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Research article

Exponential inequalities and a strong law of large numbers for END random variables under sub-linear expectations

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Abstract: The focus of our work is to investigate exponential inequalities for extended negatively dependent (END) random variables in sub-linear expectations. Through these exponential inequalities, we were able to establish the strong law of large numbers with convergence rate $O\left(n^{-1/2}\ln^{1/2} n\right)$. Our findings in sub-linear expectation spaces have extended the corresponding results previously established in probability space.

Keywords: sub-linear expectation; exponential inequality; strong law of large numbers; convergence rate; END random variables
Mathematics Subject Classification: 60F15

1. Introduction

Uncertainty presents a primary challenge for financial and risk research institutes, which has led to the classical probabilistic space theory being greatly challenged in this field. The tools of probability and expectation additivity in probability space lose their effectiveness in nonlinear financial and risk studies. As a solution to this problem of non-additivity, Peng [1–3] proposed a complete theoretical framework and axiom system of sub-linear expectation space. This proposal has garnered the attention of numerous scholars, who are eagerly studying and researching this topic. As a result, a series of new theories in sub-linear expectation spaces have been continuously proven. For instance, Zhang [4–7] has established the exponential inequality, Rosenthal's type inequality, Kolmogorov's type strong law of large numbers, strong limit theorems, and the application of the law of iterated logarithm under sub-linear expectations. Wu and Jiang [8] have also proven the strong law of numbers and Chover's law of the iterated logarithm under sub-linear expectations.

Exponential inequalities play a crucial role in the proof of strong limit theorems and provide a useful tool for establishing the convergence rate of the strong law of numbers. All kinds of exponential inequality theorems have been continuously proven in probabilistic space, such as those by Kim and Kim [9], Nooghabi and Azarnoosh [10], Xing et al. [11], Sung [12], Christofides and Hadjikyriakou [13], and Wang et al. [14]. The sub-linear expectation framework provides a good solution to the non-additive probability problem and extends many properties of probability spaces to sub-linear expectation spaces. Based on this theory, this paper establishes exponential inequalities and a strong law of large numbers with a convergence rate $O\left(n^{-1/2}\ln^{1/2}n\right)$ for unbounded END random variable sequences under sub-linear expectations. As a result, the corresponding results obtained by Wang et al. [14] have been generalized to the sub-linear expectation space context. Similarly, Tang et al. [15] have also established exponential inequalities for extended independent random variables. However, the range of the extended negatively dependent random variables that we studied is wider than that of Tang et al. [15] extended independent random variables. Furthermore, we have obtained a conclusion that they do not include each other under weaker conditions, as shown in literature [15]. We expect that our results may be applied to some practical inverse problems where randomness plays an important role; see e.g., [16–21].

The remainder of the paper is organized as follows: In Section 2, we briefly introduce the conceptual framework and properties under sub-linear expectations, as well as the necessary definitions and lemmas required for this paper. Section 3 establishes the exponential inequalities and a strong law of large numbers for unbounded END random variable sequences under sub-linear expectations. In Section 4, we provide the proof of the main results of Section 3. Finally, Section 5 presents the conclusion.

2. Preliminaries

We use the framework and notations of Peng [1–3]. Let $(\Omega, \mathcal{F})$ be a given measurable space and let $\mathcal{H}$ be a linear space of real functions defined on $(\Omega, \mathcal{F})$ such that if $X_1, \ldots, X_n \in \mathcal{H}$ then $\varphi(X_1, \ldots, X_n) \in \mathcal{H}$ for each $\varphi \in C_{l,Lip}(\mathbb{R}_n)$, where $C_{l,Lip}(\mathbb{R}_n)$ denotes the linear space of (local Lipschitz) functions φ satisfying

$$|\varphi(\mathbf{x}) - \varphi(\mathbf{y})| \leq C(1 + |\mathbf{x}|^m + |\mathbf{y}|^m)|\mathbf{x} - \mathbf{y}|, \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}_n,$$

for some $C > 0, m \in \mathbb{N}$ depending on φ. $\mathcal{H}$ is considered as a space of "random variables". In this case we denote $X \in \mathcal{H}$ is considered as a space of "random variables". We also denote $C_{b,Lip}(\mathbb{R}_n)$ to be the bounded Lipschitz functions $\varphi(x)$ satisfying

$$|\varphi(\mathbf{x})| \leq C, \ |\varphi(\mathbf{x}) - \varphi(\mathbf{y})| \leq C|\mathbf{x} - \mathbf{y}|, \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}_n,$$

for some $C > 0$, depending on φ.

Definition 2.1. (Zhang [6]). A sub-linear expectation $\hat{\mathbb{E}}$ on $\mathcal{H}$ is a function $\hat{\mathbb{E}} : \mathcal{H} \to \bar{\mathbb{R}}$ satisfying the following properties: for all $X, Y \in \mathcal{H}$, we have
(a) Monotonicity: If $X \geq Y$ then $\hat{\mathbb{E}}X \geq \hat{\mathbb{E}}Y$;
(b) Constant preserving: $\hat{\mathbb{E}}c = c$;
(c) Sub-additivity: $\hat{\mathbb{E}}(X + Y) \leq \hat{\mathbb{E}}X + \hat{\mathbb{E}}Y$ whenever $\hat{\mathbb{E}}X + \hat{\mathbb{E}}Y$ is not of the form $+\infty - \infty$ or $-\infty + \infty$;
(d) Positive homogeneity: $\hat{\mathbb{E}}(\lambda X) = \lambda\hat{\mathbb{E}}X, \lambda \geq 0$.

Here $\bar{\mathbb{R}} = [-\infty, +\infty]$. The triple $\left(\Omega, \mathcal{H}, \hat{\mathbb{E}}\right)$ is called a sub-linear expectation space. Given a sub-linear expectation $\hat{\mathbb{E}}$, let us denote the conjugate expectation $\hat{\varepsilon}$ of $\hat{\mathbb{E}}$ by

$$\hat{\varepsilon}X := -\hat{\mathbb{E}}(-X), \forall X \in \mathcal{H}.$$

From the definition, it is easily shown that $\hat{\varepsilon}(X) \leq \hat{\mathbb{E}}(X), \hat{\mathbb{E}}(X + c) = \hat{\mathbb{E}}(X) + c$ and $\hat{\mathbb{E}}(X - Y) \leq \hat{\mathbb{E}}X - \hat{\mathbb{E}}Y$ for all $X, Y \in \mathcal{H}$ with $\hat{\mathbb{E}}Y$ being finite. Further, if $\hat{\mathbb{E}}(|X|)$ is finite, then $\hat{\varepsilon}X$ and $\hat{\mathbb{E}}X$ are both finite. It is called to be countably sub-additive if $\hat{\mathbb{E}}X \leq \sum_{n=1}^{\infty} \hat{\mathbb{E}}X_n$, whenever $X \leq \sum_{n=1}^{\infty} X_n$, $X, X_n \in \mathcal{H}$ and $X \geq 0, X_n \geq 0, n \geq 1$.

Next, we introduce the capacities corresponding to the sub-linear expectations. Let $\mathcal{G} \subset \mathcal{F}$. A function $V : \mathcal{G} \to [0, 1]$ is called a capacity if

$$V(\emptyset) = 0,\ V(\Omega) = 1 \text{ and } V(A) \leq V(B) \text{ for } \forall A \subset B, A, B \in \mathcal{G}.$$

It is called to be sub-additive if $V(A \cup B) \leq V(A) + V(B)$ for all $A, B \in \mathcal{G}$ with $A \cup B \in \mathcal{G}$. It is called to be countably sub-additive if $V\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} V(A_n), \forall A_n \in \mathcal{F}$.

Let $\left(\Omega, \mathcal{H}, \hat{\mathbb{E}}\right)$ be a sub-linear space, and $\hat{\varepsilon}$ be the conjugate of $\hat{\mathbb{E}}$. We denote a pair $(\mathbb{V}, \mathcal{V})$ of capacities by

$$\mathbb{V}(A) := \inf\{\hat{\mathbb{E}}\xi; I_A \leq \xi, \xi \in \mathcal{H}\},\ \mathcal{V}(A) := 1 - \mathbb{V}(A^c),\ \forall A \in \mathcal{F},$$

where A^c is the complement set of A. Then

$$\mathbb{V}(A) := \hat{\mathbb{E}}(I_A),\ \mathcal{V}(A) := \hat{\varepsilon}(I_A),\ \text{if } I_A \in \mathcal{H}. \tag{2.1}$$

For example, if $A = \emptyset$ then $I_A = 0 \in \mathcal{H}$ and if $A = \Omega$ then $I_A = 1 \in \mathcal{H}$. Further, we have

$$\hat{\mathbb{E}}f \leq \mathbb{V}(A) \leq \hat{\mathbb{E}}g,\ \hat{\varepsilon}f \leq \mathcal{V}(A) \leq \hat{\varepsilon}g,\ \text{if} f \leq I_A \leq g,\ f, g \in \mathcal{H}. \tag{2.2}$$

It is obvious that $\mathbb{V}$ is sub-additive, but $\mathcal{V}$ and $\hat{\varepsilon}$ are not. However, we have

$$\mathcal{V}(A \cup B) \leq \mathcal{V}(A) + \mathbb{V}(B) \text{ and } \hat{\varepsilon}(X + Y) \leq \hat{\varepsilon}X + \hat{\mathbb{E}}Y. \tag{2.3}$$

Due to the fact that $\mathbb{V}(A^c \cap B^c) = \mathbb{V}(A^c \setminus B) \geq \mathbb{V}(A^c) - \mathbb{V}(B)$ and $\hat{\mathbb{E}}(-X - Y) \geq \hat{\mathbb{E}}(-X) - \hat{\mathbb{E}}Y$.

Also, we define the Choquet integrals expectations $(C_{\mathbb{V}}, C_{\mathcal{V}})$ by

$$C_V(X) = \int_0^{\infty} V(X > t)\mathrm{d}t + \int_{-\infty}^{0} [V(X \geq t) - 1]\mathrm{d}t,$$

with V being replaced by $\mathbb{V}$ and $\mathcal{V}$ respectively.

Remark 2.1. *From (2.2), for* $\forall X \in \mathcal{H}, x > 0, p > 0$, *it emerges that* $\mathbb{V}(|X| \geq y) \leq \hat{\mathbb{E}}(|X|^p)/x^p$, *which is the well-known Markov's inequality.*

Definition 2.2. (Peng [1], Zhang [4]). (i) (Identical distribution). Let X_1 and X_2 be two random vectors defined respectively in sub-linear expectation spaces $\left(\Omega, \mathcal{H}, \hat{\mathbb{E}}\right)$. They are called identically distributed, denoted by $X_1 \overset{d}{=} X_2$ if

$$\hat{\mathbb{E}}_1[\varphi(X_1)] = \hat{\mathbb{E}}_2[\varphi(X_2)],\ \forall \varphi \in C_{l,Lip}(\mathbb{R}),$$

whenever the sub-linear expectation are finite. A sequence $\{X_n, n \geq 1\}$ of random variables is said to be identically distributed if $X_i \overset{d}{=} X_1$ for each $i \geq 1$.

(ii) (Extended negatively dependent). A sequence of random variables $\{X_n, n \geq 1\}$ is named to be upper (resp. lower) extended negatively dependent if there is some dominating constant $K \geq 1$ such that

$$\hat{\mathbb{E}}\left(\prod_{i=1}^{n} \varphi_i(X_i)\right) \leq K \prod_{i=1}^{n} \hat{\mathbb{E}}\left(\varphi_i(X_i)\right), \ \forall n \geq 2,$$

whenever the non-negative functions $\varphi_i(x) \in C_{b,Lip}(\mathbb{R}), i = 1, 2, ...$, are all non-decreasing (resp. all non-increasing). They are named extended negatively dependent (END) if they are both upper extended negatively dependent and lower extended negatively dependent. It shall be noted that the extended negatively dependence of $\{X_n; n \geq 1\}$ under $\hat{\mathbb{E}}$ does not imply the extended negatively dependence under $\hat{\varepsilon}$.

It is obvious that, let $\{X_n; n \geq 1\}$ be a sequence of extended negatively dependent random variables and $f_1(x), f_2(x), ... \in C_{l,Lip}(\mathbb{R})$ are all non-decreasing (resp. all non-increasing) functions, then $\{f_n(X_n); n \geq 1\}$ is also a sequence of extended negatively dependent random variables.

In the following, let $\{X_n; n \geq 1\}$ be a sequence random variables in $\left(\Omega, \mathcal{H}, \hat{\mathbb{E}}\right)$, and $I(\cdot)$ denote an indicator function. The symbol C stands for a generic positive constant which may differ from one place to another.

To prove our results, we need the following three lemmas.

Lemma 2.1. *Let $\{X_n; n \geq 1\}$ be a sequence of END random variables in $\left(\Omega, \mathcal{H}, \hat{\mathbb{E}}\right)$ with $\hat{\mathbb{E}}X_n \leq 0$ for each $n \geq 1$. If there exists a sequence of positive numbers $\{c_n, n \geq 1\}$ such that $|X_i| \leq c_i$ for each $i \geq 1$, then for any $t > 0$ and $n \geq 1$,*

$$\hat{\mathbb{E}} \exp\left\{t \sum_{i=1}^{n} X_i\right\} \leq C \exp\left\{\frac{t^2}{2} \sum_{i=1}^{n} \mathrm{e}^{tc_i} \hat{\mathbb{E}} X_i^2\right\}.$$

Proof. It is easy to check that for all $x \in \mathbb{R}, \mathrm{e}^x \leq 1 + x + \frac{1}{2}x^2\mathrm{e}^{|x|}$. Thus, by $\hat{\mathbb{E}}X_i \leq 0$ and $|X_i| \leq c_i$ for each $i \geq 1$, we have

$$\begin{aligned}\hat{\mathbb{E}}\mathrm{e}^{tX_i} &\leq 1 + t\hat{\mathbb{E}}X_i + \frac{1}{2}t^2\hat{\mathbb{E}}\left(X_i^2\mathrm{e}^{t|X_i|}\right) \leq 1 + \frac{1}{2}t^2\hat{\mathbb{E}}\left(X_i^2\mathrm{e}^{t|X_i|}\right) \\ &\leq 1 + \frac{1}{2}t^2\mathrm{e}^{tc_i}\hat{\mathbb{E}}X_i^2 \leq \exp\left\{\frac{1}{2}t^2\mathrm{e}^{tc_i}\hat{\mathbb{E}}X_i^2\right\},\end{aligned}$$

for any $t > 0$. By Definition 2.2, we can see that

$$\hat{\mathbb{E}} \exp\left\{t \sum_{i=1}^{n} X_i\right\} \leq C \prod_{i=1}^{n} \hat{\mathbb{E}}\mathrm{e}^{tX_i} \leq C \exp\left\{\frac{t^2}{2} \sum_{i=1}^{n} \mathrm{e}^{tc_i} \hat{\mathbb{E}} X_i^2\right\}.$$

This completes the proof of Lemma 2.1. □

Lemma 2.2. (Zhang [4], Theorem 3.1). *Let $\{X_1, ..., X_n\}$ be a sequence of END random variables in $\left(\Omega, \mathcal{H}, \hat{\mathbb{E}}\right)$ with $\hat{\mathbb{E}}X_i \leq 0$ Then*

$$\mathbb{V}(S_n \geq x) \leq C\frac{B_n^2}{x^2}, \ \forall x > 0,$$

where $S_n = \sum_{i=1}^{n} X_i, B_n^2 = \sum_{i=1}^{n} \hat{\mathbb{E}}X_i^2$.

Lemma 2.3. (Borel-Cantelli's lemma, Zhang [6] Lemma 3.9). *Let $\{A_n; n \geq 1\}$ be a sequence of events in $\mathcal{F}$. Suppose that V is a countably sub-additive capacity. If $\sum_{n=1}^{\infty} V(A_n) < \infty$, then*

$$V(A_n; \text{i.o.}) = 0, \quad \text{where } \{A_n; \text{i.o.}\} = \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} A_i.$$

3. Main results

This section presents the main results of this paper. First, we provide the exponential inequalities for unbounded END random variable sequences. Then, we establish a strong law of large numbers with convergence rate $O\left(n^{-1/2} \ln^{1/2} n\right)$.

3.1. Exponential inequalities

Let $\{X_n; n \geq 1\}$ be a sequence of random variables in $\left(\Omega, \mathcal{H}, \hat{\mathbb{E}}\right)$, and $\{c_n; n \geq 1\}$ be a sequence of positive numbers. Define for $1 \leq i \leq n, n \geq 1$,

$$\begin{aligned} &X_{1,i,n} = -c_n I(X_i < -c_n) + X_i I(-c_n \leq X_i \leq c_n) + c_n I(X_i > c_n), \\ &X_{2,i,n} = (X_i - c_n) I(X_i > c_n),\ X_{3,i,n} = (X_i + c_n) I(X_i < -c_n). \end{aligned} \tag{3.1}$$

It is easy to check that $X_{1,i,n} + X_{2,i,n} + X_{3,i,n} = X_i$ for $1 \leq i \leq n, n \geq 1$ and $\{X_{1,i,n}; 1 \leq i \leq n\}$ are bounded by c_n for each fixed $n \geq 1$.

Let $f_1(x) = -cI(x < -c) + xI(-c \leq x \leq c) + cI(x > c)$, $f_2(x) = (x - c)I(x > c)$, $f_3(x) = (x + c)I(x < -c)$ for any $c > 0$, then $\{f_i(x), i = 1, 2, 3\} \in C_{l,Lip}$ and $\{f_i(x), i = 1, 2, 3\}$ is non-decreasing. So, if $\{X_n; n \geq 1\}$ are END random variables in $\left(\Omega, \mathcal{H}, \hat{\mathbb{E}}\right)$, then $\{X_{1,i,n}, X_{2,i,n}, X_{3,i,n}; 1 \leq i \leq n\}$, are also END random variables in $\left(\Omega, \mathcal{H}, \hat{\mathbb{E}}\right)$ for each fixed $n \geq 1$ due to the fact that $\{f_i(x), i = 1, 2, 3\} \in C_{l,Lip}$ and $\{f_i(x), i = 1, 2, 3\}$ is non-decreasing.

Theorem 3.1. *Let $\{X_n; n \geq 1\}$ be a sequence of END random variables in $\left(\Omega, \mathcal{H}, \hat{\mathbb{E}}\right)$ and $\{X_{1,i,n}; 1 \leq i \leq n, n \geq 1\}$ be defined by (3.1). Define $B_n^2 = \sum_{i=1}^{n} \hat{\mathbb{E}} X_i^2, n \geq 1$. Then for any $\varepsilon > 0$ such that $\varepsilon \leq 2\mathrm{e}B_n^2/c_n$ and $n \geq 1$,*

$$\mathbb{V}\left(\sum_{i=1}^{n}\left(X_{1,i,n} - \hat{\mathbb{E}}X_{1,i,n}\right) > \varepsilon\right) \leq C \exp\left\{-\frac{\varepsilon^2}{8\mathrm{e}B_n^2}\right\}, \tag{3.2}$$

$$\mathbb{V}\left(\sum_{i=1}^{n}\left(X_{1,i,n} - \hat{\varepsilon}X_{1,i,n}\right) < -\varepsilon\right) \leq C \exp\left\{-\frac{\varepsilon^2}{8\mathrm{e}B_n^2}\right\}. \tag{3.3}$$

In particular, if $\hat{\mathbb{E}}X_{1,i,n} = \hat{\varepsilon}X_{1,i,n}$, then

$$\mathbb{V}\left(\left|\sum_{i=1}^{n}\left(X_{1,i,n} - \hat{\mathbb{E}}X_{1,i,n}\right)\right| > \varepsilon\right) \leq 2C \exp\left\{-\frac{\varepsilon^2}{8\mathrm{e}B_n^2}\right\}. \tag{3.4}$$

Corollary 3.1. *Let $\{X_n; n \geq 1\}$ be a sequence of identically distributed END random variables in $\left(\Omega, \mathcal{H}, \hat{\mathbb{E}}\right)$ and $\{X_{1,i,n}; 1 \leq i \leq n, n \geq 1\}$ be defined by* (3.1)*. Then for any $\varepsilon > 0$ such that $\varepsilon \leq 2\mathrm{e}\hat{\mathbb{E}}X_1^2/c_n$,*

$$\mathbb{V}\left(\sum_{i=1}^{n}\left(X_{1,i,n} - \hat{\mathbb{E}}X_{1,i,n}\right) > n\varepsilon\right) \leq C\exp\left\{-\frac{n\varepsilon^2}{8\mathrm{e}\hat{\mathbb{E}}X_1^2}\right\},$$
$$\mathbb{V}\left(\sum_{i=1}^{n}\left(X_{1,i,n} - \hat{\varepsilon}X_{1,i,n}\right) < -n\varepsilon\right) \leq C\exp\left\{-\frac{n\varepsilon^2}{8\mathrm{e}\hat{\mathbb{E}}X_1^2}\right\}.$$

In particular, if $\hat{\mathbb{E}}X_{1,i,n} = \hat{\varepsilon}X_{1,i,n}$, then

$$\mathbb{V}\left(\left|\sum_{i=1}^{n}\left(X_{1,i,n} - \hat{\mathbb{E}}X_{1,i,n}\right)\right| > n\varepsilon\right) \leq 2C\exp\left\{-\frac{n\varepsilon^2}{8\mathrm{e}\hat{\mathbb{E}}X_1^2}\right\}.$$

Theorem 3.2. *Let $\{X_n; n \geq 1\}$ be a sequence of identically distributed END random variables in $\left(\Omega, \mathcal{H}, \hat{\mathbb{E}}\right)$ and $\left\{X_{q,i,n}; 1 \leq i \leq n, n \geq 1\right\}, q = 2, 3$ be defined by* (3.1) *with $\lim_{c\to\infty}\hat{\mathbb{E}}\left[(X_1^2 - c)^+\right] = 0$. Assume that there exists a $\delta > 0$ satisfying $\sup_{t\leq\delta}\hat{\mathbb{E}}\mathrm{e}^{t|X_1|} \leq M_\delta < \infty$, where M_δ is a positive constant depending only on δ. Then for any $\varepsilon > 0$ and $t \in (0, \delta]$,*

$$\mathbb{V}\left(\frac{1}{n}\sum_{i=1}^{n}\left(X_{q,i,n} - \hat{\mathbb{E}}X_{q,i,n}\right) > \varepsilon\right) \leq C\frac{M_\delta}{t^2\varepsilon^2 n}\mathrm{e}^{-tc_n}, \tag{3.5}$$

$$\mathbb{V}\left(\sum_{i=1}^{n}\left(X_{q,i,n} - \hat{\varepsilon}X_{q,i,n}\right) < -\varepsilon\right) \leq C\frac{M_\delta}{t^2\varepsilon^2 n}\mathrm{e}^{-tc_n}. \tag{3.6}$$

In particular, if $\hat{\mathbb{E}}X_{q,i,n} = \hat{\varepsilon}X_{q,i,n}$, then

$$\mathbb{V}\left(\frac{1}{n}\left|\sum_{i=1}^{n}\left(X_{q,i,n} - \hat{\mathbb{E}}X_{q,i,n}\right)\right| > \varepsilon\right) \leq 2C\frac{M_\delta}{t^2\varepsilon^2 n}\mathrm{e}^{-tc_n}. \tag{3.7}$$

Corollary 3.2. *Let $\{X_n; n \geq 1\}$ be a sequence of identically distributed END random variables in $\left(\Omega, \mathcal{H}, \hat{\mathbb{E}}\right)$ with $\lim_{c\to\infty}\hat{\mathbb{E}}\left[(X_1^2 - c)^+\right] = 0$ and $\hat{\mathbb{E}}\mathrm{e}^{\delta|X_1|} < \infty$ for some $\delta > 0$. Let $\left\{X_{q,i,n}; 1 \leq i \leq n, n \geq 1\right\}, q = 2, 3$ be defined by (3.1). Then for any $\varepsilon > 0$,*

$$\mathbb{V}\left(\frac{1}{n}\sum_{i=1}^{n}\left(X_{q,i,n} - \hat{\mathbb{E}}X_{q,i,n}\right) > \varepsilon\right) \leq C\frac{\hat{\mathbb{E}}\mathrm{e}^{\delta|X_1|}}{\delta^2\varepsilon^2 n}\mathrm{e}^{-\delta c_n},$$

$$\mathbb{V}\left(\sum_{i=1}^{n}\left(X_{q,i,n} - \hat{\varepsilon}X_{q,i,n}\right) < -\varepsilon\right) \leq C\frac{\hat{\mathbb{E}}\mathrm{e}^{\delta|X_1|}}{\delta^2\varepsilon^2 n}\mathrm{e}^{-\delta c_n}.$$

In particular, if $\hat{\mathbb{E}}X_{q,i,n} = \hat{\varepsilon}X_{q,i,n}$, then

$$\mathbb{V}\left(\frac{1}{n}\left|\sum_{i=1}^{n}\left(X_{q,i,n} - \hat{\mathbb{E}}X_{q,i,n}\right)\right| > \varepsilon\right) \leq 2C\frac{\hat{\mathbb{E}}\mathrm{e}^{\delta|X_1|}}{\delta^2\varepsilon^2 n}\mathrm{e}^{-\delta c_n}.$$

Theorem 3.3. *Let* $\{X_n; n \geq 1\}$ *be a sequence of identically distributed END random variables in* $\left(\Omega, \mathcal{H}, \hat{\mathbb{E}}\right)$ *with* $\lim_{c\to\infty} \hat{\mathbb{E}}\left[(X_1^2 - c)^+\right] = 0$ *and* $\hat{\mathbb{E}}e^{\delta|X_1|} < \infty$ *for some* $\delta > 0$ *and* $\{c_n; n \geq 1\}$ *be a sequence of positive numbers such that*

$$0 < c_n \leq \left(\frac{en\hat{\mathbb{E}}X_1^2}{2\delta}\right)^{1/3} \qquad for\ any \quad n \geq n_0\,, \tag{3.8}$$

where n_0 *is a positive integer. Define* $\varepsilon_n = \sqrt{8\delta e\hat{\mathbb{E}}X_1^2 c_n/n}$. *Then for* $n \geq n_0$,

$$\mathbb{V}\left(\frac{1}{n}\sum_{i=1}^{n}\left(X_i - \hat{\mathbb{E}}X_i\right) > 3\varepsilon_n\right) \leq C\left(1 + \frac{2\hat{\mathbb{E}}e^{\delta|X_1|}}{\delta^3 e\hat{\mathbb{E}}X_1^2 c_n}\right)e^{-\delta c_n}, \tag{3.9}$$

$$\mathbb{V}\left(\sum_{i=1}^{n}(X_i - \hat{\varepsilon}X_i) < -3\varepsilon_n\right) \leq C\left(1 + \frac{2\hat{\mathbb{E}}e^{\delta|X_1|}}{\delta^3 e\hat{\mathbb{E}}X_1^2 c_n}\right)e^{-\delta c_n}. \tag{3.10}$$

In particular, if $\hat{\mathbb{E}}X_i = \hat{\varepsilon}X_i$, *then*

$$\mathbb{V}\left(\frac{1}{n}\left|\sum_{i=1}^{n}\left(X_i - \hat{\mathbb{E}}X_i\right)\right| > 3\varepsilon_n\right) \leq 2C\left(1 + \frac{2\hat{\mathbb{E}}e^{\delta|X_1|}}{\delta^3 e\hat{\mathbb{E}}X_1^2 c_n}\right)e^{-\delta c_n}. \tag{3.11}$$

3.2. The strong law of large numbers

Theorem 3.4. *Let* $\{X_n; n \geq 1\}$ *be a sequence of identically distributed END random variables in* $\left(\Omega, \mathcal{H}, \hat{\mathbb{E}}\right)$ *with* $\lim_{c\to\infty} \hat{\mathbb{E}}\left[(X_1^2 - c)^+\right] = 0$ *and* $\hat{\mathbb{E}}e^{\delta|X_1|} < \infty$ *for some* $\delta > 1$. *Suppose* $\mathbb{V}$ *is countably sub-additive, then*

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} a_n\left(X_i - \hat{\mathbb{E}}X_i\right) \leq 0 \qquad \text{a.s.} \quad \mathbb{V}, \tag{3.12}$$

$$\liminf_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} a_n\left(X_i - \hat{\varepsilon}X_i\right) \geq 0 \qquad \text{a.s.} \quad \mathbb{V}. \tag{3.13}$$

In particular, if $\hat{\mathbb{E}}X_i = \hat{\varepsilon}X_i$, *then*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} a_n\left(X_i - \hat{\mathbb{E}}X_i\right) = 0 \qquad \text{a.s.} \quad \mathbb{V}, \tag{3.14}$$

where $a_n = O(n^{1/2}\ln^{-1/2} n)$.

Remark 3.1. *Here, we extend the results of Tang et al. [15] for extended independent random variables to the case of extended negatively dependent random variables. Our Theorem 3.2 to Theorem 3.4 weaken the condition of [15] from* $\hat{\mathbb{E}}e^{\delta X_1^2} < \infty$ *to* $\hat{\mathbb{E}}e^{\delta|X_1|} < \infty$. *Furthermore, in Theorem 3.3, we improves the results of [15] for* $0 < c_n \leq \left(\frac{en\hat{\mathbb{E}}X_1^2}{2\delta}\right)^{1/4}$ *to an arbitrary positive sequence satisfying (3.8) only.*

4. Proof of main results

4.1. Proofs of the exponential inequalities

Proof of Theorem 3.1. It is easily seen that $\left|X_{1,i,n} - \hat{\mathbb{E}}X_{1,i,n}\right| \leq 2c_n$ for each $1 \leq i \leq n, n \geq 1$. Noting that $(a-b)^2 \leq 2(a^2+b^2)$ and $\hat{\mathbb{E}}X_{1,i,n}^2 \leq \hat{\mathbb{E}}X_i^2$ for $1 \leq i \leq n$,

$$\hat{\mathbb{E}}\left(X_{1,i,n} - \hat{\mathbb{E}}X_{1,i,n}\right)^2 \leq 2\hat{\mathbb{E}}\left(X_{1,i,n}^2 + \left(\hat{\mathbb{E}}X_{1,i,n}\right)^2\right) \leq 4\hat{\mathbb{E}}X_{1,i,n}^2 \leq 4\hat{\mathbb{E}}X_i^2.$$

Therefore, by Lemma 2.1, we have that for any $t > 0$ and $n \geq 1$,

$$\begin{aligned}
&\hat{\mathbb{E}}\exp\left\{t\sum_{i=1}^{n}\left(X_{1,i,n} - \hat{\mathbb{E}}X_{1,i,n}\right)\right\}\\
\leq&\exp\left\{\frac{t^2}{2}\mathrm{e}^{2tc_n}\sum_{i=1}^{n}\hat{\mathbb{E}}\left(X_{1,i,n} - \hat{\mathbb{E}}X_{1,i,n}\right)^2\right\}\\
\leq&C\exp\left\{2t^2\mathrm{e}^{2tc_n}\sum_{i=1}^{n}\hat{\mathbb{E}}X_i^2\right\}\\
=&C\exp\left\{2t^2\mathrm{e}^{2tc_n}B_n^2\right\}.
\end{aligned}$$

By Markov's inequality, we have that for any $t > 0$,

$$\begin{aligned}
&\mathbb{V}\left(\sum_{i=1}^{n}\left(X_{1,i,n} - \hat{\mathbb{E}}X_{1,i,n}\right) > \varepsilon\right)\\
\leq&\mathrm{e}^{-t\varepsilon}\hat{\mathbb{E}}\exp\left\{t\sum_{i=1}^{n}\left(X_{1,i,n} - \hat{\mathbb{E}}X_{1,i,n}\right)\right\}\\
\leq&C\exp\left\{-t\varepsilon + 2t^2\mathrm{e}^{2tc_n}B_n^2\right\}.
\end{aligned}$$

Taking $t = \varepsilon/\left(4\mathrm{e}B_n^2\right)$, and noting that $2tc_n \leq 1$, we get

$$\begin{aligned}
&\mathbb{V}\left(\sum_{i=1}^{n}\left(X_{1,i,n} - \hat{\mathbb{E}}X_{1,i,n}\right) > \varepsilon\right)\\
\leq&C\exp\left\{-t\varepsilon + 2t^2\mathrm{e}^{2tc_n}B_n^2\right\}\\
\leq&C\exp\left\{-\frac{\varepsilon^2}{4\mathrm{e}B_n^2} + \frac{\varepsilon^2}{8\mathrm{e}B_n^2}\right\}\\
=&C\exp\left\{-\frac{\varepsilon^2}{8\mathrm{e}B_n^2}\right\}.
\end{aligned}$$

That is, (3.2) holds.

Obviously, $\{-X_{1,i,n}; n \geq 1\}$ is a sequence of END random variables in $\left(\Omega, \mathcal{H}, \hat{\mathbb{E}}\right)$, and also satisfies the conditions of Theorem 3.1. Considering $\{-X_{1,i,n}; n \geq 1\}$ instead of $\{X_{1,i,n}; n \geq 1\}$ in (3.2), we can get

$$\mathbb{V}\left(\sum_{i=1}^{n}\left(-X_{1,i,n} - \hat{\mathbb{E}}\left(-X_{1,i,n}\right)\right) > \varepsilon\right) \leq C\exp\left\{-\frac{\varepsilon^2}{8\mathrm{e}B_n^2}\right\}.$$

By $\hat{\varepsilon}X_{1,i,n} = -\hat{\mathbb{E}}(-X_{1,i,n})$, we have

$$\begin{aligned}
&\mathbb{V}\left(\sum_{i=1}^{n}(X_{1,i,n}-\hat{\varepsilon}X_{1,i,n})<-\varepsilon\right)\\
=&\mathbb{V}\left(\sum_{i=1}^{n}\left(-X_{1,i,n}-\hat{\mathbb{E}}\left(-X_{1,i,n}\right)\right)>\varepsilon\right)\\
\le& C\exp\left\{-\frac{\varepsilon^2}{8\mathrm{e}B_n^2}\right\}.
\end{aligned}$$

That is, (3.3) holds.

In particular, if $\hat{\mathbb{E}}X_{1,i,n} = \hat{\varepsilon}X_{1,i,n}$, (3.4) follow from (3.2), (3.3), then

$$\begin{aligned}
&\mathbb{V}\left(\left|\sum_{i=1}^{n}\left(X_{1,i,n}-\hat{\mathbb{E}}X_{1,i,n}\right)\right|>\varepsilon\right)\\
\le&\mathbb{V}\left(\sum_{i=1}^{n}\left(X_{1,i,n}-\hat{\mathbb{E}}X_{1,i,n}\right)>\varepsilon\right)+\mathbb{V}\left(\sum_{i=1}^{n}\left(X_{1,i,n}-\hat{\mathbb{E}}X_{1,i,n}\right)<-\varepsilon\right)\\
=&\mathbb{V}\left(\sum_{i=1}^{n}\left(X_{1,i,n}-\hat{\mathbb{E}}X_{1,i,n}\right)>\varepsilon\right)+\mathbb{V}\left(\sum_{i=1}^{n}(X_{1,i,n}-\hat{\varepsilon}X_{1,i,n})<-\varepsilon\right)\\
\le&2C\exp\left\{-\frac{\varepsilon^2}{8\mathrm{e}B_n^2}\right\}.
\end{aligned}$$

That completes the proof of Theorem 3.1. □

Proof of Theorem 3.2. By Lemma 2.2 and $(a-b)^2 \le 2(a^2+b^2)$, we have

$$\begin{aligned}
&\mathbb{V}\left(\frac{1}{n}\sum_{i=1}^{n}\left(X_{q,i,n}-\hat{\mathbb{E}}X_{q,i,n}\right)>\varepsilon\right)\\
=&\mathbb{V}\left(\sum_{i=1}^{n}\left(X_{q,i,n}-\hat{\mathbb{E}}X_{q,i,n}\right)>n\varepsilon\right)\\
\le& C\frac{\sum_{i=1}^{n}\hat{\mathbb{E}}\left(X_{q,i,n}-\hat{\mathbb{E}}X_{q,i,n}\right)^2}{n^2\varepsilon^2}\\
\le& C\frac{\hat{\mathbb{E}}X_{q,1,n}^2}{n\varepsilon^2}
\end{aligned}$$

Therefore, it remains only to estimate $\hat{\mathbb{E}}X_{q,1,n}^2$. Noting that $\hat{\mathbb{E}}\mathrm{e}^{tX_1} \le \hat{\mathbb{E}}\mathrm{e}^{t|X_1|} \le M_\delta$ for any $t \in (0,\delta]$. From Lemma 3.9 of Zhang [7], we can infer directly that if $\lim_{c\to\infty}\hat{\mathbb{E}}\left[(X^2-c)^+\right] = 0$, then $\hat{\mathbb{E}}(X^2) \le C_{\mathbb{V}}(X^2)$. Noting that

$$\begin{aligned}
&0 \le \lim_{c\to\infty}\hat{\mathbb{E}}\left(\left[(X_1-c_n)^2 I(X_1>c_n)-c\right]^+\right) \le \lim_{c\to\infty}\hat{\mathbb{E}}\left[(X_1^2-c)^+\right]=0,\\
&0 \le \lim_{c\to\infty}\hat{\mathbb{E}}\left(\left[(X_1+c_n)^2 I(X_1<-c_n)-c\right]^+\right) \le \lim_{c\to\infty}\hat{\mathbb{E}}\left[(X_1^2-c)^+\right]=0,
\end{aligned}$$

we have $\hat{\mathbb{E}}\left((X_1 - c_n)^2 I(X_1 > c_n)\right) \le C_{\mathbb{V}}\left((X_1 - c_n)^2 I(X_1 > c_n)\right)$ and $\hat{\mathbb{E}}\left((X_1 + c_n)^2 I(X_1 < -c_n)\right) \le C_{\mathbb{V}}\left((X_1 + c_n)^2 I(X_1 < -c_n)\right)$.

For $q = 2$, by Markov's inequality, it follows that

$$\begin{aligned}
\hat{\mathbb{E}}X_{2,1,n}^2 &= \hat{\mathbb{E}}\left((X_1 - c_n)^2 I(X_1 > c_n)\right) \le C_{\mathbb{V}}\left((X_1 - c_n)^2 I(X_1 > c_n)\right) \\
&= \int_0^\infty 2y\mathbb{V}\left(|X_1 - c_n| I(X_1 > c_n) > y\right) \mathrm{d}y \\
&= \int_0^\infty 2y\mathbb{V}\left(X_1 - c_n > y\right) \mathrm{d}y \\
&= \int_{c_n}^\infty 2(u - c_n)\mathbb{V}\left(X_1 > u\right) \mathrm{d}u \quad (y = u - c_n) \\
&\le \int_{c_n}^\infty 2(u - c_n)\mathrm{e}^{-tu}\hat{\mathbb{E}}\mathrm{e}^{tX_1} \mathrm{d}u \\
&\le M_\delta \int_{c_n}^\infty 2(u - c_n)\mathrm{e}^{-tu} \mathrm{d}u \\
&= \frac{2M_\delta \mathrm{e}^{-tc_n}}{t^2}.
\end{aligned}$$

It follows that

$$\mathbb{V}\left(\frac{1}{n}\sum_{i=1}^n \left(X_{2,i,n} - \hat{\mathbb{E}}X_{2,i,n}\right) > \varepsilon\right) \le C\frac{\hat{\mathbb{E}}X_{2,1,n}^2}{\varepsilon^2 n} \le C\frac{M_\delta \mathrm{e}^{-tc_n}}{t^2\varepsilon^2 n}.$$

For $q = 3$, by Markov's inequality, it follows that

$$\begin{aligned}
\hat{\mathbb{E}}X_{3,1,n}^2 &= \hat{\mathbb{E}}\left((X_1 + c_n)^2 I(X_1 < -c_n)\right) \le C_{\mathbb{V}}\left((X_1 + c_n)^2 I(X_1 < -c_n)\right) \\
&= \int_0^\infty 2y\mathbb{V}\left(|X_1 + c_n| I(X_1 < -c_n) > y\right) \mathrm{d}y \\
&= \int_0^\infty 2y\mathbb{V}\left(X_1 + c_n < -y\right) \mathrm{d}y \\
&= \int_0^\infty 2y\mathbb{V}\left(-X_1 > y + c_n\right) \mathrm{d}y \\
&= \int_{c_n}^\infty 2(u - c_n)\mathbb{V}\left(-X_1 > u\right) \mathrm{d}u \quad (y = u - c_n) \\
&\le \int_{c_n}^\infty 2(u - c_n)\mathrm{e}^{-tu}\hat{\mathbb{E}}\mathrm{e}^{-tX_1} \mathrm{d}u \\
&\le M_\delta \int_{c_n}^\infty 2(u - c_n)\mathrm{e}^{-tu} \mathrm{d}u \\
&= \frac{2M_\delta \mathrm{e}^{-tc_n}}{t^2}.
\end{aligned}$$

It follows that

$$\mathbb{V}\left(\frac{1}{n}\sum_{i=1}^n \left(X_{3,i,n} - \hat{\mathbb{E}}X_{3,i,n}\right) > \varepsilon\right) \le C\frac{\hat{\mathbb{E}}X_{3,1,n}^2}{\varepsilon^2 n} \le C\frac{M_\delta \mathrm{e}^{-tc_n}}{t^2\varepsilon^2 n}.$$

That is, (3.5) holds.

Obviously, $\{-X_{q,i,n}; n \geq 1\}$, $q = 2, 3$ is a sequence of END random variables in $\left(\Omega, \mathcal{H}, \hat{\mathbb{E}}\right)$, and also satisfies the conditions of Theorem 3.2. Considering $\left\{-X_{q,i,n}; n \geq 1\right\}$ instead of $\left\{X_{q,i,n}; n \geq 1\right\}$ in (3.5), we can get

$$\mathbb{V}\left(\frac{1}{n}\sum_{i=1}^{n}\left(-X_{q,i,n} - \hat{\mathbb{E}}(-X_{q,i,n})\right) > \varepsilon\right) \leq C\frac{M_\delta \mathrm{e}^{-tc_n}}{t^2\varepsilon^2 n}.$$

By $\hat{\varepsilon}X_{q,i,n} = -\hat{\mathbb{E}}(-X_{q,i,n})$, we have

$$\begin{aligned}
&\mathbb{V}\left(\frac{1}{n}\sum_{i=1}^{n}\left(X_{q,i,n} - \hat{\varepsilon}X_{q,i,n}\right) < -\varepsilon\right) \\
=&\mathbb{V}\left(\frac{1}{n}\sum_{i=1}^{n}\left(-X_{q,i,n} - \hat{\mathbb{E}}(-X_{q,i,n})\right) > \varepsilon\right) \\
\leq& C\frac{M_\delta \mathrm{e}^{-tc_n}}{t^2\varepsilon^2 n}.
\end{aligned}$$

That is, (3.6) holds.

In particular, if $\hat{\mathbb{E}}X_{q,i,n} = \hat{\varepsilon}X_{q,i,n}$, (3.7) follow (3.5) and (3.6), then

$$\begin{aligned}
&\mathbb{V}\left(\frac{1}{n}\left|\sum_{i=1}^{n}\left(X_{q,i,n} - \hat{\mathbb{E}}(-X_{q,i,n})\right)\right| > \varepsilon\right) \\
=&\mathbb{V}\left(\frac{1}{n}\sum_{i=1}^{n}\left(X_{q,i,n} - \hat{\mathbb{E}}X_{q,i,n}\right) > \varepsilon\right) + \left(\frac{1}{n}\sum_{i=1}^{n}\left(X_{q,i,n} - \hat{\varepsilon}X_{q,i,n}\right) < -\varepsilon\right) \\
\leq& 2C\frac{M_\delta \mathrm{e}^{-tc_n}}{t^2\varepsilon^2 n}.
\end{aligned}$$

That completes the proof of Theorem 3.2. □

Proof of Corollary 3.2. It is easily seen that $\sup_{t\leq\delta}\hat{\mathbb{E}}\mathrm{e}^{t|X_1|} \leq \hat{\mathbb{E}}\mathrm{e}^{\delta|X_1|} = M_\delta < \infty$, which implies the desired results immediately from Theorem 3.2. □

4.2. Proof of the strong law of large numbers

Proof of Theorem 3.3. It is easy to check that $\varepsilon_n c_n \leq 2\mathrm{e}\hat{\mathbb{E}}X_1^2$ and $n\varepsilon_n^2/\left(8\mathrm{e}\hat{\mathbb{E}}X_1^2\right) = \delta c_n$. It follows from Corollary 3.1 and Corollary 3.2 that

$$\begin{aligned}
&\mathbb{V}\left(\frac{1}{n}\sum_{i=1}^{n}\left(X_i - \hat{\mathbb{E}}X_i\right) > 3\varepsilon_n\right) \leq \mathbb{V}\left(\frac{1}{n}\sum_{i=1}^{n}\left(X_{1,i,n} - \hat{\mathbb{E}}X_{1,i,n}\right) > \varepsilon_n\right) \\
&+\mathbb{V}\left(\frac{1}{n}\sum_{i=1}^{n}\left(X_{2,i,n} - \hat{\mathbb{E}}X_{2,i,n}\right) > \varepsilon_n\right) + \mathbb{V}\left(\frac{1}{n}\sum_{i=1}^{n}\left(X_{3,i,n} - \hat{\mathbb{E}}X_{3,i,n}\right) > \varepsilon_n\right) \\
\leq& C\exp\left\{-\frac{n\varepsilon_n^2}{8\mathrm{e}\hat{\mathbb{E}}X_1^2}\right\} + C\frac{\hat{\mathbb{E}}\mathrm{e}^{\delta|X_1|}\mathrm{e}^{-\delta c_n}}{\delta^2\varepsilon_n^2 n} + C\frac{\hat{\mathbb{E}}\mathrm{e}^{\delta|X_1|}\mathrm{e}^{-\delta c_n}}{\delta^2\varepsilon_n^2 n} \\
\leq& C\left(1 + \frac{2\hat{\mathbb{E}}\mathrm{e}^{\delta|X_1|}}{\delta^3\mathrm{e}\hat{\mathbb{E}}X_1^2 c_n}\right)\mathrm{e}^{-\delta c_n},
\end{aligned}$$

and

$$\mathbb{V}\left(\frac{1}{n}\sum_{i=1}^{n}(X_i-\hat{\varepsilon}X_i)<-3\varepsilon_n\right)\leq\mathbb{V}\left(\frac{1}{n}\sum_{i=1}^{n}(X_{1,i,n}-\hat{\varepsilon}X_{1,i,n})<-\varepsilon_n\right)$$
$$+\mathbb{V}\left(\frac{1}{n}\sum_{i=1}^{n}(X_{2,i,n}-\hat{\varepsilon}X_{2,i,n})<-\varepsilon_n\right)+\mathbb{V}\left(\frac{1}{n}\sum_{i=1}^{n}(X_{3,i,n}-\hat{\varepsilon}X_{3,i,n})<-\varepsilon_n\right)$$
$$\leq C\exp\left\{-\frac{n\varepsilon_n^2}{8\mathrm{e}\hat{\mathbb{E}}X_1^2}\right\}+C\frac{\hat{\mathbb{E}}\mathrm{e}^{\delta|X_1|}\mathrm{e}^{-\delta c_n}}{\delta^2\varepsilon_n^2 n}+C\frac{\hat{\mathbb{E}}\mathrm{e}^{\delta|X_1|}\mathrm{e}^{-\delta c_n}}{\delta^2\varepsilon_n^2 n}$$
$$\leq C\left(1+\frac{2\hat{\mathbb{E}}\mathrm{e}^{\delta|X_1|}}{\delta^3\mathrm{e}\hat{\mathbb{E}}X_1^2 c_n}\right)\mathrm{e}^{-\delta c_n}.$$

That is, (3.9) and (3.10) holds.

In parrticular, if $\hat{\mathbb{E}}X_i=\hat{\varepsilon}X_i$, we have

$$\mathbb{V}\left(\frac{1}{n}\left|\sum_{i=1}^{n}\left(X_i-\hat{\mathbb{E}}X_i\right)\right|>3\varepsilon_n\right)\leq\mathbb{V}\left(\frac{1}{n}\left|\sum_{i=1}^{n}\left(X_{1,i,n}-\hat{\mathbb{E}}X_{1,i,n}\right)\right|>\varepsilon_n\right)$$
$$+\mathbb{V}\left(\frac{1}{n}\left|\sum_{i=1}^{n}\left(X_{2,i,n}-\hat{\mathbb{E}}X_{2,i,n}\right)\right|>\varepsilon_n\right)+\mathbb{V}\left(\frac{1}{n}\left|\sum_{i=1}^{n}\left(X_{3,i,n}-\hat{\mathbb{E}}X_{3,i,n}\right)\right|>\varepsilon_n\right)$$
$$\leq 2C\exp\left\{-\frac{n\varepsilon_n^2}{8\mathrm{e}\hat{\mathbb{E}}X_1^2}\right\}+2C\frac{\hat{\mathbb{E}}\mathrm{e}^{\delta|X_1|}\mathrm{e}^{-\delta c_n}}{\delta^2\varepsilon_n^2 n}+2C\frac{\hat{\mathbb{E}}\mathrm{e}^{\delta|X_1|}\mathrm{e}^{-\delta c_n}}{\delta^2\varepsilon_n^2 n}$$
$$\leq 2C\left(1+\frac{2\hat{\mathbb{E}}\mathrm{e}^{\delta|X_1|}}{\delta^3\mathrm{e}\hat{\mathbb{E}}X_1^2 c_n}\right)\mathrm{e}^{-\delta c_n}.$$

The proof of Theorem 3.3 is completed. □

Proof of Theorem 3.4. Taking $c_n=\ln n$ and $\delta>1$ in Theorem 3.3, can get the following result

$$\sum_{n=1}^{\infty}\mathbb{V}\left(\frac{1}{n}\sum_{i=1}^{n}\left(X_i-\hat{\mathbb{E}}X_i\right)>3\varepsilon_n\right)$$
$$=\sum_{n=1}^{\infty}\mathbb{V}\left(\frac{1}{n}\sum_{i=1}^{n}\left(X_i-\hat{\mathbb{E}}X_i\right)>3\sqrt{8\delta\mathrm{e}\hat{\mathbb{E}}X_1^2\ln n/n}\right)$$
$$=\sum_{n=1}^{\infty}\mathbb{V}\left(\frac{1}{n}\sum_{i=1}^{n}\left(X_i-\hat{\mathbb{E}}X_i\right)>3\sqrt{8\delta\mathrm{e}\hat{\mathbb{E}}X_1^2}\left(n^{-1/2}\ln^{1/2}n\right)\right)$$
$$\leq C\sum_{i=1}^{\infty}\left(1+\frac{2\hat{\mathbb{E}}\mathrm{e}^{\delta|X_1|}}{\delta^3\mathrm{e}\hat{\mathbb{E}}X_1^2 c_n}\right)\mathrm{e}^{-\delta c_n}$$
$$=C\left(\sum_{n=1}^{\infty}\frac{1}{n^\delta}+\frac{2\hat{\mathbb{E}}\mathrm{e}^{\delta|X_1|}}{\delta^3\mathrm{e}\hat{\mathbb{E}}X_1^2}\sum_{i=1}^{\infty}\frac{1}{n^\delta\ln n}\right)<\infty, \tag{4.1}$$

and

$$\sum_{n=1}^{\infty}\mathbb{V}\left(\frac{1}{n}\sum_{i=1}^{n}(X_i-\hat{\varepsilon}X_i)<-3\varepsilon_n\right)$$

$$
\begin{aligned}
&=\sum_{n=1}^{\infty} \mathbb{V}\left(\frac{1}{n}\sum_{i=1}^{n}(X_i - \hat{\varepsilon}X_i) > 3\sqrt{8\delta \mathrm{e}\hat{\mathbb{E}}X_1^2 \ln n/n}\right)\\
&=\sum_{n=1}^{\infty} \mathbb{V}\left(\frac{1}{n}\sum_{i=1}^{n}(X_i - \hat{\varepsilon}X_i) < -3\sqrt{8\delta \mathrm{e}\hat{\mathbb{E}}X_1^2}\left(n^{-1/2}\ln^{1/2} n\right)\right)\\
&\leq C\sum_{i=1}^{\infty}\left(1+\frac{2\hat{\mathbb{E}}\mathrm{e}^{\delta|X_1|}}{\delta^3 \mathrm{e}\hat{\mathbb{E}}X_1^2 c_n}\right)\mathrm{e}^{-\delta c_n}\\
&=C\left(\sum_{n=1}^{\infty}\frac{1}{n^{\delta}}+\frac{2\hat{\mathbb{E}}\mathrm{e}^{\delta|X_1|}}{\delta^3 \mathrm{e}\hat{\mathbb{E}}X_1^2}\sum_{i=1}^{\infty}\frac{1}{n^{\delta}\ln n}\right) < \infty. \qquad (4.2)
\end{aligned}
$$

From Eqs (4.1) and (4.2), we can obtain that $\sum_{n=1}^{\infty}\mathbb{V}\left(\frac{1}{n}\sum_{i=1}^{n} a_n\left(X_i - \hat{\mathbb{E}}X_i\right) > \varepsilon\right) < \infty$, and $\sum_{n=1}^{\infty}\mathbb{V}\left(\frac{1}{n}\sum_{i=1}^{n} a_n\left(X_i - \hat{\varepsilon}X_i\right) < -\varepsilon\right) < \infty$ for any $\varepsilon > 0$ and $a_n = O(n^{1/2}\ln^{-1/2} n)$. Then for Lemma 2.3 (Borel-Canttelli's lemma) and $\mathbb{V}$ being countably sub-additive, we have

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} a_n\left(X_i - \hat{\mathbb{E}}X_i\right) \leq 0 \qquad \text{a.s.} \quad \mathbb{V},$$

and

$$\liminf_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} a_n\left(X_i - \hat{\varepsilon}X_i\right) \geq 0 \qquad \text{a.s.} \quad \mathbb{V}.$$

That is, (3.12) and (3.13) holds.

In particular, if $\hat{\mathbb{E}}X_i = \hat{\varepsilon}X_i$, then $\sum_{n=1}^{\infty}\mathbb{V}\left(\frac{1}{n}\sum_{i=1}^{n} a_n\left|\left(X_i - \hat{\mathbb{E}}X_i\right)\right| > \varepsilon\right) < \infty$ can be obtained directly from Eqs (4.1) and (4.2). By Lemma 2.3 (Borel-Canttelli's lemma) and $\mathbb{V}$ being countably sub-additive, we have

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} a_n\left(X_i - \hat{\mathbb{E}}X_i\right) = 0 \qquad \text{a.s.} \quad \mathbb{V}.$$

That completes the proof of Theorem 3.4. □

5. Conclusions

This paper presents new results regarding exponential inequalities and a strong law of large numbers for END random variables under sub-linear expectations. These theorems extend the corresponding results in classical linear expectation space.

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Conflict of interest

In this paper, all authors disclaim any conflict of interest.

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