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Research article

Fuzzy differential subordination and superordination results for q-analogue of multiplier transformation

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Abstract: In this paper the authors combine the quantum calculus applications regarding the theories of differential subordination and superordination with fuzzy theory. These results are established by means of an operator defined as the q-analogue of the multiplier transformation. Interesting fuzzy differential subordination and superordination results are derived by the authors involving the functions belonging to a new class of normalized analytic functions in the open unit disc U which is defined and investigated here by using this q-operator.

Keywords: convex function; differential operator; q-analogue operator; fuzzy differential subordination; fuzzy best dominant; fuzzy differential superordination; fuzzy best subordinant
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1. Introduction

Numerous applications of the quantum calculus in geometric function theory have emerged in the recent years after the general context for such research was established by Srivastava in a book chapter published in 1989 [1]. Certain aspects regarding the use of quantum calculus in geometric function theory are highlighted in a recent paper [2] and other developments are emphasized in the review done by Srivastava in 2020 [3] alongside the multitude of q-operators derived by involving well-known differential and integral operators specific to geometric function theory.

Many investigations concerned q-analogue of Ruscheweyh differential operator defined in [4] and q-analogue of Sălăgean differential operator introduced in [5]. For example, differential subordinations are investigated involving a certain q-Ruscheweyh type derivative operator in [6], q-Ruscheweyh derivative operator is used for the definition and coefficient estimates investigation of a new class of analytic functions in [7] and classes of analytic univalent functions are introduced and investigated in [8] using both Ruscheweyh and Sălăgean q-analogue operators. Subordination results involving q-analogue of the Sălăgean differential operator are obtained in [9] and a generalization of the Sălăgean q-differential operator is involved in the study of certain differential subordinations in [10]. q-Bernardi integral operator is introduced in [11] and Srivastava-Attiya operator and the multiplier transformation are adapted to quantum calculus approach in [12].

In recent investigations, q-analogue of the multiplier transformation was used to define and study new subclasses of harmonic univalent functions [13] and to obtain fuzzy differential subordinations in [14].

Lotfi A. Zadeh in 1965 [15] introduced the notion of fuzzy set and in [16] and [17] are exposed different applications of this notion. In geometric function theory was introduced in 2011 [18] the concept of fuzzy subordination and the theory of fuzzy differential subordination develops since 2012 [19] when Miller and Mocanu [20] adapted the classical theory of differential subordination to the fuzzy theory. In 2017 was introduced the dual notion, fuzzy differential superordination [21]. Since then, numerously researchers studied different properties [22] of differential operators involving fuzzy differential subordinations and superordinations: Wanas operator [23, 24], generalized Noor-Sălăgean operator [25], Sălăgean and Ruscheweyh operators [26] or a linear operator [27]

In this paper, fuzzy differential subordinations and superordinations is added to the study associated to the quantum calculus for the first time using the q-analogue of the multiplier transformation [28].

The notions and preliminary known results used in the research are first introduced.

The investigation is set in the unit disc of the complex plane, $U = \{z \in \mathbb{C} : |z| < 1\}$ and involves the class of holomorphic functions in the unit disc $\mathcal{H}(U)$.

Particular subclasses of $\mathcal{H}(U)$ involved are:

$$\mathcal{A}(p,n) = \{f(z) = z^p + \sum_{j=p+n}^{\infty} a_j z^j \in \mathcal{H}(U)\},$$

with $\mathcal{A}_n = \mathcal{A}(1,n)$, $\mathcal{A} = \mathcal{A}_1 = \mathcal{A}(1,1)$ and

$$\mathcal{H}[a,n] = \{f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots \in \mathcal{H}(U)\},$$

where $n, p \in \mathbb{N}$, $a \in \mathbb{C}$.

Definition 1. *([18]) The pair $(A, \mathcal{F}_{\mathcal{A}})$ i the fuzzy subset of X, where $\mathcal{F}_{\mathcal{A}} : X \to [0,1]$ and $\mathcal{A} = \{x \in X : 0 < \mathcal{F}_{\mathcal{A}}(x) \leq 1\}$. The set $\mathcal{A}$ is the support of the fuzzy set $(\mathcal{A}, \mathcal{F}_{\mathcal{A}})$ and $\mathcal{F}_{\mathcal{A}}$ is he membership function of $(\mathcal{A}, \mathcal{F}_{\mathcal{A}})$. It is denoted $\mathcal{A} = \operatorname{supp}(\mathcal{A}, \mathcal{F}_{\mathcal{A}})$.*

Definition 2. *([18]) The function $f \in \mathcal{H}(D)$ is fuzzy subordinate to the function $g \in \mathcal{H}(D)$, denoted $f \prec_{\mathcal{F}} g$, where $D \subset \mathbb{C}$, if the following conditions hold:*
1) $f(z_0) = g(z_0)$, for $z_0 \in D$ a fixed point,
2) $\mathcal{F}_{f(D)} f(z) \leq \mathcal{F}_{g(D)} g(z)$, $z \in D$.

Definition 3. *([19, Definition 2.2]) Let h be a univalent function in U and $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$, with $h(0) = \psi(a, 0; 0) = a$. If the function p is analytic in U, having the property $p(0) = a$ and the fuzzy differential subordination holds*

$$\mathcal{F}_{\psi(\mathbb{C}^3 \times U)} \psi(p(z), zp'(z), z^2 p''(z); z) \leq \mathcal{F}_{h(U)} h(z), \quad z \in U, \tag{1.1}$$

then p is a fuzzy solution of the fuzzy differential subordination. The univalent function q is a fuzzy dominant of the fuzzy solutions of the fuzzy differential subordination, when $\mathcal{F}_{p(U)}p(z) \leq \mathcal{F}_{q(U)}q(z)$, $z \in U$, for all p satisfying (1.1). A fuzzy dominant $\widetilde{q}$ with property $\mathcal{F}_{\widetilde{q}(U)}\tilde{q}(z) \leq \mathcal{F}_{q(U)}q(z)$, $z \in U$, for all fuzzy dominants q of (1.1) is the fuzzy best dominant of (1.1).

Definition 4. *([21]) Let h be an analytic function in U and $\varphi : \mathbb{C}^3 \times U \to \mathbb{C}$. If p and $\varphi(p(z), zp'(z), z^2p''(z); z)$ are univalent functions in U and the fuzzy differential superordination hold*

$$\mathcal{F}_{h(U)}h(z) \leq \mathcal{F}_{\varphi(\mathbb{C}^3\times U)}\varphi(p(z), zp'(z), z^2p''(z); z), \quad z \in U, \tag{1.2}$$

i.e.,

$$h(z) \prec_{\mathcal{F}} \varphi(p(z), zp'(z), z^2p''(z); z), \quad z \in U,$$

then p is a fuzzy solution of the fuzzy differential superordination. An analytic function q is fuzzy subordinant of the fuzzy differential superordination when

$$\mathcal{F}_{q(U)}q(z) \leq \mathcal{F}_{p(U)}p(z), \quad z \in U,$$

for all p satisfying (1.2). A univalent fuzzy subordination $\widetilde{q}$ with property $\mathcal{F}_{q(U)}q \leq \mathcal{F}_{q(U)}\widetilde{q}$ for all fuzzy subordinate q of (1.2) is the fuzzy best subordinate of (1.2).

Definition 5. *([19]) Q denotes the set of all injective and analytic functions f on $\overline{U}\backslash E(f)$, with property $f'(\zeta) \neq 0$ for $\zeta \in \partial U\backslash E(f)$, and $E(f) = \{\zeta \in \partial U : \lim_{z\to\zeta} f(z) = \infty\}$.*

The following lemmas are useful for the proofs of the new results contained in the next sections.

Lemma 1. *([29]) Let a convex function g in U and the function*

$$h(z) = n\alpha zg'(z) + g(z),$$

with $z \in U$, n a positive integer and $\alpha > 0$.

When the function

$$g(0) + p_n z^n + p_{n+1}z^{n+1} + \cdots = p(z), \quad z \in U$$

is holomorphic in U and the fuzzy differential subordination

$$\mathcal{F}_{p(U)}\left(\alpha zp'(z) + p(z)\right) \leq \mathcal{F}_{h(U)}h(z), \quad z \in U,$$

holds, then the fuzzy differential subordination

$$\mathcal{F}_{p(U)}p(z) \leq \mathcal{F}_{g(U)}g(z)$$

holds too, and the result is sharp.

Lemma 2. *([29]) Let a convex function h such that $h(0) = a$, and $\alpha \in \mathbb{C}^*$ with $Re\ \alpha \geq 0$. When $p \in \mathcal{H}[a, n]$ and the fuzzy differential subordination*

$$\mathcal{F}_{p(U)}\left(\frac{zp'(z)}{\gamma} + p(z)\right) \leq \mathcal{F}_{h(U)}h(z), \quad z \in U,$$

holds, then the fuzzy differential subordinations

$$\mathcal{F}_{p(U)}p(z) \le \mathcal{F}_{g(U)}g(z) \le \mathcal{F}_{h(U)}h(z), \quad z \in U,$$

hold for

$$g(z) = \frac{\alpha}{nz^{\frac{\alpha}{n}}} \int_0^z h(t)t^{\frac{\alpha}{n}-1}dt, \quad z \in U.$$

Lemma 3. *([20, Corollary 2.6g.2, p. 66]) Let a convex function h such that $h(0) = a$, and $\alpha \in \mathbb{C}^*$ with $Re\ \alpha \ge 0$. When $p \in Q \cap \mathcal{H}[a, n]$, $\frac{zp'(z)}{\alpha} + p(z)$ is a univalent function in U and the fuzzy differential superordination*

$$\mathcal{F}_{h(U)}h(z) \le \mathcal{F}_{p(U)}\left(\frac{zp'(z)}{\alpha} + p(z)\right),\ z \in U,$$

holds, then the fuzzy differential superordination

$$\mathcal{F}_{g(U)}g(z) \le \mathcal{F}_{p(U)}p(z),\ \ z \in U,$$

holds too, for the convex function $g(z) = \frac{\alpha}{nz^{\frac{\alpha}{n}}} \int_0^z h(t)t^{\frac{\alpha}{n}-1}dt$, $z \in U$ the fuzzy best subordinant.

Lemma 4. *([20, Corollary 2.6g.2, p. 66]) Consider a convex function g in U and the function*

$$h(z) = \frac{zg'(z)}{\alpha} + g(z),\ \ z \in U,$$

with $\alpha \in \mathbb{C}^*$, $Re\ \alpha \ge 0$. If $p \in Q \cap \mathcal{H}\,[a, n]$, $\frac{zp'(z)}{\alpha} + p(z)$ is a univalent function in U and the fuzzy differential superordination

$$\mathcal{F}_{g(U)}\left(\frac{zg'(z)}{\alpha} + g(z)\right) \le \mathcal{F}_{p(U)}\left(\frac{zp'\,(z)}{\alpha} + p(z)\right),\ \ z \in U,$$

holds, then the fuzzy differential superordination

$$\mathcal{F}_{g(U)}g(z) \le \mathcal{F}_{p(U)}p(z),\ z \in U,$$

holds too, for $g(z) = \frac{\alpha}{nz^{\frac{\alpha}{n}}} \int_0^z h(t)t^{\frac{\alpha}{n}-1}dt$, $z \in U$ the fuzzy best subordinant.

We remind the definition of the q-analogue of the multiplier transformation:

Definition 6. *([12]) Denote by $\mathcal{I}_q^{m,l}$ the q-analogue of multiplier transformation*

$$\mathcal{I}_q^{m,l} f\,(z) := z + \sum_{k=2}^{\infty} \left(\frac{[k + l]_q}{[1 + l]_q}\right)^m a_k z^k,$$

with $l > -1$, $q \in (0, 1)$, m a real number and $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in \mathcal{A}$, $z \in U$.

Applying the properties of q-calculus we get

$$z\mathcal{D}_q\left(\mathcal{I}_q^{m,l} f\,(z)\right) = \left(1 + \frac{[l]_q}{q^l}\right)\mathcal{I}_q^{m+1,l} f\,(z) - \frac{[l]_q}{q^l}\mathcal{I}_q^{m,l} f\,(z)\,.$$

Using the q-analogue of the multiplier transformation $\mathcal{I}_q^{m,l}$ given in Definition 6, a new subclass of normalized analytic functions in the open unit disc U is introduced in Section 2 of this paper. It is proved that this new class is convex and using this property, fuzzy subordination results are investigated in the theorems of Section 2 involving functions from the newly defined class, operator $\mathcal{I}_q^{m,l}$ and Lemmas 1 and 2. In Section 3, fuzzy differential superordinations involving the operator $\mathcal{I}_q^{m,l}$ are considered for which the best subordinants are also found. Lemmas 3 and 4 are necessary for establishing the new results.

2. Fuzzy differential subordination results

The new class of normalized analytic functions in the open unit disc U is defined using the q-analogue of the multiplier transformation $\mathcal{I}_q^{m,l}$ given in Definition 6.

Definition 7. *Let $\alpha \in [0,1)$. The class $\mathcal{F}S_{m,l}^{q}(\alpha)$ consists the functions $f \in \mathcal{A}$ with property*

$$\mathcal{F}_{\left(\mathcal{I}_q^{m,l}f\right)'(U)}\left(\mathcal{I}_q^{m,l}f(z)\right)' > \alpha, \qquad z \in U. \tag{2.1}$$

The first result concerning the class $\mathcal{F}S_{m,l}^{q}(\alpha)$ establishes its convexity.

Theorem 1. *$\mathcal{F}S_{m,l}^{q}(\alpha)$ is a convex set.*

Proof. Consider the functions

$$f_j(z) = z + \sum_{k=2}^{\infty} a_{jk} z^k, \quad z \in U, \quad j = 1, 2,$$

belonging to the class $\mathcal{F}S_{m,l}^{q}(\alpha)$. It is enough to prove that the function

$$f(z) = \lambda_1 f_1(z) + \lambda_2 f_2(z)$$

belongs to the class $\mathcal{F}S_m(\delta,\alpha)$, with λ_1 and λ_2 positive real numbers such that $\lambda_1 + \lambda_2 = 1$.

The function f has the following form

$$f(z) = z + \sum_{k=2}^{\infty} (\lambda_1 a_{1k} + \lambda_2 a_{2k}) z^k, \qquad z \in U,$$

and

$$\mathcal{I}_q^{m,l} f(z) = z + \sum_{k=2}^{\infty} \left(\frac{[k+l]_q}{[1+l]_q}\right)^m (\lambda_1 a_{1k} + \lambda_2 a_{2k}) z^k, \quad z \in U. \tag{2.2}$$

Differentiating relation (2.2) we obtain

$$\left(\mathcal{I}_q^{m,l} f(z)\right)' = 1 + \sum_{k=2}^{\infty} \left(\frac{[k+l]_q}{[1+l]_q}\right)^m (\lambda_1 a_{1k} + \lambda_2 a_{2k}) k z^{k-1}, \quad z \in U.$$

We have $f'(z) = (\lambda_1 f_1 + \lambda_2 f_2)'(z) = \lambda_1 f_1'(z) + \lambda_2 f_2'(z)$, $z \in U$, and
$\left(\mathcal{I}_q^{m,l} f(z)\right)' = \left(\mathcal{I}_q^{m,l}(\lambda_1 f_1 + \lambda_2 f_2)(z)\right)' = \lambda_1\left(\mathcal{I}_q^{m,l} f_1(z)\right)' + \lambda_2\left(\mathcal{I}_q^{m,l} f_2(z)\right)'$.

Applying fuzzy theory, we obtain that

$\mathcal{F}_{\left(\mathcal{I}_q^{m,l}f\right)'(U)}\left(\mathcal{I}_q^{m,l}f(z)\right)' = \mathcal{F}_{\left(\mathcal{I}_q^{m,l}(\lambda_1 f_1+\lambda_2 f_2)\right)'(U)}\left(\mathcal{I}_q^{m,l}(\lambda_1 f_1 + \lambda_2 f_2)(z)\right)' =$
$\mathcal{F}_{\left(\mathcal{I}_q^{m,l}(\lambda_1 f_1+\lambda_2 f_2)\right)'(U)}\left(\lambda_1\left(\mathcal{I}_q^{m,l}f_1(z)\right)' + \lambda_2\left(\mathcal{I}_q^{m,l}f_2(z)\right)'\right) =$
$\frac{\mathcal{F}_{\left(\lambda_1\mathcal{I}_q^{m,l}f_1\right)'(U)}\left(\lambda_1\left(\mathcal{I}_q^{m,l}f_1(z)\right)'\right)+\mathcal{F}_{\left(\lambda_2\mathcal{I}_q^{m,l}f_2\right)'(U)}\left(\lambda_2\left(\mathcal{I}_q^{m,l}f_2(z)\right)'\right)}{2} =$
$\frac{\mathcal{F}_{\left(\mathcal{I}_q^{m,l}f_1\right)'(U)}\left(\mathcal{I}_q^{m,l}f_1(z)\right)'+\mathcal{F}_{\left(\mathcal{I}_q^{m,l}f_2\right)'(U)}\left(\mathcal{I}_q^{m,l}f_2(z)\right)'}{2}$.

Since $f_1, f_2 \in \mathcal{F}S_{m,l}^{q}(\alpha)$ we have $\alpha < \mathcal{F}_{\left(\mathcal{I}_q^{m,l}f_1\right)'(U)}\left(\mathcal{I}_q^{m,l}f_1(z)\right)' \leq 1$ and $\alpha < \mathcal{F}_{\left(\mathcal{I}_q^{m,l}f_2\right)'(U)}\left(\mathcal{I}_q^{m,l}f_2(z)\right)' \leq 1$, $z \in U$. Therefore $\alpha < \frac{\mathcal{F}_{\left(\mathcal{I}_q^{m,l}f_1\right)'(U)}\left(\mathcal{I}_q^{m,l}f_1(z)\right)'+\mathcal{F}_{\left(\mathcal{I}_q^{m,l}f_2\right)'(U)}\left(\mathcal{I}_q^{m,l}f_2(z)\right)'}{2} \leq 1$ and we obtain that $\alpha < \mathcal{F}_{\left(\mathcal{I}_q^{m,l}f\right)'(U)}\left(\mathcal{I}_q^{m,l}f(z)\right)' \leq 1$, which means that $f \in \mathcal{F}S_{m,l}^{q}(\alpha)$ and $\mathcal{F}S_{m,l}^{q}(\alpha)$ is convex. □

We next investigate a series of fuzzy differential subordinations involving the convex functions of the class $S_{m,l}^{q}(\alpha)$ and the q-analogue of the multiplier transformation $\mathcal{I}_q^{m,l}$.

Theorem 2. *Considering g a convex function, we define the function*

$$h(z) = \frac{zg'(z)}{a+2} + g(z), \quad a > 0, \ z \in U. \tag{2.3}$$

For $f \in \mathcal{F}S_{m,l}^{q}(\alpha)$ consider

$$F(z) = \frac{a+2}{z^{a+1}} \int_0^z t^a f(t)\, dt, \quad z \in U, \tag{2.4}$$

then the fuzzy differential subordination

$$\mathcal{F}_{\left(\mathcal{I}_q^{m,l} f\right)'(U)} \left(\mathcal{I}_q^{m,l} f(z)\right)' \leq \mathcal{F}_{h(U)} h(z), \quad z \in U, \tag{2.5}$$

implies the sharp fuzzy differential subordination

$$\mathcal{F}_{\left(\mathcal{I}_q^{m,l} F\right)'(U)} \left(\mathcal{I}_q^{m,l} F(z)\right)' \leq \mathcal{F}_{g(U)} g(z), \quad z \in U.$$

Proof. Relation (2.4) can be written as following

$$z^{a+1} F(z) = (a+2) \int_0^z t^a f(t)\, dt, \tag{2.6}$$

and differentiating it, we have

$$zF'(z) + (a+1) F(z) = (a+2) f(z) \tag{2.7}$$

and

$$z\left(\mathcal{I}_q^{m,l} F(z)\right)' + (a+1) \mathcal{I}_q^{m,l} F(z) = (a+2) \mathcal{I}_q^{m,l} f(z), \quad z \in U. \tag{2.8}$$

Differentiating the last relation, we get

$$\frac{z\left(\mathcal{I}_q^{m,l} F(z)\right)''}{a+2} + \left(\mathcal{I}_q^{m,l} F(z)\right)' = \left(\mathcal{I}_q^{m,l} f(z)\right)', \quad z \in U, \tag{2.9}$$

and the fuzzy subordination (2.5) can be written in the form

$$\mathcal{F}_{\left(\mathcal{I}_q^{m,l} F\right)'(U)} \left(\frac{z\left(\mathcal{I}_q^{m,l} F(z)\right)''}{a+2} + \left(\mathcal{I}_q^{m,l} F(z)\right)'\right) \leq \mathcal{F}_{g(U)} \left(\frac{zg'(z)}{a+2} + g(z)\right). \tag{2.10}$$

Denoting

$$p(z) = \left(\mathcal{I}_q^{m,l} F(z)\right)' \in \mathcal{H}[1,1], \tag{2.11}$$

fuzzy differential subordination (2.10) has the following form

$$\mathcal{F}_{p(U)} \left(\frac{zp'(z)}{a+2} + p(z)\right) \leq \mathcal{F}_{g(U)} \left(\frac{zg'(z)}{a+2} + g(z)\right), \quad z \in U.$$

Applying Lemma 1 we get

$$\mathcal{F}_{p(U)}p(z) \leq \mathcal{F}_{g(U)}g(z),$$

that means

$$\mathcal{F}_{\left(\mathcal{I}_q^{m,l}F\right)'(U)}\left(\mathcal{I}_q^{m,l}F(z)\right)' \leq F_{g(U)}g(z), \quad z \in U$$

for g the fuzzy best dominant. □

Theorem 3. *Denoting*

$$I_a(f)(z) = \frac{a+2}{z^{a+1}}\int_0^z t^a f(t)\,dt, \ \ a > 0, \tag{2.12}$$

then

$$I_a\left[S_{m,l}^{q}(\alpha)\right] \subset S_{m,l}^{q}(\alpha^*), \tag{2.13}$$

where

$$\alpha^* = 2\alpha - 1 + (a+2)(2-2\alpha)\int_0^1 \frac{t^{a+1}}{t+1}dt. \tag{2.14}$$

Proof. We obtain using the hypothesis of Theorem 3 for the convex function $h(z) = \frac{1+(2\alpha-1)z}{1+z}$ and following the same steps like the proof of Theorem 2 the fuzzy differential subordination

$$\mathcal{F}_{p(U)}\left(\frac{zp'(z)}{a+2} + p(z)\right) \leq F_{h(U)}h(z),$$

with p defined by relation (2.11).

Applying Lemma 2 we get the fuzzy differential subordinations

$$\mathcal{F}_{p(U)}p(z) \leq \mathcal{F}_{g(U)}g(z) \leq \mathcal{F}_{h(U)}h(z),$$

equivalently with

$$\mathcal{F}_{\left(\mathcal{I}_q^{m,l}F\right)'(U)}\left(\mathcal{I}_q^{m,l}F(z)\right)' \leq \mathcal{F}_{g(U)}g(z) \leq \mathcal{F}_{h(U)}h(z),$$

where

$$g(z) = \frac{a+2}{z^{a+2}}\int_0^z t^{a+1}\frac{1+(2\alpha-1)t}{1+t}dt =$$

$$2\alpha - 1 + \frac{(a+2)(2-2\alpha)}{z^{a+2}}\int_0^z \frac{t^{a+1}}{t+1}dt.$$

Taking account that g is a convex function with $g(U)$ symmetric to the real axis, we obtain

$$\mathcal{F}_{\left(\mathcal{I}_q^{m,l}F\right)'(U)}\left(\mathcal{I}_q^{m,l}F(z)\right)' \geq \min_{|z|=1}\mathcal{F}_{g(U)}g(z) = \mathcal{F}_{g(U)}g(1) =$$

$$\alpha^* = 2\alpha - 1 + (a+2)(2-2\alpha)\int_0^1 \frac{t^{a+1}}{t+1}.$$

□

Theorem 4. *Considering the convex function g such that* $g(0) = 1$, *we define the function*

$$h(z) = zg'(z) + g(z), \qquad z \in U.$$

If $f \in \mathcal{A}$ *verifies the fuzzy subordination*

$$\mathcal{F}_{\left(\mathcal{I}_q^{m,l} f\right)'(U)} \left(\mathcal{I}_q^{m,l} f(z)\right)' \le \mathcal{F}_{h(U)} h(z), \qquad z \in U, \tag{2.15}$$

then the sharp fuzzy differential subordination

$$\mathcal{F}_{\mathcal{I}_q^{m,l} f(U)} \frac{\mathcal{I}_q^{m,l} f(z)}{z} \le F_{g(U)} g(z), \qquad z \in U$$

holds.

Proof. Considering

$$p(z) = \frac{\mathcal{I}_q^{m,l} f(z)}{z} = \frac{z + \sum_{k=2}^{\infty} \left(\frac{[k+l]_q}{[1+l]_q}\right)^m a_k z^k}{z} = 1 + p_1 z + p_2 z^2 + ..., \qquad z \in U,$$

evidently $p \in \mathcal{H}[1, 1]$, so we can write

$$zp(z) = \mathcal{I}_q^{m,l} f(z), \qquad z \in U,$$

and differentiating it we get

$$\left(\mathcal{I}_q^{m,l} f(z)\right)' = zp'(z) + p(z), \qquad z \in U.$$

The fuzzy subordination (2.15) take the form

$$\mathcal{F}_{p(U)} \left(zp'(z) + p(z)\right) \le \mathcal{F}_{g(U)} \left(zg'(z) + g(z)\right), \qquad z \in U,$$

and applying Lemma 1, we get

$$\mathcal{F}_{p(U)} p(z) \le \mathcal{F}_{g(U)} g(z), \qquad z \in U,$$

that means

$$\mathcal{F}_{\mathcal{I}_q^{m,l} f(U)} \frac{\mathcal{I}_q^{m,l} f(z)}{z} \le \mathcal{F}_{g(U)} g(z), \qquad z \in U.$$

□

Theorem 5. *Considering the convex function h with the property* $h(0) = 1$, *for* $f \in \mathcal{A}$ *that verifies the fuzzy subordination*

$$\mathcal{F}_{\left(\mathcal{I}_q^{m,l} f\right)'(U)} \left(\mathcal{I}_q^{m,l} f(z)\right)' \le \mathcal{F}_{h(U)} h(z), \qquad z \in U, \tag{2.16}$$

we get the fuzzy differential subordination

$$\mathcal{F}_{\mathcal{I}_q^{m,l} f(U)} \frac{\mathcal{I}_q^{m,l} f(z)}{z} \le \mathcal{F}_{g(U)} g(z), \qquad z \in U,$$

for the convex function $g(z) = \frac{1}{z} \int_0^z h(t)dt$, *which is the fuzzy best dominant.*

Proof. Denote
$$p(z)=\frac{\mathcal{I}_q^{m,l}f(z)}{z}=1+\sum_{k=2}^{\infty}\left(\frac{[k+l]_q}{[1+l]_q}\right)^m a_k z^{k-1}\in\mathcal{H}[1,1],\quad z\in U.$$

Differentiating it the relation, we get
$$\left(\mathcal{I}_q^{m,l}f(z)\right)'=zp'(z)+p(z),\qquad z\in U$$
and fuzzy differential subordination (2.16) has the following form
$$\mathcal{F}_{p(U)}\left(zp'(z)+p(z)\right)\le\mathcal{F}_{h(U)}h(z),\qquad z\in U.$$

After applying Lemma 2, we get
$$\mathcal{F}_{p(U)}p(z)\le\mathcal{F}_{g(U)}g(z),\quad z\in U,$$
written as
$$\mathcal{F}_{\mathcal{I}_q^{m,l}f(U)}\frac{\mathcal{I}_q^{m,l}f(z)}{z}\le F_{g(U)}g(z),\quad z\in U$$
for $g(z)=\frac{1}{z}\int_0^z h(t)dt$ the fuzzy best dominant. □

Theorem 6. *Considering a convex function g such that $g(0)=1$, we define the function $h(z)=zg'(z)+g(z)$, $z\in U$. When $f\in\mathcal{A}$ verifies the fuzzy subordination*
$$\mathcal{F}_{\mathcal{I}_q^{m,l}f(U)}\left(\frac{z\mathcal{I}_q^{m+1,l}f(z)}{\mathcal{I}_q^{m,l}f(z)}\right)'\le\mathcal{F}_{h(U)}h(z),\quad z\in U,\tag{2.17}$$
then we get the sharp fuzzy differential subordination
$$\mathcal{F}_{\mathcal{I}_q^{m,l}f(U)}\frac{\mathcal{I}_q^{m+1,l}f(z)}{\mathcal{I}_q^{m,l}f(z)}\le\mathcal{F}_{g(U)}g(z),\quad z\in U.$$

Proof. Denote
$$p(z)=\frac{\mathcal{I}_q^{m+1,l}f(z)}{\mathcal{I}_q^{m,l}f(z)}=\frac{z+\sum_{k=2}^{\infty}\left(\frac{[k+l]_q}{[1+l]_q}\right)^{m+1}a_kz^k}{z+\sum_{k=2}^{\infty}\left(\frac{[k+l]_q}{[1+l]_q}\right)^{m}a_kz^k}.$$

Differentiating it we get $p'(z)=\frac{\left(\mathcal{I}_q^{m+1,l}f(z)\right)'}{\mathcal{I}_q^{m,l}f(z)}-p(z)\frac{\left(\mathcal{I}_q^{m,l}f(z)\right)'}{\mathcal{I}_q^{m,l}f(z)}$ written as $zp'(z)+p(z)=\left(\frac{z\mathcal{I}_q^{m+1,l}f(z)}{\mathcal{I}_q^{m,l}f(z)}\right)'$.

Fuzzy differential subordination (2.17) has the following form, for $z\in U$,
$$\mathcal{F}_{p(U)}\left(zp'(z)+p(z)\right)\le\mathcal{F}_{g(U)}\left(zg'(z)+g(z)\right),$$
and applying Lemma 1, we get the fuzzy differential subordination, for $z\in U$,
$$\mathcal{F}_{p(U)}p(z)\le\mathcal{F}_{g(U)}g(z),$$
written as
$$\mathcal{F}_{\mathcal{I}_q^{m,l}f(U)}\frac{\mathcal{I}_q^{m+1,l}f(z)}{\mathcal{I}_q^{m,l}f(z)}\le\mathcal{F}_{g(U)}g(z).$$
□

3. Fuzzy differential superordination results

In this section, fuzzy differential superordinations are investigated concerning q-analogue of the multiplier transformation $\mathcal{I}_q^{m,l}$ and its derivatives of first and second order. The fuzzy best subordinant is given for each of the investigated fuzzy differential superordinations.

Theorem 7. *Considering $f \in \mathcal{A}$ and a convex function h in U such that $h(0) = 1$, denote $F(z) = \frac{a+2}{z^{a+1}} \int_0^z t^a f(t)\,dt$, $z \in U$, Re $a > -2$, and assume that $\left(\mathcal{I}_q^{m,l} f(z)\right)'$ is a univalent function in U, $\left(\mathcal{I}_q^{m,l} F(z)\right)' \in \mathcal{Q} \cap \mathcal{H}[1,1]$. If the fuzzy differential superordination*

$$\mathcal{F}_{h(U)} h(z) \leq \mathcal{F}_{\left(\mathcal{I}_q^{m,l} f\right)'(U)} \left(\mathcal{I}_q^{m,l} f(z)\right)', \quad z \in U, \tag{3.1}$$

states, then we get the fuzzy differential superordination

$$\mathcal{F}_{g(U)} g(z) \leq \mathcal{F}_{\left(\mathcal{I}_q^{m,l} F\right)'(U)} \left(\mathcal{I}_q^{m,l} F(z)\right)', \quad z \in U,$$

with the convex function $g(z) = \frac{a+2}{z^{a+2}} \int_0^z h(t) t^{a+1} dt$ the fuzzy best subordinant.

Proof. Using the relation $z^{a+1} F(z) = (a+2) \int_0^z t^a f(t)\,dt$ from Theorem 2 and differentiating it we can write $zF'(z) + (a+1) F(z) = (a+2) f(z)$ in the following form $z\left(\mathcal{I}_q^{m,l} F(z)\right)' + (a+1) \mathcal{I}_q^{m,l} F(z) = (a+2) \mathcal{I}_q^{m,l} f(z)$, $z \in U$, which after differentiating it again has the form

$$\frac{z\left(\mathcal{I}_q^{m,l} F(z)\right)''}{a+2} + \left(\mathcal{I}_q^{m,l} F(z)\right)' = \left(\mathcal{I}_q^{m,l} f(z)\right)', \quad z \in U.$$

Using the last relation, the fuzzy differential superordination (3.1) can be written

$$\mathcal{F}_{h(U)} h(z) \leq \mathcal{F}_{\left(\mathcal{I}_q^{m,l} F\right)'(U)} \left(\frac{z\left(\mathcal{I}_q^{m,l} F(z)\right)''}{a+2} + \left(\mathcal{I}_q^{m,l} F(z)\right)' \right). \tag{3.2}$$

Define

$$p(z) = \left(\mathcal{I}_q^{m,l} F(z)\right)', \quad z \in U, \tag{3.3}$$

and replacing (3.3) in (3.2) we have $\mathcal{F}_{h(U)} h(z) \leq \mathcal{F}_{p(U)} \left(\frac{zp'(z)}{a+2} + p(z)\right)$, $z \in U$. Applying Lemma 3 considering $n = 1$ and $\alpha = a+2$, it yields $\mathcal{F}_{g(U)} g(z) \leq \mathcal{F}_{p(U)} p(z)$, equivalently with $\mathcal{F}_{g(U)} g(z) \leq \mathcal{F}_{\left(\mathcal{I}_q^{m,l} F\right)'(U)} \left(\mathcal{I}_q^{m,l} F(z)\right)'$, $z \in U$, with the fuzzy best subordinant $g(z) = \frac{a+2}{z^{a+2}} \int_0^z h(t) t^{a+1} dt$ convex function. □

Theorem 8. *Considering a convex function g, we define the function $h(z) = \frac{zg'(z)}{a+2} + g(z)$, with Re $a > -2$, $z \in U$. For $f \in \mathcal{A}$, denote $F(z) = \frac{a+2}{z^{a+1}} \int_0^z t^a f(t)\,dt$, $z \in U$ and assume that $\left(\mathcal{I}_q^{m,l} f(z)\right)'$ is univalent in U and $\left(\mathcal{I}_q^{m,l} F(z)\right)' \in \mathcal{Q} \cap \mathcal{H}[1,1]$. When the fuzzy differential superordination*

$$\mathcal{F}_{h(U)} h(z) \leq \mathcal{F}_{\left(\mathcal{I}_q^{m,l} f\right)'(U)} \left(\mathcal{I}_q^{m,l} f(z)\right)', \quad z \in U, \tag{3.4}$$

states, then we get the fuzzy differential superordination

$$\mathcal{F}_{g(U)} g(z) \leq \mathcal{F}_{\left(\mathcal{I}_q^{m,l} F\right)'(U)} \left(\mathcal{I}_q^{m,l} F(z)\right)', \quad z \in U,$$

for $g(z) = \frac{a+2}{z^{a+2}} \int_0^z h(t) t^{a+1} dt$ the fuzzy best subordinant.

Proof. Considering $p(z) = \left(\mathcal{I}_q^{m,l}F(z)\right)'$, $z \in U$, following the the proof of Theorem 7 we can write the fuzzy differential superordination (3.4) in the following form

$$\mathcal{F}_{g(U)}\left(\frac{zg'(z)}{a+2} + g(z)\right) \leq \mathcal{F}_{p(U)}\left(\frac{zp'(z)}{a+2} + p(z)\right), \quad z \in U.$$

Applying Lemma 4 for $\alpha = a + 2$ and $n = 1$, we obtain the fuzzy differential superordination $\mathcal{F}_{g(U)}g(z) \leq \mathcal{F}_{p(U)}p(z) = \mathcal{F}_{\left(\mathcal{I}_q^{m,l}F\right)'(U)}\left(\mathcal{I}_q^{m,l}F(z)\right)'$, $z \in U$, having $g(z) = \frac{a+2}{z^{a+2}}\int_0^z h(t)t^{a+1}dt$ the fuzzy best subordinant. □

Theorem 9. *For $f \in \mathcal{A}$ denote $F(z) = \frac{a+2}{z^{a+1}}\int_0^z t^a f(t)\,dt$, $z \in U$, and $h(z) = \frac{1+(2\alpha-1)z}{1+z}$, where Re $a > -2$, $\alpha \in [0,1)$. Assume that $\left(\mathcal{I}_q^{m,l}f(z)\right)'$ is univalent in U, $\left(\mathcal{I}_q^{m,l}F(z)\right)' \in Q \cap \mathcal{H}[1,1]$ and the fuzzy differential superordination*

$$\mathcal{F}_{h(U)}h(z) \leq \mathcal{F}_{\left(\mathcal{I}_q^{m,l}f\right)'(U)}\left(\mathcal{I}_q^{m,l}f(z)\right)', \ z \in U, \tag{3.5}$$

is satisfied, then the fuzzy differential superordination

$$\mathcal{F}_{g(U)}g(z) \leq \mathcal{F}_{\left(\mathcal{I}_q^{m,l}F\right)'(U)}\left(\mathcal{I}_q^{m,l}F(z)\right)', \quad z \in U,$$

is satisfied for the convex function $g(z) = 2\alpha - 1 + \frac{(a+2)(2-2\alpha)}{z^{a+2}}\int_0^z \frac{t^{a+1}}{t+1}dt$, $z \in U$ as the fuzzy best subordinant.

Proof. Denoting $p(z) = \left(\mathcal{I}_q^{m,l}F(z)\right)'$, following the proof of Theorem 7, the fuzzy differential superordination (3.5) can be written as $\mathcal{F}_{h(U)}h(z) \leq \mathcal{F}_{p(U)}\left(\frac{zp'(z)}{a+2} + p(z)\right)$, $z \in U$.

Applying Lemma 3, we get the fuzzy differential superordination $\mathcal{F}_{g(U)}g(z) \leq \mathcal{F}_{p(U)}p(z)$, with $g(z) = \frac{a+2}{z^{a+2}}\int_0^z \frac{1+(2\alpha-1)t}{1+t}t^{a+1}dt = 2\alpha - 1 + \frac{(a+2)(2-2\alpha)}{z^{a+2}}\int_0^z \frac{t^{a+1}}{t+1}dt$ and $\mathcal{F}_{g(U)}g(z) \leq \mathcal{F}_{\left(\mathcal{I}_q^{m,l}F\right)'(U)}\left(\mathcal{I}_q^{m,l}F(z)\right)'$, $z \in U$, g is the fuzzy best subordinant and it is convex. □

Theorem 10. *For $f \in \mathcal{A}$, consider a convex function h such that $h(0) = 1$ and assume that $\left(\mathcal{I}_q^{m,l}f(z)\right)'$ is univalent and $\frac{\mathcal{I}_q^{m,l}f(z)}{z} \in Q \cap \mathcal{H}[1,1]$. When the fuzzy differemtial superordination*

$$\mathcal{F}_{h(U)}h(z) \leq \mathcal{F}_{\left(\mathcal{I}_q^{m,l}f\right)'(U)}\left(\mathcal{I}_q^{m,l}f(z)\right)', \qquad z \in U, \tag{3.6}$$

holds, then the following fuzzy differential superordination

$$\mathcal{F}_{g(U)}g(z) \leq \mathcal{F}_{\mathcal{I}_q^{m,l}f(U)}\frac{\mathcal{I}_q^{m,l}f(z)}{z}, \qquad z \in U,$$

is satisfied, for the convex function $g(z) = \frac{1}{z}\int_0^z h(t)dt$ the fuzzy best subordinant.

Proof. Denoting $p(z) = \frac{\mathcal{I}_q^{m,l}f(z)}{z} = \frac{z+\sum_{k=2}^{\infty}\left(\frac{[k+l]_q}{[1+l]_q}\right)^m a_k z^k}{z} \in \mathcal{H}[1,1]$, $z \in U$, we can write $\mathcal{I}_q^{m,l}f(z) = zp(z)$, and differentiating it we have $\left(\mathcal{I}_q^{m,l}f(z)\right)' = zp'(z) + p(z)$, $z \in U$.

With this notation, the fuzzy differential superordination (3.6) becomes $\mathcal{F}_{h(U)}h(z) \leq \mathcal{F}_{p(U)}(zp'(z) + p(z))$, $z \in U$, and applying Lemma 3, we get $\mathcal{F}_{g(U)}g(z) \leq \mathcal{F}_{p(U)}p(z) = \mathcal{F}_{\mathcal{I}_q^{m,l}f((U)}\frac{\mathcal{I}_q^{m,l}f(z)}{z}$, $z \in U$, for $g(z) = \frac{1}{z}\int_0^z h(t)dt$ the fuzzy best subordinant and convex. □

Theorem 11. *Considering a convex function g in U we define the function h by* $h(z) = zg'(z) + g(z)$*. Assume* $\left(\mathcal{I}_q^{m,l} f(z)\right)'$ *is univalent,* $\frac{\mathcal{I}_q^{m,l} f(z)}{z} \in \mathcal{Q} \cap \mathcal{H}[1,1]$ *for* $f \in \mathcal{A}$ *and the fuzzy superordination*

$$\mathcal{F}_{g(U)}\left(zg'(z) + g(z)\right) \leq \mathcal{F}_{\left(\mathcal{I}_q^{m,l} f\right)'(U)}\left(\mathcal{I}_q^{m,l} f(z)\right)', \qquad z \in U, \tag{3.7}$$

is satisfied, then the fuzzy differential supeordination

$$\mathcal{F}_{g(U)} g(z) \leq \mathcal{F}_{\mathcal{I}_q^{m,l} f(U)} \frac{\mathcal{I}_q^{m,l} f(z)}{z}, \qquad z \in U,$$

is satisfied for $g(z) = \frac{1}{z}\int_0^z h(t)dt$ *the fuzzy best subordinant.*

Proof. Taking account the proof of Theorem 10 for $p(z) = \frac{\mathcal{I}_q^{m,l} f(z)}{z}$, the fuzzy superordination (3.7) can be written in the following form $\mathcal{F}_{g(U)}\left(zg'(z) + g(z)\right) \leq \mathcal{F}_{p(U)}\left(zp'(z) + p(z)\right), z \in U$.

Applying Lemma 4, we get the fuzzy differential superordination $\mathcal{F}_{g(U)} g(z) \leq \mathcal{F}_{p(U)} p(z)$, equivalently with $\mathcal{F}_{g(U)} g(z) \leq \mathcal{F}_{\mathcal{I}_q^{m,l} f(U)} \frac{\mathcal{I}_q^{m,l} f(z)}{z}$, where $g(z) = \frac{1}{z}\int_0^z h(t)dt$ the fuzzy best subordinant. □

Theorem 12. *Let* $h(z) = \frac{1+(2\alpha-1)z}{1+z}$ *with* $0 \leq \alpha < 1$, $z \in U$*. For* $f \in \mathcal{A}$ *suppose that* $\left(\mathcal{I}_q^{m,l} f(z)\right)'$ *is univalent and* $\frac{\mathcal{I}_q^{m,l} f(z)}{z} \in \mathcal{Q} \cap \mathcal{H}[1,1]$*. If the fuzzy differential superordination*

$$\mathcal{F}_{h(U)} h(z) \leq \mathcal{F}_{\left(\mathcal{I}_q^{m,l} f\right)'(U)}\left(\mathcal{I}_q^{m,l} f(z)\right)', \qquad z \in U, \tag{3.8}$$

holds, then we get the following fuzzy differential superordination

$$\mathcal{F}_{g(U)} g(z) \leq \mathcal{F}_{\mathcal{I}_q^{m,l} f(U)} \frac{\mathcal{I}_q^{m,l} f(z)}{z}, \qquad z \in U,$$

where the fuzzy best subordinant is the convex function $g(z) = 2\alpha - 1 + 2(1-\alpha)\frac{\ln(1+z)}{z}$, $z \in U$*.*

Proof. Following the proof of Theorem 10 for $p(z) = \frac{\mathcal{I}_q^{m,l} f(z)}{z}$, the fuzzy superordination (3.8) takes the form $\mathcal{F}_{h(U)} h(z) \leq \mathcal{F}_{p(U)}\left(zp'(z) + p(z)\right), z \in U$.

Applying Lemma 3, we get the following fuzzy differential superordination $\mathcal{F}_{g(U)} g(z) \leq \mathcal{F}_{p(U)} p(z)$, which can be written as $\mathcal{F}_{g(U)} g(z) \leq \mathcal{F}_{\mathcal{I}_q^{m,l} f(U)} \frac{\mathcal{I}_q^{m,l} f(z)}{z}$, $z \in U$. The function $g(z) = \frac{1}{z}\int_0^z \frac{1+(2\alpha-1)t}{1+t}dt = 2\alpha - 1 + \frac{2(1-\alpha)}{z}\ln(z+1)$ is the best subordinant and it is convex. □

Theorem 13. *Considering a convex function h such that* $h(0) = 1$*, for* $f \in \mathcal{A}$ *suppose that* $\left(\frac{z\mathcal{I}_q^{m+1,l} f(z)}{\mathcal{I}_q^{m,l} f(z)}\right)'$ *is univalent and* $\frac{\mathcal{I}_q^{m+1,l} f(z)}{\mathcal{I}_q^{m,l} f(z)} \in \mathcal{Q} \cap \mathcal{H}[1,1]$*. If the fuzzy differential superordination*

$$\mathcal{F}_{h(U)} h(z) \leq \mathcal{F}_{\mathcal{I}_q^{m,l} f(U)}\left(\frac{z\mathcal{I}_q^{m+1,l} f(z)}{\mathcal{I}_q^{m,l} f(z)}\right)', \qquad z \in U, \tag{3.9}$$

holds, then we get the following fuzzy differential superordination

$$\mathcal{F}_{g(U)} g(z) \leq \mathcal{F}_{\mathcal{I}_q^{m,l} f(U)} \frac{\mathcal{I}_q^{m+1,l} f(z)}{\mathcal{I}_q^{m,l} f(z)}, \qquad z \in U,$$

with the fuzzy best subordinant is the convex function $g(z) = \frac{1}{z}\int_0^z h(t)dt$*.*

Proof. Let $p(z)=\frac{\mathcal{I}_q^{m+1,l}f(z)}{\mathcal{I}_q^{m,l}f(z)}$, after differentiating it we can write $p'(z)=\frac{\left(\mathcal{I}_q^{m+1,l}f(z)\right)'}{\mathcal{I}_q^{m,l}f(z)}-p(z)\frac{\left(\mathcal{I}_q^{m,l}f(z)\right)'}{\mathcal{I}_q^{m,l}f(z)}$ in the form $zp'(z)+p(z)=\left(\frac{z\mathcal{I}_q^{m+1,l}f(z)}{\mathcal{I}_q^{m,l}f(z)}\right)'$.

Fuzzy differential superordination (3.9) for $z\in U$, becomes $\mathcal{F}_{h(U)}h(z)\le\mathcal{F}_{p(U)}(zp'(z)+p(z))$.

Applying Lemma 3, we obtain the following fuzzy differential superordination $\mathcal{F}_{g(U)}g(z)\le\mathcal{F}_{p(U)}p(z)=\mathcal{F}_{\mathcal{I}_q^{m,l}f(U)}\frac{\mathcal{I}_q^{m+1,l}f(z)}{\mathcal{I}_q^{m,l}f(z)}$, $z\in U$, for the fuzzy best subordinant $g(z)=\frac{1}{z}\int_0^z h(t)dt$ convex. □

Theorem 14. *Consider a convex function g and the function h defined by $h(z)=zg'(z)+g(z)$. For $f\in\mathcal{A}$ assume that $\left(\frac{z\mathcal{I}_q^{m+1,l}f(z)}{\mathcal{I}_q^{m,l}f(z)}\right)'$ is univalent and $\frac{\mathcal{I}_q^{m+1,l}f(z)}{\mathcal{I}_q^{m,l}f(z)}\in Q\cap\mathcal{H}[1,1]$. If the fuzzy differential superordination*

$$\mathcal{F}_{g(U)}h(z)=\mathcal{F}_{g(U)}(zg'(z)+g(z))\le\mathcal{F}_{\mathcal{I}_q^{m,l}f(U)}\left(\frac{z\mathcal{I}_q^{m+1,l}f(z)}{\mathcal{I}_q^{m,l}f(z)}\right)',\qquad z\in U,\tag{3.10}$$

states, then we get the fuzzy differential superordination

$$\mathcal{F}_{g(U)}g(z)\le\mathcal{F}_{\mathcal{I}_q^{m,l}f(U)}\frac{\mathcal{I}_q^{m+1,l}f(z)}{\mathcal{I}_q^{m,l}f(z)},\qquad z\in U,$$

and the fuzzy best subordinant is $g(z)=\frac{1}{z}\int_0^z h(t)dt$.

Proof. Following the proof of Theorem 13 for $p(z)=\frac{\mathcal{I}_q^{m+1,l}f(z)}{\mathcal{I}_q^{m,l}f(z)}$, the fuzzy superordination (3.10) has the form $\mathcal{F}_{g(U)}h(z)=\mathcal{F}_{g(U)}(zg'(z)+g(z))\le\mathcal{F}_{p(U)}(zp'(z)+p(z))$, $z\in U$.

Applying Lemma 4, it yields $\mathcal{F}_{g(U)}g(z)\le\mathcal{F}_{p(U)}p(z)$, equivalently with $\mathcal{F}_{g(U)}g(z)=\mathcal{F}_{g(U)}\left(\frac{1}{z}\int_0^z h(t)dt\right)\le\mathcal{F}_{\mathcal{I}_q^{m,l}f(U)}\frac{\mathcal{I}_q^{m+1,l}f(z)}{\mathcal{I}_q^{m,l}f(z)}$, $z\in U$, and the fuzzy best subordinant is g. □

Theorem 15. *Consider $h(z)=\frac{1+(2\alpha-1)z}{1+z}$, with $0\le\alpha<1$. For $f\in\mathcal{A}$ asume that $\left(\frac{z\mathcal{I}_q^{m+1,l}f(z)}{\mathcal{I}_q^{m,l}f(z)}\right)'$ is univalent and $\frac{\mathcal{I}_q^{m+1,l}f(z)}{\mathcal{I}_q^{m,l}f(z)}\in Q\cap\mathcal{H}[1,1]$. If the fuzzy differential superordination*

$$\mathcal{F}_{h(U)}h(z)\le\mathcal{F}_{\mathcal{I}_q^{m,l}f(U)}\left(\frac{z\mathcal{I}_q^{m+1,l}f(z)}{\mathcal{I}_q^{m,l}f(z)}\right)',\qquad z\in U,\tag{3.11}$$

holds, then the fuzzy differential superordination

$$\mathcal{F}_{g(U)}g(z)\le\mathcal{F}_{\mathcal{I}_q^{m,l}f(U)}\frac{\mathcal{I}_q^{m+1,l}f(z)}{\mathcal{I}_q^{m,l}f(z)},\qquad z\in U,$$

holds, and the fuzzy best subordinant is the convex function $g(z)=2\alpha-1+2(1-\alpha)\frac{\ln(1+z)}{z}$, $z\in U$.

Proof. Using the notation $p(z)=\frac{\mathcal{I}_q^{m+1,l}f(z)}{\mathcal{I}_q^{m,l}f(z)}$, the fuzzy differential superordination (3.11) can be written $\mathcal{F}_{h(U)}h(z)\le\mathcal{F}_{p(U)}(zp'(z)+p(z))$, $z\in U$.

Applying Lemma 3, we get the fuzzy differential superordination $\mathcal{F}_{g(U)}g(z) \leq \mathcal{F}_{p(U)}p(z)$, equivalently with $\mathcal{F}_{g(U)}g(z) \leq \mathcal{F}_{I_q^{m,l}f(U)}\frac{I_q^{m+1,l}f(z)}{I_q^{m,l}f(z)}$, $z \in U$.

The best subordinant is the convex function $g(z) = \frac{1}{z}\int_0^z \frac{1+(2\alpha-1)t}{1+t}dt = 2\alpha - 1 + 2(1-\alpha)\frac{1}{z}\ln(z+1)$, $z \in U$. □

4. Conclusions

The new results proved in this paper are related to a new class of analytic normalized functions in U, $\mathcal{FS}_{m,l}^{q}(\alpha)$, given in Definition 7. using the q-analogue of the multiplier transformation $I_q^{m,l}$ shown in Definition 6 involving fuzzy theory. In Section 2 of the paper, the class is introduced and its convexity property is proved. Using this attribute of the functions belonging to class $\mathcal{FS}_{m,l}^{q}(\alpha)$, sharp fuzzy differential subordinations are next investigated in five theorems. In Theorem 2 the fuzzy best dominant for the fuzzy differential subordination is also provided and in Theorem 3 a certain inclusion relation is proved for the class $\mathcal{FS}_{m,l}^{q}(\alpha)$. In Section 3 of the paper, fuzzy differential superordinations are established in the nine theorems involving the q-analogue of the multiplier transformation $I_q^{m,l}$, its first derivative $\left(I_q^{m,l}f(z)\right)'$, second derivative $\left(I_q^{m,l}f(z)\right)''$ and the expression $\frac{zI_q^{m+1,l}f(z)}{I_q^{m,l}f(z)}$ and its derivative.

For future studies, the fuzzy subordination and superordiantion results obtained here can inspire investigations where other q-operators are used instead of q-analogue of the multiplier transformation $I_q^{m,l}f(z)$. Also, since the fuzzy best dominant of the fuzzy differential subordination in Theorem 2 is given, and the fuzzy best subordinants are provided for the fuzzy differential superordinations studied in Section 3, conditions for univalence of the operator $I_q^{m,l}f(z)$ investigated here could be further obtained. Other classes of univalent functions could be defined using the q-analogue of the multiplier transformation $I_q^{m,l}f(z)$ and different subordiantion relations. Coefficient estimates could also be searched for the class $S_{m,l}^{q}(\alpha)$.

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Conflict of interest

The authors declare that they have no competing interests. The authors drafted the manuscript, read and approved the final manuscript.

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