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Research article

Discrete generalized Darboux transformation and rational solutions for the three-field Blaszak-Marciniak lattice equation

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Abstract: Under consideration is the discrete three-field Blaszak-Marciniak lattice equation. Firstly, this discrete equation is mapped to the continuous nonlinear equations under the continuous limit. Secondly, the generalized $(m, 3N - m)$-fold Darboux transformation of this discrete equation is constructed and established. Finally, by applying the resulting Darboux transformation, some singular rational solutions and mixed exponential-rational solutions are presented, in particular, their limit state analysis and singular trajectories are analyzed graphically. These results may be helpful to explain some relevant physical phenomena.

Keywords: three-field Blaszak-Marciniak equation; continuous limit; discrete generalized $(m, 3N - m)$-fold Darboux transformation; rational solutions; limit state analysis
Mathematics Subject Classification: 35Q51, 35Q35, 37K40

1. Introduction

In 1994, Blaszak and Marciniak have studied R-matrix forms of many integrable lattice system and also have proposed some different forms of the Blaszak-Marciniak (BM) lattice equations, which have enriching bi-Hamiltonian structures [1]. One of BM lattice equations has the following form [1]

$$\begin{cases} p_{n,t} = q_{n-1}r_{n-1} - q_n r_n, \\ q_{n,t} = r_{n+1} - r_{n-1}, \\ r_{n,t} = r_n(p_n - p_{n+1}), \end{cases} \tag{1.1}$$

where $p_n = p(n,t), q_n = q(n,t), r_n = r(n,t)$ stand for the potential functions with respect to space variable n and time variable t. Moreover, in Ref [1], the Lax pair of Eq (1.1) is given by

$$\chi_{n+1} = J_n\chi_n = \begin{pmatrix} 0 & 1 & 0 \\ p_n - \lambda & q_n & 1 \\ r_n & 0 & 0 \end{pmatrix}\chi_n, \tag{1.2}$$

$$\chi_{n,t} = L_n\chi_n = \begin{pmatrix} 0 & 0 & 1 \\ r_n & 0 & 0 \\ -q_{n-1}r_{n-1} & r_{n-1} & \lambda - p_n \end{pmatrix}\chi_n, \tag{1.3}$$

where λ is a time-independent spectral parameter, $\chi_n = (\eta_n, \gamma_n, \psi_n)^{\mathrm{T}}$ is a basic solution of (1.2) and (1.3) (T is transposing a vector or matrix). The compatibility condition $J_{n,t} = (L_{n+1})\ J_n - J_nL_n$ of (1.2) and (1.3) leads to Eq (1.1). In Ref [2], an infinite number of conservation laws of Eq (1.1) have been presented through a systematic method. In Ref [3], the new integrable symplectic map and involutive system of conserved integrals of Eq (1.1) have been obtained. In Ref [4], a method of constructing the Hamiltonian structure of Eq (1.1) has been described from conservation laws. In Ref [5], some soliton solutions, rational solutions and Bäcklund transformations of Eq (1.1) have been obtained by using the Hirota bilinear form. In Ref [6], the isospectral multi-Hamiltonian structure of Eq (1.1) has been obtained. In Ref [7], the authors have developed a sequence of master symmetries and commutable generalized symmetries for Eq (1.1). In Ref [8], the Hamiltonian structures and their relationship with the conservation laws of Eq (1.1) have been studied. In Ref [9], the N-fold Darboux transformation (DT) and explicit solutions for Eq (1.1) in terms of the determinant have been investigated. In Ref [10], the authors have constructed the matrix Lax representations of all the three-field and four-field BM lattice equations covering Eq (1.1).

Finding exact solutions of nonlinear equations is an important research subject [11–16]. Some methods of finding the exact solutions have been proposed and developed such as the Hirota transformation [5, 11, 12], Painlevé analysis [14, 17, 18], DT [9, 15, 19–21] and Bäcklund transformation [16, 22, 23]. In Refs [24, 25], one of the authors of this paper has proposed a generalized $(m, N-m)$-fold DT, which is taken as a generalization of N-fold DT. This generalized DT can not only give soliton solutions, but also can give some rational solutions and mixed interaction solutions. This generalized method is first used to solve some nonlinear equations with 2×2 Lax pair, and later has been successfully extended to nonlinear equations with 3×3 and 4×4 Lax pair [26, 27]. However, there are specific difficulties that need to be overcome from 2×2 Lax pair to 3×3 Lax pair. It should be emphasized that nonlinear equations with 3×3 Lax pair are more difficult to solve than 2×2 Lax pair, which is worthy of further research. However, as far as we know, there is still no relevant research work about the continuous limit, the discrete discrete generalized $(m, 3N-m)$-fold DT, various rational solutions and mixed exponential-rational solutions of Eq (1.1). Therefore, in this paper, we will extend the generalized DT method to make further research on Eq (1.1) with 3×3 Lax pair.

This article is organized below. In Section 2, we convert Eq (1.1) into the new continuous equations by using the continuous limit technique. In Section 3, based on the known Lax pair (1.2) and (1.3), we will construct the discrete generalized DT of Eq (1.1). In Section 4, some rational solutions and mixed exponential-rational solutions will be given by using the resulting generalized DT. Meanwhile, we will use asymptotic analysis to study limit state analysis of rational solutions, and their singular trajectories are shown graphically. Finally, a few conclusions are summarized.

2. Continuous equations related to Eq (1.1)

The continuous limit of discrete integrable equations is an essential research area [28]. Below we will investigate some continuous equations related to discrete Eq (1.1) via the continuous limit technique.

(i) If the limit conditions

$$\begin{cases} p_n = 1 - p[(n+t)\varepsilon, \varepsilon t] + O(\varepsilon) \equiv 1 - p(x,\tau) + O(\varepsilon), \\ q_n = 1 - q[(n+t)\varepsilon, \varepsilon t] + O(\varepsilon) \equiv 1 - q(x,\tau) + O(\varepsilon), \\ r_n = -r[(n+t)\varepsilon, \varepsilon t] + O(\varepsilon) \equiv -r(x,\tau) + O(\varepsilon), \end{cases} \tag{2.1}$$

and

$$\begin{cases} p_{n\pm1} = 1 - p(x \pm \varepsilon, \tau) + O(\varepsilon) = 1 - (p \pm p_x\varepsilon + \frac{1}{2}p_{xx}\varepsilon^2 \pm \frac{1}{6}p_{xxx}\varepsilon^3) + O(\varepsilon), \\ q_{n\pm1} = 1 - q(x \pm \varepsilon, \tau) + O(\varepsilon) = 1 - (q \pm q_x\varepsilon + \frac{1}{2}q_{xx}\varepsilon^2 \pm \frac{1}{6}q_{xxx}\varepsilon^3) + O(\varepsilon), \\ r_{n\pm1} = -r(x \pm \varepsilon, \tau) + O(\varepsilon) = -(r \pm r_x\varepsilon + \frac{1}{2}r_{xx}\varepsilon^2 \pm \frac{1}{6}r_{xxx}\varepsilon^3) + O(\varepsilon), \end{cases} \tag{2.2}$$

are used, Eq (1.1) is transformed into

$$\begin{cases} (p_\tau + p_x + r_x - qr_x - rq_x)\varepsilon + O(\varepsilon^2) = 0, \\ (q_\tau + q_x - 2r_x)\varepsilon + O(\varepsilon^2) = 0, \\ (r_\tau + r_x - rp_x)\varepsilon + O(\varepsilon^2) = 0. \end{cases} \tag{2.3}$$

As we put τ as t and ignore $O(\varepsilon^2)$ of Eq (2.3), Eq (1.1) can actually converge to a new continuous nonlinear equation.

(ii) When the limit conditions

$$\begin{cases} p_n = p[(n+t)\varepsilon, \varepsilon t] + O(\varepsilon) \equiv p(x,\tau) + O(\varepsilon), \\ q_n = q[(n+t)\varepsilon, \varepsilon t] + O(\varepsilon) \equiv q(x,\tau) + O(\varepsilon), \\ r_n = r[(n+t)\varepsilon, \varepsilon t] + O(\varepsilon) \equiv r(x,\tau) + O(\varepsilon), \end{cases} \tag{2.4}$$

and

$$\begin{cases} p_{n\pm1} = p(x \pm \varepsilon, \tau) + O(\varepsilon) = p \pm p_x\varepsilon + \frac{1}{2}p_{xx}\varepsilon^2 \pm \frac{1}{6}p_{xxx}\varepsilon^3 + O(\varepsilon), \\ q_{n\pm1} = q(x \pm \varepsilon, \tau) + O(\varepsilon) = q \pm q_x\varepsilon + \frac{1}{2}q_{xx}\varepsilon^2 \pm \frac{1}{6}q_{xxx}\varepsilon^3 + O(\varepsilon), \\ r_{n\pm1} = r(x \pm \varepsilon, \tau) + O(\varepsilon) = r \pm r_x\varepsilon + \frac{1}{2}r_{xx}\varepsilon^2 \pm \frac{1}{6}r_{xxx}\varepsilon^3 + O(\varepsilon), \end{cases} \tag{2.5}$$

are applied, Eq (1.1) is transformed into

$$\begin{cases} (p_\tau + p_x)\varepsilon^2 + O(\varepsilon^3) = 0, \\ (q_\tau + q_x - 2r_x)\varepsilon^2 + O(\varepsilon^3) = 0, \\ (r_\tau + r_x)\varepsilon^2 + O(\varepsilon^3) = 0. \end{cases} \tag{2.6}$$

When we ignore $O(\varepsilon^3)$, Eq (1.1) can actually converge to a linear equation.

3. Discrete generalized DT method

In the section, we study the discrete generalized $(m, 3N-m)$-fold DT of Eq (1.1). To this aim, we consider the following gauge transformation

$$\tilde{\chi}_n = W_n \chi_n, \tag{3.1}$$

where W_n is the 3×3 Darboux matrix. Based on the knowledge of the DT, $\tilde{\chi}_n$ needs to satisfy

$$\tilde{\chi}_{n+1} = \tilde{J}_n \tilde{\chi}_n = W_{n+1} J_n W_n^{-1} \tilde{\chi}_n, \qquad \tilde{\chi}_{n,t} = \tilde{L}_n \tilde{\chi}_n = (W_{n,t} + W_n L_n) W_n^{-1} \tilde{\chi}_n, \tag{3.2}$$

where $\tilde{J}_n, \tilde{L}_n$ keep the consistent forms as J_n, L_n except for replacing the old p_n, q_n, r_n with the new $\tilde{p}_n$, $\tilde{q}_n$, $\tilde{r}_n$. In addition, in Ref [9], a certain Darboux matrix W_n has been constructed as

$$W_n = \begin{pmatrix} \lambda^N + \sum_{j=0}^{N-1} A_n^{(j)} \lambda^j & \sum_{j=0}^{N-1} B_n^{(j)} \lambda^j & \sum_{j=0}^{N-1} C_n^{(j)} \lambda^j \\ -F_n^{(N-1)} \lambda^N + \sum_{j=0}^{N-1} D_n^{(j)} \lambda^j & \lambda^N + \sum_{j=0}^{N-1} E_n^{(j)} \lambda^j & \sum_{j=0}^{N-1} F_n^{(j)} \lambda^j \\ \sum_{j=0}^{N-1} G_n^{(j)} \lambda^j & \sum_{j=0}^{N-1} H_n^{(j)} \lambda^j & (1 + C_n^{(N-1)}) \lambda^N + \sum_{j=0}^{N-1} K_n^{(j)} \lambda^j \end{pmatrix}, \tag{3.3}$$

where N is the positive integer, $A_n^{(j)}$, $B_n^{(j)}$, $C_n^{(j)}$, $D_n^{(j)}$, $E_n^{(j)}$, $F_n^{(j)}$, $G_n^{(j)}$, $H_n^{(j)}$ and $K_n^{(j)}$ are functions of the variables n, t. In this work, we will continue to use Darboux matrix W_n in (3.3) to give a new generalized DT, which will be used to give new rational and mixed interaction solutions. By virtue of the relationship between $\tilde{J}_n$ and J_n in (3.2), we have

$$\begin{cases} \tilde{p}_n = p_n + A_n^{(N-1)} - E_{n+1}^{(N-1)} + (q_n + B_n^{(N-1)} - F_{n+1}^{(N-1)})F_n^{(N-1)}, \\ \tilde{q}_n = q_n + B_n^{(N-1)} - F_{n+1}^{(N-1)}, \\ \tilde{r}_n = -H_{n+1}^{(N-1)} + r_n(1 + C_{n+1}^{(N-1)}). \end{cases} \tag{3.4}$$

If we use m spectral parameters λ to solve these $9N$ unknown functions, it is necessary to expand $W_n(\lambda_i+\varepsilon)\chi_n(\lambda_i+\varepsilon)$, where ε is an arbitrary very small parameter, $\chi_n(\lambda_i+\varepsilon) = \chi_n^{(0)}(\lambda_i) + \chi_n^{(1)}(\lambda_i)\varepsilon + \chi_n^{(2)}(\lambda_i)\varepsilon^2 + \chi_n^{(3)}(\lambda_i)\varepsilon^3 + \cdots$, $W_n(\lambda_i+\varepsilon) = W_n^{(0)} + W_n^{(1)}\varepsilon + \cdots + W_n^{(z_i)}\varepsilon^{z_i}$ and z_i satisfies $3N = m + \sum_{i=1}^{m} z_i$. If we let $W_n(\lambda_i+\varepsilon)\chi_n(\lambda_i+\varepsilon) = 0$, then we can obtain

$$\begin{cases} W_n^{(0)}(\lambda_i)\chi_n^{(0)}(\lambda_i) = 0, \\ W_n^{(0)}(\lambda_i)\chi_n^{(1)}(\lambda_i) + W_n^{(1)}(\lambda_i)\chi_n^{(0)}(\lambda_i) = 0, \\ W_n^{(0)}(\lambda_i)\chi_n^{(2)}(\lambda_i) + W_n^{(1)}(\lambda_i)\chi_n^{(1)}(\lambda_i) + W_n^{(2)}(\lambda_i)\chi_n^{(0)}(\lambda_i) = 0, \\ \cdots\cdots, \\ \sum_{j=0}^{z_i} W_n^{(j)}(\lambda_i)\chi_n^{(z_i-j)}(\lambda_i) = 0, \end{cases}$$

where

$$A_n^{(N-1)} = \frac{\Delta A_n^{(N-1)}}{\Delta_1},\ B_n^{(N-1)} = \frac{\Delta B_n^{(N-1)}}{\Delta_1},\ C_n^{(N-1)} = \frac{\Delta C_n^{(N-1)}}{\Delta_1}, E_n^{(N-1)} = \frac{\Delta E_n^{(N-1)}}{\Delta_2},\ F_n^{(N-1)} = \frac{\Delta F_n^{(N-1)}}{\Delta_2},\ H_n^{(N-1)} = \frac{\Delta H_n^{(N-1)}}{\Delta_1},$$

with Δ_1=det$([\Delta_1^{(1)},\ \Delta_1^{(2)},\ ...,\ \Delta_1^{(m)}])^{\mathrm{T}}$, Δ_2=det$([\Delta_2^{(1)},\ \Delta_2^{(2)},\ \cdots,\ \Delta_2^{(m)}])^{\mathrm{T}}$, $\Delta_1^{(i)} = (\Delta_{1,j,h}^{(i)})_{(z_i+1)\times 3N}$, $\Delta_2^{(i)} = (\Delta_{2,j,h}^{(i)})_{(z_i+1)\times 3N}$, in which $\Delta_{1,j,h}^{(i)}$, $\Delta_{2,j,h}^{(i)}$ $(1 \le j \le z_i+1,\ 0 \le h \le 3N,\ \ 0 \le i \le m)$ are given as

$$\Delta_{1,j,h}^{(i)} = \begin{cases} \sum_{k=0}^{j-1} C_{N-h}^{k}\lambda_i^{N-h-k}\eta_{i,n}^{(j-1-k)} & \text{as} \quad 1 \le j \le z_i+1,\ 1 \le h \le N, \\ \sum_{k=0}^{j-1} C_{2N-h}^{k}\lambda_i^{2N-h-k}\gamma_{i,n}^{(j-1-k)} & \text{as} \quad 1 \le j \le z_i+1,\ 1 \le h \le 2N, \\ \sum_{k=0}^{j-1} C_{3N-h}^{k}\lambda_i^{3N-h-k}\psi_{i,n}^{(j-1-k)} & \text{as} \quad 1 \le j \le z_i+1,\ 2N+1 \le h \le 3N, \end{cases}$$

$$\Delta_{2,j,h}^{(i)} = \begin{cases} \sum_{k=0}^{j-1} C_{N-h}^{k}\lambda_i^{N-h-k}\eta_{i,n}^{(j-1-k)} & \text{as} \quad 1 \le j \le 1+z_i,\ 1 \le h \le N, \\ \sum_{k=0}^{j-1} C_{2N-h}^{k}\lambda_i^{2N-h-k}\gamma_{i,n}^{(j-1-k)} & \text{as} \quad 1 \le j \le 1+z_i,\ N+1 \le h \le 2N, \\ \sum_{k=0}^{j-1} C_{3N-h}^{k}\lambda_i^{3N-h-k}\psi_{i,n}^{(j-1-k)} - C_{3N}^{2h-N-2}\sum_{k=0}^{j-1} C_{3N-h+1}^{k}\lambda_i^{3N-h-k+1}\eta_{i,n}^{(j-1-k)} & \text{as} \quad 1 \le j \le 1+z_i,\ 2N+1 \le h \le 3N, \end{cases}$$

where $\Delta A_n^{(N-1)}$, $\Delta B_n^{(N-1)}$ and $\Delta C_n^{(N-1)}$ are derived by replacing the first, $(N+1)$th and $(2N+1)$th columns of Δ_1 with $(a^{(1)}, a^{(2)}, \cdots, a^{(m)})^{\mathrm{T}}$, respectively, with $a^{(i)} = (a_j^{(i)})_{(z_i+1)\times 1}$, in which $a_j^{(i)} = -\sum_{k=0}^{j-1} C_N^k\lambda_i^{N-k}\eta_{i,n}^{(j-1-k)}$, $\Delta E_n^{(N-1)}$ and $\Delta F_n^{(N-1)}$ are derived by substituting the $(N+1)$th and $(2N+1)$th columns of Δ_2, respectively, by $(b^{(1)}, b^{(2)}, \cdots, b^{(m)})^{\mathrm{T}}$ with $b^{(i)} = (b_j^{(i)})_{(z_i+1)\times 1}$, in which $b_j^{(i)} = -\sum_{k=0}^{j-1} C_N^k\lambda_i^{N-k}\gamma_{i,n}^{(j-1-k)}$ $(1 \le j \le z_i+1,\ 1+N \le h \le 3N)$. $\Delta H_n^{(N-1)}$ is derived by replacing the first column to Δ_1 by $(d^{(1)}, d^{(2)}, \cdots, d^{(m)})^{\mathrm{T}}$ with $d^{(i)} = (d_j^{(i)})_{(z_i+1)\times 1}$, in which $d_j^{(i)} = -(1 + C_n^{(N-1)})\sum_{k=0}^{j-1} C_N^k\lambda_i^{N-k}\psi_{i,n}^{(j-1-k)}$.

The expression (3.4) with m spectral parameters are referred to as the generalized $(m, 3N-m)$-fold DT, where m stands for the number of the used spectral parameters, N denotes the order of DT, and $3N-m$ is the total order number of Taylor expansion of χ_n used. It should be noted that when $m = 3N$ and

$z_i = 0$, the generalized $(m, 3N - m)$-fold DT becomes the $(3N, 0)$-fold DT, which includes the usual $3N$-fold DT in Ref [9]. If the Taylor expansion is not used, the $(3N, 0)$-fold DT is just the usual $3N$-fold DT, which only can give soliton solutions. If we use the Taylor expansion in the $(3N, 0)$-fold DT, we can give some rational solutions and mixed solutions. When $1 \le m < 3N$, we can derive more kinds of exact solutions. In the next section, we will discuss the special cases of this method.

4. Exact rational solutions and mixed exponential-rational solutions

By inserting the initial solutions $p_n = \frac{2}{3}$, $q_n = 1$, $r_n = \frac{1}{27}$ of Eq (1.1) into Lax pair (1.2) and (1.3), we can give a basic solution with $\lambda = \lambda_k$ as

$$\chi_n = \begin{pmatrix} \eta_n \\ \gamma_n \\ \psi_n \end{pmatrix} = \begin{pmatrix} C_{k,1}\tau_1^n e^{\frac{t}{27\tau_1}+\delta(\varepsilon)} + C_{k,2}\tau_2^n e^{\frac{t}{27\tau_2}+\delta(\varepsilon)} + C_{k,3}\tau_3^n e^{\frac{t}{27\tau_3}-\delta(\varepsilon)} \\ C_{k,1}\tau_1^{n+1} e^{\frac{t}{27\tau_1}+\delta(\varepsilon)} + C_{k,2}\tau_2^{n+1} e^{\frac{t}{27\tau_2}+\delta(\varepsilon)} + C_{k,3}\tau_3^{n+1} e^{\frac{t}{27\tau_3}-\delta(\varepsilon)} \\ \frac{1}{27}C_{k,1}\tau_1^{n-1} e^{\frac{t}{27\tau_1}+\delta(\varepsilon)} + \frac{1}{27}C_{k,2}\tau_2^{n-1} e^{\frac{t}{27\tau_2}+\delta(\varepsilon)} + \frac{1}{27}C_{k,3}\tau_3^{n-1} e^{\frac{t}{27\tau_3}-\delta(\varepsilon)} \end{pmatrix}, \tag{4.1}$$

where $\delta(\varepsilon) = (36 - 36\lambda_k + 12\sqrt{12\lambda_k^3 - 27\lambda_k^2 + 18\lambda_k - 3})^{\frac{1}{3}} \sum_{j=0}^{3N-1} m_j \varepsilon^{3j}$, $m_j\,(j = 0, 1, \cdots, 3N-1)$, $C_{k,1}$, $C_{k,2}$, $C_{k,3}$ are arbitrary constants, while τ_1, τ_2 and τ_3 satisfy $\tau^3 - \tau^2 + (\lambda_k - \frac{2}{3})\tau - \frac{1}{27} = 0$ respectively. Below, two kinds of Taylor series expansions of χ_n at $\lambda = \lambda_1 = 1$ are displayed, and other λ_k do not make Taylor series expansion.

• **Type I.** When $C_{1,1} = 0$, $C_{1,2} = -C_{1,3} = \frac{I}{\varepsilon}$, where I is an imaginary unit, the first three coefficients of the χ_n expansion are given by

$$\chi_n^{(0)} = \begin{pmatrix} \eta_n^{(0)} \\ \gamma_n^{(0)} \\ \psi_n^{(0)} \end{pmatrix} = -\frac{1}{27} e^{\frac{t}{9}} 3^{-n+\frac{1}{6}} \begin{pmatrix} 9\xi \\ 3\xi + 27 \\ \xi - 9 \end{pmatrix},$$

$$\chi_n^{(1)} = \begin{pmatrix} \eta_n^{(1)} \\ \gamma_n^{(1)} \\ \psi_n^{(1)} \end{pmatrix}, \quad \chi_n^{(2)} = \begin{pmatrix} \eta_n^{(2)} \\ \gamma_n^{(2)} \\ \psi_n^{(2)} \end{pmatrix},$$

in which

$\eta_n^{(1)} = \frac{1}{5832} e^{\frac{t}{9}} 3^{-n+\frac{1}{6}} [\xi^4 - 18\xi^3 + 27\xi^2 + 270\xi + 243t^2 + (54\xi^2 - 810\xi + 2430)t]$,

$\gamma_n^{(1)} = \frac{1}{17496} e^{\frac{t}{9}} 3^{-n+\frac{1}{6}} [\xi^4 + 18\xi^3 + 27\xi^2 - 702\xi - 1944 + 243t^2 + (54\xi^2 + 162\xi - 486)t]$,

$\psi_n^{(1)} = \frac{1}{52488} e^{\frac{t}{9}} 3^{-n+\frac{1}{6}} [\xi^4 - 54\xi^3 + 999\xi^2 - 7506\xi + 19440 + 243t^2 + (54\xi^2 - 1782\xi + 14094)t]$,

$\eta_n^{(2)} = -\frac{1}{99202320} e^{\frac{t}{9}} 3^{-n+\frac{1}{6}} [\xi^7 - 63\xi^6 + 945\xi^5 + 6615\xi^4 - 180306\xi^3 + 285768\xi^2 + 2566080\xi + (76545\xi - 1377810)t^3 + (8505\xi^3 - 382725\xi^2 + 5051970\xi - 18921924)t^2 + (189\xi^5 - 11340\xi^4 + 212625\xi^3 - 1234926\xi^2 - 673596\xi + 11547360)t]$,

$\gamma_n^{(2)} = -\frac{1}{297606960} e^{\frac{t}{9}} 3^{-n+\frac{1}{6}} [\xi^7 - 756\xi^5 - 1890\xi^4 + 134379\xi^3 + 561330\xi^2 - 4414824\xi - 14696640 + (76545\xi - 688905)t^3 + (8505\xi^3 - 153090\xi^2 + 229635\xi + 1745226)t^2 + (189\xi^5 - 2835\xi^4 - 42525\xi^3 + 372519\xi^2 + 1898316\xi - 2781864)t]$,

$\psi_n^{(2)} = -\frac{1}{892820880} e^{\frac{t}{9}} 3^{-n+\frac{1}{6}} [\xi^7 - 126\xi^6 + 6048\xi^5 - 137970\xi^4 + 1495179\xi^3 - 5960304\xi^2 - 8640108\xi + 80831520 + (76545\xi - 2066715)t^3 + (8505\xi^3 - 612360\xi^2 + 14007735\xi - 101590524)t^2 + (189\xi^5 - 19845\xi^4 + 773955\xi^3 - 13864851\xi^2 + 112490532\xi - 322984908)t]$,

where $\xi = 9n - t$, other $\chi_n^{(j)} (j = 3, 4, 5 \cdots)$ are not displayed here.

• **Type II.** If $C_{1,1}=1$, $C_{1,2}=C_{1,3}=0$, the first three coefficients of the χ_n expansion are given by

$$\chi_n^{(0)}=\begin{pmatrix}\eta_n^{(0)}\\ \gamma_n^{(0)}\\ \psi_n^{(0)}\end{pmatrix}=e^{\frac{t}{9}}3^{-n-2}\begin{pmatrix}9\\3\\1\end{pmatrix},$$

$$\chi_n^{(1)}=\begin{pmatrix}\eta_n^{(1)}\\ \gamma_n^{(1)}\\ \psi_n^{(1)}\end{pmatrix}=-\frac{1}{2}e^{\frac{t}{9}}3^{-n-5}\begin{pmatrix}\xi^3-9\xi^2+324m_0\xi+27\xi t-162t\\ \frac{1}{3}\xi^3+6\xi^2+108m_0\xi+974m_0+27\xi+9\xi t+27t\\ \frac{1}{9}\xi^3-4\xi^2+36m_0\xi-324m_0+45\xi-162+3\xi t-45t\end{pmatrix},$$

$$\chi_n^{(2)}=\begin{pmatrix}\eta_n^{(2)}\\ \gamma_n^{(2)}\\ \psi_n^{(2)}\end{pmatrix}$$

in which

$$\begin{aligned}\eta_n^{(2)}=&\ \tfrac{1}{4723920}e^{\frac{t}{9}}3^{-n}[\xi^6-45\xi^5+1620m_0\xi^4+405\xi^4-29160m_0\xi^3+3645\xi^3+524880m_0^2\xi^2+43740m_0\xi^2\\&-39366\xi^2-1574640m_0^2\xi+787320m_0\xi+18895680m_0^3+10935t^3+(3645\xi^2+393660m_0-98415\xi\\&+590490)t^2+(135\xi^4+87480m_0\xi^2-5670\xi^3+4723920m_0^2-1312200m_0\xi+69255\xi^2+3936600m_0\\&-236196\xi)t],\\ \gamma_n^{(2)}=&\ \tfrac{1}{14171760}e^{\frac{t}{9}}3^{-n}[\xi^6+9\xi^5-405\xi^4+1620m_0\xi^4+29160m_0\xi^3-3645\xi^3+26244\xi^2+524880m_0^2\xi^2\\&+7873200m_0^2\xi+43740m_0\xi^2-787320m_0\xi+236196\xi+18895680m_0^3+28343520m_0^2+10935t^3(\\&+3645\xi^2+393660m_0-32805\xi)t^2+(135\xi^4+87480m_0\xi^2-810\xi^3+4723920m_0^2+262440m_0\xi\\&-18225\xi^2-787320m_0+26244\xi+236196)t],\\ \psi_n^{(2)}=&\ \tfrac{1}{42515280}e^{\frac{t}{9}}3^{-n}[\xi^6-99\xi^5+1620m_0\xi^4+3645\xi^4-87480m_0\xi^3-61965\xi^3+524880m_0^2\xi^2\\&-11022480m_0^2\xi+1618380m_0\xi^2-11809800m_0\xi+485514\xi^2+28343520m_0-1417176\xi\\&+18895680m_0^3+56687040m_0^2+10935t^3+(3645\xi^2+393660m_0-164025\xi+1771470)t^2+(135\xi^4\\&+87480m_0\xi^2-10530\xi^3+4723920m_0^2-2886840m_0\xi+287955\xi^2+22832280m_0-3254256\xi\\&+12754584)t],\end{aligned}$$

where $\xi=9n-t$, and other $\chi_n^{(j)}(j\geq 2)$ are omitted.

In the following, we will obtain various rational solutions of Eq (1.1) by applying generalized DT only when $N=1$. It should be noted here that the definition of order number of rational solutions is based on the used highest order number of Taylor expansion.

4.1. *Exact first-order rational and mixed exponential-rational solutions*

As $m=3$, we need to use two spectral parameters. Specifically, we choose **Type I** in (4.1) for $\lambda_1=1$, $\lambda_2=\frac{1}{4}$ corresponds to the coefficients $C_{2,1}=1$, $C_{2,2}=C_{2,3}=0$, while choosing $C_{3,2}=1$, $C_{3,1}=C_{3,3}=0$ for $\lambda_3=\frac{19}{27}$, then we can give the simplest rational solutions of Eq (1.1) as

$$\tilde{p}_n=\frac{2}{3}-\frac{36}{(2\xi-15)(2\xi-33)},\quad \tilde{q}_n=1+\frac{216}{(2\xi+3)(2\xi-33)},\quad \tilde{r}_n=\frac{1}{27}-\frac{12}{(2\xi-15)^2},\tag{4.2}$$

whose numerator and denominator are quadratic polynomials. We call (4.2) the first-order rational solutions because we only use the first coefficient of the Taylor expansion.

Noted that $\tilde{p}_n$ owns two singular lines:

$$2\xi - 15 = 0, \qquad 2\xi - 33 = 0,$$

$\tilde{q}_n$ owns two singular lines:

$$2\xi + 3 = 0, \qquad 2\xi - 33 = 0,$$

$\tilde{r}_n$ owns one singular line:

$$2\xi - 15 = 0.$$

To easier understand (4.2), their structure plots are displayed in Figure 1.

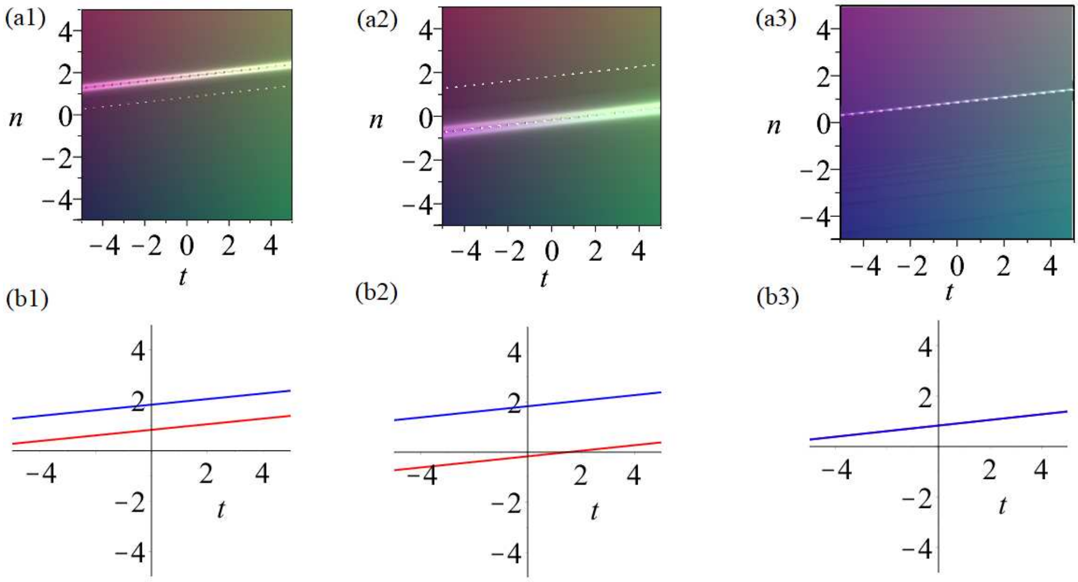

Figure 1. First-order rational solutions (4.2): $(a1)$–$(a3)$ Contour plots of $\tilde{p}_n$, $\tilde{q}_n$ and $\tilde{r}_n$, respectively; $(b1)$–$(b3)$ Singular trajectories of $\tilde{p}_n$, $\tilde{q}_n$ and $\tilde{r}_n$ corresponding to $(a1)$–$(a3)$, respectively.

When the coefficients $C_{k,1}$, $C_{k,2}$, $C_{k,3}$ corresponding to λ_2 and λ_3 are not all zeroes, the mixed exponential-rational solutions of Eq (1.1) can be given as

$$\tilde{p}_n = \frac{2}{3} + A_n^{(0)} - E_{n+1}^{(0)} + (1 + B_n^{(0)} - F_{n+1}^{(0)})F_n^{(0)}, \ \tilde{q}_n = 1 + B_n^{(0)} - F_{n+1}^{(0)}, \ \tilde{r}_n = \frac{1}{27} - H_{n+1}^{(0)} + \frac{1}{27}C_{n+1}^{(0)}, \quad (4.3)$$

where $A_n^{(0)} = \frac{\Delta A_n^{(0)}}{\Delta_1}$, $B_n^{(0)} = \frac{\Delta B_n^{(0)}}{\Delta_1}$, $C_{n+1}^{(0)} = \frac{\Delta C_{n+1}^{(0)}}{\Delta_1}$, $E_{n+1}^{(0)} = \frac{\Delta E_{n+1}^{(0)}}{\Delta_2}$, $F_n^{(0)} = \frac{\Delta F_n^{(0)}}{\Delta_2}$, $H_{n+1}^{(0)} = \frac{\Delta H_{n+1}^{(0)}}{\Delta_1}$,
in which

$$\Delta A_n^{(0)} = \begin{vmatrix} -\lambda_1\eta_n^{(0)} & \gamma_n^{(0)} & \psi_n^{(0)} \\ -\lambda_2\eta_n(\lambda_2) & \gamma_n(\lambda_2) & \psi_n(\lambda_2) \\ -\lambda_3\eta_n(\lambda_3) & \gamma_n(\lambda_3) & \psi_n(\lambda_3) \end{vmatrix}, \ \Delta B_n^{(0)} = \begin{vmatrix} \eta_n^{(0)} & -\lambda_1\eta_n^{(0)} & \psi_n^{(0)} \\ \eta_n(\lambda_2) & -\lambda_2\eta_n(\lambda_2) & \psi_n(\lambda_2) \\ \eta_n(\lambda_3) & -\lambda_3\eta_n(\lambda_3) & \psi_n(\lambda_3) \end{vmatrix},$$

$$\Delta F_n^{(0)} = \begin{vmatrix} \eta_n^{(0)} & \gamma_n^{(0)} & -\lambda_1\gamma_n^{(0)} \\ \eta_n(\lambda_2) & \gamma_n(\lambda_2) & -\lambda_2\gamma_n(\lambda_2) \\ \eta_n(\lambda_3) & \gamma_n(\lambda_3) & -\lambda_3\gamma_n(\lambda_3) \end{vmatrix}, \ \Delta_2 = \begin{vmatrix} \eta_n^{(0)} & \gamma_n^{(0)} & \psi_n^{(0)} - \lambda_1\eta_n^{(0)} \\ \eta_n(\lambda_2) & \gamma_n(\lambda_2) & \psi_n(\lambda_2) - \lambda_2\eta_n(\lambda_2) \\ \eta_n(\lambda_3) & \gamma_n(\lambda_3) & \psi_n(\lambda_3) - \lambda_3\eta_n(\lambda_3) \end{vmatrix},$$

$$\Delta_1=\begin{vmatrix}\eta_n^{(0)} & \gamma_n^{(0)} & \psi_n^{(0)}\\ \eta_n(\lambda_2) & \gamma_n(\lambda_2) & \psi_n(\lambda_2)\\ \eta_n(\lambda_3) & \gamma_n(\lambda_3) & \psi_n(\lambda_3)\end{vmatrix},\ \Delta E_{n+1}^{(0)}=\begin{vmatrix}\eta_{n+1}^{(0)} & -\lambda_1\gamma_{n+1}^{(0)} & \psi_{n+1}^{(0)}-\lambda_1\eta_{n+1}^{(0)}\\ \eta_{n+1}(\lambda_2) & -\lambda_2\gamma_{n+1}(\lambda_2) & \psi_{n+1}(\lambda_2)-\lambda_2\eta_{n+1}(\lambda_2)\\ \eta_{n+1}(\lambda_3) & -\lambda_3\gamma_{n+1}(\lambda_3) & \psi_{n+1}(\lambda_3)-\lambda_3\eta_{n+1}(\lambda_3)\end{vmatrix},$$

$$\Delta C_{n+1}^{(0)}=\begin{vmatrix}\eta_{n+1}^{(0)} & \gamma_{n+1}^{(0)} & -\lambda_1\eta_{n+1}^{(0)}\\ \eta_{n+1}(\lambda_2) & \gamma_{n+1}(\lambda_2) & -\lambda_2\eta_{n+1}(\lambda_2)\\ \eta_{n+1}(\lambda_3) & \gamma_{n+1}(\lambda_3) & -\lambda_3\eta_{n+1}(\lambda_3)\end{vmatrix},\ \Delta H_{n+1}^{(0)}=\begin{vmatrix}\eta_{n+1}^{(0)} & -(1+C_{n+1}^{(0)})\lambda_1\psi_{n+1}^{(0)} & \psi_{n+1}^{(0)}\\ \eta_{n+1}(\lambda_2) & -(1+C_{n+1}^{(0)})\lambda_2\psi_{n+1}(\lambda_2) & \psi_{n+1}(\lambda_2)\\ \eta_{n+1}(\lambda_3) & -(1+C_{n+1}^{(0)})\lambda_3\psi_{n+1}(\lambda_3) & \psi_{n+1}(\lambda_3)\end{vmatrix}.$$

Here we only list the determinant expressions of the mixed exponential-rational solutions of Eq (1.1) and do not discuss their structures.

4.2. Exact second-order rational and mixed exponential-rational solutions

As $m = 2$, we need to use two spectral parameters. Here we discuss two cases:

Case 1. We just use the above second type expansion Type II of (4.1) for $\lambda_1 = 1$, and we choose $\lambda_2 = \frac{1}{4}$, whose corresponding coefficients are $C_{2,1} = 1$, $C_{2,2} = C_{2,3} = 0$, then the second-order rational solutions of Eq (1.1) can be given as

$$\begin{cases}\tilde{p}_n=\dfrac{2}{3}+\dfrac{9(-2\xi^2+48\xi+18t+216m_0-333)}{(\xi^2-24\xi+9t+108m_0+81)(\xi^2-6\xi+9t+108m_0-54)},\\ \tilde{q}_n=1+\dfrac{108(\xi^2+3\xi-9t-108m_0-36)}{(\xi^2+12\xi+9t+108m_0-27)(\xi^2-24\xi+9t+108m_0+81)},\\ \tilde{r}_n=\dfrac{1}{27}+\dfrac{3(-2\xi^2+12\xi+18t+216m_0-63)}{(\xi^2-6\xi+9t+108m_0-54)^2}.\end{cases}\tag{4.4}$$

Whose numerator and denominator are the fourth-order polynomials. We call (4.4) the second-order rational solutions owing to $m = 2$. Noted that $\tilde{p}_n$ has the four singular curves:

$$\xi-12-3\sqrt{7-12m_0-t}=0,\ \xi-12+3\sqrt{7-12m_0-t}=0,$$
$$\xi-3-3\sqrt{7-12m_0-t}=0,\ \xi-3+3\sqrt{7-12m_0-t}=0,$$

$\tilde{q}_n$ has the four singular curves:

$$\xi+6-3\sqrt{7-12m_0-t}=0,\ \xi+6+3\sqrt{7-12m_0-t}=0,$$
$$\xi-12-3\sqrt{7-12m_0-t}=0,\ \xi-12+3\sqrt{7-12m_0-t}=0,$$

$\tilde{r}_n$ owns the two singular curves:

$$\xi-3-3\sqrt{7-12m_0-t}=0,\ \ \xi-3+3\sqrt{7-12m_0-t}=0.$$

The structure plots of solution (4.4) are displayed in Figures 2 and 3. In solution (4.4), m_0 is an arbitrary parameter to control the position of the singular curves, which means that the rational solutions can be moved to the position we need by changing m_0. Figure 2 shows the structures with $m_0 = 0$, while Figure 3 shows the structures after translation position with $m_0 = 2$.

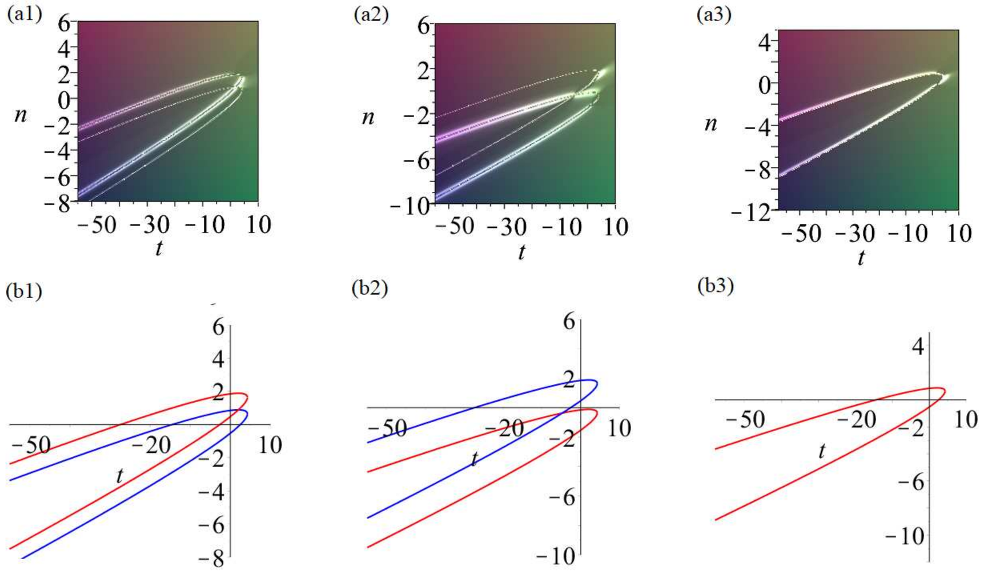

Figure 2. Second-order rational solutions (4.4): $(a1)$–$(a3)$ Contour plots of $\tilde{p}_n$, $\tilde{q}_n$ and $\tilde{r}_n$ with $m_0 = 0$, respectively; $(b1)$–$(b3)$ Singular trajectory plots of $\tilde{p}_n$, $\tilde{q}_n$ and $\tilde{r}_n$ with $m_0 = 0$, respectively.

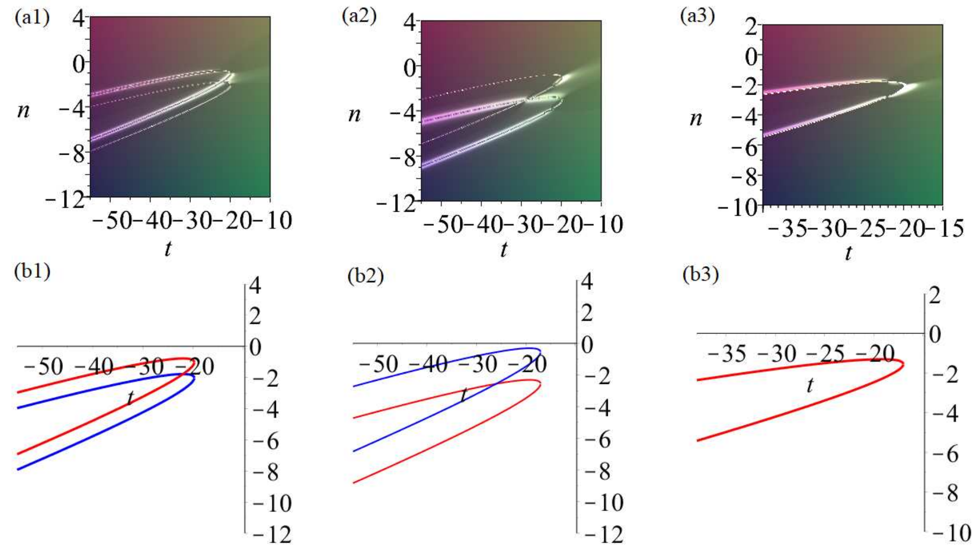

Figure 3. Position translational second-order rational solutions (4.4): $(a1)$–$(a3)$ Contour plots of $\tilde{p}_n$, $\tilde{q}_n$ and $\tilde{r}_n$ with $m_0 = 2$, respectively; $(b1)$–$(b3)$ Singular trajectory plots of $\tilde{p}_n$, $\tilde{q}_n$ and $\tilde{r}_n$ with $m_0 = 2$, respectively.

It should be noted that if at least two of $C_{2,1}$, $C_{2,2}$, $C_{2,3}$ are not zeroes corresponding to the spectral parameter $\lambda_2 = \frac{1}{4}$, we can obtain the mixed exponential-rational solutions of Eq (1.1) as

$$\tilde{p}_n = \frac{2}{3} + A_n^{(0)} - E_{n+1}^{(0)} + (1 + B_n^{(0)} - F_{n+1}^{(0)})F_n^{(0)},\ \tilde{q}_n = 1 + B_n^{(0)} - F_{n+1}^{(0)},\ \tilde{r}_n = \frac{1}{27} - H_{n+1}^{(0)} + \frac{1}{27}C_{n+1}^{(0)}, \quad (4.5)$$

where

$$A_n^{(0)} = \frac{\Delta A_n^{(0)}}{\Delta_1},\ B_n^{(0)} = \frac{\Delta B_n^{(0)}}{\Delta_1},\ C_{n+1}^{(0)} = \frac{\Delta C_{n+1}^{(0)}}{\Delta_1},\ E_{n+1}^{(0)} = \frac{\Delta E_{n+1}^{(0)}}{\Delta_2},\ F_n^{(0)} = \frac{\Delta F_n^{(0)}}{\Delta_2},\ H_{n+1}^{(0)} = \frac{\Delta H_{n+1}^{(0)}}{\Delta_1},$$

in which

$$\Delta A_n^{(0)} = \begin{vmatrix} -\lambda_1\eta_n^{(0)} & \gamma_n^{(0)} & \psi_n^{(0)} \\ -\lambda_1\eta_n^{(1)} - \eta_n^{(0)} & \gamma_n^{(1)} & \psi_n^{(1)} \\ -\lambda_2\eta_n(\lambda_2) & \gamma_n(\lambda_2) & \psi_n(\lambda_2) \end{vmatrix},\ \Delta B_n^{(0)} = \begin{vmatrix} \eta_n^{(0)} & -\lambda_1\eta_n^{(0)} & \psi_n^{(0)} \\ \eta_n^{(1)} & -\lambda_1\eta_n^{(1)} - \eta_n^{(0)} & \psi_n^{(1)} \\ \eta_n(\lambda_2) & -\lambda_2\eta_n(\lambda_2) & \psi_n(\lambda_2) \end{vmatrix},$$

$$\Delta F_n^{(0)} = \begin{vmatrix} \eta_n^{(0)} & \gamma_n^{(0)} & -\lambda_1\gamma_n^{(0)} \\ \eta_n^{(1)} & \gamma_n^{(1)} & -\lambda_1\gamma_n^{(1)} - \gamma_n^{(0)} \\ \eta_n(\lambda_2) & \gamma_n(\lambda_2) & -\lambda_2\gamma_n(\lambda_2) \end{vmatrix},\ \Delta_2 = \begin{vmatrix} \eta_n^{(0)} & \gamma_n^{(0)} & \psi_n^{(0)} - \lambda_1\eta_n^{(0)} \\ \eta_n^{(1)} & \gamma_n^{(1)} & \psi_n^{(1)} - \lambda_1\eta_n^{(1)} - \eta_n^{(0)} \\ \eta_n(\lambda_2) & \gamma_n(\lambda_2) & \psi_n(\lambda_2) - \lambda_2\eta_n(\lambda_2) \end{vmatrix},$$

$$\Delta_1 = \begin{vmatrix} \eta_n^{(0)} & \gamma_n^{(0)} & \psi_n^{(0)} \\ \eta_n^{(1)} & \gamma_n^{(1)} & \psi_n^{(1)} \\ \eta_n(\lambda_2) & \gamma_n(\lambda_2) & \psi_n(\lambda_2) \end{vmatrix},\ \Delta E_{n+1}^{(0)} = \begin{vmatrix} \eta_{n+1}^{(0)} & -\lambda_1\gamma_{n+1}^{(0)} & \psi_{n+1}^{(0)} - \lambda_1\eta_{n+1}^{(0)} \\ \eta_{n+1}^{(1)} & -\lambda_1\gamma_{n+1}^{(1)} - \gamma_{n+1}^{(0)} & \psi_{n+1}^{(1)} - \lambda_1\eta_{n+1}^{(1)} - \eta_{n+1}^{(0)} \\ \eta_{n+1}(\lambda_2) & -\lambda_2\gamma_{n+1}(\lambda_2) & \psi_{n+1}(\lambda_2) - \lambda_2\eta_{n+1}(\lambda_2) \end{vmatrix},$$

$$\Delta C_{n+1}^{(0)} = \begin{vmatrix} \eta_{n+1}^{(0)} & \gamma_{n+1}^{(0)} & -\lambda_1\eta_{n+1}^{(0)} \\ \eta_{n+1}^{(1)} & \gamma_{n+1}^{(1)} & -\lambda_1\eta_{n+1}^{(1)} - \eta_{n+1}^{(0)} \\ \eta_{n+1}(\lambda_2) & \gamma_{n+1}(\lambda_2) & -\lambda_2\eta_{n+1}(\lambda_2) \end{vmatrix},\ \Delta H_{n+1}^{(0)} = \begin{vmatrix} \eta_{n+1}^{(0)} & -(1+C_{n+1}^{(0)})\lambda_1\psi_{n+1}^{(0)} & \psi_{n+1}^{(0)} \\ \eta_{n+1}^{(1)} & -(1+C_{n+1}^{(0)})\lambda_1\psi_{n+1}^{(1)} - (1+C_{n+1}^{(0)})\psi_{n+1}^{(0)} & \psi_{n+1}^{(1)} \\ \eta_{n+1}(\lambda_2) & -(1+C_{n+1}^{(0)})\lambda_2\psi_{n+1}(\lambda_2) & \psi_{n+1}(\lambda_2) \end{vmatrix}.$$

Here we list the determinant expressions of these mixed solutions, and draw their structure plots as shown in Figure 4.

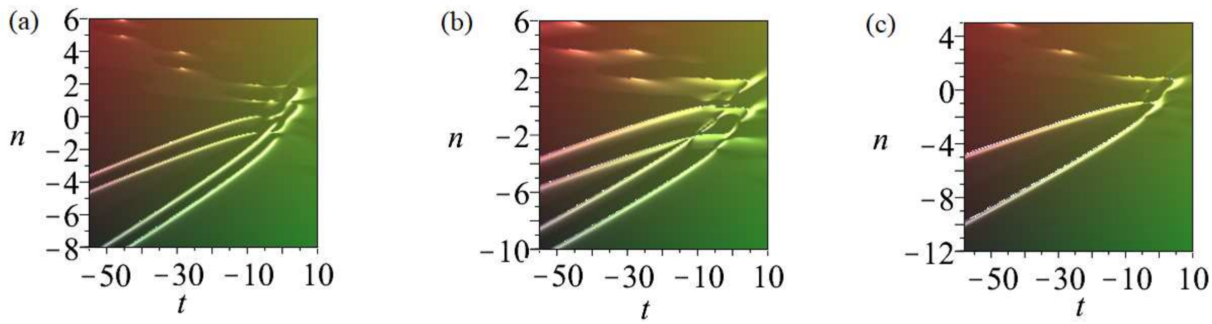

Figure 4. Mixed exponential-rational solutions (4.5): (a)–(c) Contour plots of $\tilde{p}_n$, $\tilde{q}_n$ and $\tilde{r}_n$, respectively.

Case 2. If we use the above first type expansion Type I of (4.1) for λ_1, and we choose $\lambda_2 = \frac{1}{4}$ with the coefficients $C_{2,1} = 1$, $C_{2,2} = C_{2,3} = 0$, we can get the new second-order rational solutions of Eq (1.1) as

$$\tilde{p}_n = \frac{2}{3} - \frac{18M_1}{M_2M_3},\ \tilde{q}_n = 1 + \frac{108M_4}{M_5M_6},\ \tilde{r}_n = \frac{1}{27} - \frac{6M_7}{M_8^2}, \tag{4.6}$$

where

$M_1 = 2\xi^6 - 108\xi^5 + 1899\xi^4 - 11070\xi^3 - 3078\xi^2 + 1458t^3 + (810\xi^2 - 8748\xi + 10935)t^2 + (18\xi^4 - 432\xi^3 - 2592\xi^2 + 39366\xi - 56133)t + 194643\xi - 459270,$

$M_2 = \xi^4 - 12\xi^3 - 126\xi^2 + 243\xi - 81t^2 + (18\xi^2 - 81)t + 1458,$

$M_3 = \xi^4 - 48\xi^3 + 684\xi^2 - 3321\xi - 81t^2 + (18\xi^2 - 324\xi + 1377)t + 4374,$

$M_4 = 2\xi^6 - 18\xi^5 - 504\xi^4 + 7236\xi^3 - 9639\xi^2 - 217242\xi + 1458t^3 + (810\xi^2 - 4374\xi - 21870)t^2$
$+(18\xi^4 - 756\xi^3 + 1782\xi^2 + 48114\xi + 48843)t + 196830,$
$M_5 = \xi^4 - 48\xi^3 + 684\xi^2 - 3321\xi - 81t^2 + (18\xi^2 - 324\xi + 1377)t + 4374,$
$M_6 = \xi^4 + 24\xi^3 + 36\xi^2 - 2025\xi - 81t^2 + (18\xi^2 + 324\xi + 1377)t - 8748,$
$M_7 = 2\xi^6 - 36\xi^5 - 153\xi^4 + 5454\xi^3 - 7452\xi^2 - 120285\xi + 1458t^3 + (810\xi^2 - 2916\xi - 15309)t^2$
$+(18\xi^4 - 432\xi^3 - 2592\xi^2 + 18954\xi + 35721)t + 249318,$
$M_8 = \xi^4 - 12\xi^3 - 126\xi^2 + 243\xi - 81t^2 + (18\xi^2 - 81)t + 1458.$

Below we use asymptotic analysis technique to analyze the limit states of solutions (4.6), we define $\alpha_1 = \xi - 3\sqrt{(\sqrt{2}-1)t}$, $(t > 0)$, $\alpha_2 = \xi - 3\sqrt{(\sqrt{2}+1)(-t)}$, $(t < 0)$, then we can work out the asymptotic states of solutions (4.6) when $t \to \infty$.

When $\xi = \alpha_1 + 3\sqrt{(\sqrt{2}-1)t}$, $t \to +\infty$:

$$\begin{aligned} \tilde{p}_n &\to \frac{2}{3} - \frac{18}{2\alpha_1^2 - 30\alpha_1 + 6\sqrt{2}\alpha_1 + 81 - 45\sqrt{2}}, \\ \tilde{q}_n &\to 1 + \frac{108}{2\alpha_1^2 - 12\alpha_1 + 6\sqrt{2}\alpha_1 - 135 - 18\sqrt{2}}, \\ \tilde{r}_n &\to \frac{1}{27} - \frac{6}{2\alpha_1^2 - 12\alpha_1 + 6\sqrt{2}\alpha_1 + 27 - 18\sqrt{2}}. \end{aligned} \tag{4.7}$$

When $\xi = \alpha_2 + 3\sqrt{(\sqrt{2}+1)(-t)}$, $t \to -\infty$:

$$\begin{aligned} \tilde{p}_n &\to \frac{2}{3} - \frac{18}{2\alpha_2^2 - 6\sqrt{2}\alpha_2 - 30\alpha_2 + 45\sqrt{2} + 81}, \\ \tilde{q}_n &\to 1 + \frac{108}{2\alpha_2^2 - 6\sqrt{2}\alpha_2 - 12\alpha_2 + 18\sqrt{2} - 135}, \\ \tilde{r}_n &\to \frac{1}{6} - \frac{6}{2\alpha_2^2 - 6\sqrt{2}\alpha_2 - 12\alpha_2 + 18\sqrt{2} + 27}. \end{aligned} \tag{4.8}$$

Noted that $\tilde{p}_n$ has four singularities:

$$2\alpha_1 - 24 + 3\sqrt{2} = 0,\ 2\alpha_1 - 6 + 3\sqrt{2} = 0,\ -2\alpha_2 + 24 + 3\sqrt{2} = 0,\ -2\alpha_2 + 6 + 3\sqrt{2} = 0,$$

$\tilde{q}_n$ has singularities at four curves:

$$2\alpha_1 + 12 + 3\sqrt{2} = 0,\ 2\alpha_1 - 24 + 3\sqrt{2} = 0,\ -2\alpha_2 - 12 + 3\sqrt{2} = 0,\ -2\alpha_2 + 24 + 3\sqrt{2} = 0,$$

$\tilde{r}_n$ possesses singularities at two curves:

$$2\alpha_1 - 6 + 3\sqrt{2}=0,\ -2\alpha_2 + 6 + 3\sqrt{2} = 0.$$

Next, we plot these singular curves and compare them with contour plots of solutions (4.6). Near $t = 0$, the interaction structures of solutions (4.6) are relatively complicated, and we only consider the relatively large range of t (i.e., $n^2+t^2 \geq 25$). Figure 5(a1)–(a3) show the contour plots of solutions (4.6), Figure 5(b1)–(b3) display the singular trajectories. By comparing them, we find that these singular trajectories are completely consistent with the exact solutions, which also shows the accuracy of our asymptotic analysis.

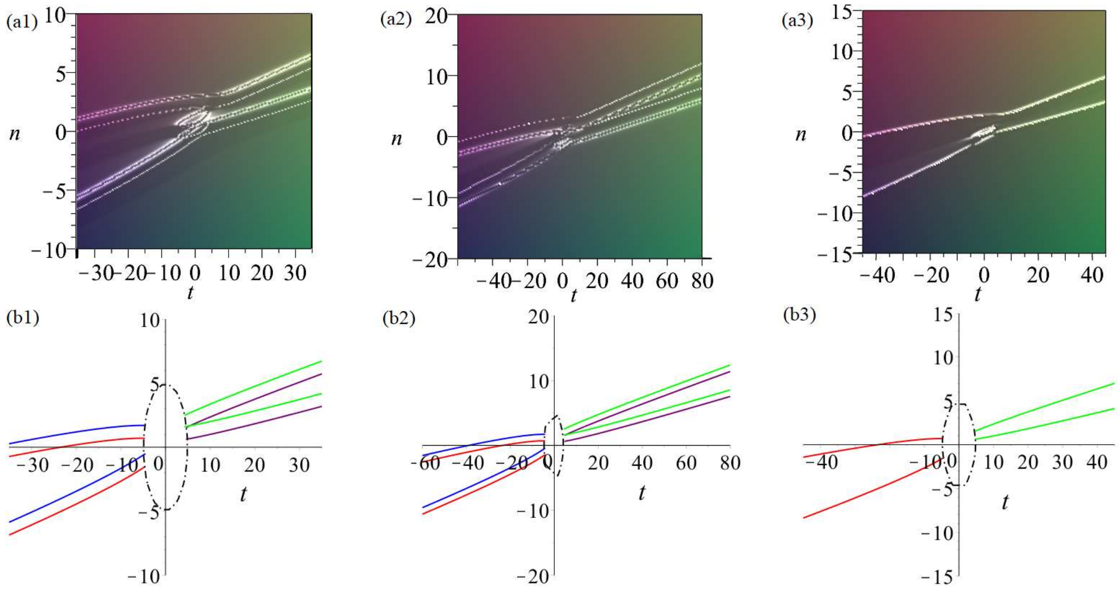

Figure 5. Second-order rational solutions (4.6): $(a1)$–$(a3)$ Contour plots of $\tilde{p}_n$, $\tilde{q}_n$ and $\tilde{r}_n$, respectively; $(b1)$–$(b3)$ Singular trajectory plots of the limit states of $\tilde{p}_n$, $\tilde{q}_n$ and $\tilde{r}_n$, respectively.

4.3. Exact third-order rational solutions

When $m = 1$, we just utilize one spectral parameter λ_1 with **Type II** expansion, then we can obtain the third-order rational solutions of Eq (1.1) as

$$\tilde{p}_n = \frac{2}{3} + A_n^{(0)} - E_{n+1}^{(0)} + (1 + B_n^{(0)} - F_{n+1}^{(0)})F_n^{(0)},\ \tilde{q}_n = 1 + B_n^{(0)} - F_{n+1}^{(0)},\ \tilde{r}_n = \frac{1}{27} - H_{n+1}^{(0)} + \frac{1}{27}C_{n+1}^{(0)}, \quad (4.9)$$

which can be simplified as

$$\tilde{p}_n = \frac{2}{3} - \frac{54N_1}{N_2N_3},\quad \tilde{q}_n = 1 + \frac{324N_4}{N_5N_6},\quad \tilde{r}_n = \frac{1}{27} + \frac{N_7}{N_8^2}, \quad (4.10)$$

with

$$\begin{aligned} N_1 = {} & \xi^{10} - 25\xi^9 - 150\xi^8 + 6390\xi^7 - 3807\xi^6 - 455625\xi^5 + 656100\xi^4 + 8266860\xi^3 + 8503056\xi^2 - 492075t^5 + (-273375\xi^2 \\ & -820125\xi)t^4 + (-12150\xi^4 - 72900\xi^3 - 4483350\xi^2 + 8135640\xi + 59442660)t^3 + (810\xi^6 - 8910\xi^5 - 230850\xi^4 \\ & -386370\xi^3 + 481140\xi^2 + 73614420\xi + 46294416)t^2 + (45\xi^8 - 900\xi^7 + 4590\xi^6 - 26892\xi^5 - 380295\xi^4 + 2332800\xi^3 \\ & +22438620\xi^2 + 54797472\xi)t, \\ N_2 = {} & \xi^6 - 33\xi^5 + 315\xi^4 - 675\xi^3 - 2916\xi^2 + 8748\xi + 3645t^3 + (405\xi^2 - 6075\xi + 58320)t^2 + (45\xi^4 - 810\xi^3 + 405\xi^2 \\ & +26244\xi + 8748)t, \\ N_3 = {} & \xi^6 + 21\xi^5 + 45\xi^4 - 1485\xi^3 - 10206\xi^2 - 17496\xi + 3645t^3 + (405\xi^2 + 1215\xi + 36450)t^2 + (45\xi^4 + 810\xi^3 + 405\xi^2 \\ & -32076\xi - 17496)t, \end{aligned}$$

$$N_4 = \xi^{10} + 50\xi^9 + 570\xi^8 - 5760\xi^7 - 115587\xi^6 + 70470\xi^5 + 6502680\xi^4 + 7610760\xi^3 - 114318864\xi^2 - 170061120\xi$$
$$-492075t^5 + (-273375\xi^2 - 2296350\xi - 17714700)t^4 + (-12150\xi^4 + 145800\xi^3 - 2515050\xi^2 - 50913360\xi$$
$$-198010980)t^3 + (810\xi^6 + 46980\xi^5 + 279450\xi^4 - 2682720\xi^3 - 48070260\xi^2 - 244069200\xi - 605606544)t^2$$
$$+(45\xi^8 + 1800\xi^7 + 19170\xi^6 + 123768\xi^5 - 15795\xi^4 - 17700120\xi^3 - 122428260\xi^2 - 153055008\xi - 170061120)t,$$
$$N_5 = \xi^6 - 33\xi^5 + 315\xi^4 - 675\xi^3 - 2916\xi^2 + 8748\xi + 3645t^3 + (405\xi^2 - 6075\xi + 58320)t^2 + (45\xi^4 - 810\xi^3 + 405\xi^2$$
$$+26244\xi + 8748)t,$$
$$N_6 = \xi^6 + 75\xi^5 + 2205\xi^4 + 31725\xi^3 + 223074\xi^2 + 612360\xi + 3645t^3 + (405\xi^2 + 8505\xi + 80190)t^2 + (45\xi^4 + 2430\xi^3$$
$$+44145\xi^2 + 303264\xi + 612360)t,$$
$$N_7 = -18\xi^{10} - 630\xi^9 - 3780\xi^8 + 79380\xi^7 + 826686\xi^6 - 2208870\xi^5 - 34904520\xi^4 - 7085880\xi^3 + 187067232\xi^2$$
$$+8857350t^5 + (4920750\xi^2 + 50191650\xi + 212576400)t^4 + (218700\xi^4 + 1312200\xi^3 + 80700300\xi^2 + 774722880\xi$$
$$+1395918360)t^3 + (-14580\xi^6 - 539460\xi^5 - 1093500\xi^4 + 62067060\xi^3 + 647964360\xi^2 + 2047819320\xi$$
$$+187067232)t^2 + (-810\xi^8 - 22680\xi^7 - 199260\xi^6 - 740664\xi^5 + 9994590\xi^4 + 176884560\xi^3$$
$$+644815080\xi^2 + 374134464\xi)t,$$
$$N_8 = \xi^6 + 21\xi^5 + 45\xi^4 - 1485\xi^3 - 10206\xi^2 - 17496\xi + 3645t^3 + (405\xi^2 + 1215\xi + 36450)t^2 + (45\xi^4 + 810\xi^3$$
$$+405\xi^2 - 32076\xi - 17496)t.$$

In the same way, we also investigate limit state of solutions (4.10). Let $\beta = \xi - \sqrt{[3(80+30\sqrt{6})^{\frac{1}{3}} + \frac{30}{(80+30\sqrt{6})^{\frac{1}{3}}} + 15](-t)}$, $(t < 0)$, as $t \to -\infty$, the limit states of solutions (4.10) are given as below:

$$\tilde{p}_n \to \frac{2}{3} - \frac{3600}{W_1 W_2}, \quad \tilde{q}_n \to 1 + \frac{21600}{W_1 W_3}, \quad \tilde{r}_n \to \frac{1}{27} - \frac{1200}{W_2^2}, \tag{4.11}$$

with

$$W_1 = 3\sqrt{6}(80+30\sqrt{6})^{\frac{2}{3}} - 8(80+30\sqrt{6})^{\frac{2}{3}} - 10(80+30\sqrt{6})^{\frac{1}{3}} + 20\beta - 110,$$
$$W_2 = 3\sqrt{6}(80+30\sqrt{6})^{\frac{2}{3}} - 8(80+30\sqrt{6})^{\frac{2}{3}} - 10(80+30\sqrt{6})^{\frac{1}{3}} + 20\beta + 70,$$
$$W_3 = 3\sqrt{6}(80+30\sqrt{6})^{\frac{2}{3}} - 8(80+30\sqrt{6})^{\frac{2}{3}} - 10(80+30\sqrt{6})^{\frac{1}{3}} + 20\beta + 250.$$

From the asymptotic expressions (4.11), we can find that $\tilde{p}_n$ owns two singular curves:

$3\sqrt{6}(80+30\sqrt{6})^{\frac{2}{3}} - 8(80+30\sqrt{6})^{\frac{2}{3}} - 10(80+30\sqrt{6})^{\frac{1}{3}} + 20\beta - 110 = 0$ and $3\sqrt{6}(80+30\sqrt{6})^{\frac{2}{3}} - 8(80+30\sqrt{6})^{\frac{2}{3}} - 10(80+30\sqrt{6})^{\frac{1}{3}} + 20\beta + 70 = 0$.

$\tilde{q}_n$ owns two singular curves:

$3\sqrt{6}(80+30\sqrt{6})^{\frac{2}{3}} - 8(80+30\sqrt{6})^{\frac{2}{3}} - 10(80+30\sqrt{6})^{\frac{1}{3}} + 20\beta - 110 = 0$ and $3\sqrt{6}(80+30\sqrt{6})^{\frac{2}{3}} - 8(80+30\sqrt{6})^{\frac{2}{3}} - 10(80+30\sqrt{6})^{\frac{1}{3}} + 20\beta + 250 = 0$.

$\tilde{r}_n$ has one singular curves:

$$3\sqrt{6}(80+30\sqrt{6})^{\frac{2}{3}} - 8(80+30\sqrt{6})^{\frac{2}{3}} - 10(80+30\sqrt{6})^{\frac{1}{3}} + 20\beta + 70 = 0.$$

We observe that the solutions (4.10) tend to their backgrounds as $t \to +\infty$ respectively, so we will only consider $t < 0$ here. Figure 6 shows the contour plots and singular trajectory plots outside the scope of $n^2 + t^2 = 25$, from which we can clearly see that the singular trajectories are completely consistent with exact solutions, which also demonstrates the accuracy of our asymptotic analysis results.

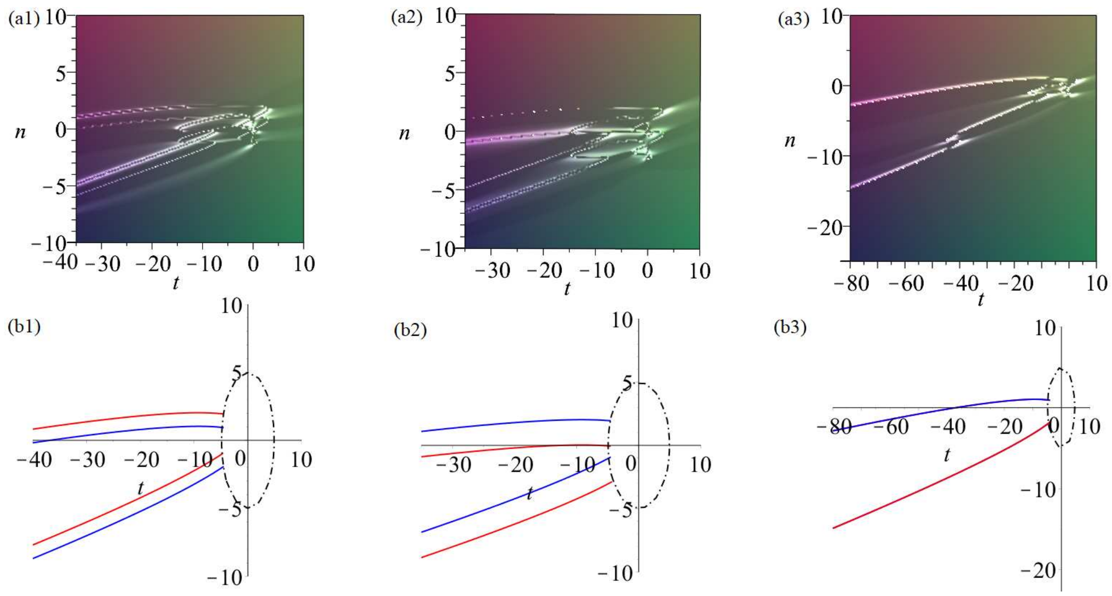

Figure 6. Third-order rational solutions (4.10): $(a1)$–$(a3)$ Contour plots of $\tilde{p}_n$, $\tilde{q}_n$ and $\tilde{r}_n$, respectively; $(b1)$–$(b3)$ Singular trajectory plots of the limit states of $\tilde{p}_n$, $\tilde{q}_n$ and $\tilde{r}_n$, respectively.

5. Conclusions

In this article, the discrete three-field BM lattice Eq (1.1) has been further studied. Some new achievements have been obtained: (i) BM lattice Eq (1.1) has been mapped to the new continuous Eqs (2.3) and (2.6) under the continuous limit; (ii) The generalized $(m, 3N - m)$-fold DT including the previous N-fold DT in Ref [9] has been constructed for Eq (1.1), and the generalized DT can not only give soliton solutions, but also can give rational and mixed exponential-rational solutions. (iii) By applying the resulting generalized DT, some rational solutions and mixed exponential-rational solutions have been exactly obtained, and the structure plots and limit state analysis of rational solutions are investigated graphically (see Figures 1–6). These new findings might be useful in understanding some physical phenomena.

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Conflict of interest

The authors declare that they have no competing interests.

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