Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(7): 15712–15722.
DOI: 10.3934/math.2023802
Received: 17 December 2022
Revised: 11 April 2023
Accepted: 11 April 2023
Published: 28 April 2023

Research article

Frames associated with an operator in spaces with an indefinite metric

Osmin Ferrer Villar[1,*], **Jesús Domínguez Acosta**[1,2] **and Edilberto Arroyo Ortiz**[1]

[1] Department of Mathematics, University of Sucre, Cra. 28 # 5-267, Puerta Roja, Sincelejo, Sucre, Colombia

[2] Corporacin Universitaria del Caribe CECAR, Cra. Troncal de Occidente Km 1-Via Corozal, Sincelejo, Sucre, Colombia

* **Correspondence:** Email: osmin.ferrer@unisucre.edu.co; Tel: +57604795255.

Abstract: In the present paper, we study frames associated with an operator ($\mathcal{W}$-frames) in Krein spaces, and we give the definition of frames associated with an operator depending on the adjoint of the operator in the Krein space (Definition 4.1). We prove that the definition given in [A. Mohammed, K. Samir, N. Bounader, K-frames for Krein spaces, *Ann. Funct. Anal.*, **14** (2023), 10.], which depends on the adjoint of the operator in the associated Hilbert space, is a consequence of our definition. We prove that our definition is independent of the fundamental decomposition (Theorem 4.1) and that having $\mathcal{W}$-frames for the Krein space necessarily gives $\mathcal{W}$-frames for the Hilbert spaces that compose the Krein space (Theorem 4.4). We also prove that orthogonal projectors generate new operators with their respective frames (Theorem 4.2). We prove an equivalence theorem for $\mathcal{W}$-frames (Theorem 4.3), without depending on the fundamental symmetry as usually given in Hilbert spaces.

Keywords: indefinite metric; Krein space; frames; $\mathcal{W}$-frames
Mathematics Subject Classification: 42C15, 46C05, 46C20

1. Introduction

The frame theory for Hilbert spaces has its origin in [7] and was developed by I. Daubechies in [4, 5]. Frames can be considered as “overcomplete bases”, and their overcompleteness makes them more flexible than orthonormal bases. They have proven to be a powerful tool, for example, in signal processing and wavelet analysis [10].

In [8] a definition of frames for Krein spaces was established by replacing the positive definite inner product in the definition of a frame for a Hilbert space by an indefinite inner product, and it is shown that the theory of frames for Krein spaces and the theory of frames for associated Hilbert spaces are analogous. GăvruÅ£a in [9] defined K-frames in Hilbert spaces as a generalization of frames, which allows one to precisely reconstruct the images of a bounded linear operator on a Hilbert space. In [11] Mohammed, Samir and Bounader defined K-frames in Krein spaces using the adjoint of the operator on the Hilbert space associated with the Krein space and presented an equivalence result for K-frames depending on the fundamental symmetry ([11], Proposition 3.14).

In this paper, we give a definition (Definition 4.1) of $\mathcal{W}$-frames in Krein spaces which does not depend directly on the adjoint of the operator in the associated Hilbert space. Instead, it depends on the adjoint of the operator on the Krein space, and we prove that the definition given in [11] is a consequence of ours. Following Wagner, Ferrer and Esmeral in [8], we prove that the definition given in this investigation is independent of the fundamental decomposition and that having $\mathcal{W}$-frames for the Krein space necessarily gives $\mathcal{W}$-frames for the Hilbert spaces that compose this space. We also prove that the orthogonal projectors generate new operators with their respective associated frames.

2. Preliminaries

Theorem 2.1. *[6] Let $(H_1, \langle\cdot,\cdot\rangle_1)$, $(H_2, \langle\cdot,\cdot\rangle_2)$ and $(H, \langle\cdot,\cdot\rangle)$ be Hilbert spaces and $W_1 \in B(H_1, H)$, $W_2 \in B(H_2, H)$ be bounded operators. The following statements are equivalent:*

(i) $R(W_1) \subset R(W_2)$;

(ii) $W_1 W_1^* \leq \lambda^2 W_2 W_2^*$ *for some* $\lambda \geq 0$;

(iii) There exists a bounded operator $X \in B(H_1, H_2)$ *such that* $W_1 = W_2 X$.

Definition 2.1. *[1, 2] A space $\mathcal{K}$ with an indefinite inner product $[\cdot,\cdot]$ that admits a fundamental decomposition of the form*

$$\mathcal{K} = \mathcal{K}^+ [\dot{+}] \mathcal{K}^-,$$

such that $(\mathcal{K}^+, [\cdot,\cdot])$ and $(\mathcal{K}^-, -[\cdot,\cdot])$ are Hilbert spaces, is called a Krein space, which we denote as $(\mathcal{K}, [\cdot,\cdot])$.

Definition 2.2. *[1, 2] Let $(\mathcal{K}, [\cdot,\cdot])$ be a Krein space with a decomposition $\mathcal{K} = \mathcal{K}^- [\dot{+}] \mathcal{K}^+$, and two operators are defined*

$$\mathcal{P}^+ : \mathcal{K} \longrightarrow \mathcal{K}^+, \;\; \mathcal{P}^- : \mathcal{K} \longrightarrow \mathcal{K}^-,$$

naturally, respectively, for $\mathcal{P}^+(x) = x^+$ and $\mathcal{P}^-(x) = x^-$ for all $x \in \mathcal{K}$, where $x^+ \in \mathcal{K}^+$, $x^- \in \mathcal{K}^-$ and $x = x^+ + x^-$. The operators $\mathcal{P}^+$ and $\mathcal{P}^-$ are known as fundamental projectors.

The operator $\mathcal{J} : \mathcal{K} \longrightarrow \mathcal{K}$ defined by $\mathcal{J} = \mathcal{P}^+ - \mathcal{P}^-$, that is,

$$\mathcal{J} x = \mathcal{P}^+ x - \mathcal{P}^- x = x^+ - x^-, \;\; \text{for all } x \in \mathcal{K},$$

is called the fundamental symmetry of Krein space $\mathcal{K}$.

Remark 2.1. *For a Krein space with fundamental decomposition $\mathcal{K} = \mathcal{K}^- [\dot{+}] \mathcal{K}^+$ and a fundamental symmetry $\mathcal{J}$, from now on we will write it $(\mathcal{K} = \mathcal{K}^+ [\dot{+}] \mathcal{K}^-, [\cdot,\cdot], \mathcal{J})$.*

Proposition 2.1. *[1, 2] Let $(\mathcal{K} = \mathcal{K}^+ [\dot{+}] \mathcal{K}^-, [\cdot,\cdot], \mathcal{J})$ be a Krein space, and then $\mathcal{J}$ is invertible, $\mathcal{J}^2 = I$, $\mathcal{J}^{-1} = \mathcal{J}$, and $\mathcal{J}$ is symmetric, isometric and a self-adjoint operator.*

Definition 2.3. *[1, 2] Let $(\mathcal{K} = \mathcal{K}^{+}[\dot{+}]\mathcal{K}^{-}, [\cdot,\cdot], \mathcal{J})$ be a Krein space. We define the function $[\cdot,\cdot]_{\mathcal{J}} : \mathcal{K}\times\mathcal{K} \longrightarrow \mathbb{C}$ for*

$$[x, y]_{\mathcal{J}} = [\mathcal{J}x, y], \quad x, y \in \mathcal{K}.$$

This function is called the $\mathcal{J}$-inner product.

Note that if we have another fundamental decomposition, then we will have another fundamental symmetry and consequently another $\mathcal{J}$-inner product.

Definition 2.4. *[1,2] The fundamental symmetry $\mathcal{J}$ associated with Krein space $(\mathcal{K} = \mathcal{K}^{+}[\dot{+}]\mathcal{K}^{-}, [\cdot,\cdot])$ induces a norm in $\mathcal{K}$ defined by*

$$\|x\|_{\mathcal{J}} := \sqrt{[x, x]_{\mathcal{J}}}, \ \text{ for all } \ x \in \mathcal{K},$$

and this norm is called the $\mathcal{J}$-norm of $\mathcal{K}$. Explicitly,

$$\|x\|_{\mathcal{J}} = ([x^{+}, x^{+}] - [x^{-}, x^{-}])^{1/2}, \ \text{ for all } \ x \in \mathcal{K}.$$

Remark 2.2. *It defines*

$$\|x^{+}\|_{+} = \sqrt{[x^{+}, x^{+}]}, \ x^{+} \in \mathcal{K}^{+} \quad \textit{and} \quad \|x^{-}\|_{-} = \sqrt{-[x^{-}, x^{-}]}, \ x^{-} \in \mathcal{K}^{-}.$$

From now on, the topology studied in Krein spaces will be directly related to the $\mathcal{J}$-norm of $\mathcal{K}$.

Theorem 2.2. *[1] Let $(\mathcal{K}, [\cdot,\cdot])$ be a Krein space and let*

$$\mathcal{K} = \mathcal{K}_1^{+}[\dot{+}]\mathcal{K}_1^{-}, \ \mathcal{K} = \mathcal{K}_2^{+}[\dot{+}]\mathcal{K}_2^{-},$$

be two fundamental decompositions. If $\mathcal{J}_1$ and $\mathcal{J}_2$ are the respective fundamental symmetries, it follows that $\|\cdot\|_{\mathcal{J}_1}$ and $\|\cdot\|_{\mathcal{J}_2}$ are equivalent norms.

Theorem 2.3. *[2] Let $(\mathcal{K} = \mathcal{K}^{+}[\dot{+}]\mathcal{K}^{-}, [\cdot,\cdot], \mathcal{J})$ be a Krein space. Then, $(\mathcal{K}, [\cdot,\cdot]_{\mathcal{J}})$ is a Hilbert space.*

Definition 2.5. *[1] Let $(\mathcal{K}_1 = \mathcal{K}_1^{+}[\dot{+}]\mathcal{K}_1^{-}, [\cdot,\cdot]_1)$ and $(\mathcal{K}_2 = \mathcal{K}_2^{+}[\dot{+}]\mathcal{K}_2^{-}, [\cdot,\cdot]_2)$ be Krein spaces. The adjoint of the linear operator $\mathcal{W} : \mathcal{K}_1 \longrightarrow \mathcal{K}_2$, is the unique linear operator $W^{[*]} : Dom(\mathcal{W}^{[*]}) \subset \mathcal{K}_2 \longrightarrow \mathcal{K}_1$ such that*

$$[Wk_1, k_2]_2 = [k_1, W^{[*]}k_2]_1, \quad \textit{for all } \ k_1 \in \mathcal{K}_1,$$

$k_2 \in Dom(\mathcal{W}^{[*]})$, and $\mathcal{W}^{*\mathcal{J}} : Dom(W^{*\mathcal{J}}) \subset \mathcal{K}_2 \longrightarrow \mathcal{K}_1$ such that

$$[Wk_1, k_2]_{\mathcal{J}_2} = [k_1, W^{*\mathcal{J}}k_2]_{\mathcal{J}_1}, \ \textit{for all } \ k_1 \in \mathcal{K}_1 \textit{ and } k_2 \in Dom(W^{*\mathcal{J}}).$$

Theorem 2.4. *[1] Let $(\mathcal{K} = \mathcal{K}^{+}[\dot{+}]\mathcal{K}^{-}, [\cdot,\cdot], \mathcal{J})$ be a Krein space and $W \in L(\mathcal{K})$ be a bounded linear operator. If $W^{[*]}$ and $W^{*\mathcal{J}}$ are the adjoints in the Krein and Hilbert spaces, respectively, then $W^{[*]} = \mathcal{J}W^{*\mathcal{J}}\mathcal{J}$.*

From the above result we get $W^{*\mathcal{J}} = IW^{*\mathcal{J}}I = \mathcal{J}\mathcal{J}W^{*\mathcal{J}}\mathcal{J}\mathcal{J} = \mathcal{J}(\mathcal{J}W^{*\mathcal{J}}\mathcal{J})\mathcal{J} = \mathcal{J}W^{[*]}\mathcal{J}$.

Lemma 2.1. *[8] Let $(\mathcal{K} = \mathcal{K}^{+}[\dot{+}]\mathcal{K}^{-}, [\cdot,\cdot], \mathcal{J})$ be a Krein space and $\mathcal{P}$ be an orthogonal projector that commutes with $\mathcal{J}$. Then, the spaces $\mathcal{P}\mathcal{K}$ and $(I - \mathcal{P})\mathcal{K}$ are Krein spaces with fundamental symmetries $\mathcal{P}\mathcal{J}$ and $(I - \mathcal{P})\mathcal{J}$, respectively.*

Example 2.1. *[8] Now, $\ell_2(\mathbb{N})$ can also be seen as a Krein space with an inner product whose inner $\mathcal{J}$-product coincides with the usual one. In this sense we define the following mapping:*

$$[\cdot,\cdot]_{\ell_2} : \ell_2(\mathbb{N}) \times \ell_2(\mathbb{N}) \to \mathbb{C},\ [\{\alpha_n\}_{n\in\mathbb{N}}, \{\beta_n\}_{n\in\mathbb{N}}]_{\ell_2} := \sum_{n\in\mathbb{N}} (-1)^n \alpha_n \overline{\beta_n},$$

for all $\{\alpha_n\}_{n\in\mathbb{N}}, \{\beta_n\}_{n\in\mathbb{N}} \in \ell_2(\mathbb{N})$. Thus, if $\{e_n\}_{n\in\mathbb{N}}$ is the canonical orthonormal basis of $\ell_2(\mathbb{N})$, then $\ell_2(\mathbb{N})$ accepts the following fundamental decomposition:

$$\ell_2(\mathbb{N}) = \ell_2^+(\mathbb{N})[\dot{+}]\ell_2^-(\mathbb{N}),$$

where $\ell_2^+(\mathbb{N}) = \overline{span}\{e_{2n} : n \in \mathbb{N}\}$ and $\ell_2^-(\mathbb{N}) = \overline{span}\{e_{2n+1} : n \in \mathbb{N}\}$ with associated fundamental symmetry

$$\mathcal{J}_{\ell_2} : (\ell_2(\mathbb{N}), [\cdot,\cdot]_{\ell_2}) \to (\ell_2(\mathbb{N}), [\cdot,\cdot]_{\ell_2}),$$

given by $\mathcal{J}_{\ell_2}(\{\alpha_n\}_{n\in\mathbb{N}}) = \{(-1)^n \alpha_n\}_{n\in\mathbb{N}}$ for all $\{\alpha_n\}_{n\in\mathbb{N}} \in \ell_2(\mathbb{N})$. Therefore, $[\cdot,\cdot]_{\mathcal{J}_{\ell_2}} = \langle\cdot,\cdot\rangle_{\ell_2}$.

From now on whenever we see $\ell_2(\mathbb{N})$ as Krein space we will understand that it is endowed with a fundamental symmetry $\mathcal{J}_{\ell_2}$ such that $[\cdot,\cdot]_{\mathcal{J}_{\ell_2}} = \langle\cdot,\cdot\rangle_{\ell_2}$. An example of such is the one developed above, and more trivial is the symmetry given by the identity operator on $\ell_2(\mathbb{N})$. Thus we will write $\pounds_2(\mathbb{N})$ instead of $\ell_2(\mathbb{N})$ when viewed as Krein space with such properties and the fundamental symmetry by $\mathcal{J}_{\pounds_2}$, to avoid confusion.

3. Frames in indefinite metric spaces

The following results were established in [8] for Wagner, Ferrer and Esmeral.

Definition 3.1. *Let $(\mathcal{K} = \mathcal{K}^+[\dot{+}]\mathcal{K}^-, [\cdot,\cdot], \mathcal{J})$ be a Krein space and $\mathcal{N} \subseteq \mathbb{N}$. A sequence $\{x_n\}_{n\in\mathcal{N}} \subset \mathcal{K}$ is called a frame for $\mathcal{K}$ if there exist constants $0 < A \leq B < \infty$ such that*

$$A\,\|x\|_{\mathcal{J}}^2 \leq \sum_{n\in\mathcal{N}} |[x, x_n]|^2 \leq B\,\|x\|_{\mathcal{J}}^2 \quad \text{for } x \in \mathcal{K}.$$

Definition 3.2. *Let $(\mathcal{K} = \mathcal{K}^+[\dot{+}]\mathcal{K}^-, [\cdot,\cdot], \mathcal{J})$ and $(\pounds_2(\mathbb{N}), [\cdot,\cdot]_{\mathcal{J}_{\pounds_2}}, \mathcal{J}_{\pounds_2})$ be Krein spaces, such that $[\cdot,\cdot]_{\mathcal{J}_{\pounds_2}}$ coincides with the standard inner product $\langle\cdot,\cdot\rangle$ defined in $\ell_2(\mathbb{N})$. Given a frame $\{x_n\}_{n\in\mathbb{N}}$ for $\mathcal{K}$, the linear mapping*

$$T : \pounds_2(\mathbb{N}) \longrightarrow \mathcal{K}, \quad T(\{\alpha_n\}_{n\in\mathbb{N}}) = \sum_{n\in\mathbb{N}} \alpha_n x_n$$

is called a pre-frame operator.

Remark 3.1. *The adjoint of T is given by*

$$T^{[*]}k = \mathcal{J}_{\pounds_2}(\{[k, x_n]\}_{n\in\mathbb{N}}), \quad \text{for } k \in \mathcal{K}.$$

In fact, for all $\{\alpha_n\}_{n\in\mathbb{N}} \in \pounds_2(\mathbb{N})$ and $k \in \mathcal{K}$, we have

$$\begin{aligned}[T(\{\alpha_n\}_{n\in\mathbb{N}}), k] &= \left[\sum_{n\in\mathbb{N}} \alpha_n x_n, k\right] = \sum_{n\in\mathbb{N}} [\alpha_n x_n, k] = \sum_{n\in\mathbb{N}} \alpha_n [x_n, k] = \sum_{n\in\mathbb{N}} \alpha_n \overline{[k, x_n]} \\ &= \langle \{\alpha_n\}_{n\in\mathbb{N}}, \{[k, x_n]\}_{n\in\mathbb{N}} \rangle_{\ell_2} = [\{\alpha_n\}_{n\in\mathbb{N}}, \mathcal{J}_{\pounds_2}(\{[k, x_n]\}_{n\in\mathbb{N}})].\end{aligned}$$

Definition 3.3. *Let* $(\mathcal{K} = \mathcal{K}^+[\dot{+}]\mathcal{K}^-, [\cdot,\cdot], \mathcal{J})$ *and* $(\pounds_2(\mathbb{N}), [\cdot,\cdot]_{\mathcal{J}_{\pounds_2}}, \mathcal{J}_{\pounds_2})$ *be Krein spaces, so that* $[\cdot,\cdot]_{\mathcal{J}_{\pounds_2}}$ *coincides with the standard inner product* $\langle\cdot,\cdot\rangle$ *defined in* $\ell_2(\mathbb{N})$, *and* $\{x_n\}_{n\in\mathbb{N}} \subset \mathcal{K}$ *is a frame for* $\mathcal{K}$. *The operator*

$$S := T\,\mathcal{J}_{\pounds_2}T^{[*]}$$

is called the frame operator.

Following the definition of frames in spaces with an indefinite metric introduced in [8] by Wagner, Ferrer and Esmeral, in [11] the K-frames in Krein spaces are defined as follows.

Definition 3.4. *Let* $(\mathcal{K} = \mathcal{K}^+[\dot{+}]\mathcal{K}^-, [\cdot,\cdot], \mathcal{J})$ *be a Krein space and* $\mathcal{W} : \mathcal{K} \to \mathcal{K}$ *be a bounded operator. It is said that* $\{x_n\}_{n\in\mathbb{N}}$ *is a* $\mathcal{W}$*-frame for* $\mathcal{K}$ *if there exist constants* $A, B > 0$ *such that*

$$A\|\mathcal{W}^{*\mathcal{J}}x\|_{\mathcal{J}}^2 \leq \sum_{n\in\mathbb{N}} |\,[x, x_n]|^2 \leq B\|x\|_{\mathcal{J}}^2, \quad \textit{for all } \ x \in \mathcal{K}.$$

$\mathcal{W}^{*\mathcal{J}}$ is the adjoint in the Hilbert space associated with the Krein space $(\mathcal{K} = \mathcal{K}^+[\dot{+}]\mathcal{K}^-, [\cdot,\cdot], \mathcal{J})$.

4. Frames associated with an operator in spaces of indefinite metrics

In this section we give a definition similar to the previous one, using the adjoint of Krein space and showing that the one given in [11] is a consequence of our own.

Definition 4.1. *Let* $(\mathcal{K} = \mathcal{K}^+[\dot{+}]\mathcal{K}^-, [\cdot,\cdot], \mathcal{J})$ *be a Krein space and* $\mathcal{W} : \mathcal{K} \to \mathcal{K}$ *be a bounded operator. It is said that* $\{x_n\}_{n\in\mathbb{N}}$ *is a* $\mathcal{W}$*-frame for* $\mathcal{K}$ *if there exist constants* $A, B > 0$ *such that*

$$A\|\mathcal{W}^{[*]}x\|_{\mathcal{J}}^2 \leq \sum_{n\in\mathbb{N}} |\,[x, x_n]|^2 \leq B\|x\|_{\mathcal{J}}^2, \quad \textit{for all } \ x \in \mathcal{K}.$$

Remark 4.1.

$$A\|\mathcal{W}^{*\mathcal{J}}x\|_{\mathcal{J}}^2 = A\|\mathcal{J}\mathcal{W}^{[*]}\mathcal{J}x\|_{\mathcal{J}}^2 = A\|\mathcal{W}^{[*]}\mathcal{J}x\|_{\mathcal{J}}^2 \leq \sum_{n\in\mathbb{N}} |\,[\mathcal{J}x, x_n]|^2 = \sum_{n\in\mathbb{N}} |\,[x, x_n]_{\mathcal{J}}|^2 \leq B\|\mathcal{J}x\|_{\mathcal{J}}^2 = B\|x\|_{\mathcal{J}}^2.$$

Therefore,

$$A\|\mathcal{W}^{*\mathcal{J}}x\|_{\mathcal{J}}^2 \leq \sum_{n\in\mathbb{N}} |\,[x, x_n]|^2 \leq B\|x\|_{\mathcal{J}}^2, \quad \textit{for all } \ x \in \mathcal{K}.$$

Example 4.1. *We consider the vector space* $\mathbb{C}^2$ *over* $\mathbb{C}$, *with the usual sum and product and the function* $[\cdot,\cdot] : \mathbb{C}^2 \times \mathbb{C}^2 \longrightarrow \mathbb{C}$ *given by*

$$[(x_1, y_1), (x_2, y_2)] = x_1\overline{x_2} - y_1\overline{y_2}. \tag{4.1}$$

Well, it turns out that the space with inner product $(\mathbb{C}^2, [\cdot,\cdot])$ *is a Krein space with fundamental decomposition* $\mathbb{C}^2 = \mathcal{K}^+[\dot{+}]\mathcal{K}^-$, *where* $\mathcal{K}^+ = \{(x, 0) : \ x \in \mathbb{C}\}$ *and* $\mathcal{K}^- = \{(0, y) : \ y \in \mathbb{C}\}$. *Then, the fundamental symmetry is given by*

$$\mathcal{J}((x, y)) = \mathcal{P}^+(x, y) - \mathcal{P}^-(x, y) = (x, -y).$$

Let us consider the operator $\mathcal{W} : \mathbb{C}^2 \to \mathbb{C}^2$ *defined by* $\mathcal{W}((x, y)) = (y, -x)$, *which is self-adjoint, and* $\{x_n\}_{n=1}^5 = \{(i, 0), (i, 0), (0, -i), (0, -i), (0, -i)\}$.

Let $x = (m, r) \in \mathbb{C}^2$*, and then,* $\|(m, r)\|^2_{\mathcal{J}} = [(m, r), (m, r)]_{\mathcal{J}} = [\mathcal{J}(m, r), (m, r)] = [(m, -r), (m, r) = m\overline{m} - (-r)\overline{r} = |m|^2 + |r|^2$.

Then,

$$\sum_{n=1}^{5} |[x, x_n]|^2 = 2|[(m, r), (i, 0)]|^2 + 3|[(m, r), (0, -i)]|^2 = 2|-mi|^2 + 3|-ri|^2$$
$$= 2|m|^2 + 3|r|^2 \leq 3(|m|^2 + |r|^2) = 3\|(m, r)\|^2_J = 3\|x\|^2_{\mathcal{J}}.$$

Also,

$$\|\mathcal{W}^{[*]}x\|^2_{\mathcal{J}} = [\mathcal{W}^{[*]}x, \mathcal{W}^{[*]}x]_{\mathcal{J}} = [\mathcal{W}^{[*]}(m, r), \mathcal{W}^{[*]}(m, r)]_{\mathcal{J}} = [(r, -m), (r, -m)]_{\mathcal{J}} = [\mathcal{J}(r, -m), (r, -m)]$$
$$= [(r, m), (r, -m)] = r\overline{r} - m(\overline{-m}) = |r|^2 + |m|^2 \leq 2|m|^2 + 3|r|^2.$$

Thus, $\|\mathcal{W}^{[*]}x\|^2_{\mathcal{J}} = |m|^2 + |r|^2 \leq 2|m|^2 + 3|r|^2 = \sum_{n=1}^{4} |[x, x_n]|^2 \leq 3(|m|^2 + |r|^2) = 3\|x\|^2_{\mathcal{J}}$.
Consequently, $\{x_n\}_{n=1}^{5} = \{(i, 0), (i, 0), (0, -i), (0, -i), (0, -i)\}$ *is a* $\mathcal{W}$*-frame for* $\mathbb{C}^2$.

The definition of K-frames given in [11], which is an adaptation of the definition of frames given in [8], was presented apparently depending on the fundamental symmetry. We will show below that the $\mathcal{W}$-frames according to the definition given in this paper are independent of the fundamental decomposition of the Krein space in question.

Theorem 4.1. *Let* $(\mathcal{K}, [\cdot, \cdot])$ *be a Krein space with fundamental decompositions* $\mathcal{K} = \mathcal{K}_1^+[\dot{+}]\mathcal{K}_1^-$, $\mathcal{K} = \mathcal{K}_2^+[\dot{+}]\mathcal{K}_2^-$ *and fundamental symmetries* $\mathcal{J}_1$, $\mathcal{J}_2$*, respectively, and* $\mathcal{W} : \mathcal{K} \to \mathcal{K}$ *is a bounded operator. If* $\{x_n\}_{n\in\mathbb{N}}$ *is a frame for* $\mathcal{W}$ *with respect to* $\mathcal{J}_1$*, then* $\{x_n\}_{n\in\mathbb{N}}$ *is a frame for* $\mathcal{W}$ *with respect to* $\mathcal{J}_2$.

Proof. Let $\{x_n\}_{n\in\mathbb{N}} \subset \mathcal{K}$ be a frame for $\mathcal{W}$, in $(\mathcal{K} = \mathcal{K}_1^+[\dot{+}]\mathcal{K}_1^-, [\cdot, \cdot], \mathcal{J}_1)$, and then there exist constants $A, B > 0$ such that $A\|\mathcal{W}^{[*]}x\|^2_{\mathcal{J}_1} \leq \sum_{n\in\mathbb{N}} |[x, x_n]|^2 \leq B\|x\|^2_{\mathcal{J}_1}, \forall x \in \mathcal{K}$.

Since the norms $\|\cdot\|_{\mathcal{J}_1}$ and $\|\cdot\|_{\mathcal{J}_2}$ are equivalents, there exist constants $C, D > 0$ such that

$$C\|x\|_{\mathcal{J}_1} \leq \|x\|_{\mathcal{J}_2} \leq D\|x\|_{\mathcal{J}_1} \text{ for all } x \in \mathcal{K}. \tag{4.2}$$

Since $\mathcal{W}^{[*]}x \in \mathcal{K}$ for all $x \in \mathcal{K}$,

$$C\|\mathcal{W}^{[*]}x\|_{\mathcal{J}_1} \leq \|\mathcal{W}^{[*]}x\|_{\mathcal{J}_2} \leq D\|\mathcal{W}^{[*]}x\|_{\mathcal{J}_1} \text{ for all } x \in \mathcal{K}. \tag{4.3}$$

Thus,

$$\frac{A}{D}\|\mathcal{W}^{[*]}x\|_{\mathcal{J}_2} \leq A\|\mathcal{W}^{[*]}x\|^2_{\mathcal{J}_1} \leq \sum_{n\in\mathbb{N}} |[x, x_n]|^2 \leq B\|x\|^2_{\mathcal{J}_1} \leq \frac{B}{C}\|x\|^2_{\mathcal{J}_2}, \forall x \in \mathcal{K}.$$

Consequently $\{x_n\}_{n\in\mathbb{N}}$ is a *frame* for $\mathcal{W}$ in $(\mathcal{K} = \mathcal{K}_2^+[\dot{+}]\mathcal{K}_2^-, [\cdot, \cdot], \mathcal{J}_2)$.

Proposition 4.1. *Let* $(\mathcal{K} = \mathcal{K}^+[\dot{+}]\mathcal{K}^-, [\cdot, \cdot], \mathcal{J})$ *be a Krein space and* $\mathcal{P}$ *be an orthogonal projection that commutes with* $\mathcal{J}$*. Then,* $\|\mathcal{P}x\|_{\mathcal{P}\mathcal{J}} = \|x\|_{\mathcal{J}}$ *for all* $x \in \mathcal{K}$.

Proof. Let $x \in \mathcal{K}$, and then $\|\mathcal{P}x\|^2_{\mathcal{P}\mathcal{J}} = [\mathcal{P}x, \mathcal{P}x]_{\mathcal{P}\mathcal{J}} = [\mathcal{P}\mathcal{J}\mathcal{P}x, \mathcal{P}x] = [\mathcal{J}\mathcal{P}\mathcal{P}x, \mathcal{P}x] = [\mathcal{J}\mathcal{P}^2x, \mathcal{P}x] = [\mathcal{J}\mathcal{P}x, \mathcal{P}x] = [\mathcal{P}x, \mathcal{P}x]_{\mathcal{J}} = [x, x]_{\mathcal{J}} = \|x\|^2_{\mathcal{J}}$. Consequently, $\|\mathcal{P}x\|_{\mathcal{P}\mathcal{J}} = \|x\|_{\mathcal{J}}$ for all $x \in \mathcal{K}$.

The following result shows that orthogonal projectors in spaces of indefinite metric preserve $\mathcal{W}$-frames.

Theorem 4.2. *Let* $(\mathcal{K} = \mathcal{K}^+[\dot{+}]\mathcal{K}^-, [\cdot,\cdot], \mathcal{J})$ *be a Krein space,* $\mathcal{W} : \mathcal{K} \to \mathcal{K}$ *is a bounded operator in* $\mathcal{K}$*, and* $\mathcal{P}$ *is an orthogonal projection that commutes with* $\mathcal{J}$*. If* $\{x_n\}_{n\in\mathbb{N}}$ *is a* $\mathcal{W}$ *- frame for* $\mathcal{K}$*, then* $\{\mathcal{P}x_n\}_{n\in\mathbb{N}}$ *is a* $\mathcal{WP}$*-frame for* $\mathcal{PK}$*.*

Proof. The subspace $\mathcal{PK}$ of $\mathcal{K}$ is a Krein space with fundamental symmetry $\mathcal{PJ}$ (see [8]). Since $\{x_n\}_{n\in\mathbb{N}}$ is a $\mathcal{W}$-frame for $\mathcal{K}$, there exist constants $A, B > 0$ such that

$$A\|\mathcal{W}^{[*]}x\|^2_{\mathcal{J}} \le \sum_{n\in\mathbb{N}} |[x, x_n]|^2 \le B\|x\|^2_{\mathcal{J}}, \text{ for all } x \in \mathcal{K}. \tag{4.4}$$

Also, if t belongs to $\mathcal{PK}$, then there exists $k \in \mathcal{K}$ such that $t = \mathcal{P}k$.

Since $\mathcal{P}k \in \mathcal{K}$ for (4.4), we have $A\|\mathcal{W}^{[*]}\mathcal{P}k\|^2_{\mathcal{J}} \le \sum_{n\in\mathbb{N}} |[\mathcal{P}k, x_n]|^2 \le B\|\mathcal{P}k\|^2_{\mathcal{J}}$.

So,

$$\begin{aligned} A\|(\mathcal{WP})^{[*]}t\|^2_{\mathcal{PJ}} &= A\|\mathcal{P}^{[*]}\mathcal{W}^{[*]}t\|^2_{\mathcal{PJ}} = A\|\mathcal{P}\mathcal{W}^{[*]}t\|^2_{\mathcal{PJ}} = A\|\mathcal{W}^{[*]}t\|^2_{\mathcal{J}} = A\|\mathcal{W}^{[*]}\mathcal{P}k\|^2_{\mathcal{J}} \\ &\le \sum_{n\in\mathbb{N}} |[\mathcal{P}k, x_n]|^2 = \sum_{n\in\mathbb{N}} |[\mathcal{P}^2k, x_n]|^2 = \sum_{n\in\mathbb{N}} |[\mathcal{P}k, \mathcal{P}x_n]|^2 = \sum_{n\in\mathbb{N}} |[t, \mathcal{P}x_n]|^2 \le B\|\mathcal{P}k\|^2_{\mathcal{J}} \\ &= B\|\mathcal{PP}k\|^2_{\mathcal{PJ}} = B\|\mathcal{P}^2k\|^2_{\mathcal{PJ}} = B\|\mathcal{P}k\|^2_{\mathcal{PJ}} = B\|t\|^2_{\mathcal{PJ}}, \text{ for all } t \in \mathcal{PK}. \end{aligned}$$

Thus, $A\|(\mathcal{WP})^{[*]}t\|^2_{\mathcal{PJ}} \le \sum_{n\in\mathbb{N}} |[t, \mathcal{P}x_n]|^2 \le B\|t\|^2_{\mathcal{PJ}}$, *for all* $t \in \mathcal{PK}$.

Proposition 4.2. *Let* $(\mathcal{K} = \mathcal{K}^+[\dot{+}]\mathcal{K}^-, [\cdot,\cdot], \mathcal{J})$ *be a Krein space. If* $\{x_n = x_n^+ + x_n^-\}_{n\in\mathbb{N}} \subset \mathcal{K}$ *is a Bessel sequence for* $(\mathcal{K} = \mathcal{K}^+[\dot{+}]\mathcal{K}^-, [\cdot,\cdot])$*, then* $\{x_n^+\}_{n\in\mathbb{N}}$ *and* $\{x_n^-\}_{n\in\mathbb{N}}$ *are Bessel sequences for* $(\mathcal{K}^+, [\cdot,\cdot])$ *and* $(\mathcal{K}^-, -[\cdot,\cdot])$*, respectively.*

Proof. Since $\{x_n = x_n^+ + x_n^-\}_{n\in\mathbb{N}}$ is a Bessel sequence for $(\mathcal{K} = \mathcal{K}^+[\dot{+}]\mathcal{K}^-, [\cdot,\cdot])$, there exists a constant $B > 0$ such that

$$\sum_{n\in\mathbb{N}} |[x, x_n]|^2 \le B\,\|x\|^2_{\mathcal{J}}, \text{ for all } x \in \mathcal{K}. \tag{4.5}$$

Let $x^+ \in \mathcal{K}^+ \subset \mathcal{K}$ and $x^- \in \mathcal{K}^- \subset \mathcal{K}$, and then for (4.5) we have that

$$\sum_{n\in\mathbb{N}} \left|[x^+, x_n]\right|^2 \le B\left\|x^+\right\|^2_{\mathcal{J}} \;\; \textit{and} \;\; \sum_{n\in\mathbb{N}} \left|[x^-, x_n]\right|^2 \le B\left\|x^-\right\|^2_{\mathcal{J}}.$$

As $[x^+, x_n] = [x^+, x_n^+ + x_n^-] = [x^+, x_n^+] + [x^+, x_n^-] = [x^+, x_n^+] + 0 = [x^+, x_n^+]$ and $[x^-, x_n] = [x^-, x_n^+ + x_n^-] = [x^-, x_n^+] + [x^-, x_n^-] = 0 + [x^-, x_n^-] = [x^-, x_n^-]$, then,

$$\sum_{n\in\mathbb{N}} \left|[x^+, x_n^+]\right|^2 = \sum_{n\in\mathbb{N}} \left|[x^+, x_n]\right|^2 \le B\left\|x^+\right\|^2_{\mathcal{J}}, \;\; \textit{for all} \;\; x^+ \in \mathcal{K}^+,$$

and

$$\sum_{n\in\mathbb{N}} \left|-[x^-, x_n^-]\right|^2 = \sum_{n\in\mathbb{N}} \left|[x^-, x_n^-]\right|^2 = \sum_{n\in\mathbb{N}} \left|[x^-, x_n]\right|^2 \le B\left\|x^-\right\|^2_{\mathcal{J}}, \;\; \textit{for all} \;\; x^- \in \mathcal{K}^-.$$

Thus, $\{x_n^+\}_{n\in\mathbb{N}}$ and $\{x_n^-\}_{n\in\mathbb{N}}$ are Bessel sequences for $(\mathcal{K}^+, [\cdot,\cdot])$ and $(\mathcal{K}^-, -[\cdot,\cdot])$, respectively.

The following result was presented in [11] with the restriction on the images of a sequence, under the fundamental symmetry.

In this paper we present and show a result where it is observed that such a restriction is not necessary. The result holds as usual in the Hilbert spaces for any sequence of Krein space.

Theorem 4.3. *Let $(\mathcal{K} = \mathcal{K}^+[\dot{+}]\mathcal{K}^-, [\cdot,\cdot], \mathcal{J})$ be a Krein space, $\{x_n\}_{n\in\mathbb{N}} \subset \mathcal{K}$, and $\mathcal{W} : \mathcal{K} \to \mathcal{K}$ is a bounded operator. Then, the following statements are equivalent.*

(i) $\{x_n\}_{n\in\mathbb{N}}$ is a Bessel sequence for $(\mathcal{K} = \mathcal{K}^+[\dot{+}]\mathcal{K}^-, [\cdot,\cdot], \mathcal{J})$, and there exists a sequence of Bessel $\{y_n\}_{n\in\mathbb{N}}$ for $(\mathcal{K} = \mathcal{K}^+[\dot{+}]\mathcal{K}^-, [\cdot,\cdot], \mathcal{J})$, such that $\mathcal{W}x = \sum_{n\in\mathbb{N}}[x, y_n]x_n$ for all $x \in \mathcal{K}$.

(ii) $\{x_n\}_{n\in\mathbb{N}}$ is a $\mathcal{W}$−frame for $(\mathcal{K} = \mathcal{K}^+[\dot{+}]\mathcal{K}^-, [\cdot,\cdot], \mathcal{J})$.

Proof. $(i) \to (ii)$

Suppose that $\{x_n\}_{n\in\mathbb{N}}$ is a Bessel sequence for $(\mathcal{K} = \mathcal{K}^+[\dot{+}]\mathcal{K}^-, [\cdot,\cdot], \mathcal{J})$ and that there exists a Bessel sequence $\{y_n\}_{n\in\mathbb{N}}$ for $(\mathcal{K} = \mathcal{K}^+[\dot{+}]\mathcal{K}^-, [\cdot,\cdot], \mathcal{J})$, such that $\mathcal{W}x = \sum_{n\in\mathbb{N}}[x, y_n]x_n$ for all $x \in \mathcal{K}$.

Since $\{x_n\}_{n\in\mathbb{N}}$, $\{y_n\}_{n\in\mathbb{N}}$ are Bessel sequences for $(\mathcal{K} = \mathcal{K}^+[\dot{+}]\mathcal{K}^-, [\cdot,\cdot], \mathcal{J})$, there exist $M, B > 0$ such that

$$\sum_{n\in\mathbb{N}} |[x, x_n]|^2 \leq B\,\|x\|_{\mathcal{J}}^2 \quad \textit{and} \quad \sum_{n\in\mathbb{N}} \left|[x, y_n]\right|^2 \leq M\,\|x\|_{\mathcal{J}}^2 \quad \text{for all} \quad x \in \mathcal{K}. \tag{4.6}$$

It remains to prove that there exists $A > 0$ such that $A\|\mathcal{W}^{[*]}x\|_{\mathcal{J}}^2 \leq \sum_{n\in\mathbb{N}} |[x, x_n]|^2$ for all $x \in \mathcal{K}$.

Since $\mathcal{J}$ is an isometry in the Hilbert space $(\mathcal{K}, [\cdot,\cdot]_{\mathcal{J}})$, we have

$$\begin{aligned}
\|\mathcal{W}^{[*]}x\|_{\mathcal{J}} =& \|\mathcal{J}\mathcal{W}^{[*]}x\|_{\mathcal{J}} = \sup_{\|y\|_{\mathcal{J}}=1}\left\{\left|\left[\mathcal{J}\mathcal{W}^{[*]}x, y\right]_{\mathcal{J}}\right|\right\} = \sup_{\|y\|_{\mathcal{J}}=1}\left\{\left|\left[\mathcal{J}^2\mathcal{W}^{[*]}x, y\right]\right|\right\} = \sup_{\|y\|_{\mathcal{J}}=1}\left\{\left|\left[I\mathcal{W}^{[*]}x, y\right]\right|\right\} \\
=& \sup_{\|y\|_{\mathcal{J}}=1}\left\{\left|\left[\mathcal{W}^{[*]}x, y\right]\right|\right\} = \sup_{\|y\|_{\mathcal{J}}=1}\left\{\left|[x, \mathcal{W}y]\right|\right\} = \sup_{\|y\|_{\mathcal{J}}=1}\left\{\left|\left[x, \sum_{n\in\mathbb{N}}[y, y_n]\,x_n\right]\right|\right\} \\
=& \sup_{\|y\|_{\mathcal{J}}=1}\left\{\left|\sum_{n\in\mathbb{N}}\overline{[y, y_n]}\,[x, x_n]\right|\right\} \leq \sup_{\|y\|_{\mathcal{J}}=1}\left\{\sum_{n\in\mathbb{N}}\left|\overline{[y, y_n]}\,[x, x_n]\right|\right\} \\
\leq& \sup_{\|y\|_{\mathcal{J}}=1}\left\{\left[\sum\left|[y_n, y]\right|^2\right]^{1/2}\left[\sum|[x, x_n]|^2\right]^{1/2}\right\} \leq \sup_{\|y\|_{\mathcal{J}}=1}\left\{\left[M\|y\|_{\mathcal{J}}^2\right]^{1/2}\left[\sum|[x, x_n]|^2\right]^{1/2}\right\} \\
\leq& \sup_{\|y\|_{\mathcal{J}}=1}\left\{M^{1/2}\|y\|_{\mathcal{J}}\left[\sum|[x, x_n]|^2\right]^{1/2}\right\} = M^{1/2}\left[\sum|[x, x_n]|^2\right]^{1/2}\sup_{\|y\|_{\mathcal{J}}=1}\{\|y\|_{\mathcal{J}}\} \\
=& M^{1/2}\left[\sum|[x, x_n]|^2\right]^{1/2}.
\end{aligned}$$

So, $\|\mathcal{W}^{[*]}x\|_{\mathcal{J}}^2 \leq M\sum_{n\in\mathbb{N}}|[x, x_n]|^2$ for all $x \in \mathcal{K}$. This implies

$$\frac{1}{M}\|\mathcal{W}^{[*]}x\|_{\mathcal{J}}^2 \leq \sum|[x, x_n]|^2 \quad \text{for all} \quad x \in \mathcal{K}. \tag{4.7}$$

We consider $0 < \frac{1}{M} = A$, and using (4.6 and 4.7) we have

$$A\|\mathcal{W}^{[*]}x\|_{\mathcal{J}}^2 \leq \sum_{n\in\mathbb{N}}|\,[x, x_n]\,|^2 \leq B\|x\|_{\mathcal{J}}^2, \quad \text{for all} \quad x \in \mathcal{K}.$$

ii) $\rightarrow$ *i*) Suppose that $\{x_n\}_{n\in\mathbb{N}}$ is a $\mathcal{W}$−frame for $(\mathcal{K} = \mathcal{K}^+[\dot{+}]\mathcal{K}^-, [\cdot,\cdot], \mathcal{J})$, and then there exist constants $A, B > 0$ such that

$$A\|\mathcal{W}^{[*]}x\|_{\mathcal{J}}^2 \leq \sum_{n\in\mathbb{N}} |[x, x_n]|^2 \leq B\|x\|_{\mathcal{J}}^2, \quad \text{for all } \ x \in \mathcal{K}.$$

From the above inequality we have that $\sum_{n\in\mathbb{N}} |[x, x_n]|^2 \leq B\|x\|_{\mathcal{J}}^2$ for all $x \in \mathcal{K}$, i.e., $\{x_n\}_{n\in\mathbb{N}}$ is a Bessel sequence for $(\mathcal{K} = \mathcal{K}^+[\dot{+}]\mathcal{K}^-, [\cdot,\cdot], \mathcal{J})$.

In [8] the authors showed that the operator $T : \pounds_2(\mathbb{N}) \rightarrow \mathcal{K}$ given by $T(\{a_n\}_{n\in\mathbb{N}}) = \sum_{n\in\mathbb{N}} a_n x_n$, is well defined and bounded, and also $\|T\|_{\mathcal{J}} \leq \sqrt{B}$. Since $\mathcal{W}$ and T are bounded operators, and T is an epimorphism (see [3]), $R(\mathcal{W}) \subset R(T) = \mathcal{K}$. By Theorem 2.1 there exists the bounded linear operator $M : (\mathcal{K}, [\cdot,\cdot]_J) \rightarrow \ell_2(\mathbb{N})$ such that $\mathcal{W} = TM$.

We consider

$$F_n : (\mathcal{K}, [\cdot,\cdot]_{\mathcal{J}}) \rightarrow \mathbb{C}, \qquad F_n(x) = (Mx)_n = a_n^x.$$

Since $Mx \in \ell_2(\mathbb{N})$, and we write $(Mx)_n$ to indicate the terms of the sequence Mx.

We define $a^x = Mx$. We have

$$|F_n(x)| = |a_n^x| \leq \left(\sum_{n\in\mathbb{N}} |a_n^x|^2\right)^{1/2} = \|a^x\|_{\ell^2} = \|Mx\|_{\ell_2} \leq \|M\|\,\|x\|_{\mathcal{J}}.$$

Therefore, for each $n \in \mathbb{N}$, $F_n : \mathcal{K} \rightarrow \mathbb{C}$ are continuous linear functionals. From the Riesz representation theorem for Krein spaces (see [1]), it follows that there exists $\{y_n\}_{n\in\mathbb{N}} \subset \mathcal{K}$ such that $a_n^x = F_n(x) = [x, y_n]$ for all $x \in \mathcal{K}$.

Then, for $x \in \mathcal{K}$, $\mathcal{W}x = TMx = T(Mx) = T(\{a_n^x\}_{n\in\mathbb{N}}) = \sum_{n\in\mathbb{N}} a_n^x x_n = \sum_{n\in\mathbb{N}} [x, y_n]x_n$. So, $\mathcal{W}x = \sum_{n\in\mathbb{N}} [x, y_n]x_n$ for all $x \in \mathcal{K}$.

It remains to prove that $\{y_n\}_{n\in\mathbb{N}} \subset \mathcal{K}$ is a Bessel sequence. In effect, $\sum_{n\in\mathbb{N}} \left|[x, y_n]\right|^2 = \sum_{n\in\mathbb{N}} |a_n^x|^2 = \|a^x\|_{\ell_2}^2 \leq \|M\|^2\|x\|_{\mathcal{J}}^2$, and therefore, $\{y_n\}_{n\in\mathbb{N}}$ is a Bessel sequence for $(\mathcal{K} = \mathcal{K}^+[\dot{+}]\mathcal{K}^-, [\cdot,\cdot], \mathcal{J})$.

As an application of the previous theorem, using the fundamental projectors below, we obtain frames associated with these projectors for the subspaces that compose the Krein space.

Theorem 4.4. *Let $(\mathcal{K} = \mathcal{K}^+[\dot{+}]\mathcal{K}^-, [\cdot,\cdot])$ be a Krein space with fundamental symmetry $\mathcal{J}$, and $\mathcal{W} : \mathcal{K} \rightarrow \mathcal{K}$ is a bounded operator. If the sequence $\{x_n = x_n^+ + x_n^-\}_{n\in\mathbb{N}}$ is a $\mathcal{W}$-frame for $\mathcal{K}$, then $\{x_n^+\}_{n\in\mathbb{N}}$ and $\{x_n^-\}_{n\in\mathbb{N}}$ are $\mathcal{P}^+\mathcal{W}$ and $\mathcal{P}^-\mathcal{W}$ frames for $(\mathcal{K}^+, [\cdot,\cdot])$ and $(\mathcal{K}^+, -[\cdot,\cdot])$, respectively.*

Proof. Since $\{x_n = x_n^+ + x_n^-\}_{n\in\mathbb{N}}$ is a $\mathcal{W}$-frame for $\mathcal{K}$, then there exists a Bessel sequence $\{y_n = y_n^+ + y_n^-\}_{n\in\mathbb{N}}$ for $\mathcal{K}$ such that for all $x \in \mathcal{K}$ we have that $\mathcal{W}x = \sum_{n\in\mathbb{N}} [x, y_n]x_n$.

Since $\{x_n = x_n^+ + x_n^-\}_{n\in\mathbb{N}}$ and $\{y_n = y_n^+ + y_n^-\}_{n\in\mathbb{N}}$ are Bessel sequences for $\mathcal{K}$ by Proposition 4.2 $\{x_n^+\}_{n\in\mathbb{N}}$, $\{y_n^+\}_{n\in\mathbb{N}}$ are Bessel sequences for $(\mathcal{K}^+, [\cdot,\cdot])$. $\{x_n^-\}_{n\in\mathbb{N}}$, $\{y_n^-\}_{n\in\mathbb{N}}$ are Bessel sequences for $(\mathcal{K}^-, -[\cdot,\cdot])$.

Let $x^+ \in \mathcal{K}^+ \subset \mathcal{K}$ *and* $x^- \in \mathcal{K}^- \subset \mathcal{K}$, and then

$$\mathcal{W}x^+ = \sum_{n\in\mathbb{N}} [x^+, y_n]x_n = \sum_{n\in\mathbb{N}} [x^+, y_n^+]x_n \quad \textit{and} \quad \mathcal{W}x^- = \sum_{n\in\mathbb{N}} [x^-, y_n]x_n = \sum_{n\in\mathbb{N}} [x^-, y_n^-]x_n.$$

Additionally,

$$\mathcal{P}^{+}\mathcal{W}x^{+} = \mathcal{P}^{+}(\mathcal{W}x^{+}) = \mathcal{P}^{+}\left(\sum_{n\in\mathbb{N}}[x^{+}, y_n^{+}]x_n\right) = \sum_{n\in\mathbb{N}}[x^{+}, y_n^{+}]\mathcal{P}^{+}(x_n) = \sum_{n\in\mathbb{N}}[x^{+}, y_n^{+}]x_n^{+},$$

and,

$$\mathcal{P}^{-}\mathcal{W}x^{-} = \mathcal{P}^{-}(\mathcal{W}x^{-}) = \mathcal{P}^{-}\left(\sum_{n\in\mathbb{N}}[x^{-}, y_n^{-}]x_n\right) = \sum_{n\in\mathbb{N}}[x^{-}, y_n^{-}]\mathcal{P}^{-}(x_n) = \sum_{n\in\mathbb{N}}([x^{-}, y_n^{-}])x_n^{-}.$$

Theorem 4.3 ensures that $\{x_n^{+}\}_{n\in\mathbb{N}}$ and $\{x_n^{-}\}_{n\in\mathbb{N}}$ are $\mathcal{P}^{+}\mathcal{W}$ and $\mathcal{P}^{-}\mathcal{W}$ frames for $(\mathcal{K}^{+}, [\cdot,\cdot])$ and $(\mathcal{K}^{-}, -[\cdot,\cdot])$, respectively.

5. Conclusions

The $\mathcal{W}$-frames in Krein spaces are well defined, and they are a generalization of the K-frames in Hilbert spaces introduced by GăvruÅ£a in [9]. The $\mathcal{W}$-frames are independent of the decomposition of the Krein space. By having $\mathcal{W}$-frames for a Krein space one necessarily has $\mathcal{W}$-frames for the Hilbert spaces that compose the Krein space, and the orthogonal projectors project $\mathcal{W}$-frames on $\mathcal{W}\mathcal{P}$-frames.

Conflict of interest

The authors declare that they have no conflict of interest in this work.

References

1. T. Azizov, I. S. Iokhvidov, *Linear operators in spaces with an indefinite metric*, John Wiley & Sons, 1989.
2. J. Bognár, *Indefinite inner product spaces*, Berlin: Springer, 1974.
3. O. Christensen, *An introduction to frames and riesz bases*, Boston: Birkhäuser, 2003.
4. I. Daubechies, A. Grossmann, Y. Meyer, Painless nonorthogonal expansions, *J. Math. Phys.*, **27** (1986), 1271–1283. https://doi.org/10.1063/1.527388
5. I. Daubechies, The wavelet transform, time-frequency localization and signal analysis, *IEEE T. Inform. Theory*, **36** (1990), 961–1005. https://doi.org/10.1109/18.57199
6. R. Douglas, On majorization, factorization and range inclusion of operators on Hilbert space, *Proc. Amer. Math. Soc.*, **17** (1966), 413–415. https://doi.org/10.2307/2035178
7. R. Duffin, A. Schaeffer, A class of nonharmonic Fourier series, *Trans. Amer. Math. Soc.*, **72** (1952), 341–366. https://doi.org/10.1090/S0002-9947-1952-0047179-6
8. K. Esmeral, O. Ferrer, E. Wagner, Frames in Krein spaces arising from a non-regular W-metric, *Banach J. Math. Anal.*, **9** (2015), 1–16. http://doi.org/10.15352/bjma/09-1-1
9. L. GăvruÅ£a, Frames for operators, *Appl. Comput. Harmon. A.*, **32** (2012), 139–144. https://doi.org/10.1016/j.acha.2011.07.006

10. K. Grchenig, *Foundations of time-frequency analysis*, Springer Science & Business Media, 2001.

11. A. Mohammed, K. Samir, N. Bounader, K-frames for Krein spaces, *Ann. Funct. Anal.*, **14** (2023), 10. https://doi.org/10.1007/s43034-022-00223-3

AIMS Press