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Research article

Quantile hedging for contingent claims in an uncertain financial environment

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Abstract: This paper first studies the quantile hedging problem of contingent claims in an uncertain market model. A special kind of no-arbitrage, that is, the absence of immediate profit, is characterized. Instead of the traditional no-arbitrage targeting the whole market, the absence of immediate profit depends on the confidence level of the portfolio manager for hedging risk. We prove that the condition of absence of immediate profit holds if and only if the initial price of each risky asset lies between the α-optimistic value and α-pessimistic value of its discounted price at the end of the period. The bounds of the minimal quantile hedging price are derived under the criterion of no-arbitrage in this paper, that is, the absence of immediate profit. Moreover, numerical experiments are implemented to verify that the condition of absence of immediate profit can be a good substitute for the traditional no-arbitrage, since the latter is difficult to achieve. Thus, it may provide a better principle of pricing due to the flexibility from the optional confidence level for the market participants in the increasingly complex financial market.

Keywords: quantile hedging; uncertain market; no-arbitrage; absence of immediate profit; hedging risk
Mathematics Subject Classification: 90C70, 91G20, 91G80

1. Introduction

It is pretty classical in mathematical finance to study the hedging problem of a contingent claim h_T at time T by using a self-financing portfolio process $(V_t)_{0\leq t\leq T}$ such that $V_T \geq h_T$. Recall that the minimal hedging price in a frictionless market is studied [1, 2] under the traditional no-arbitrage condition [3]. The main results and research opportunities based on hedge accounting-related studies are identified in [4].

A critical problem of perfect hedging, where the inequality $V_T \geq h_T$ holds almost everywhere, is that the required hedging cost is too high from a practical point of view. Thus, it is necessary to relax the requirement of common hedging. The quantile hedging problem is studied by minimizing the hedging cost such that the probability of successful hedging is at least α, where α is the confidence level chosen by the investors [5]. Due to the superior performance in catering to the objective needs of financial markets, quantile hedging has been widely studied since it was proposed [6–10].

The current research on perfect hedging and quantile hedging is basically carried out under the framework of classical probability theory. Investors can clearly master the uncertain state in the future. More importantly, investors can estimate the probability of occurrence from historical data. However, the complexity of financial markets and the limitation of information resources make it not easy to grasp the probability of financial variables since the investors often cannot obtain sufficient sample data smoothly. Economic uncertainty will also have an impact on portfolio prices [11, 12]. Instead of the probability estimation from large amounts of historical data, more investors construct their own degrees of belief about certain financial events according to the experience of industry experts. A new type of axiomatic mathematics system, uncertainty theory, has been built to model human belief degree [13].

Uncertainty theory has been fully developed since its creation. In particular, it has been applied to the field of financial research, and an uncertain stock model was proposed [14]. An equivalent theorem of no-arbitrage condition for Liu's uncertain stock model was derived [15]. The pricings of various derivatives, such as European options, American options, Asian options, currency options, lookback options, credit default swaps and stock loans, have been widely studied under uncertain environments [16–24]. Despite the relatively late starting, uncertain financial research has become an important branch of mathematical finance and has great prospects in the future.

What is different from the pricing theory in the sense of traditional probability is that the prices of derivative securities in the current literature on uncertain financial research are directly defined as the expected value of a discounted payoff based on the given uncertain measure. Although it is convenient to solve the related uncertain differential equations, the prices obtained from the above approach may not be equitable in the real market due to the lack of rigorous derivation of pricing theory, such as hedging and quantile hedging. Actually, the option parity formula is invalid for the current option pricing model in uncertain markets, that is to say, this option pricing method can obtain stable arbitrage opportunities in the market [25]. From this point of view, it seems to be necessary to study the hedging or quantile hedging of a contingent claim in uncertain financial markets.

The goal of this paper is to first study the quantile hedging problem in an uncertain market model, where the prices of underlying assets at the end of the period are uncertain variables instead of random variables, by considering, holistically, the following two facts: (1) a high cost of perfect hedging makes it necessary to consider quantile hedging; (2) uncertain finance can better model the real market than the stochastic finance since most investors construct a belief degree about financial events from the experts' experience. A contingent claim is said to be quantile-hedged if the relief degree of being covered is at least the confidence level α by starting with an initial capital, which is called the hedging price, and trading a (hedging) strategy. And, the valuing of a contingent claim is to minimize the initial capital in the class of hedging strategies.

Based on the above quantile hedging model, this paper mainly characterizes a special kind of arbitrage opportunity, which is called immediate profit. Actually, an immediate profit means that the investors can hedge the zero contingent claim by starting from a negative price [26]. We consider the absence of immediate profit (AIP) in the proposed quantile hedging model. Obviously, AIP is not discussed for the entire market, since it is related to the confidence level α chosen by the portfolio managers.

The implication of this paper is to build a more popular principle of pricing for the portfolio managers than the traditional no-arbitrage condition, that is, the AIP with the flexibility from the optional confidence level, in the increasingly complex financial market. Actually, we will show, by some numerical experiments, that the traditional no-arbitrage condition is difficult to be satisfied in the real market, especially in the uncertain financial market. This paper may provide a new idea for the uncertain financial research. The results can be applied in the aspects of financial asset pricing, portfolio management, optimal reinsurance and so on.

This paper is organized as follows. Section 2 builds the quantile hedging model and introduces the concept of AIP in a single-period uncertain market. In Section 3, the equivalent condition for AIP is obtained. Moreover, the bounds of the minimal quantile hedging price are derived under the AIP condition. At last, numerical applications of the AIP condition are discussed in Section 4.

2. Model and definitions

Recall that some basic definitions and useful results in uncertain theory, such as uncertain variable, uncertain process and uncertain reliability analysis, can be found in [13, 14, 27, 28]. Consider a single-period uncertain market model based on the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$, where $\mathcal{L}$ is the σ-algebra on the nonempty set Γ and $\mathcal{M}$ is the uncertain measure. Suppose that there are m risky assets with the price vector $S_0 = (S_0^1, S_0^2, \cdots, S_0^m)$ at the initial time and the discounted price vector $S_T = (S_T^1, S_T^2, \cdots, S_T^m)$ at the end of the period, where S_T is an m-dimensional uncertain vector. It is known that S_T is an uncertain vector if and only if $S_T^1, S_T^2, \cdots, S_T^m$ are uncertain variables. Some assumptions and notations are listed as follows:

- The prices $S_T^1, S_T^2, \cdots, S_T^m$ are supposed to be independent and the uncertainty distributions are $\Phi_1, \Phi_2, \cdots, \Phi_m$, respectively.
- The prices S_t^i are supposed to be non-negative for all $t = 0, T$ and $i = 1, 2, \cdots, m$.
- Real-valued uncertain variable h_T represents the payoff of a contingent claim with maturity T, and h_T is supposed to be non-negative.
- The uncertain variable 1_Λ is
$$1_\Lambda(\gamma) = \begin{cases} 1, & \text{if } \gamma \in \Lambda, \\ 0, & \text{if } \gamma \notin \Lambda, \end{cases}$$
where Λ is an event, that is, an element in the σ-algebra $\mathcal{L}$.
- $\mathbb{R}_+ := \{x \in \mathbb{R} | x \geq 0\}$ and $\mathbb{R}_- := \{x \in \mathbb{R} | x \leq 0\}$ are respectively non-negative and non-positive real number sets.
- Denote the set of all risky assets indices as $\mathbb{N}_0 := \{1, 2, \cdots, m\}$.

Definition 2.1. *The contingent claim h_T is quantile-hedged if there exists an initial capital $P \in \mathbb{R}$ and a strategy $x \in \mathbb{R}^m$ such that*
$$\mathcal{M}\{P + x \cdot \Delta S_T - h_T \geq 0\} \geq \alpha, \tag{2.1}$$
where $\alpha \in (\frac{1}{2}, 1)$ is the given confidence level.

An initial capital $P \in \mathbb{R}$, starting from which allows achievement of the quantile hedging of the contingent claim h_T, may be regarded as the possible price of h_T. In this way, the initial capital $P \in \mathbb{R}$ in Definition 2.1 is called the quantile hedging price of h_T, and $x \in \mathbb{R}^m$ is the corresponding hedging strategy. Let $\mathcal{P}(h_T)$ be the set of all quantile hedging prices, that is,

$$\mathcal{P}(h_T) := \{P \in \mathbb{R} | \ \exists\, x \in \mathbb{R}^m \ s.t.\ \mathcal{M}\{P + x \cdot \Delta S_T - h_T \geq 0\} \geq \alpha\}.$$

Without loss of generality, this paper assumes that the set $\mathcal{P}(h_T)$ is non-empty, i.e., $\mathcal{P}(h_T) \neq \emptyset$.

Definition 2.2. *The minimal quantile hedging price of the contingent claim h_T is defined as*

$$P^* := \inf_{x \in \mathbb{R}^m} \mathcal{P}(h_T). \tag{2.2}$$

If the zero contingent claim is considered, i.e., $h_T = 0$, it is obvious to see that $0 \in \mathcal{P}(0)$. Recall that AIP requires the minimal super-hedging price of the zero claim to be zero [26]. This new type of no-arbitrage concept is defined as follows in the sense of quantile hedging, which implies that it is impossible to successfully achieve the quantile hedging of the zero claim with a negative price.

Definition 2.3. *AIP holds if*

$$\mathcal{P}(0) \cap \mathbb{R}_- = \{0\}. \tag{2.3}$$

3. Main results

3.1. Fundamental theorem of asset pricing

Theorem 3.1. *The condition of AIP holds if and only if*

$$(S_T^i)_{\sup}(\alpha) \leq S_0^i \leq (S_T^i)_{\inf}(\alpha), \ \forall\, i = 1, 2, \cdots, m, \tag{3.1}$$

where $(S_T^i)_{\sup}(\alpha)$ and $(S_T^i)_{\inf}(\alpha)$ are the α-optimistic value and α-pessimistic value of S_T^i, respectively.

Proof. By considering the case where $h_T = 0$, it can be obtained that

$$\mathcal{P}(0) = \{P \in \mathbb{R} | \ \exists\, x \in \mathbb{R}^m \ s.t.\ \mathcal{M}\{P + x \cdot \Delta S_T \geq 0\} \geq \alpha\}.$$

($\Leftarrow$) Define the following function:

$$R(S_T^1, S_T^2, \cdots, S_T^m) := P + x \cdot \Delta S_T.$$

Then, the zero contingent claim is super-hedged if and only if $R(S_T^1, S_T^2, \cdots, S_T^m) \geq 0$. The reliability index supporting that the zero contingent claim can be super-hedged is

$$Reliability = \mathcal{M}\{R(S_T^1, S_T^2, \cdots, S_T^m) \geq 0\}.$$

Then, the zero contingent claim is quantile-hedged if and only if $Reliability \geq \alpha$. Denote

$$N_1 := \{i \in \mathbb{N}_0 \mid x_i > 0\}$$

and

$$N_2 := \{i \in \mathbb{N}_0 \mid x_i < 0\};$$

it has

$$R(S_T^1, S_T^2, \cdots, S_T^m) = P + \sum_{i=1}^{m} x_i(S_T^i - S_0^i) = P + \sum_{i\in N_1} x_i(S_T^i - S_0^i) + \sum_{i\in N_2} x_i(S_T^i - S_0^i).$$

For the case where $N_1 \cup N_2 = \emptyset$, i.e., $x_i = 0$ for all $1 \le i \le m$, the AIP condition holds trivially. Indeed, the quantile hedging of the zero contingent claim implies that $\mathcal{M}\{P \ge 0\} \ge \alpha$, where $\alpha \in (\frac{1}{2}, 1)$. Thereby, it must have $P \ge 0$ due to the fact that the value of $\mathcal{M}\{P \ge 0\}$ is either 1 or 0.

Next, the case where $N_1 \cup N_2 \ne \emptyset$ is considered. It is obvious that $R(y_1, y_2, \cdots, y_m)$ is strictly increasing w.r.t. y_i if $i \in N_1$, and strictly decreasing w.r.t. y_i if $i \in N_2$. From the reliability index theorem [28], the reliability index is

$$Reliability = \beta,$$

where β is the root of

$$P + \sum_{i\in N_1} x_i[\Phi_i^{-1}(1-\beta) - S_0^i] + \sum_{i\in N_2} x_i[\Phi_i^{-1}(\beta) - S_0^i] = 0.$$

Note that the quantile hedging implies that $Reliability = \beta \ge \alpha$. Furthermore, since inverse uncertainty distribution is a monotone increasing function on [0, 1], it holds that

$$\begin{aligned} P &= \sum_{i\in N_1} x_i[S_0^i - \Phi_i^{-1}(1-\beta)] + \sum_{i\in N_2} x_i[S_0^i - \Phi_i^{-1}(\beta)] \\ &\ge \sum_{i\in N_1} x_i[S_0^i - \Phi_i^{-1}(1-\alpha)] + \sum_{i\in N_2} x_i[S_0^i - \Phi_i^{-1}(\alpha)] \\ &\ge 0, \end{aligned}$$

since (3.1) implies that $\Phi_i^{-1}(1-\alpha) \le S_0^i \le \Phi_i^{-1}(\alpha)$. Thereby, the AIP condition holds.

($\Rightarrow$) Assume that AIP holds and there exists $i_0 \in \{1, 2, \cdots, m\}$ such that $S_0^{i_0} < (S_T^{i_0})_{\sup}(\alpha)$, i.e.,

$$S_0^{i_0} < \Phi_{i_0}^{-1}(1-\alpha).$$

By starting from the quantile hedging price

$$P = S_0^{i_0} - \Phi_{i_0}^{-1}(1-\alpha) < 0 \tag{3.2}$$

and taking the strategy x as

$$x_i = \begin{cases} 1, & i = i_0, \\ 0, & otherwise, \end{cases} \tag{3.3}$$

it has

$$P + x\cdot\Delta S_T = P + \sum_{i=1}^{m} x_i(S_T^i - S_0^i) = S_0^{i_0} - \Phi_{i_0}^{-1}(1-\alpha) + S_T^{i_0} - S_0^{i_0} = S_T^{i_0} - \Phi_{i_0}^{-1}(1-\alpha).$$

Since $\mathcal{M}\{S_T^{i_0} \le \Phi_{i_0}^{-1}(1-\alpha)\} = 1-\alpha$, it has

$$\mathcal{M}\{P + x\cdot\Delta S_T \ge 0\} = \mathcal{M}\{S_T^{i_0} \ge \Phi_{i_0}^{-1}(1-\alpha)\} = \alpha.$$

That is to say, the quantile hedging of the zero contingent claim can be achieved by starting from a negative price (3.2) and trading a strategy (3.3), which is contradicted with the AIP condition.

A contradiction can be also obtained by the similar arguments for the case where AIP holds and there exists $i_0 \in \{1, 2, \cdots, m\}$ such that $S_0^{i_0} > (S_T^{i_0})_{\inf}(\alpha)$. Thus, the AIP condition must imply that (3.1) holds for all $i = 1, 2, \cdots, m$.

3.2. Bounds of minimal quantile hedging price

The bounds of the minimal quantile hedging price, i.e., the interval of arbitrage-free prices are studied in this section. Here, the arbitrage-free property precisely refers to the AIP.

Next, we can imitate the concept of the almost sure supremum [29] to define the almost sure infimum of a real-valued uncertain variable.

Definition 3.1. *A number, "$a.s.\inf\xi$", is called the almost sure infimum of a real-valued uncertain variable ξ if*
1) $\mathcal{M}\{\xi < a.s.\inf\xi\} = 0$;
2) $\mathcal{M}\{\xi \le c\} > 0$ for every $c > a.s.\inf\xi$.

It is trivial to hold that $a.s.\inf\xi \le \xi \le a.s.\sup\xi$. Then, the bounds of the minimal quantile hedging price are showed in the following theorem.

Theorem 3.2. *The condition of AIP holds if and only if the minimal quantile hedging price of a contingent claim h_T satisfies*

$$B_l^* \le P^* \le B_u^*, \tag{3.4}$$

where

$$B_l^* = \sum_{i=1}^{m} S_0^i \left(a.s.\inf\frac{h_T}{mS_T^i}\right),\ \ B_u^* = \sum_{i=1}^{m} S_0^i \left(a.s.\sup\frac{h_T}{mS_T^i}\right).$$

Proof. ($\Leftarrow$) The sufficiency is trivial. Indeed, (3.4) implies that $P^* \ge 0$ as $B_l^* \ge 0$. In this case, the AIP condition $\mathcal{P}(0) \cap \mathbb{R}_- = \{0\}$ trivially holds.

($\Rightarrow$) The necessity is to be proved by the two steps.

Step 1: First, recall that

$$P^* = \inf_{x\in\mathbb{R}^m} \mathcal{P}(h_T),$$

where

$$\mathcal{P}(h_T) = \{P \in \mathbb{R} |\ \exists\ x \in \mathbb{R}^m\ s.t.\ \mathcal{M}\{P + x\cdot\Delta S_T - h_T \ge 0\} \ge \alpha\}.$$

In fact,

$$P + x\cdot\Delta S_T - h_T = P - \sum_{i=1}^{m} x_i S_0^i + \sum_{i=1}^{m} x_i S_T^i - h_T = P - \sum_{i=1}^{m} x_i S_0^i + \sum_{i=1}^{m} S_T^i \left(x_i - \frac{h_T}{mS_T^i}\right).$$

Note that the following fact

$$a.s.\inf\frac{h_T}{mS_T^i} \le \frac{h_T}{mS_T^i} \le a.s.\sup\frac{h_T}{mS_T^i}$$

holds for each $i = 1, 2, \cdots, m$; then,

$$x_i - a.s.\sup\frac{h_T}{mS_T^i} \le x_i - \frac{h_T}{mS_T^i} \le x_i - a.s.\inf\frac{h_T}{mS_T^i},\ \ \forall\ i = 1, 2, \cdots, m.$$

Thus, it can be deduced that

$$R_u(S_T^1, S_T^2, \cdots, S_T^m) \le P + x\cdot\Delta S_T - h_T \le R_l(S_T^1, S_T^2, \cdots, S_T^m), \tag{3.5}$$

where

$$R_u(S_T^1, S_T^2, \cdots, S_T^m) := P - \sum_{i=1}^{m} x_i S_0^i + \sum_{i=1}^{m} S_T^i \left(x_i - a.s. \sup \frac{h_T}{mS_T^i} \right),$$

and

$$R_l(S_T^1, S_T^2, \cdots, S_T^m) := P - \sum_{i=1}^{m} x_i S_0^i + \sum_{i=1}^{m} S_T^i \left(x_i - a.s. \inf \frac{h_T}{mS_T^i} \right).$$

Furthermore, the two useful sets are introduced as

$$\mathcal{B}_u := \left\{ P \in \mathbb{R} | \ \exists \ x \in \mathbb{R}^m \ s.t. \ \mathcal{M}\left\{ R_u(S_T^1, S_T^2, \cdots, S_T^m) \geq 0 \right\} \geq \alpha \right\}$$

and

$$\mathcal{B}_l := \left\{ P \in \mathbb{R} | \ \exists \ x \in \mathbb{R}^m \ s.t. \ \mathcal{M}\left\{ R_l(S_T^1, S_T^2, \cdots, S_T^m) \geq 0 \right\} \geq \alpha \right\}.$$

Then, it can be easily observed from (3.5) that

$$\mathcal{B}_u \subseteq \mathcal{P}(h_T) \subseteq \mathcal{B}_l$$

such that

$$\inf_{x \in \mathbb{R}^m} \mathcal{B}_l \leq \inf_{x \in \mathbb{R}^m} \mathcal{P}(h_T) \leq \inf_{x \in \mathbb{R}^m} \mathcal{B}_u. \tag{3.6}$$

Step 2: Next, the infimums of the sets $\mathcal{B}_l$ and $\mathcal{B}_u$ are computed. Denote

$$J_1 := \left\{ i \in \mathbb{N}_0 \mid x_i > a.s. \inf \frac{h_T}{mS_T^i} \right\}$$

and

$$J_2 := \left\{ i \in \mathbb{N}_0 \mid x_i < a.s. \inf \frac{h_T}{mS_T^i} \right\}.$$

a) For the case where $J_1 \cup J_2 = \emptyset$, i.e., $x_i = a.s. \inf \frac{h_T}{mS_T^i}$ for all $1 \leq i \leq m$, it has

$$R_l(S_T^1, S_T^2, \cdots, S_T^m) = P - \sum_{i=1}^{m} S_0^i \left(a.s. \inf \frac{h_T}{mS_T^i} \right)$$

such that

$$\mathcal{B}_l = \left\{ P \in \mathbb{R} | \ P \geq \sum_{i=1}^{m} S_0^i \left(a.s. \inf \frac{h_T}{mS_T^i} \right) = B_l^* \right\}.$$

In this way, B_l^* is actually the infimum of the set $\mathcal{B}_l$.

b) For the case where $J_1 \cup J_2 \neq \emptyset$, it can be observed that $R_l(y_1, y_2, \cdots, y_m)$ is strictly increasing w.r.t. y_i if $i \in J_1$, and strictly decreasing w.r.t. y_i if $i \in J_2$. From the reliability index theorem, it has

$$\mathcal{M}\left\{ R_l(S_T^1, S_T^2, \cdots, S_T^m) \geq 0 \right\} = \beta,$$

where β is the root of

$$P = \sum_{i=1}^{m} x_i S_0^i + \sum_{i \in J_1} \Phi_i^{-1}(1-\beta) \left(x_i - a.s. \inf \frac{h_T}{mS_T^i} \right) + \sum_{i \in J_2} \Phi_i^{-1}(\beta) \left(x_i - a.s. \inf \frac{h_T}{mS_T^i} \right). \tag{3.7}$$

Then, the set $\mathcal{B}_l$ can be equivalently written as

$$\mathcal{B}_l = \{P \in \mathbb{R} |\, \exists\, x \in \mathbb{R}^m \ s.t.\, \beta \geq \alpha\}\,.$$

It can be proved that $\beta \geq \alpha$ if and only if the following inequality holds, i.e.,

$$P \geq \sum_{i=1}^{m} x_i S_0^i + \sum_{i\in J_1} \Phi_i^{-1}(1-\alpha)\left(x_i - a.s.\inf \frac{h_T}{mS_T^i}\right) + \sum_{i\in J_2} \Phi_i^{-1}(\alpha)\left(x_i - a.s.\inf \frac{h_T}{mS_T^i}\right). \tag{3.8}$$

Indeed, the necessity is obvious, as $\beta \geq \alpha$ implies that

$$\Phi_i^{-1}(\alpha) \leq \Phi_i^{-1}(\beta)$$

and

$$\Phi_i^{-1}(1-\alpha) \geq \Phi_i^{-1}(1-\beta)$$

hold for all $i = 1, 2, \cdots, m$. Thereby, it can be obtained from (3.7) that the inequality (3.8) holds. On the contrary, the sufficiency is to prove that $\beta \geq \alpha$ under the assumption (3.8). Actually, the inequality (3.8) implies that there exist some $i_0 \in \mathbb{N}_0$ satisfying $\Phi_{i_0}^{-1}(\alpha) \leq \Phi_{i_0}^{-1}(\beta)$. Otherwise, it must have $\Phi_i^{-1}(\alpha) > \Phi_i^{-1}(\beta)$, $\forall\, i = 1, 2, \cdots, m$. Furthermore, $\Phi_i^{-1}(1-\alpha) \leq \Phi_i^{-1}(1-\beta)$, $\forall\, i = 1, 2, \cdots, m$. In this case, we can see that

$$P < \sum_{i=1}^{m} x_i S_0^i + \sum_{i\in J_1} \Phi_i^{-1}(1-\alpha)\left(x_i - a.s.\inf \frac{h_T}{mS_T^i}\right) + \sum_{i\in J_2} \Phi_i^{-1}(\alpha)\left(x_i - a.s.\inf \frac{h_T}{mS_T^i}\right),$$

which is contradicted with the assumption (3.8). Thereby, $\beta \geq \alpha$ can be obtained from the assertion that $\Phi_{i_0}^{-1}(\alpha) \leq \Phi_{i_0}^{-1}(\beta)$.

Now, the problem of solving the infimum of $\mathcal{B}_l$ can be transferred into the optimization, i.e.,

$$\inf_{x\in\mathbb{R}^m} \mathcal{B}_l = \inf_{x\in\mathbb{R}^m} f(x),$$

where

$$f(x) := \sum_{i=1}^{m} x_i S_0^i - \sum_{i\in J_1} \Phi_i^{-1}(1-\alpha)\left(x_i - a.s.\inf \frac{h_T}{mS_T^i}\right) - \sum_{i\in J_2} \Phi_i^{-1}(\alpha)\left(x_i - a.s.\inf \frac{h_T}{mS_T^i}\right).$$

Actually, the function $f(x)$ can be written as

$$\begin{aligned} f(x) &= \sum_{i\in J_1}\left[x_i S_0^i - \Phi_i^{-1}(1-\alpha)\left(x_i - a.s.\inf \frac{h_T}{mS_T^i}\right)\right] \\ &\quad + \sum_{i\in J_2}\left[x_i S_0^i - \Phi_i^{-1}(\alpha)\left(x_i - a.s.\inf \frac{h_T}{mS_T^i}\right)\right] \\ &\quad + \sum_{i\in \mathbb{N}_0\setminus(J_1\cup J_2)} S_0^i\left(a.s.\inf \frac{h_T}{mS_T^i}\right) \\ &= \sum_{i\in J_1} g_i(x_i) + \sum_{i\in J_2} k_i(x_i) + \sum_{i\in \mathbb{N}_0\setminus(J_1\cup J_2)} S_0^i\left(a.s.\inf \frac{h_T}{mS_T^i}\right), \end{aligned}$$

where

$$g_i(x) := x\left[S_0^i - \Phi_i^{-1}(1-\alpha)\right] + \Phi_i^{-1}(1-\alpha)\left(a.s.\inf \frac{h_T}{mS_T^i}\right),\ x \in \mathbb{R},\ i \in J_1,$$

and

$$k_i(x) := x\left[S_0^i - \Phi_i^{-1}(\alpha)\right] + \Phi_i^{-1}(1-\alpha)\left(a.s.\inf \frac{h_T}{mS_T^i}\right),\ x \in \mathbb{R},\ i \in J_2.$$

From Theorem 3.1, the AIP condition holds if and only if

$$\Phi_i^{-1}(1-\alpha) \le S_0^i \le \Phi_i^{-1}(\alpha),\ \forall\, i = 1, 2, \cdots, m.$$

Obviously, $g_i(x)$ is a non-decreasing function w.r.t. x and $k_i(x)$ is a non-increasing function w.r.t. x. Thus, for each $i \in J_1$, it has

$$\inf_{x_i \in \mathbb{R}} g_i(x_i) = \inf_{x_i > a.s.\inf \frac{h_T}{mS_T^i}} g_i(x_i) = g_i\left(a.s.\inf \frac{h_T}{mS_T^i}\right) = S_0^i\left(a.s.\inf \frac{h_T}{mS_T^i}\right),$$

and for each $i \in J_2$, it has

$$\inf_{x_i \in \mathbb{R}} k_i(x_i) = \inf_{x_i < a.s.\inf \frac{h_T}{mS_T^i}} k_i(x_i) = k_i\left(a.s.\inf \frac{h_T}{mS_T^i}\right) = S_0^i\left(a.s.\inf \frac{h_T}{mS_T^i}\right).$$

In this way, we can see that

$$\begin{aligned}
\inf_{x \in \mathbb{R}^m} \mathcal{B}_l &= \inf_{x \in \mathbb{R}^m} f(x) \\
&= \sum_{i \in J_1} S_0^i\left(a.s.\inf \frac{h_T}{mS_T^i}\right) + \sum_{i \in J_2} S_0^i\left(a.s.\inf \frac{h_T}{mS_T^i}\right) \\
&\quad + \sum_{i \in \mathbb{N}_0 \backslash (J_1 \cup J_2)} S_0^i\left(a.s.\inf \frac{h_T}{mS_T^i}\right) \\
&= \sum_{i=1}^{m} S_0^i\left(a.s.\inf \frac{h_T}{mS_T^i}\right) \\
&= B_l^*.
\end{aligned}$$

Finally, it can be obtained that the infimum of the set $\mathcal{B}_l$ is B_l^*. And, the infimum of the set $\mathcal{B}_u$ can be computed to be B_u^* by similar arguments. Thus, it can be finally deduced from (3.6) that $B_l^* \le P^* \le B_u^*$.

4. Numerical applications

This section considers the AIP condition in an uncertain stock model with multiple stocks [14], where the stock prices are supposed to be independent. In detail, the market consists of one bond and m stocks. The bond price B_t and the stock prices X_{it} are given as

$$\begin{cases} dB_t = rB_t dt, \\ dX_{it} = \mu_i X_{it} dt + \sigma_i X_{it} dC_t,\ i = 1, 2, \cdots, m, \end{cases} \tag{4.1}$$

where r is the risk-free rate, μ_i and σ_i are respectively the drift coefficients and the diffusion coefficients, $i = 1, 2, \cdots, m$, and C_t is a canonical process.

The following corollary is a direct application of Theorem 3.1 in a special single-period uncertain market model where the discounted stock prices at time T are determined by (4.1).

Corollary 4.1. *In the single-period uncertain market model with the stock price $S_0 = (S_0^1, S_0^2, \cdots, S_0^m)$ at time 0 and the discounted stock price $S_T = (S_T^1, S_T^2, \cdots, S_T^m)$ at time T, where $S_T^i = e^{-rT} X_{iT}$ and X_{iT} are determined by (4.1) for all $i = 1, 2, \cdots, m$, the AIP condition holds if and only if*

$$\left|\frac{\mu_i - r}{\sigma_i}\right| \leq \frac{\sqrt{3}}{\pi} \ln\left(\frac{\alpha}{1-\alpha}\right), \ \forall\ i = 1, 2, \cdots, m. \tag{4.2}$$

Proof. It can be deduced from (4.1) that, for every $i = 1, 2, \cdots, m$,

$$\ln\left(\frac{X_{iT}}{X_{i0}}\right) = \mu_i T + \sigma_i \int_0^T dC_t \sim \mathcal{N}\left(\mu_i T, \sigma_i T\right),$$

so that

$$\ln\left(\frac{S_T^i}{S_0^i}\right) = \ln\left(\frac{X_{iT}}{X_{i0}}\right) - rT \sim \mathcal{N}\left(\mu_i T - rT, \sigma_i T\right).$$

Thus, it can be easily deduced that the discounted stock price S_T^i is a log-normal uncertain variable, that is,

$$S_T^i \sim \mathcal{LOGN}\left(\mu_i T - rT + \ln(S_0^i), \sigma_i T\right).$$

Then, the α-optimistic value and α-pessimistic value of S_T^i can be obtained to be

$$(S_T^i)_{\sup}(\alpha) = e^{\mu_i T - rT + \ln(S_0^i)} \left(\frac{1-\alpha}{\alpha}\right)^{\frac{\sqrt{3}\sigma_i T}{\pi}},$$

and

$$(S_T^i)_{\inf}(\alpha) = e^{\mu_i T - rT + \ln(S_0^i)} \left(\frac{\alpha}{1-\alpha}\right)^{\frac{\sqrt{3}\sigma_i T}{\pi}}.$$

Thus, Theorem 3.1 implies that the AIP condition holds if and only if, for all $i = 1, 2, \cdots, m$,

$$e^{\mu_i T - rT + \ln(S_0^i)} \left(\frac{1-\alpha}{\alpha}\right)^{\frac{\sqrt{3}\sigma_i T}{\pi}} \ \leq\ S_0^i \ \leq\ e^{\mu_i T - rT + \ln(S_0^i)} \left(\frac{\alpha}{1-\alpha}\right)^{\frac{\sqrt{3}\sigma_i T}{\pi}}.$$

By simple computations, it can be easily deduced that the AIP condition holds if and only if (4.2) holds.

Recall that the classical no-arbitrage condition for a multi-factor uncertain stock model is characterized in [15], which is described as the no-arbitrage determinant theorem. When the prices of stocks are determined by the one canonical process, it is easy to deduce that the no-arbitrage condition holds if and only if

$$\frac{\mu_1 - r}{\sigma_1} = \frac{\mu_2 - r}{\sigma_2} = \cdots = \frac{\mu_m - r}{\sigma_m}. \tag{4.3}$$

By comparing the equivalent condition of AIP (4.2) and no-arbitrage (4.3) in Liu's uncertain stock model with multiple stocks, it can be observed that the criterion of classical no-arbitrage is established for the whole market since it just needs to judge whether all of the stocks have the same value of $\frac{\mu-r}{\sigma}$. The criterion of AIP depends on the threshold $\frac{\sqrt{3}}{\pi}\ln\left(\frac{\alpha}{1-\alpha}\right)$, which may vary with the confidence level α chosen by the portfolio managers.

Next, we show that the AIP condition is valid in the real market. Especially, the numerical examples show that the AIP condition can be a good substitute for the traditional no-arbitrage, since the latter is difficult to be achieved.

Consider the stock model (4.1) with three stocks, i.e., the bond price B_t and the stock prices X_{it}, $i = 1, 2, 3$, that are determined by

$$\begin{cases} dB_t = rB_t dt, \\ dX_{1t} = \mu_1 X_{1t} dt + \sigma_1 X_{1t} dC_t, \\ dX_{2t} = \mu_2 X_{2t} dt + \sigma_2 X_{2t} dC_t, \\ dX_{3t} = \mu_3 X_{3t} dt + \sigma_3 X_{3t} dC_t. \end{cases} \tag{4.4}$$

Example 4.1. Three stocks, i.e., Junshi Biosciences (688180.SH), Sinovac Biotec (688136.SH) and Mabwell (688062.SH), were chosen from the Shanghai Stock Exchange. We adopted the α-path method [30] to estimate the parameters μ_i and σ_i, $i = 1, 2, 3$ by the closing prices from January to August, 2022. The risk-free rate r is chosen as the one-year treasury bond rate in that month. The values of parameters are shown in Table 1.

Table 1. The values of parameters r, μ and σ.

	Jan.	Feb.	Mar.	Apr.	May	Jun.	Jul.	Aug.
r	0.0056	0.0056	0.0058	0.0056	0.0053	0.0056	0.0053	0.0047
μ_1	0.0009	-0.0055	0.0104	-0.0239	0.0159	0.0224	0.0123	-0.0165
σ_1	0.0229	0.0136	0.0136	0.0188	0.0175	0.0131	0.0149	0.0103
μ_2	-0.0028	0.0058	-0.0037	0.0136	0.0255	0.0182	0.0128	-0.0248
σ_2	0.0063	0.0074	0.0096	0.0181	0.0142	0.0129	0.0092	0.0154
μ_3	0.1216	0.0063	0.0218	-0.0137	0.0012	0.0169	0.0105	-0.0164
σ_3	0.0877	0.0083	0.0199	0.0087	0.0039	0.0095	0.0102	0.0142

The values of $\frac{\mu-r}{\sigma}$ for three stocks are shown in Figure 1. It can be observed that the traditional no-arbitrage condition was not satisfied since the equalities (4.3) were invalid for all 8 months. The AIP conditions were checked for Stock 1 in Figure 2 (with the confidence level $\alpha = 95\%$) and Figure 3 (with the confidence level $\alpha = 98\%$). The AIP conditions were checked for Stock 2 in Figure 4 (with the confidence level $\alpha = 95\%$) and Figure 5 (with the confidence level $\alpha = 98\%$). The AIP conditions were checked for Stock 3 in Figure 6 (with the confidence level $\alpha = 95\%$) and Figure 7 (with the confidence level $\alpha = 98\%$).

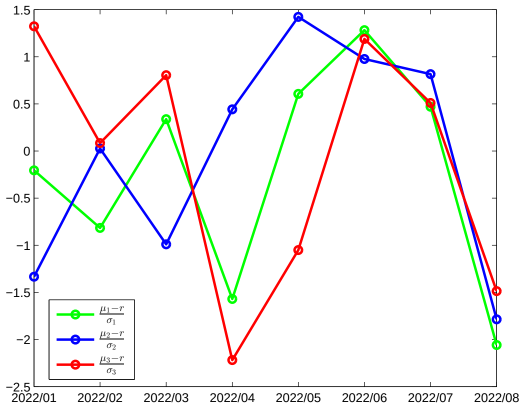

Figure 1. The values of $\frac{\mu-r}{\sigma}$ for three stocks.

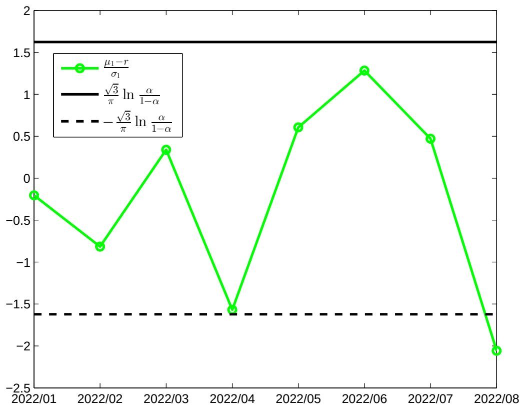

Figure 2. Stock 1 (Junshi) with $\alpha = 95\%$.

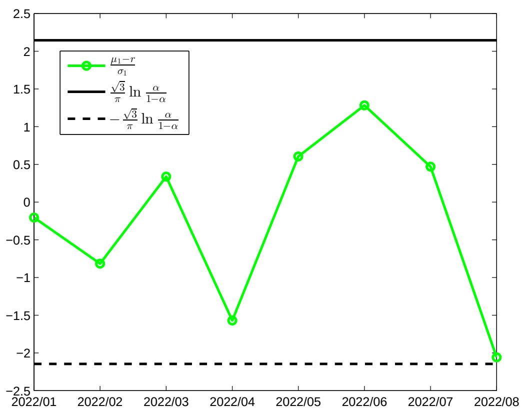

Figure 3. Stock 1 (Junshi) with $\alpha = 98\%$.

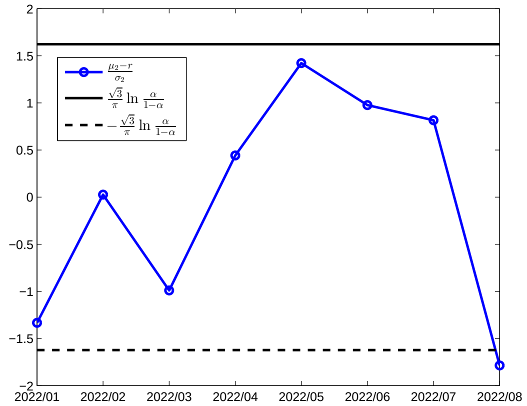

Figure 4. Stock 2 (Sinovac) with $\alpha = 95\%$.

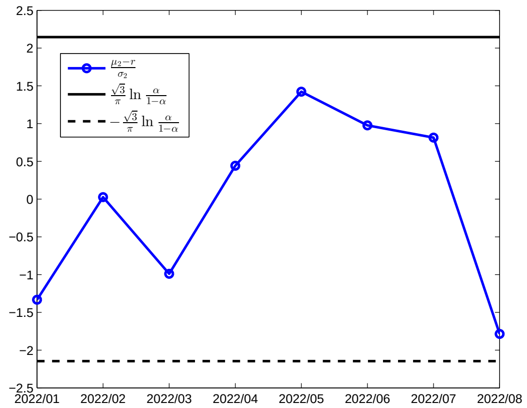

Figure 5. Stock 2 (Sinovac) with $\alpha = 98\%$.

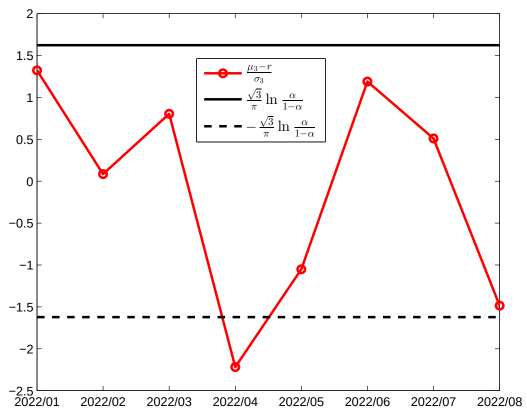

Figure 6. Stock 3 (Mabwell) with $\alpha = 95\%$.

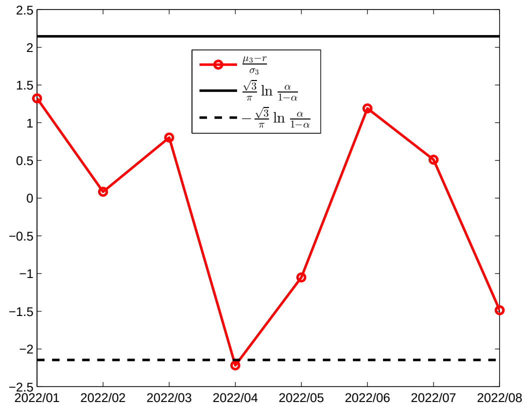

Figure 7. Stock 3 (Mabwell) with $\alpha = 98\%$.

We can observe that the market satisfied the AIP condition with $\alpha = 95\%$ except for April and August. And, the market satisfied the AIP condition at $\alpha = 98\%$, except for April. Actually, the prices of Stock 3 in April fluctuated greatly from 14.16 CNY to 20 CNY and possessed a maximum yield of 8.5%, which may have led to the violation of the AIP condition in April.

Example 4.2. Next, the other three stocks, i.e., China National Gold Group Gold Jewelery Co., Ltd. (600916.SH), Chow Tai Seng Jewelery Co., Ltd. (002867.SZ) and Guangdong Chj Industry Co., Ltd. (002345.SZ), were chosen. Similarly, we adopted the α-path method [30] to estimate the parameters μ_i and σ_i, $i = 1, 2, 3$ by the closing prices from January to August, 2022. The risk-free rate r was chosen as the one-year treasury bond rate in that month. The values of parameters are shown in Table 2.

Table 2. The values of parameters r, μ and σ for the other three stocks.

	Jan.	Feb.	Mar.	Apr.	May	Jun.	Jul.	Aug.
r	0.0056	0.0056	0.0058	0.0056	0.0053	0.0056	0.0053	0.0047
μ_1	-0.0021	-0.0025	-0.0163	0.0234	-0.0203	-0.0028	0.0077	-0.0182
σ_1	0.0026	0.0145	0.0056	0.0219	0.0133	0.0049	0.0049	0.0250
μ_2	0.0161	0.0098	-0.0161	-0.0030	0.0757	-0.0111	-0.0122	-0.0171
σ_2	0.0120	0.0067	0.0109	0.0064	0.0468	0.0085	0.0046	0.0113
μ_3	0.0451	0.0086	-0.0059	0.0008	-0.0157	0.0076	-0.0044	-0.0207
σ_3	0.0314	0.0064	0.0080	0.0085	0.0113	0.0132	0.0051	0.0203

The values of $\frac{\mu - r}{\sigma}$ for these three stocks are shown in Figure 8. It can be observed that the traditional no-arbitrage condition was not satisfied since the equalities (4.3) were invalid for all 8 months. The AIP conditions were checked for Stock 1 in Figure 9 (with the confidence level $\alpha = 95\%$) and Figure 10 (with the confidence level $\alpha = 98\%$). The AIP conditions were checked for Stock 2 in Figure 11 (with the confidence level $\alpha = 95\%$) and Figure 12 (with the confidence level $\alpha = 98\%$). The AIP conditions were checked for Stock 3 in Figure 13 (with the confidence level $\alpha = 95\%$) and Figure 14 (with the confidence level $\alpha = 98\%$).

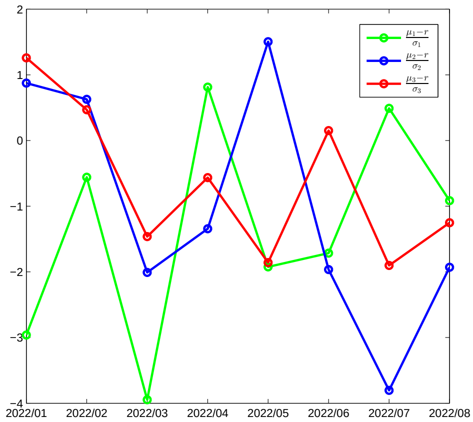

Figure 8. The values of $\frac{\mu-r}{\sigma}$ for the other three stocks.

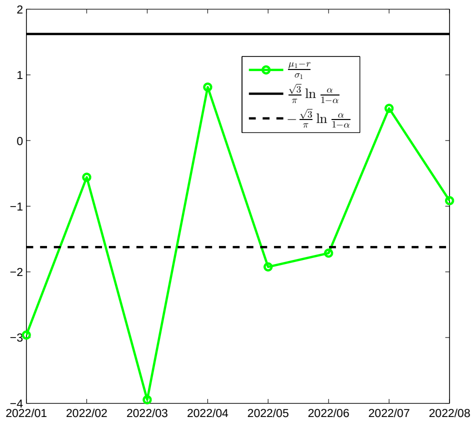

Figure 9. Stock 1 (China National Gold) with $\alpha = 95\%$.

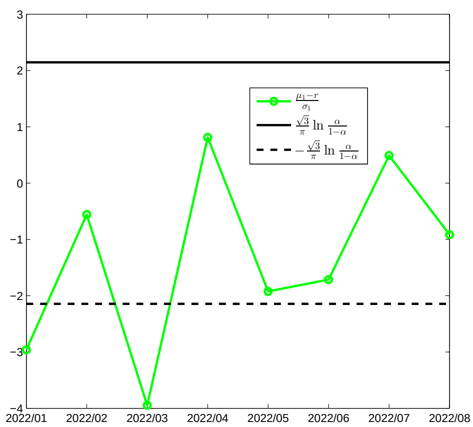

Figure 10. Stock 1 (China National Gold) with $\alpha = 98\%$.

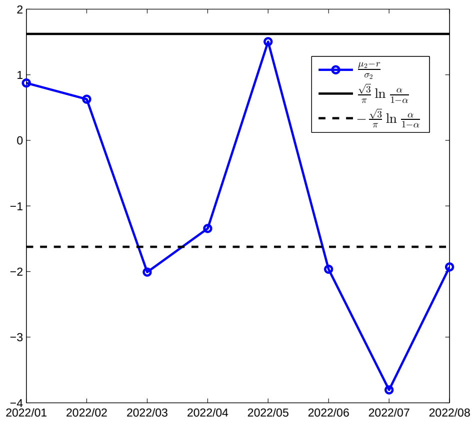

Figure 11. Stock 2 (Chow Tai Seng) with $\alpha = 95\%$.

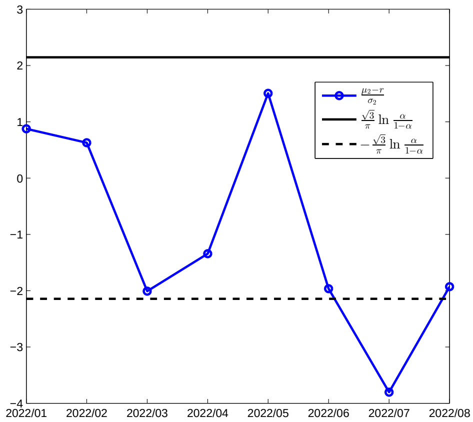

Figure 12. Stock 2 (Chow Tai Seng) with $\alpha = 98\%$.

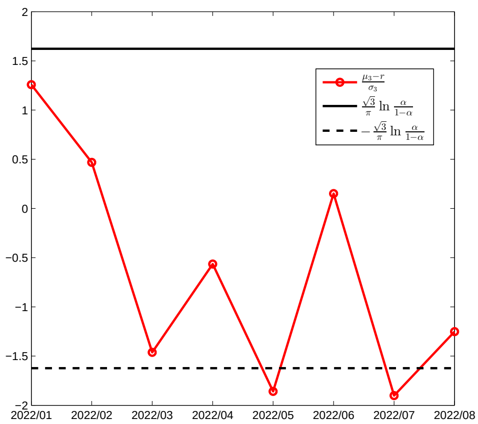

Figure 13. Stock 3 (Guangdong Chj) with $\alpha = 95\%$.

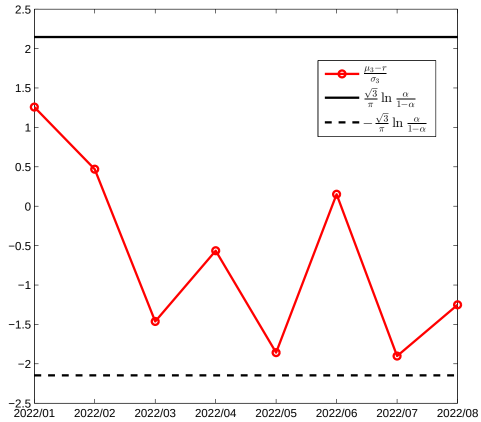

Figure 14. Stock 3 (Guangdong Chj) with $\alpha = 98\%$.

We can observe that the AIP condition with $\alpha = 95\%$ was difficult to be satisfied, except for February and April. But, the market satisfied the AIP condition with $\alpha = 98\%$, except for January, March and July. Thus, a higher confidence level could be considered by portfolio managers compared with the market in Example 4.1.

5. Conclusions

This paper investigates the quantile hedging problem in a single-period uncertain market model, where the discounted prices of risky assets at the end of the period are uncertain variables. An equivalent condition for a special kind of no-arbitrage, AIP, has been characterized. That is, the initial price of each risky asset lies between the α-optimistic value and α-pessimistic value of its discounted price at the end of the period. Moreover, the bounds of the minimal quantile hedging price have been derived under the criterion of AIP. The numerical experiments show that the AIP condition can be a good substitute for the traditional no-arbitrage in the real market due to the flexibility from the optional confidence level. In the following research, we will aim to address the quantile hedging problem in a multi-period uncertain market model, and even a time-continuous uncertain market model. On the other hand, we may consider certain factors in the quantile hedging model, such as outliers in forecasting [31], quantitative easing effectiveness [32] and so on.

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Conflict of interest

All authors declare no conflicts of interest regarding the publication of this paper.

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