eta\Delta N(x,\tau), \quad x \in (0, l\pi),\ \tau > 0, \\ \frac{\partial}{\partial \vec{n}} M(x,\tau) = \frac{\partial}{\partial \vec{n}} N(x,\tau) = 0, \quad x \in (0, l\pi),\ \tau > 0, \\ M(x,0) = M_0(x) \geq 0, \quad N(x,0) = N_0(x) \geq 0, \quad x \in (0, l\pi). \end{cases} \tag{4.1}$$

For system (4.1) and according to [19], also we get

$$D(\mu) = \begin{pmatrix} \hat{\delta} + \mu_1 & 0 \\ 0 & \eta \end{pmatrix}, \quad L(\mu) = \begin{pmatrix} \sigma_0 & \pi \\ (\sigma_0 + \mu_2) & -(\sigma_0 + \mu_2) \end{pmatrix}, \tag{4.2}$$

and

$$F(\varphi, \mu) = \begin{pmatrix} \beta(\varphi_1 + M^*)\left(1 - \frac{(\varphi_1 + M^*)}{L}\right) - \frac{\gamma(\varphi_1+M^*)^\alpha(\varphi_2+M^*)}{1+\gamma\tau_h(\varphi_1+M^*)^\alpha} - \sigma_0\varphi_1 + a_1\varphi_2 \\ (\sigma_0 + \mu_2)(\varphi_2 + M^*)\left(1 - \frac{\varphi_2+M^*}{\varphi_1+M^*}\right) - (\sigma_0 + \mu_2)\varphi_1 + (\sigma_0 + \mu_2)\varphi_2 \end{pmatrix}, \tag{4.3}$$

where

$$\pi = -\frac{\gamma M^*}{1 + \gamma\tau_h M^*}, \quad \text{and} \quad \varphi = (\varphi_1, \varphi_2)^T \in X.$$

It follows that

$$D(0) = \begin{pmatrix} \hat{\delta} & 0 \\ 0 & \eta \end{pmatrix}, \quad D_1(\mu) = \begin{pmatrix} 2\mu_1 & 0 \\ 0 & 0 \end{pmatrix}, \quad L(0) = \begin{pmatrix} \sigma_0 & \pi \\ \sigma_0 & -\sigma_0 \end{pmatrix}, \quad L_1(\mu) = \begin{pmatrix} 0 & 0 \\ 2\mu_2 & -2\mu_2 \end{pmatrix},$$

and

$$Q(\varphi,\chi) = \begin{pmatrix} a_{11}\varphi_1\chi_1 + a_{12}(\varphi_1\chi_2 + \varphi_2\chi_1) + a_{13}\varphi_2\chi_2 \\ a_{21}\varphi_1\chi_1 + a_{22}(\varphi_1\chi_2 + \varphi_2\chi_1) + a_{23}\varphi_2\chi_2 \end{pmatrix},$$

$$C(\varphi,\chi,\nu)=\begin{pmatrix} b_{11}\varphi_1\chi_1\nu_1 + b_{12}(\varphi_1\chi_1\nu_2 + \varphi_1\chi_2\nu_1 + \varphi_2\chi_1\nu_1) \\ +b_{13}(\varphi_1\chi_2\nu_2 + \varphi_2\chi_1\nu_2 + \varphi_2\chi_2\nu_1) + b_{14}\varphi_2\chi_2\nu_2 \\ b_{21}\varphi_1\chi_1\nu_1 + b_{22}(\varphi_1\chi_1\nu_2 + \varphi_1\chi_2\nu_1 + \varphi_2\chi_1\nu_1) \\ +b_{23}(\varphi_1\chi_2\nu_2 + \varphi_2\chi_1\nu_2 + \varphi_2\chi_2\nu_1) + b_{24}\varphi_2\chi_2\nu_2 \end{pmatrix},$$

where

$$\chi = (\chi_1,\chi_2)^T \in X, \quad \nu = (\nu_1,\nu_2)^T \in X.$$

The coefficients a_{ij} and b_{ij} are given as

$$a_{11} = \frac{\alpha(1-\alpha)\gamma(M^*)^{\alpha-1} + \left(2\gamma^2\alpha^2\tau_h + \alpha(1-\alpha)\right)(M^*)^{\alpha-2}}{(1+\gamma\tau_h(M^*)^{\alpha})^3},\ a_{12} = \frac{-\alpha\gamma(M^*)^{\alpha-1}}{(1+\gamma\tau_h(M^*)^{\alpha})^2},$$

$$a_{21} = -2\frac{\sigma_0}{M^*},\ a_{22} = 2\frac{\sigma_0}{M^*},\ a_{23} = -2\frac{\sigma_0}{M^*},$$

$$b_{11}=\frac{-\alpha(\alpha-1)(\alpha-2)\gamma(M^*)^{\alpha-2}(1+\gamma\tau_h(M^*)^{\alpha})^2+6\alpha^2\gamma^2\tau_h(\alpha-1)(M^*)^{\alpha-2}(1+\gamma\tau_h(M^*)^{\alpha})-6\alpha^3\gamma^3\tau_h(M^*)^{3\alpha-2}}{(1+\gamma\tau_h(M^*)^{\alpha})^4},$$

$$b_{12} = \frac{\alpha(1-\alpha)\gamma(M^*)^{\alpha-2}(1+\gamma\tau_h(M^*)^{\alpha}) + 2\alpha^2\gamma^2\tau_h(M^*)^{2\alpha-2}}{(1+\gamma\tau_h(M^*)^{\alpha})^3},$$

$$b_{21} = 6\frac{\sigma_0}{(M^*)^2},\ b_{22} = -4\frac{\sigma_0}{(M^*)^2},\ b_{23} = 2\frac{\sigma_0}{(M^*)^2},$$

$$a_{13} = b_{13} = b_{14} = b_{24} = 0.$$

Now, we get the corresponding characteristic matrices as

$$\mathbb{D}_k(\lambda) = \begin{pmatrix} \lambda + \delta\mu_k - \sigma_0 & -\pi \\ -\sigma_0 & \lambda + \eta\mu_k + \sigma_0 \end{pmatrix}, \quad k \in \mathbb{N}.$$

Clearly, $\lambda = \pm i\omega$ with $\omega = \sqrt{Det_0}$, are eigenvalues of $\mathbb{D}_0(\lambda)$, and $\lambda = 0$ is a simple eigenvalues for $\mathbb{D}_{\hat{\delta}}(\lambda)$, while other eigenvalues have negative real parts. From Theorem 3.8 and by using a simple calculate we can obtain

$$\varphi_1 = \begin{pmatrix} 1 \\ \frac{\sigma_0}{\hat{\delta}\mu_{\hat{\delta}}+\sigma_0} \end{pmatrix}, \quad \varphi_2 = \begin{pmatrix} 1 \\ \frac{\sigma_0-i\omega}{-\pi} \end{pmatrix},$$

and

$$\chi_1 = \begin{pmatrix} \frac{\hat{\delta}\mu_{\hat{\delta}}+\sigma_0}{(1+\hat{\delta})\mu_{\hat{\delta}}} \\ \frac{(\mu_{\hat{\delta}}-\sigma_0)(\hat{\delta}\mu_{\hat{\delta}}+\sigma_0)}{(1+\hat{\delta})\mu_{\hat{\delta}}\sigma_0} \end{pmatrix}, \quad \chi_2 = \begin{pmatrix} \frac{-\pi\sigma_0}{-\pi\sigma_0+(\omega+i\sigma_0)^2} \\ \frac{-\pi(i\omega-\sigma_0)}{-\pi\sigma_0+(\omega+i\sigma_0)^2} \end{pmatrix}.$$

Then, by the procedure developed in [19, 20], the normal form restricted on central manifold at T-H bifurcation singularity is

$$\begin{cases} \dot{Z}_1 = m_1(\mu)Z_1 + m_{200}Z_1^2 + m_{011}Z_2\overline{Z}_2 \\ \quad +m_{300}Z_1^3 + m_{111}Z_1Z_2\overline{Z}_2 + \text{h.o.t.}, \\ \dot{Z}_2 = i\omega Z_2 + n_2(\mu)Z_2 + n_{110}Z_1Z_2 \\ \quad +n_{210}Z_1^2Z_2 + n_{021}Z_2^2\overline{Z}_2 + \text{h.o.t.}, \\ \dot{\overline{Z}}_2 = -i\omega\overline{Z}_2 + \overline{n}_2(\mu)\overline{Z}_2 + \overline{n}_{110}Z_1\overline{Z}_2 \\ \quad +\overline{n}_{210}Z_1^2\overline{Z}_2 + \overline{n}_{021}Z_2\overline{Z}_2^2 + \text{h.o.t.}. \end{cases} \tag{4.4}$$

where the calculation of $m_1(\mu),\ m_{200},\ m_{011},\ m_{300},\ m_{111},\ n_2(\mu),\ n_{110},\ n_{210},\ n_{021}$ are given in "Appendix".

By using the new parameter transformation $Z_1 = r,\ Z_2 = \rho\cos\vartheta - i\rho\sin\vartheta$, then we get

$$\begin{cases} \dot{r} = m_1(\mu)r + m_{300}r^3 + m_{111}r\rho^2, \\ \dot{\rho} = Re(n_2(\mu))\rho + Re(n_{210})\rho r^2 + Re(n_{021})\rho^2. \end{cases} \tag{4.5}$$

5. Numerical simulation

Here, we will provide some figure for illustrating the obtained results. In fact we investigate the following cases:

Figure 7: In this figure we set $\beta = 0.5$, $L = 15$, $\gamma = 1.5$, $\alpha = 0.7$, $\tau_h = 1.7$, $\sigma = 3.1$, $\delta = 0.02$, $\eta = 0.04$, $l = 1$ and the initial data $M(x, 0) = 0.5 + 0.1\cos(3x)$, $N(x, 0) = 0.5 + 0.1\cos(2x)$. Here, we obtain the stability of the nonhomogeneous steady state.

Figure 8: In this figure we consider the following values $\beta = 3.1$, $L = 50$, $\gamma = 1.5$, $\alpha = 0.66$, $\tau_h = 0.01$, $\sigma = 1.01$, $\delta = 0.01$, $\eta = 0.04$, $l = 1$ and the initial data $M(x, 0) = 0.5 + 0.1\cos(3x)$, $N(x, 0) = 0.5 + 0.1\cos(2x)$. Here, we arrive to the stability of the nonhomogeneous steady state.

Figure 9: In this graphical representation we set $\beta = 1.1$, $L = 50$, $\gamma = 1.5$, $\alpha = 0.66$, $\tau_h = 0.01$, $\sigma = 0.51$, $\delta = 0.01$, $\eta = 0.04$, $l = 1$ and the data $M(x, 0) = 0.5+0.1\cos(3x)$, $N(x, 0) = 0.5+0.1\cos(2x)$. We arrive at stability of the nonhomogeneous steady state.

Figure 10: Here we choose the set of values $\beta = 1.1$, $L = 50$, $\gamma = 1.5$, $\alpha = 0.66$, $\tau_h = 0.01$, $\sigma = 0.1$, $\delta = 0.01$, $\eta = 0.04$, $l = 1$ and the initial data $M(x, 0) = 0.5 + 0.1\cos(3x)$, $N(x, 0) = 0.5 + 0.1\cos(2x)$. Here, we arrive to stability of nonhomogeneous periodic solutions.

Figure 11: Here we choose the set of values $\beta = 1.51$, $L = 10$, $\gamma = 1.5$, $\alpha = 0.66$, $\tau_h = 0.01$, $\sigma = 0.051$, $\delta = 0.01$, $\eta = 0.04$, $l = 1$ and the data $M(x, 0) = 0.5 + 0.1\cos(3x)$, $N(x, 0) = 0.5 + 0.1\cos(2x)$. Here, we arrive to stability of nonhomogeneous periodic solutions.

Figure 12: Here, we choose the set of values values $\beta = 1.51$, $L = 10$, $\gamma = 1.5$, $\alpha = 0.66$, $\tau_h = 0.01$, $\sigma = 0.08$, $\delta = 0.09$, $\eta = 0.01$, $l = 1$ and the data $M(x, 0) = 0.5+0.1\cos(3x)$, $N(x, 0) = 0.5+0.1\cos(2x)$. Here, we arrive to stability of nonhomogeneous periodic solutions.

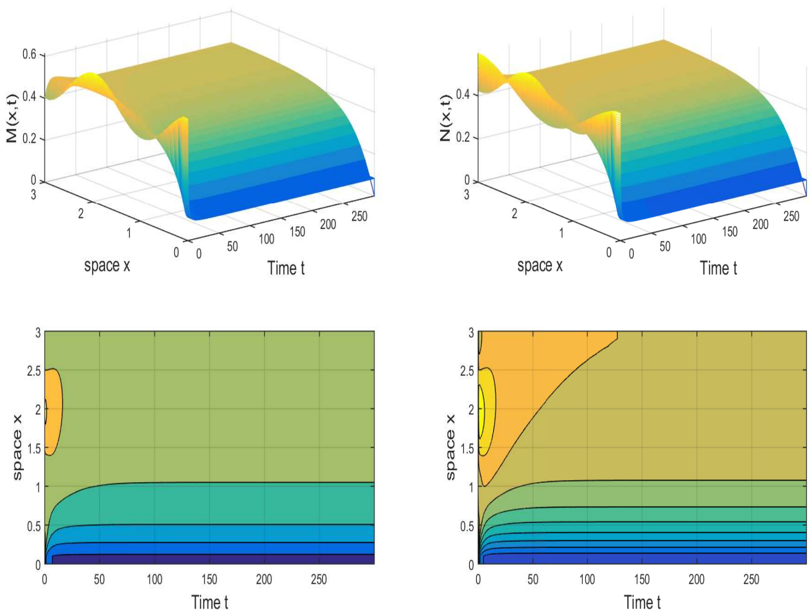

Figure 7. The stability of the non homogeneous steady state, which is obtained in the case of T-instability.

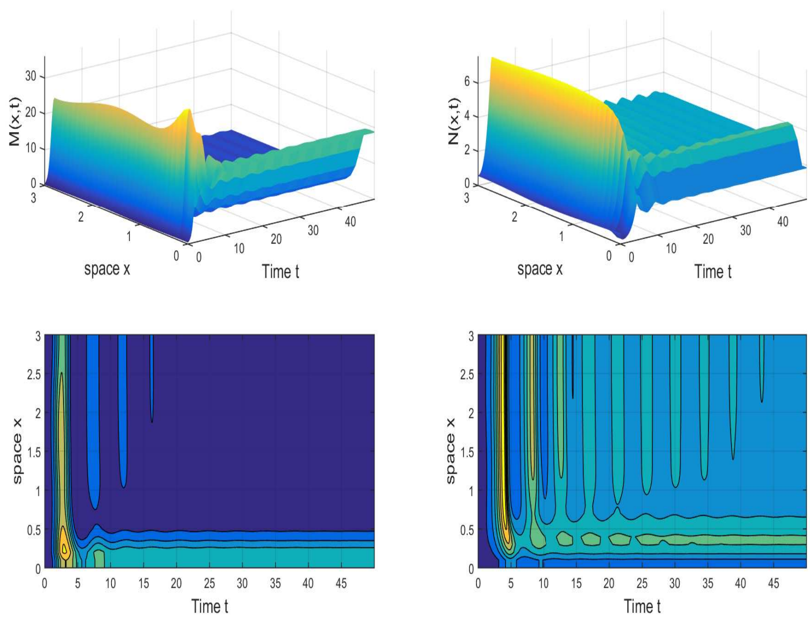

Figure 8. The stability of the non homogeneous steady state, which is obtained in the case of T-instability.

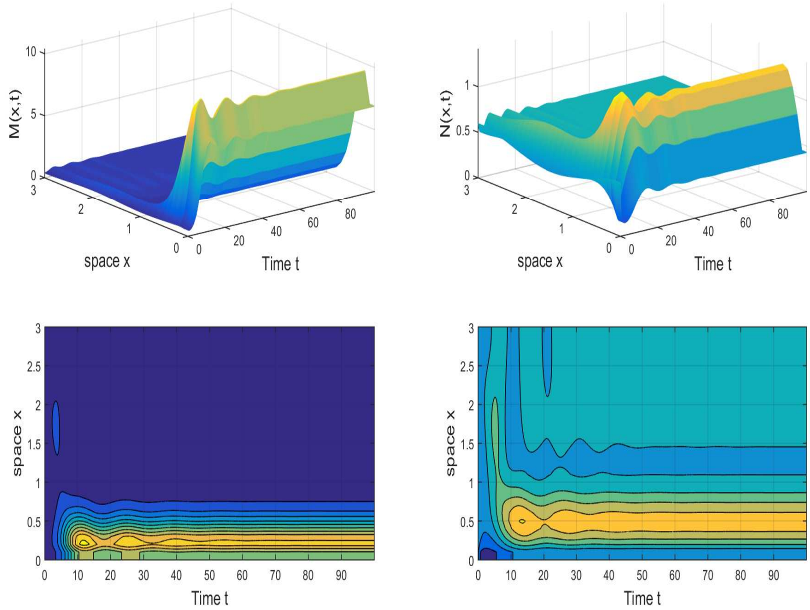

Figure 9. Non homogeneous distribution of the prey and predators, which is obtained in the case of T-instability.

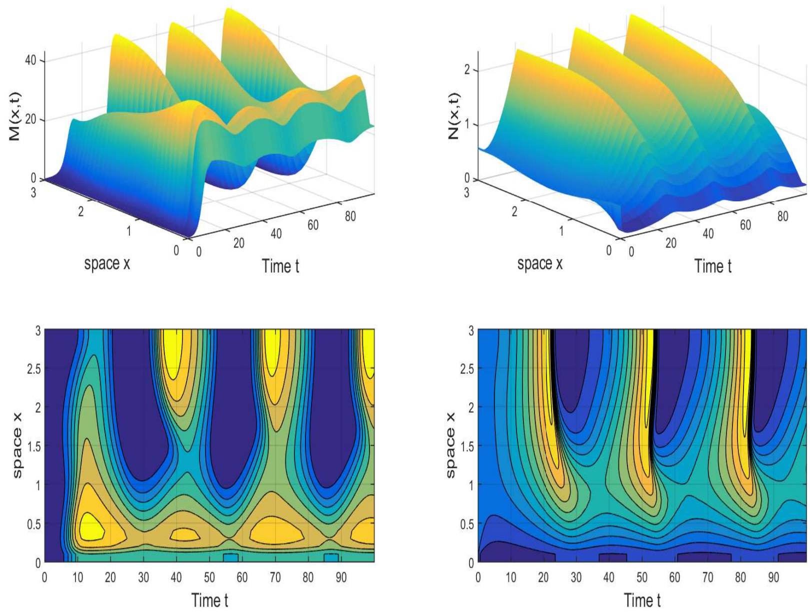

Figure 10. Non homogeneous distribution of the prey and predators with periodic patterns, which is obtained in the can of the existence Hopf bifurcation (see Theorem 3.5).

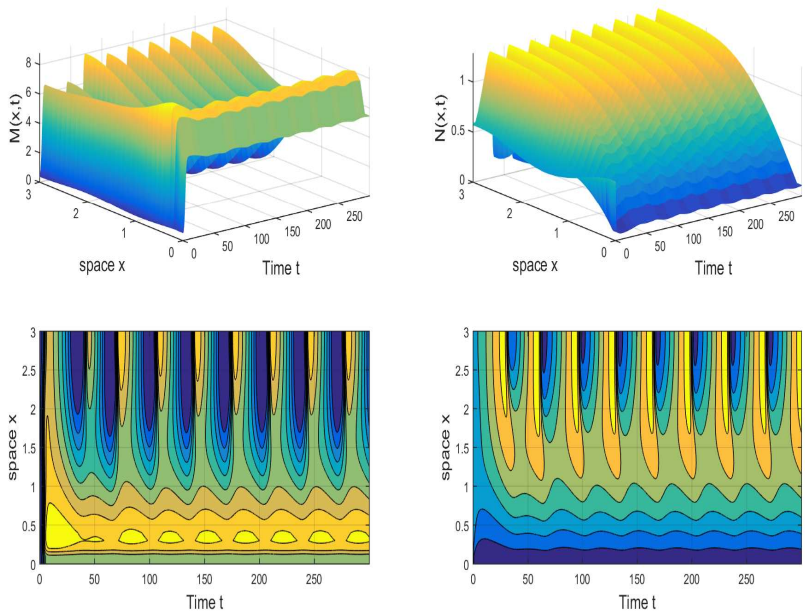

Figure 11. Non homogeneous distribution of the prey and predators with periodic patterns, which is obtained in the can of the existence Hopf bifurcation (see Theorem 3.5).

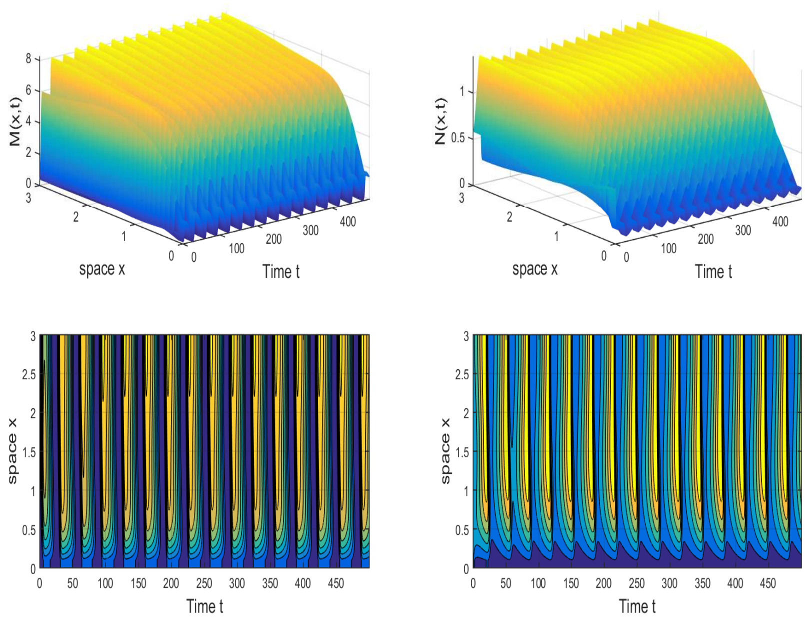

Figure 12. Non homogeneous distribution of the prey and predators with periodic patterns, which is obtained in the can of the existence Hopf bifurcation (see Theorem 3.5).

6. Discussion

In this research, we investigated a spatiotemporal P-P model with Leslie-Gower for modeling the saturation of the predator increasing in terms of the density of the prey. The reason behind considering such approximation is to highlight that the evolution of the consumers is affected directly by the density of the resources. The similarity points between the two models, in the absence of the Leslie-Gower scheme interaction functional (1.1), and the presence of this last (1.2), as the occurrence of H-bifurcation in the absence and the presence of diffusion in two studied models. The disagreement between the two considered models consists of the presence of T-instability for (1.1) and the existence of this last in the diffusive model (1.2). Our study was focused on distinguishing the influence of this case of interaction on the value of T-H bifurcation. As it is been highlighted in Figure 4, σ (increasing rate for predator) generated by considering the Leslie-Gower scheme interaction functional has a big effect on the existence of T-patterns, and hence it influences the existence of T-H bifurcation. No one can neglect the role of herd behavior in modeling much natural behavior, and the considered model can fit many cases in different species as fish population (sardines), gnus and buffalos which intermingle with the lions, hyenas, which highlights the importance of considering such as approximation.

In fact, there are many scenarios that can behold as the persistence of the two categories with nonhomogeneous patches as it is been shown in Figures 7 and 8, or in nonhomogeneous and periodic patters as the Figures 10–12. These scenarios generated by the presence of the Leslie-Gower scheme functional response (more precisely the T-H bifurcation), which shows the huge importance of considering such as approximation.

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Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix

Calculations of $m_1(\mu)$, m_{200}, m_{011}, m_{300}, m_{111}, $n_2(\mu)$, n_{110}, n_{210}, n_{021}. Here, we are in the position to give the expressions of $m_1(\mu)$, m_{200}, m_{011}, m_{300}, m_{111}, $n_2(\mu)$, n_{110}, n_{210}, n_{021}. We will put only the formulas of these parameters. For more details about the method of calculation we refer the authors to [19, 20].

$$m_1(\mu) = \frac{1}{2}\chi_1\left(L_1(\mu)\varphi_1 - \mu_{\hat{\delta}}D_1(\mu)\varphi_1\right),$$

$$m_{200} = m_{011} = m_{110} = 0,$$

$$n_2(\mu) = \frac{1}{2}\chi_2\left(L_1(\mu)\varphi_2 - 0D_1(\mu)\varphi_2\right),$$

$$m_{300} = \frac{1}{4}\chi_1 C_{\varphi_1\varphi_1\varphi_1} + \frac{1}{\omega}\chi_1 Re\left[iQ_{\varphi_1\varphi_2}\chi_2\right]Q_{\varphi_1\varphi_1} + \chi_1 Q_{\varphi_1\left(h^0_{200}+\frac{1}{\sqrt{2}}h^{2\hat{\delta}}_{200}\right)}.$$

$$m_{111} = \chi_1 C_{\varphi_1\varphi_1\overline{\varphi}_1} + \frac{2}{\omega}\chi_1 Re\left[iQ_{\varphi_1\varphi_2}\chi_2\right]Q_{\varphi_1\overline{\varphi}_1} + \chi_1\left(Q_{\varphi_1\left(h^0_{011}+\frac{1}{\sqrt{2}}h^{\hat{\delta}}_{200}\right)} + Q_{\varphi_2 h^{\hat{\delta}}_{101}} + Q_{\overline{\varphi}_2 h^{\hat{\delta}}_{110}}\right),$$

$$n_{210} = \frac{1}{2}\chi_2 C_{\varphi_1\varphi_1\varphi_2} + \frac{1}{2i\omega}\chi_2\left(2Q_{\varphi_1\varphi_1}\chi_1 Q_{\varphi_1\varphi_2} + \left(-Q_{\varphi_2\varphi_2}\chi_2 + Q_{\varphi_2\overline{\varphi}_2}\overline{\chi}_2\right)Q_{\varphi_1\varphi_1}\right) + \chi_2\left(Q_{\varphi_1 h^{\hat{\delta}}_{110}} + Q_{\varphi_2 h^0_{200}}\right),$$

$$n_{021} = \frac{1}{2}\chi_2 C_{\varphi_2\varphi_2\overline{\varphi}_2} + \frac{1}{4i\omega}\chi_2\left(\frac{2}{3}Q_{\overline{\varphi}_2\overline{\varphi}_2}\overline{\chi}_2 Q_{\varphi_2\varphi_2} + \left(-2Q_{\varphi_2\varphi_2}\chi_2 + 4Q_{\varphi_2\overline{\varphi}_2}\overline{\chi}_2\right)Q_{\varphi_2\overline{\varphi}_2}\right) + \chi_2\left(Q_{\varphi_2 h^0_{011}} + Q_{\overline{\varphi}_2 h^0_{020}}\right),$$

where

$$h^0_{200} = -\frac{1}{2}L^{-1}(0)Q_{\varphi_1\varphi_1} + \frac{1}{2i\omega}\left(\varphi_2\chi_2 - \overline{\varphi}_2\overline{\chi}_2\right)Q_{\varphi_1\varphi_1},$$

$$h^{2\hat{\delta}}_{200} = -\frac{1}{2\sqrt{2}}\left[L(0) + diag(-4\mu_{\hat{\delta}} - 4\hat{\delta}\mu_{\hat{\delta}}\right]^{-1} \times Q_{\varphi_1\varphi_1},$$

$$h^0_{011} = -L^{-1}(0)Q_{\varphi_2\overline{\varphi}_2} + \frac{1}{i\omega}\left(\varphi_2\chi_2 - \overline{\varphi}_2\overline{\chi}_2\right)Q_{\varphi_2\overline{\varphi}_2},$$

$$h_{020}^{0} = \frac{1}{2}\left[2i\omega - L(0)\right]^{-1} Q_{\varphi_2\varphi_2} - \frac{1}{2i\omega}\left(\varphi_2\chi_2 - \frac{1}{3}\overline{\varphi}_2\overline{\chi}_2\right) Q_{\varphi_2\varphi_2},$$

$$h_{200}^{\hat{\delta}} = \left[i\omega I - \left(L(0) - diag(-\mu_{\hat{\delta}} - \hat{\delta}\mu_{\hat{\delta}})\right)\right]^{-1} \times Q_{\varphi_1\varphi_2} - \frac{1}{i\omega}\varphi_1\chi_1 Q_{\varphi_1\varphi_2},$$

and

$$h_{002}^{0} = \overline{h_{020}^{0}}, \quad h_{101}^{\hat{\delta}} = \overline{h_{101}^{\hat{\delta}}}, \quad h_{200}^{2\hat{\delta}} = 0.$$

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