AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(7): 15633–15650.
DOI: 10.3934/math.2023798
Received: 22 February 2023
Revised: 05 April 2023
Accepted: 12 April 2023
Published: 27 April 2023

Research article

European option pricing problem based on a class of Caputo-Hadamard uncertain fractional differential equation

Hanjie Liu, Yuanguo Zhu* **and Yiyu Liu**

School of Mathematics and Statistics, Nanjing University of Science and Technology, Nanjing 210094, Jiangsu, China

* **Correspondence:** Email: ygzhu@njust.edu.cn.

Abstract: Uncertain fractional differential equation (UFDE) is very suitable for describing the dynamic change in uncertain environments. In this paper, we consider the European option pricing problem by applying the Caputo-Hadamard UFDEs to simulate the dynamic change of stock price. First, an uncertain stock model with the mean-reverting process is studied, and the European option pricing formulas are given. Then, the effect of uncertain interference on the bond is considered, and the corresponding European option pricing formulas are presented. Finally, some numerical examples are given to illustrate the effectiveness of pricing formulas.

Keywords: uncertainty theory; uncertain fractional differential equation; mean-reverting process; European option; uncertain interference
Mathematics Subject Classification: 34A08, 45G15, 91G30

1. Introduction

Option pricing problem is one of the most important problems in financial mathematics. An option is a contract that gives the holder the right to buy or sell an asset at a fixed price on or before a specified date. The earliest option pricing model was proposed by Black and Scholes [1] in 1973 and has always attracted the attention of many scholars. He and Zhu [2] deduced the pricing formulas of the European option with stochastic volatility. Wu et al. [3] studied a European option pricing problem under a partial information market. Guo et al. [4] discussed the pricing problem of geometric Asian options under the condition of subdiffusion Brownian motion. Golbabai and Nikan [5] presented the pricing problem of double barrier options when the underlying price change is viewed as a fractal transmission system. Li and Wang [6] studied the valuation of European option bid and asked for prices under the mixed fractional Brownian motion. Zhang and Wang [7] proposed a new bond pricing model and explained its advantages by adding the bond market factor to the original pricing model. Sheybani and Buygi [8] introduced a new model for option pricing by the Black-Scholes option pricing idea and compared it with the equilibrium option pricing model.

As we all know that if the sample data is sufficient, the probability distribution of random phenomena can be established with the probability theory. However, when the sample data is insufficient or the historical data cannot effectively predict the future situation, some experts in related fields should be invited to evaluate the reliability of an event, and then apply the experience of experts to predict the future trend. In order to accurately describe this vague concept, Liu established the uncertainty theory in 2007 [9] and refined it in 2010 [10]. In order to better describe the dynamic changes in uncertain environments, Liu [11] proposed the definition of uncertain processes and uncertain differential equations. Subsequently, Liu [12] introduced a new calculus and applied it to the fields of finance, control, filtering and dynamical systems. In addition, many scholars have further studied uncertain differential equations, see [13–16].

Fractional calculus is very suitable for describing the processes with memory and heredity. A preliminary study on fractional calculus can be found in Oldham and Spanier [17] and Samko et al. [18]. Some references on fractional differential equations can be seen in [19–21]. Zhu [22] introduced the concept of UFDEs, and gave the analytic solutions for some special Riemann-Liouville and Caputo UFDEs. Zhu [23] proved the existence and uniqueness theorem of solutions of UFDEs. Different from the traditional differential operator $\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^p$, Hadamard [24] introduced another differential operator $\left(t\frac{\mathrm{d}}{\mathrm{d}t}\right)^p$ and named it as Hadamard fractional calculus. Gambo et al. [25] and Jarad et al. [26] extended the study of Hadamard fractional calculus to the Caputo-Hadamard environment. Gohar et al. [27] studied the existence and uniqueness for solution of Caputo-Hadamard fractional differential equations. Liu et al. [28] proposed the definition of Caputo-Hadamard UFDEs, and gave the analytic solution of Caputo-Hadamard UFDEs. Subsequently, Liu et al. [29] connected the Caputo-Hadamard UFDE with Caputo-Hadamard fractional differential equation by the concept of α-path.

Based on the uncertainty theory, Liu [30] discussed some applications about uncertain differential equations in financial markets. Since then, many scholars applied different differential equations to simulate the dynamic changes of stock prices in uncertain financial markets. Chen et al. [31] proposed an uncertain stock model with periodic dividends. Gao et al. [32] studied the American barrier option pricing formulas for the currency model in uncertain environments. Jin et al. [33] proposed an uncertain stock model with the Caputo UFDE and studied the American option pricing formulas. Lu et al. [34] established an uncertain stock model with the mean-reverting process, and gave the European option pricing formulas. Lu et al. [35] studied the Asian option pricing formulas with expected and optimistic values, respectively. Peng and Yao [36] proposed a new stock model with a mean-reverting process and gave some option pricing formulas. Sun and Chen [37] proposed an Asian option model suitable for uncertain financial markets.

In the financial market, the dynamic changes of stock price can be affected by many factors. When the ordinary differential equations are used to describe the price fluctuations, it is often necessary to construct extremely complex differential equations. Moreover, some empirical parameters in the differential equations may be inconsistent with the actual situation. However, the Caputo-Hadamard UFDEs has the advantages of simple modeling and clear physical meaning of parameters, thus it becomes an important tool for complex system modeling. Furthermore, the Caputo-Hadamard UFDEs can well describe the memory properties of dynamic systems in uncertain environment, which can well meet the special requirements of dynamic systems with heredity properties.

We apply the Caputo-Hadamard UFDEs to describe the dynamic changes of stock price, and present a new uncertain stock model. Consider the effect of uncertain interference on the bond, a new uncertain stock model is constructed by using the uncertain differential equation and Caputo-Hadamard UFDE to describe the dynamic changes of bond price and stock price, respectively. The composition of this paper is as follows: in Section 2, some basic concepts and lemmas in uncertainty theory and fractional calculus are reviewed. In Section 3, the European option pricing formulas and some numerical examples are given. In Section 4, a new uncertain stock model is constructed, and the validity of the corresponding European option pricing formulas is illustrated through the numerical experiments. The last section gives the conclusion of this paper.

2. Preliminary

In this section, some basic concepts and lemmas of the uncertainty theory will be introduced, such as uncertain measure, uncertain variable, uncertain process. More detailed information can refer to [9, 10, 21, 26, 28, 29].

2.1. Uncertainty theory

Let Γ be a nonempty set and $\mathcal{L}$ be a σ-algebra over Γ. Define an uncertain measure $\mathcal{M}$ on the σ-algebra $\mathcal{L}$. Each element Λ in $\mathcal{L}$ is called an event. The set function $\mathcal{M}$ from $\mathcal{L}$ to $[0, 1]$ is called an uncertain measure which satisfies three axioms: (i) $\mathcal{M}\{\Gamma\} = 1$ for the universal set Γ; (ii) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any event Λ; (iii) $\mathcal{M}\{\bigcup_{i=1}^{\infty} \Lambda_i\} \leq \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}$ for every countable sequence of events $\Lambda_1, \Lambda_2, \cdots$.

The triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space. The product uncertain measure $\mathcal{M}$ was defined by [10], thus producing the fourth axiom of uncertainty theory. Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \cdots$. The product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying $\mathcal{M}\{\prod_{k=1}^{\infty} \Lambda_k\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\}$, where Λ_k are arbitrarily chosen events from $\mathcal{L}_k$ for $k = 1, 2, \cdots$, respectively.

An uncertain variable is a function ξ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers such that $\{\xi \in B\} = \{\gamma \in \Gamma | \xi(\gamma) \in B\}$ is an event for any Borel set B of real numbers. The uncertainty distribution $\Phi(x)$ of an uncertain variable ξ is defined by $\Phi(x) = \mathcal{M}\{\xi \leq x\}$ for any real number x. The expected value of uncertain variable ξ is defined by

$$E[\xi] = \int_0^{+\infty} \mathcal{M}\{\xi \geq r\} \mathrm{d}r - \int_{-\infty}^0 \mathcal{M}\{\xi \leq r\} \mathrm{d}r,$$

which indicated that at least one of the two integrals is finite. If the expected value of uncertain variable ξ exists, then

$$E[\xi] = \int_0^1 \Phi^{-1}(x) \mathrm{d}x,$$

where $\Phi^{-1}(x)$ is the inverse uncertainty distribution of uncertain variable ξ. Correspondingly the variance of uncertain variable ξ is defined by $V[\xi] = E\left[(\xi - E[\xi])^2\right]$. A normal uncertain variable ξ, denoted by $\xi \sim \mathcal{N}(e, \sigma)$ with expected value e and variance σ^2, has the uncertainty distribution

$$\Phi(x) = \left(1 + \exp\left(\frac{\pi(e - x)}{\sqrt{3}\sigma}\right)\right)^{-1}, \quad x \in R.$$

An uncertain process C_t is said to be a Liu process if it satisfies that: (i) $C_0 = 0$ and almost all sample paths are Lipschitz continuous; (ii) C_t has stationary and independent increments; (iii) every increment $C_{s+t} - C_s$ is a normal uncertain variable with expected value 0 and variance t^2, denoted by $C_{s+t} - C_s \sim \mathcal{N}(0, t)$. Furthermore, C_t has the uncertainty distribution

$$\Phi_t(x) = \left(1 + \exp\left(-\frac{\pi x}{\sqrt{3}t}\right)\right)^{-1},$$

and the inverse uncertainty distribution

$$\Phi_t^{-1}(\alpha) = \frac{t\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}.$$

Based on Liu process, the uncertain calculus was proposed. For any partition of closed interval $[a, b]$ with $a = t_1 < t_2 < \cdots < t_{k+1} = b$, the mesh is written as $\Delta = \max_{1\le i\le k} |t_{i+1} - t_i|$. The uncertain integral of X_t with respect to C_t is $\int_a^b X_t \mathrm{d}C_t = \lim_{\Delta\to 0} \sum_{i=1}^k X_{t_i} \cdot (C_{t_{i+1}} - C_{t_i})$, provided that the limit exists almost surely and is finite. In this case, the uncertain process X_t is said to be integrable. Let f and g are two integrable function and C_t is a Liu process. Then $\mathrm{d}X_t = f(t, X_t)\,\mathrm{d}t + g(t, X_t)\,\mathrm{d}C_t$ is called an uncertain differential equation. A solution X_t of the uncertain differential equation is an uncertain process, which is equivalent to a solution of the uncertain integral equation

$$X_t = X_0 + \int_0^t f(s, X_s)\,\mathrm{d}s + \int_0^t g(s, X_s)\,\mathrm{d}C_s.$$

Lemma 2.1 ([10]). *Let $X_{1t}, X_{2t}, \cdots, X_{nt}$ be independent uncertain processes with regular uncertainty distributions $\Phi_{1t}, \Phi_{2t}, \cdots, \Phi_{nt}$, respectively. If $f(x_1, x_2, \cdots, x_n)$ is continuous, strictly increasing with respect to $x_1, x_2, \cdots, x_m$ and strictly decreasing with respect to x_{m+1}, $x_{m+2}, \cdots, x_n$, then $f(X_{1t}, X_{2t}, \cdots, X_{nt})$ has the inverse uncertainty distribution*

$$\Phi_t^{-1}(\alpha) = f\left(\Phi_{1t}^{-1}(\alpha), \cdots, \Phi_{mt}^{-1}(\alpha), \Phi_{m+1,t}^{-1}(1-\alpha), \cdots, \Phi_{nt}^{-1}(1-\alpha)\right).$$

2.2. Fractional calculus

In this subsection, we present some definitions and properties of Hadamard and Caputo-Hadamard fractional integrals and derivatives. Unless otherwise stated in this paper, we always assume that $\delta = t\frac{\mathrm{d}}{\mathrm{d}t}$, the fractional order p be a real number with $0 < n-1 < p \le n$, where n is a positive integer.

Definition 2.1 ([21]). *The Hadamard integral of function $f(t)$ is defined by*

$$\mathcal{J}_{a^+}^p f(t) = \frac{1}{\Gamma(p)} \int_a^t \left(\log \frac{t}{s}\right)^{p-1} f(s) \frac{\mathrm{d}s}{s}, \quad 0 < a < t,$$

where the Gamma function is defined by the integral $\Gamma(p) = \int_0^\infty e^{-t} t^{p-1} \mathrm{d}t$.

Definition 2.2 ([21]). *The Hadamard derivative of function $f(t)$ is defined by*

$${}^H\mathcal{D}_{a^+}^p f(t) = \frac{1}{\Gamma(n-p)} \left(t\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_a^t \left(\log \frac{t}{s}\right)^{n-p-1} f(s) \frac{\mathrm{d}s}{s}, \quad 0 < a < t.$$

Definition 2.3 ([26]). *The Caputo-Hadamard derivative of function $f(t)$ is defined as follows:*
(i) If $p \notin \mathrm{N}^+$, the Caputo-Hadamard derivative can be represented as

$$ {}^{CH}\mathcal{D}^{p}_{a^+} f(t) = \frac{1}{\Gamma(n-p)} \int_a^t \left(\log \frac{t}{s}\right)^{n-p-1} \delta^n f(s) \frac{\mathrm{d}s}{s}, \quad 0 < a < t. $$

(ii) If $p \in \mathrm{N}^+$, then

$$ {}^{CH}\mathcal{D}^{p}_{a^+} f(t) = \delta^n f(t), \quad 0 < a < t. $$

Definition 2.4 ([28]). *Suppose that $f, g : [a, \infty) \times \mathrm{R} \to \mathrm{R}$ are two continuous functions. Then the fractional differential equation with initial conditions*

$$ \begin{cases} {}^{CH}\mathcal{D}^{p}_{a+} X_t = f(t, X_t) + g(t, X_t) \frac{\mathrm{d}C_t}{\mathrm{d}t}, & 0 < a \le t, \\ \delta^k X_t\big|_{t=a} = x_k, \quad k = 0, 1, \cdots, n-1 & \end{cases} \tag{2.1} $$

is called a Caputo-Hadamard UFDE. The solution X_t of (2.1) is an uncertain process such that

$$ X_t = \sum_{k=0}^{n-1} \frac{\left(\log \frac{t}{a}\right)^k}{\Gamma(k+1)} x_k + \frac{1}{\Gamma(p)} \int_a^t \left(\log \frac{t}{s}\right)^{p-1} f(s, X_s) \frac{\mathrm{d}s}{s} + \frac{1}{\Gamma(p)} \int_a^t \left(\log \frac{t}{s}\right)^{p-1} g(s, X_s) \frac{\mathrm{d}C_s}{s}. $$

Lemma 2.2 ([28]). *Assume that the coefficients $f(t, x)$, $g(t, x) : [a, \infty) \times \mathrm{R} \to \mathrm{R}$ in (2.1) satisfy the Lipschitz condition*

$$ |f(t, x) - f(t, z)| + |g(t, x) - g(t, z)| \le L|x - z|, \quad t \ge a > 0 $$

and the linear growth condition

$$ |f(t, x)| + |g(t, x)| \le L(1 + |x|), \quad \forall x \in \mathrm{R}, \quad t \ge a > 0, L \ge 0. $$

Then the Caputo-Hadamard UFDE (2.1) has a unique solution.

Remark 1. *Denote the ${}^{H}\mathcal{D}^{p}_{1+}$, ${}^{CH}\mathcal{D}^{p}_{1+}$ and $\mathcal{J}^{p}_{1+}$ by the abbreviations ${}^{H}\mathcal{D}^{p}$, ${}^{CH}\mathcal{D}^{p}$ and $\mathcal{J}^{p}$, respectively.*

Lemma 2.3 ([28]). *Suppose that a be constant, $b(t)$ and $\sigma(t)$ be two continuous functions on $[1, T]$. The Caputo-Hadamard UFDE with initial conditions*

$$ \begin{cases} {}^{CH}\mathcal{D}^{p} X_t = aX_t + b(t) + \sigma(t) \frac{\mathrm{d}C_t}{\mathrm{d}t}, & t \in [1, T], \\ \delta^k X_t\big|_{t=1} = x_k, \quad k = 0, 1, \cdots, n-1 & \end{cases} $$

has a solution X_t such that

$$ \begin{aligned} X_t = &\sum_{k=0}^{n-1} x_k (\log t)^k E_{p,k+1}\left(a(\log t)^p\right) + \int_1^t \left(\log \frac{t}{s}\right)^{p-1} E_{p,p}\left(a\left(\log \frac{t}{s}\right)^p\right) b(s) \frac{\mathrm{d}s}{s} \\ &+ \int_1^t \left(\log \frac{t}{s}\right)^{p-1} E_{p,p}\left(a\left(\log \frac{t}{s}\right)^p\right) \sigma(s) \frac{\mathrm{d}C_s}{s}, \end{aligned} \tag{2.2} $$

where the Mittag-Leffler function is defined by $E_{p,q}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(pk+q)}$, $\quad t \in \mathrm{C}, p > 0, q > 0$.

Lemma 2.4 ([29]). *The Caputo-Hadamard UFDE (2.1) has an α-path X_t^α which satisfies the following fractional differential equation*

$$\begin{cases} {}^{CH}\mathcal{D}_{a+}^{p} X_t^\alpha = f\left(t, X_t^\alpha\right) + \left|g\left(t, X_t^\alpha\right)\right| \Phi^{-1}(\alpha), & t \geq a > 0, \\ \delta^k X_t\Big|_{t=a} = x_k, \quad k = 0, 1, \cdots, n-1, \end{cases} \tag{2.3}$$

where $\Phi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}$, $\alpha \in (0, 1)$ is the inverse standard normal uncertainty distribution.

Lemma 2.5 ([29]). *Let X_t and X_t^α be unique solution and α-path of the Caputo-Hadamard UFDE (2.1), respectively. Then*

$$\begin{cases} \mathcal{M}\{X_t \leq X_t^\alpha, \forall t \in (a, T)\} = \alpha, \\ \mathcal{M}\{X_t > X_t^\alpha, \forall t \in (a, T)\} = 1 - \alpha. \end{cases} \tag{2.4}$$

Furthermore, the solution X_t of (2.1) has the inverse uncertain distribution $\Phi_t^{-1}(\alpha) = X_t^\alpha$.

3. Uncertain stock model

In this section, we will present a new uncertain stock model by applying the Caputo-Hadamard UFDEs to simulate the dynamic changes of stock price in uncertain financial markets. Based on the proposed uncertain stock model, the pricing formulas of the European call option and the European put option are given. Suppose that X_t is the bond price and Y_t is the stock price at time t. The uncertain stock model with mean-reverting process satisfy the following equations

$$\begin{cases} dX_t = rX_t dt, \\ {}^{CH}\mathcal{D}^p Y_t = (m - aY_t) + \sigma \frac{dC_t}{dt}, \quad t \in [1, T], \\ \delta^k Y_1 = y_k, \quad k = 0, 1, \ldots n-1, \end{cases} \tag{3.1}$$

where r represents the riskless interest rate, r, m, a, σ are some positive numbers.

3.1. European call option

European option is the option that the buyer must exercise on the expiration time of the option. Consider the general uncertain stock model, a European call option is a contract that gives the holder the right to buy a stock at an expiration time T for a strike price K. The stock price at expiration time T would be Y_T, and the profit that the holder get from buying the stock would be $(Y_T - K)^+$. Considering the time value of money resulted from the bond, the present value of the profit is $\exp(-rT)(Y_T - K)^+$. Let f_c is the price of this contract, then the net return of the investors is $(-f_c + \exp(-rT)(Y_T - K)^+)$. The net return of the banks is opposite. The most reasonable pricing is that the investors and banks have the same expected return. Thus $f_c = \exp(-rT)E\left[(Y_T - K)^+\right]$.

Definition 3.1 ([10]). *Assume a European call option has a strike price K and an expiration time T. Then the European call option price based on model (3.1) is*

$$f_c = \exp(-rT)E\left[(Y_T - K)^+\right],$$

where Y_T is the stock price at time T.

Theorem 3.1 (European call option pricing formula)**.** *Assume a European call option for the uncertain stock model (3.1) has a strike price K and an expiration time T. Then the European call option price is*

$$f_c = \exp(-rT) \int_0^1 \left(\sum_{k=0}^{n-1} y_k (\log T)^k E_{p,(k+1)} \left(-a(\log T)^p\right) \right.$$
$$\left. + \left(m + |\sigma| \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha} \right) (\log T)^p \, E_{p,p+1} \left(-a(\log T)^p\right) - K \right)^+ d\alpha. \tag{3.2}$$

Proof. The α-path of (3.1) is the solution of the corresponding Caputo-Hadamard fractional differential equation

$$\begin{cases} {}^{CH}\mathcal{D}^p Y_t^\alpha = (m - aY_t^\alpha) + |\sigma| \Phi^{-1}(\alpha), \quad t \in [1, T], \\ \delta^k Y_1 = y_k, \quad k = 0, 1, \ldots n-1. \end{cases}$$

According to Eq (2.2), we have

$$Y_t^\alpha = \sum_{k=0}^{n-1} y_k (\log t)^k E_{p,(k+1)} \left(-a \left(\log t\right)^p\right) + \left(m + |\sigma| \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha} \right) (\log t)^p E_{p,p+1} \left(-a \left(\log t\right)^p\right).$$

Obviously, $(Y_T - K)^+$ is increasing with respect to Y_T, then $(Y_T - K)^+$ has the inverse uncertainty distribution $\left(Y_T^\alpha - K\right)^+$. According to the definition of the expected value, we get

$$\begin{aligned} f_c &= \exp(-rT) E\left[(Y_T - K)^+\right] \\ &= \exp(-rT) \int_0^1 \left(Y_T^\alpha - K\right)^+ d\alpha \\ &= \exp(-rT) \int_0^1 \left(\sum_{k=0}^{n-1} y_k (\log T)^k E_{p,(k+1)} \left(-a(\log T)^p\right) \right. \\ &\quad \left. + \left(m + |\sigma| \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha} \right) (\log T)^p \, E_{p,p+1} \left(-a(\log T)^p\right) - K \right)^+ d\alpha. \end{aligned}$$

Thus, the pricing formula (3.2) of the European call option is proved. The proof ends. □

Remark 2. *The pricing formula of the European call option can also be given by the method in [34].*

Example 3.1. Suppose that the dynamic change of stock price follow the uncertain stock model (3.1), the current stock price is $y_0 = 30$, the instantaneous growth rate is $y_1 = 2$, and the riskless interest rate is $r = 2.68\%$ per annum. In addition, let $T = 3$, $m = 0.1$, $a = 0.06$, and $\sigma = 7.5$, the strike price $K = 31$. According to the European call option pricing formula (3.2), the price f_c of the European call option with different fractional order p $(0 < p \leq 2)$ can be effectively calculated, as shown in Table 1 and Figure 1.

Table 1. The price of the European call option with different fractional order p.

p	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
f_c	1.5957	1.6824	1.7502	1.7988	1.8285	1.8398	1.8333	1.8102	1.7719	1.7199
p	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
f_c	2.4485	2.3772	2.2976	2.2118	2.1214	2.0283	1.9340	1.8399	1.7472	1.6572

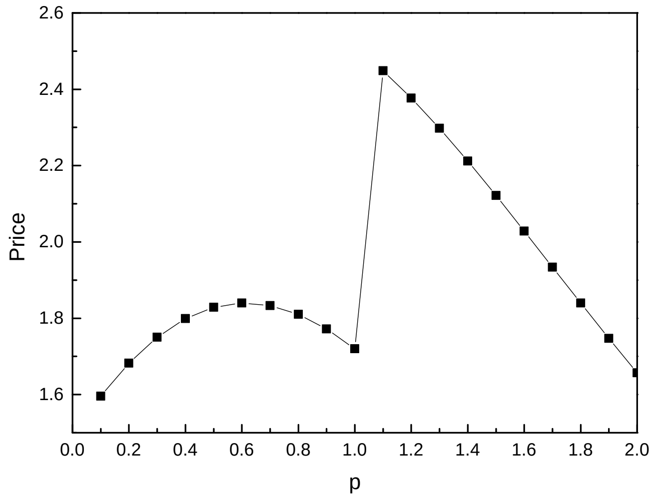

Figure 1. The price of the European call option with different fractional order p.

As can be seen from Table 1 and Figure 1, the price of the European call option decreases monotonically in the interval $[0.7, 1.0]$ and $[1.1, 2.0]$, and increases monotonically in the interval $[0.1, 0.6]$. The price of the European call option decreases faster in the interval $[1.1, 2.0]$ than the interval $[0.7, 1.0]$. This result is consistent with the variation of European call option price with fractional order p in the pricing formula (3.2).When the fractional order p changes from 1 to 1.1, the integer order differential equation is transformed into the fractional differential equation to simulate the dynamic change of stock price in the uncertain stock model (3.1). At this time, the dynamic system is endowed with the property of memory, and the stock price is affected by the historical price in obvious price fluctuation. From a mathematical point of view, since the influence of initial condition y_1, the price of the European call option will increase significantly.

3.2. European put option

A European put option is a contract that gives the holder the right to sell a stock at an expiration time T for a strike price K. The stock price at expiration time T would be Y_T, and the profit that the holder get from buying the stock would be $(K - Y_T)^+$. Considering the time value of money resulted from the bond, the present value of the profit is $\exp(-rT)\,(K - Y_T)^+$. Let f_p is the price of this contract, then the net return of the investors is $\left(-f_p + \exp(-rT)\,(K - Y_T)^+\right)$. The net return of the banks is opposite. The most reasonable pricing is that the investors and banks have the same expected return. Thus $f_p = \exp(-rT)E\left[(K - Y_T)^+\right]$.

Definition 3.2 ([10])**.** *Assume a European put option has a strike price K and an expiration time T. Then the European put option price based on model (3.1) is*

$$f_p = \exp(-rT)E\left[(K - Y_T)^+\right],$$

where Y_T is the stock price at time T.

Theorem 3.2 (European put option pricing formula)**.** *Assume a European put option for the uncertain* *stock model (3.1) has a strike price K and an expiration time T. Then the European put option price is*

$$f_p = \exp(-rT)\int_0^1 \Bigg(K - \sum_{k=0}^{n-1} y_k(\log T)^k E_{p,(k+1)}\left(-a(\log T)^p\right) - \left(m + |\sigma|\frac{\sqrt{3}}{\pi}\ln\frac{1-\alpha}{\alpha}\right)(\log T)^p\, E_{p,p+1}\left(-a(\log T)^p\right)\Bigg)^+ d\alpha. \tag{3.3}$$

Proof. The α-path of (3.1) is the solution of the corresponding Caputo-Hadamard fractional differential equation

$$\begin{cases} {}^{CH}\mathcal{D}^p Y_t^\alpha = (m - aY_t^\alpha) + |\sigma|\Phi^{-1}(\alpha), \quad t \in [1, T], \\ \delta^k Y_1 = y_k, \quad k = 0, 1, \dots n-1. \end{cases}$$

According to Eq (2.2), we have

$$Y_t^\alpha = \sum_{k=0}^{n-1} y_k(\log t)^k E_{p,(k+1)}\left(-a\,(\log t)^p\right) + \left(m + |\sigma|\frac{\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right)(\log t)^p E_{p,p+1}\left(-a\,(\log t)^p\right).$$

Obviously, $(K - Y_T)^+$ is decreasing with respect to Y_T, then $(K - Y_T)^+$ has the inverse uncertainty distribution $\left(K - Y_T^{1-\alpha}\right)^+$. According to the definition of the expected value, we get

$$\begin{aligned} f_p &= \exp(-rT)E\left[(K - Y_T)^+\right] \\ &= \exp(-rT)\int_0^1 \left(K - Y_T^{1-\alpha}\right)^+ d\alpha \\ &= \exp(-rT)\int_0^1 \Bigg(K - \sum_{k=0}^{n-1} y_k(\log T)^k E_{p,(k+1)}\left(-a(\log T)^p\right) \\ &\quad - \left(m + |\sigma|\frac{\sqrt{3}}{\pi}\ln\frac{1-\alpha}{\alpha}\right)(\log T)^p\, E_{p,p+1}\left(-a(\log T)^p\right)\Bigg)^+ d\alpha. \end{aligned}$$

Thus, the pricing formula (3.3) of the European put option is proved. The proof ends. □

Remark 3. *The pricing formula of the European put option can also be given by the method in [34].*

Example 3.2. Suppose that the dynamic change of stock price follow the uncertain stock model (3.1), the current stock price is $y_0 = 30$, the instantaneous growth rate is $y_1 = -1$, and the riskless interest rate is $r = 2.68\%$ per annum. In addition, let $T = 3$, $m = 0.1$, $a = 0.06$, and $\sigma = 7.5$, the strike price $K = 29$. According to the European put option pricing formula (3.3), the price f_p of the European put option with different fractional order p $(0 < p \le 2)$ can be effectively calculated, as shown in Table 2 and Figure 2.

Table 2. The price of the European put option with different fractional order p.

p	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
f_p	2.9716	3.1275	3.2489	3.3359	3.3889	3.4089	3.3974	3.3563	3.2878	3.1948
p	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
f_p	3.6439	3.5115	3.3636	3.2036	3.0345	2.8595	2.6815	2.5029	2.3259	2.1526

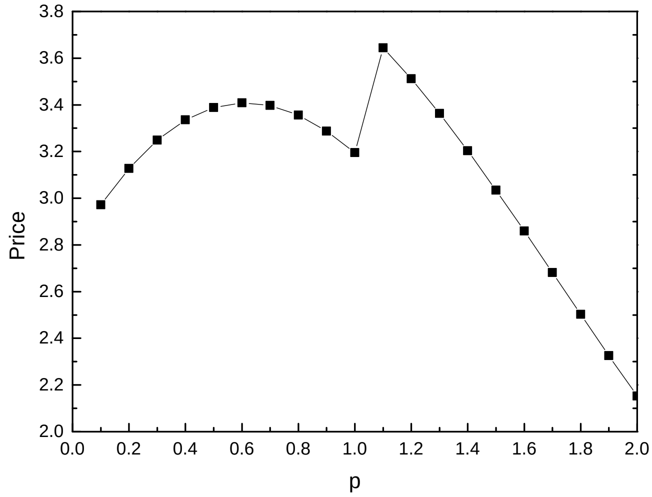

Figure 2. The price of the European put option with different fractional order p.

As can be seen from Table 2 and Figure 2, the price of the European put option decreases monotonically in the interval $[0.7, 1.0]$ and $[1.1, 2.0]$, and increases monotonically in the interval $[0.1, 0.6]$. The price of the European put option decreases faster in the interval $[1.1, 2.0]$ than the interval $[0.7, 1.0]$. This result is consistent with the variation of European put option price with fractional order p in the pricing formula (3.3). When the fractional order p changes from 1 to 1.1, the integer order differential equation is transformed into the fractional differential equation to simulate the dynamic change of stock price in the uncertain stock model (3.1). At this time, the dynamic system is endowed with the property of memory, and the stock price is affected by the historical price in obvious price fluctuation. From a mathematical point of view, since the influence of initial condition y_1, the price of the European put option will increase significantly.

4. The effect of uncertain interference on the bond

In this section, we mainly consider the effect of uncertain factors on the bond. The change of bond price in uncertain financial market will result in the corresponding change in the time value of money generated. Based on the ordinary differential equation $dX_t = rX_tdt$, we will add the uncertain interference term as dC_t. That is, the bond price X_t follows the uncertain differential equation

$$dX_t = rX_tdt + sX_tdC_t.$$

When the time value of the money generated by the bond changes, we need to reformulate the European option pricing formulas. On the basis of the uncertain stock model (3.1), a new uncertain stock model involving the bond price X_t and the stock price Y_t is introduced. Let X_t and Y_t satisfy the following equations

$$\begin{cases} dX_t = rX_tdt + sX_tdC_{1t}, \\ {}^{CH}\mathcal{D}^pY_t = (m - aY_t) + \sigma\frac{dC_{2t}}{dt}, \quad t \in [1, T], \\ \delta^kY_1 = y_k, \quad k = 0, 1, \ldots n - 1, \end{cases} \tag{4.1}$$

where r, s, m, a, σ are some positive numbers, C_{1t} and C_{2t} are two independent Liu processes.

4.1. European call option

Since the bond price X_t and the stock price Y_t are described by different differential equations driven by two independent Liu processes, respectively. Then the bond price X_t and the stock price Y_t are two independent uncertain processes. Take into account the uncertainty of the money value over time, thus the expected value of the time value of money generated by the bond is used to the present value return. Then the present value of return can be given by $E\left[\exp(-rT - sC_{1T})\,(Y_T - K)^+\right]$. Let f_c is the price of this contract. Then the net return of the investors is $(-f_c + E\left[\exp(-rT - sC_{1T})\,(Y_T - K)^+\right])$. The net return of the banks is opposite. The most reasonable pricing is that the investors and banks have the same expected return. Thus $f_c = E\left[\exp(-rT - sC_{1T})\,(Y_T - K)^+\right]$.

Definition 4.1. *Assume a European call option has a strike price K and an expiration time T. Then the European call option price based on model (4.1) is*

$$f_c = E\left[\exp(-rT - sC_{1T})\,(Y_T - K)^+\right],$$

where Y_T is the stock price at time T.

Theorem 4.1 (European call option pricing formula)**.** *Assume a European call option for the uncertain stock model (4.1) has a strike price K and an expiration time T. Then the European call option price is*

$$f_c = \int_0^1 \left[\exp\left(-rT - \frac{sT\sqrt{3}}{\pi}\ln\frac{1-\alpha}{\alpha}\right)\right]\left[\omega + \theta\frac{\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right]^+ d\alpha, \tag{4.2}$$

where

$$\omega = \sum_{k=0}^{n-1} y_k(\log T)^k E_{p,(k+1)}\left(-a(\log T)^p\right) + m(\log T)^p E_{p,p+1}\left(-a(\log T)^p\right) - K,$$
$$\theta = |\sigma|(\log T)^p E_{p,p+1}\left(-a(\log T)^p\right).$$

Proof. Note that the inverse uncertainty distribution of Liu process C_{1t} is

$$\Phi_{1t}^{-1}(\alpha) = \frac{t\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}.$$

Then $\exp(-rt - sC_{1t})$ has an inverse uncertainty distribution

$$\Psi_{1t}^{-1}(\alpha) = \exp\left(-rt - \frac{st\sqrt{3}}{\pi}\ln\frac{1-\alpha}{\alpha}\right).$$

The solution of (4.1) can be given by

$$Y_t = \sum_{k=0}^{n-1} y_k\,(\log t)^k\, E_{p,(k+1)}\left(-a\,(\log t)^p\right) + m\,(\log t)^p\, E_{p,p+1}\left(-a\,(\log t)^p\right)$$
$$+\,\sigma\int_1^t \left(\log\frac{t}{s}\right)^{p-1} E_{p,p}\left(-a\left(\log\frac{t}{s}\right)^p\right)\frac{\mathrm{d}C_{2s}}{s}.$$

Then $(Y_T - K)^+$ has an inverse uncertainty distribution is

$$\Psi_{2T}^{-1}(\alpha) = \left[\sum_{k=0}^{n-1} y_k(\log T)^k E_{p,(k+1)}\left(-a(\log T)^p\right) + m(\log T)^p E_{p,p+1}\left(-a(\log T)^p\right)\right.$$
$$\left. + |\sigma|(\log T)^p E_{p,p+1}\left(-a(\log T)^p\right) \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha} - K\right]^+ .$$

It follows from Lemma 2.1 that $\exp(-rT - sC_{1T})\,(Y_T - K)^+$ has an inverse uncertainty distribution is

$$\Psi_T^{-1}(\alpha) = \left[\exp\left(-rT - \frac{sT\sqrt{3}}{\pi}\ln\frac{1-\alpha}{\alpha}\right)\right]\left[\sum_{k=0}^{n-1} y_k(\log T)^k E_{p,(k+1)}\left(-a(\log T)^p\right)\right.$$
$$\left. + m(\log T)^p E_{p,p+1}\left(-a(\log T)^p\right) + |\sigma|(\log T)^p E_{p,p+1}\left(-a(\log T)^p\right) \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha} - K\right]^+ .$$

According to the definition of the expected value, we have

$$\begin{aligned} f_c &= E\left[\exp(-rT - sC_{1T})\,(Y_T - K)^+\right] \\ &= \int_0^1 \left[\exp\left(-rT - \frac{sT\sqrt{3}}{\pi}\ln\frac{1-\alpha}{\alpha}\right)\right]\left[\omega + \theta\frac{\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right]^+ d\alpha, \end{aligned}$$

where

$$\begin{aligned} \omega &= \sum_{k=0}^{n-1} y_k(\log T)^k E_{p,(k+1)}\left(-a(\log T)^p\right) + m(\log T)^p E_{p,p+1}\left(-a(\log T)^p\right) - K, \\ \theta &= |\sigma|(\log T)^p E_{p,p+1}\left(-a(\log T)^p\right). \end{aligned}$$

Thus, the pricing formula (4.2) of the European call option is proved. The proof ends. □

Example 4.1. Suppose that the dynamic change of stock price follow the uncertain stock model (4.1), which the parameters as follows: $y_0 = 30$, $y_1 = 2$, $r = 0.0268$, $s = 0.015$, $T = 3$, $m = 0.1$, $a = 0.06$, $\sigma = 7.5$, $K = 31$. According to the European call option pricing formula (4.2), the price of the European call option with different fractional order p ($0 < p \leq 2$) can be effectively calculated, as shown in Table 3 and Figure 3.

Table 3. The price of the European call option with different fractional order p.

p	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
f_c	1.7144	1.8074	1.8800	1.9321	1.9639	1.9759	1.9690	1.9443	1.9033	1.8476
p	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
f_c	2.6037	2.5279	2.4433	2.3517	2.2550	2.1550	2.0532	1.9512	1.8500	1.7509

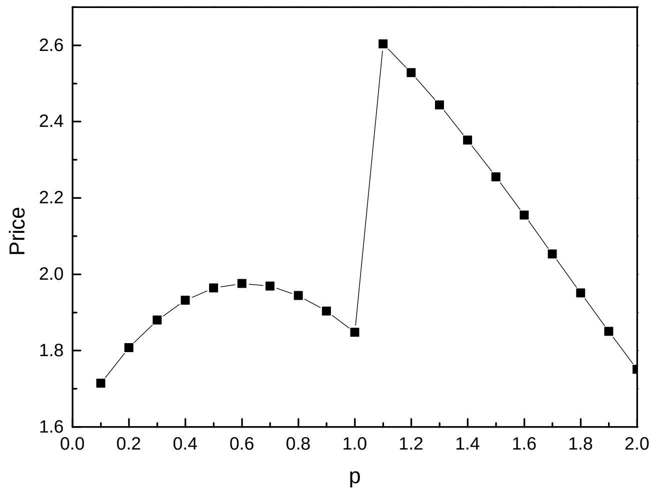

Figure 3. The price of the European call option with different fractional order p.

As can be seen from Table 3 and Figure 3, the change trend of the European call option price is similar to that in Example 3.1. The price of the European call option decreases monotonically in the interval $[0.7, 1.0]$ and $[1.1, 2.0]$, and increases monotonically in the interval $[0.1, 0.6]$. The price of the European call option decreases faster in the interval $[1.1, 2.0]$ than the interval $[0.7, 1.0]$. This result is consistent with the variation of European call option price with fractional order p in the pricing formula (4.2).When the fractional order p changes from 1 to 1.1, the integer order differential equation is transformed into the fractional differential equation to simulate the dynamic change of stock price in the uncertain stock model (4.1). At this time, the dynamic system is endowed with the property of memory, and the stock price is affected by the historical price in obvious price fluctuation. From a mathematical point of view, since the influence of initial condition y_1, the price of the European call option will increase significantly.

After adding the uncertain interference term to the bond price, a new stock model is proposed by applying different differential equations to describe the dynamic changes of the bond price and the stock price, respectively. The pricing formula of the European call option can be given by two different uncertain stock models (3.1) and (4.1), respectively. When the uncertain factors occur, they will inevitably have an effect on the normal economy. Uncertain interferences usually have a negative effect on the bond price, and the bond prices usually depreciate after being disrupted by the uncertain interference. Thus, the price of the European call option is higher by the uncertain stock model (4.1) than the uncertain stock model (3.1), which is in line with the actual situation. The price of the European call options predicted by two uncertain stock models have similar dynamic trends for different fractional order. Furthermore, it also shows that the influence of the uncertain factors on the bond price will not affect the change trend for the price of the European call option.

4.2. European put option

Since the bond price X_t and the stock price Y_t are described by different differential equations driven by two independent Liu processes, respectively. Then the bond price X_t and the stock price Y_t are two independent uncertain processes. Take into account the uncertainty of the money value over time, thus the expected value of the time value of money generated by the bond is used to the present value return.

Then the present value of return can be given by $E\left[\exp(-rT - sC_{1T})(K - Y_T)^+\right]$. Let f_p is the price of this contract. Then the net return of the investors is $\left(-f_p + E\left[\exp(-rT - sC_{1T})(K - Y_T)^+\right]\right)$. The net return of the banks is opposite. The most reasonable pricing is that the investors and banks have the same expected return. Thus $f_p = E\left[\exp(-rT - sC_{1T})(K - Y_T)^+\right]$.

Definition 4.2. *Assume a European put option has a strike price K and an expiration time T. Then the European put option price based on model (4.1) is*

$$f_p = E\left[\exp(-rT - sC_{1T})(K - Y_T)^+\right],$$

where Y_T is the stock price at time T.

Theorem 4.2 (European put option pricing formula)**.** *Assume a European put option for the uncertain stock model (4.1) has a strike price K and an expiration time T. Then the European put option price is*

$$f_p = \int_0^1 \left[\exp\left(-rT - \frac{sT\sqrt{3}}{\pi}\ln\frac{1-\alpha}{\alpha}\right)\right]\left[\zeta - \theta\frac{\sqrt{3}}{\pi}\ln\frac{1-\alpha}{\alpha}\right]^+ d\alpha, \tag{4.3}$$

where

$$\zeta = K - \sum_{k=0}^{n-1} y_k(\log T)^k E_{p,(k+1)}\left(-a(\log T)^p\right) - m(\log T)^p E_{p,p+1}\left(-a(\log T)^p\right),$$
$$\theta = |\sigma|(\log T)^p E_{p,p+1}\left(-a(\log T)^p\right).$$

Proof. It follows from Lemma 2.1 that $\exp(-rt - sC_{1t})$ has an inverse uncertainty distribution

$$\Psi_{1t}^{-1}(\alpha) = \exp\left(-rt - \frac{st\sqrt{3}}{\pi}\ln\frac{1-\alpha}{\alpha}\right),$$

and $(K - Y_T)^+$ has an inverse uncertainty distribution is

$$\Psi_{2T}^{-1}(\alpha) = \Bigg[K - \sum_{k=0}^{n-1} y_k(\log T)^k E_{p,(k+1)}\left(-a(\log T)^p\right) - m(\log T)^p E_{p,p+1}\left(-a(\log T)^p\right)$$
$$-|\sigma|(\log T)^p E_{p,p+1}\left(-a(\log T)^p\right)\frac{\sqrt{3}}{\pi}\ln\frac{1-\alpha}{\alpha}\Bigg]^+.$$

It follows from Lemma 2.1 that $\exp(-rT - sC_{1T})(K - Y_T)^+$ has an inverse uncertainty distribution is

$$\Psi_T^{-1}(\alpha) = \left[\exp\left(-rT - \frac{sT\sqrt{3}}{\pi}\ln\frac{1-\alpha}{\alpha}\right)\right]\Bigg[K - \sum_{k=0}^{n-1} y_k(\log T)^k E_{p,(k+1)}\left(-a(\log T)^p\right)$$
$$-m(\log T)^p E_{p,p+1}\left(-a(\log T)^p\right) - |\sigma|(\log T)^p E_{p,p+1}\left(-a(\log T)^p\right)\frac{\sqrt{3}}{\pi}\ln\frac{1-\alpha}{\alpha}\Bigg]^+.$$

According to the definition of the expected value, we have

$$\begin{aligned} f_c &= E\left[\exp(-rT - sC_{1T})(Y_T - K)^+\right] \\ &= \int_0^1 \left[\exp\left(-rT - \frac{sT\sqrt{3}}{\pi}\ln\frac{1-\alpha}{\alpha}\right)\right]\left[\zeta - \theta\frac{\sqrt{3}}{\pi}\ln\frac{1-\alpha}{\alpha}\right]^+ d\alpha, \end{aligned}$$

where

$$\zeta = K - \sum_{k=0}^{n-1} y_k(\log T)^k E_{p,(k+1)}\left(-a(\log T)^p\right) - m(\log T)^p E_{p,p+1}\left(-a(\log T)^p\right),$$

$$\theta = |\sigma|(\log T)^p E_{p,p+1}\left(-a(\log T)^p\right).$$

Thus, the pricing formula (4.3) of the European put option is proved. The proof ends. □

Example 4.2. Suppose that the dynamic change of stock price follow the uncertain stock model (4.1), which the parameters as follows: $y_0 = 30$, $y_1 = -1$, $r = 0.0268$, $s = 0.015$, $T = 3$, $m = 0.1$, $a = 0.06$, $\sigma = 7.5$, $K = 29$. According to the European put option pricing formula (4.3), the price of the European put option with different fractional order p ($0 < p \leq 2$) can be effectively calculated, as shown in Table 4 and Figure 4.

Table 4. The price of the European put option with different fractional order p.

p	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
f_p	3.1487	3.3151	3.4450	3.5383	3.5952	3.6167	3.6044	3.5602	3.4868	3.3871
p	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
f_p	3.8470	3.7056	3.5482	3.3781	3.1990	3.0139	2.8258	2.6372	2.4506	2.2678

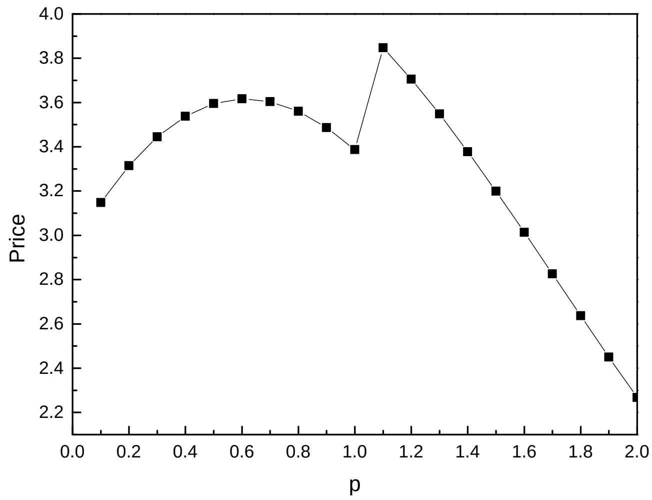

Figure 4. The price of the European put option with different fractional order p.

As can be seen from Table 4 and Figure 4, the change trend of the European put option price is similar to that in Example 3.2. The price of the European put option decreases monotonically in the interval $[0.7, 1.0]$ and $[1.1, 2.0]$, and increases monotonically in the interval $[0.1, 0.6]$. The price of the European put option decreases faster in the interval $[1.1, 2.0]$ than the interval $[0.7, 1.0]$. This result is consistent with the variation of European put option price with fractional order p in the pricing formula (4.3). When the fractional order p changes from 1 to 1.1, the integer order differential equation is transformed into the fractional differential equation to simulate the dynamic change of stock price in the uncertain stock model (4.1). At this time, the dynamic system is endowed with the property of memory, and the stock price is affected by the historical price in obvious price fluctuation. From a mathematical point of view, since the influence of initial condition y_1, the price of the European put option will increase significantly.

After adding the uncertain interference term to the bond price, a new stock model is proposed by applying different differential equations to describe the dynamic changes of the bond price and the stock price, respectively. The pricing formula of the European put option can be given by two different uncertain stock models (3.1) and (4.1), respectively. When the uncertain factors occur, they will inevitably have an effect on the normal economy. Uncertain interferences usually have a negative effect on the bond price, and the bond prices usually depreciate after being disrupted by the uncertain interference. Thus, the price of the European put option is higher by the uncertain stock model (4.1) than the uncertain stock model (3.1), which is in line with the actual situation. The price of the European put options predicted by two uncertain stock models have similar dynamic trends for different fractional order. Furthermore, it also shows that the influence of the uncertain factors on the bond price will not affect the change trend for the price of the European put option.

5. Conclusions

In this paper, we investigate the European option pricing problem based on Caputo-Hadamard UFDE in an uncertain environment, and give the European call option pricing formula and the European put option pricing formula. The dynamic changes of the European option price for the different fractional orders p is illustrated by the numerical experiments. Subsequently, the effect of uncertain interference on the bond price is studied. At the same time, the uncertain differential equation and Caputo-Hadamard UFDEs are used to simulate the dynamic change of the bond price and the stock price, respectively, a new uncertain stock model with mean-reverting is constructed. After adding the uncertain interference on the bond price, the pricing formulas of the European call option and the European put option are given. In order to illustrate the validity of the pricing formulas, the price of the European call option and the European put option for the different fractional orders p are calculated, respectively. Through the comparison of the European option price for two uncertain stock model, the price of the European option is usually higher when the bond price be disturbed by the uncertain interference. The change trend of stock price is not usually affected by the time value of money generated by the bond. In future work, we will continue to study the option pricing problem through different stock models.

Conflict of interest

We declare that we have no financial and personal relationships with other people or organizations that can inappropriately influence our work.

References

1. F. Black, M. Scholes, The pricing of options and corporate liabilities, *J. Polit. Econ.*, **81** (1973), 637–654. https://doi.org/10.1086/260062

2. X. J. He, S. P. Zhu, Pricing European options with stochastic volatility under the minimal entropy martingale measure, *Eur. J. Appl. Math.*, **27** (2016), 233–247. https://doi.org/10.1017/S0956792515000510
3. M. Wu, J. Lu, N. Huang, On European option pricing under partial information, *Appl. Math.*, **61** (2016), 61–77. https://doi.org/10.1007/s10492-016-0122-1
4. Z. Guo, X. Wang, Y. Zhang, Option pricing of geometric Asian options in a subdiffusive Brownian motion regime, *AIMS Math.*, **5** (2020), 5332–5343. https://doi.org/10.3934/math.2020342
5. A. Golbabai, O. Nikan, A computational method based on the moving least-squares approach for pricing double barrier options in a time-fractional Black-Scholes model, *Comput. Econ.*, **55** (2020), 119–141. https://doi.org/10.1007/s10614-019-09880-4
6. Z. Li, X. T. Wang, Valuation of bid and ask prices for European options under mixed fractional Brownian motion, *AIMS Math.*, **6** (2021), 7199–7214. https://doi.org/10.3934/math.2021422
7. H. Zhang, G. Wang, Reversal effect and corporate bond pricing in China, *Pac.-Basin Finance J.*, **70** (2021), 101664. https://doi.org/10.1016/j.pacfin.2021.101664
8. H. R. Sheybani, M. O. Buygi, Equilibrium-based Black-Scholes option pricing in electricity markets, *IEEE Syst. J.*, **16** (2022), 5413–5423. https://doi.org/10.1109/JSYST.2021.3131938
9. B. Liu, *Uncertainty theory*, 2 Eds., Berlin: Springer-Verlag, 2007. https://doi.org/10.1007/978-3-540-73165-8
10. B. Liu, *Uncertainty theory: a branch of mathematics for modeling human uncertainty*, Berlin: Springer-Verlag, 2010. https://doi.org/10.1007/978-3-642-13959-8
11. B. Liu, Fuzzy process, hybrid process and uncertain process, *J. Uncertain Syst.*, **2** (2008), 3–16.
12. B. Liu, Some research problems in uncertainy theory, *J. Uncertain Syst.*, **3** (2009), 3–10.
13. X. Chen, B. Liu, Existence and uniqueness theorem for uncertain differential equation, *Fuzzy. Optim. Decis. Making*, **9** (2010), 69–81. https://doi.org/10.1007/s10700-010-9073-2
14. X. Chen, D. A. Ralescu, Liu process and uncertain calculus, *J. Uncertain Anal. Appl.*, **1** (2013), 1–12. https://doi.org/10.1186/2195-5468-1-3
15. B. Liu, Extreme value theorems of uncertain process with application to insurance risk model, *Soft Comput.*, **17** (2013), 549–556. https://doi.org/10.1007/s00500-012-0930-5
16. K. Yao, Extreme values and integral of solution of uncertain differential equation. *J. Uncertain Anal. Appl.*, **1** (2013), 1–21. https://doi.org/10.1186/2195-5468-1-2
17. K. B. Oldham, J. Spanier, *The fractional calculus*, New York: Academic Press, 1974.
18. S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional integrals and derivatives, theory and applications*, New York: Gordon and Breach Science Publishers, 1993.
19. K. S. Miller, B. Ross, *An introduction to the fractional calculus and differential equations*, New York: Wiley, 1993.
20. I. Podlubny, *Fractional differential equation*, San Diego: Academic Press, 1999.
21. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, Amsterdam: Elsevier Science, 2006.
22. Y. Zhu, Uncertain fractional differential equations and an interest rate model, *Math. Method. Appl. Sci.*, **38** (2015), 3359–3368. https://doi.org/10.1002/mma.3335

23. Y. Zhu, Existence and uniqueness of the solution to uncertain fractional differential equation, *J. Uncertain Anal. Appl.*, **3** (2015), 1–11. https://doi.org/10.1186/s40467-015-0028-6
24. J. Hadamard, *Essai sur l'etude des fonctions donnees par leur developpment de Taylor*, Gauthier-Villars, 1892.
25. Y. Y. Gambo, F. Jarad, D. Baleanu, T. Abdeljawad, On Caputo modification of the Hadamard fractional derivatives. *Adv. Differ. Equ.*, **2014** (2014), 1–12. https://doi.org/10.1186/1687-1847-2014-10
26. F. Jarad, T. Abdeljawad, D. Baleanu, Caputo-type modification of the Hadamard fractional derivatives, *Adv. Differ. Equ.*, **2012** (2012), 1–8. https://doi.org/10.1186/1687-1847-2012-142
27. M. Gohar, C. Li, C. Yin, On Caputo-Hadamard fractional differential equations, *Int. J. Comput. Math.*, **97** (2020), 1459–1483. https://doi.org/10.1080/00207160.2019.1626012
28. Y. Liu, Y. Zhu, Z. Lu, On Caputo-Hadamard uncertain fractional differential equations, *Chaos, Soliton. Fract.*, **146** (2021), 110894. https://doi.org/10.1016/j.chaos.2021.110894
29. Y. Liu, H. Liu, Y. Zhu, An approach for numerical solutions of Caputo-Hadamard uncertain fractional differential equations, *Fractal Fract.*, **6** (2022), 1–14. https://doi.org/10.3390/fractalfract6120693
30. B. Liu, Toward uncertain finance theory, *J. Uncertain Anal. Appl.*, **1** (2013), 1–15. https://doi.org/10.1186/2195-5468-1-1
31. X. Chen, Y. Liu, D. A. Ralescu, Uncertain stock model with periodic dividends, *Fuzzy Optim. Decis. Making*, **12** (2013), 111–123. https://doi.org/10.1007/s10700-012-9141-x
32. R. Gao, K. Liu, Z. Li, L. Lang, American barrier option pricing formulas for currency model in uncertain environment, *J. Syst. Sci. Complex.*, **35** (2022), 283–312. https://doi.org/10.1007/s11424-021-0039-y
33. T. Jin, Y. Sun, Y. Zhu, Extreme values for solution to uncertain fractional differential equation and application to American option pricing model, *Phys. A*, **534** (2019), 122357. https://doi.org/10.1016/j.physa.2019.122357
34. Z. Lu, H. Yan, Y. Zhu, European option pricing model based on uncertain fractional differential equation, *Fuzzy Optim. Decis. Making*, **18** (2019), 199–217. https://doi.org/10.1007/s10700-018-9293-4
35. Z. Lu, Y. Zhu, B. Li, Critical value-based Asian option pricing model for uncertain financial markets, *Phys. A*, **525** (2019), 694–703. https://doi.org/10.1016/j.physa.2019.04.022
36. J. Peng, K. Yao, A new option pricing model for stocks in uncertainty markets, *Int. J. Oper. Res.*, **8** (2011), 18–26.
37. J. Sun, X. Chen, Asian option pricing formula for uncertain financial market, *J. Uncertain Anal. Appl.*, **3** (2015), 1–11. https://doi.org/10.1186/s40467-015-0035-7