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Research article

On a class of nonlinear rational systems of difference equations

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Abstract: In this paper, we construct and formulate the solutions and periodicity character of the following nonlinear rational systems of difference equations:

$$S_{n+1} = \frac{T_n S_{n-2}}{S_{n-2} + T_{n-1}}, \quad T_{n+1} = \frac{S_n T_{n-2}}{\pm T_{n-2} \pm S_{n-1}}, \qquad n = 0, 1, 2, ..., \tag{0.1}$$

where the initial conditions $s_{-2}, s_{-1}, s_0, t_{-2}, t_{-1}, t_0$ are positive real numbers. Moreover, some mathematical programs are used to support our theoretical results of each system in this paper.

Keywords: difference equations; periodic solutions; system of difference equations
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1. Introduction

Nonlinear difference equations and systems have lately appropriated the interest of numerous researchers. In fact, these types of equations have various implementation not just in mathematics but in describing some natural life phenomena, which that appear in engineering, economics, ecology, and so on. This can be due to the reality that these phenomena can be represented as modelled by using systems of difference equations. Wherefore, it has enticed the attention of a huge number of scholars and researchers over the past few years. Lately, there has been a great number of published papers growing constantly in this scope. From amid of many newly published papers in this area, we can present some of these studied as following:

In [1] Elsayed et al. have got the solutions expressions for the following nonlinear system of difference equations

$$R_{n+1} = \frac{a_1 T_{n-1} S_{n-1}}{R_{n-1} + S_{n-1} + T_{n-1}},\ S_{n+1} = \frac{a_2 T_{n-1} R_{n-1}}{R_{n-1} + S_{n-1} + T_{n-1}},\ T_{n+1} = \frac{a_3 R_{n-1} S_{n-1}}{R_{n-1} + S_{n-1} + T_{n-1}}.$$

Also, the authors in [2] obtained the forms of the solutions and periodic nature of the following rational systems of difference equations

$$x_{n+1} = \frac{y_{n-1} z_n}{z_n \pm x_{n-2}}, y_{n+1} = \frac{z_{n-1} x_n}{x_n \pm y_{n-2}}, z_{n+1} = \frac{x_{n-1} y_n}{y_n \pm z_{n-2}}.$$

El-Dessoky et al. [3], built the form of the solutions of the following nonlinear systems

$$z_{n+1} = \frac{z_n t_{n-1}}{\pm t_n \pm t_{n-1}},\ t_{n+1} = \frac{t_n z_{n-1}}{\pm z_n \pm z_{n-1}}.$$

The author in [4] obtained the forms of the analytical solutions of the following systems

$$x_{n+1} = \frac{x_{n-1} y_{n-3}}{y_{n-1}(\pm 1 \pm x_{n-1} y_{n-3})},\ y_{n+1} = \frac{y_{n-1} x_{n-3}}{x_{n-1}(\pm 1 \pm y_{n-1} x_{n-3})}.$$

Alzubaidi and Almatrafi [5] discovered the solutions' structures of the following dynamical systems of difference equations:

$$x_{n+1} = \frac{y_{n-5} x_{n-8}}{y_{n-2}(-1 - y_{n-5} x_{n-8})},\ y_{n+1} = \frac{x_{n-5} y_{n-8}}{x_{n-2}(\pm 1 \pm x_{n-5} y_{n-8})}.$$

The form of the solutions of the following third order systems

$$x_{n+1} = \frac{y_n z_{n-1}}{y_n \pm x_{n-2}}, y_{n+1} = \frac{z_n x_{n-1}}{z_n \pm y_{n-2}}, z_{n+1} = \frac{x_n y_{n-1}}{x_n \pm z_{n-2}},$$

were obtained by Alayachi et al. [6].

Moreover, in [7] the authors have explored the structure solutions of the following fifth order systems of recursive equations

$$x_{n+1} = \frac{x_{n-3} y_{n-4}}{y_n(1 + x_{n-1} y_{n-2} x_{n-3} y_{n-4})},\ y_{n+1} = \frac{y_{n-3} x_{n-4}}{x_n(\pm 1 \pm y_{n-1} x_{n-2} y_{n-3} x_{n-4})}.$$

Abdulkhaliq and Shoaib in [8] highlighted on the dynamics of the systems

$$u_{p+1} = \frac{u_p}{\xi + u_{p-1} v_{p-1} w_{p-1}}, v_{p+1} = \frac{v_p}{\eta + u_{p-1} v_{p-1} w_{p-1}}, w_{p+1} = \frac{w_p}{\zeta + u_{p-1} v_{p-1} w_{p-1}}.$$

Also, Din [9] investigated the behaviour solutions of a Lotka-Volterra model

$$x_{n+1} = \frac{\alpha x_n - \beta x_n y_n}{1 + \gamma x_n}, y_{n+1} = \frac{\delta y_n + \epsilon x_n y_n}{1 + \eta y_n}.$$

Finally, the forms and expressions of the solution of the difference equations systems

$$x_{n+1} = \frac{y_n y_{n-2}}{\pm x_{n-3} \pm y_{n-2}}, y_{n+1} = \frac{x_n x_{n-2}}{\pm y_{n-3} \pm x_{n-2}}$$

has been founded by authors in [10].

For more interesting papers on this scope can be seen in [11–21].

More precisely, the forms of exact solutions of some models of nonlinear systems of difference equations cannot be sometimes extracted. Therefore, the explore of exact solutions and other behaviors of a higher order of nonlinear systems of difference equations is quite challenging and valuable due to the importance of its applications.

Motivated by the remindered studies above , the major purpose of this paper is to construct a general form of the solutions and periodicity of the model of each systems, and so our main contributions in this regard involves:

- Obtaining the forms of the exact solutions of each systems by using manual iterations to get the final formula of the solutions.
- Investigate the periodicity of the solution of each systems of difference equations.
- Using Fibonacci sequence to formulate the exact solutions of some systems.
- Confirming our theoretical results graphically and obtain the numerical results to explain the behaviours of the solutions by using some mathematical programming such as MATLAB.

2. Main results

2.1. *On the system:* $S_{n+1} = \dfrac{T_n S_{n-2}}{S_{n-2} + T_{n-1}}, \quad T_{n+1} = \dfrac{S_n T_{n-2}}{T_{n-2} + S_{n-1}}$

In this section, we construct a specific form of the solutions of the following system

$$s_{n+1} = \frac{t_n s_{n-2}}{s_{n-2} + t_{n-1}}, \quad t_{n+1} = \frac{s_n t_{n-2}}{t_{n-2} + s_{n-1}}, \tag{2.1}$$

with positive real number initial values.

Theorem 1. Let $\{s_n, t_n\}_{n=-2}^{\infty}$ be a solution of system (2.1) and assume that $s_{-2} = c, s_{-1} = b, s_0 = a, t_{-2} = g, t_{-1} = e, t_0 = d$. Then for $n = 0, 1, ...,$

$$s_{4n-2} = c \prod_{i=0}^{n-1} \frac{(gf_{2i} + bf_{2i-1})(bf_{2i} + df_{2i-1})(cf_{2i-1} + ef_{2i-2})(ef_{2i-1} + af_{2i-2})}{(gf_{2i+1} + bf_{2i})(bf_{2i+1} + df_{2i})(cf_{2i} + ef_{2i-1})(ef_{2i} + af_{2i-1})},$$

$$s_{4n-1} = b \prod_{i=0}^{n-1} \frac{(gf_{2i+1} + bf_{2i})(bf_{2i-1} + df_{2i-2})(cf_{2i} + ef_{2i-1})(ef_{2i} + af_{2i-1})}{(gf_{2i+2} + bf_{2i+1})(bf_{2i} + df_{2i-1})(cf_{2i+1} + ef_{2i})(ef_{2i+1} + af_{2i})},$$

$$s_{4n} = a \prod_{i=0}^{n-1} \frac{(gf_{2i} + bf_{2i-1})(bf_{2i} + df_{2i-1})(cf_{2i+1} + ef_{2i})(ef_{2i+1} + af_{2i})}{(gf_{2i+1} + bf_{2i})(bf_{2i+1} + df_{2i})(cf_{2i+2} + ef_{2i+1})(ef_{2i+2} + af_{2i+1})},$$

$$s_{4n+1} = \frac{cd}{(c+e)} \prod_{i=0}^{n-1} \frac{(gf_{2i+1} + bf_{2i})(bf_{2i+1} + df_{2i})(cf_{2i+2} + ef_{2i+1})(ef_{2i} + af_{2i-1})}{(gf_{2i+2} + bf_{2i+1})(bf_{2i+2} + df_{2i+1})(cf_{2i+3} + ef_{2i+2})(ef_{2i+1} + af_{2i})},$$

$$t_{4n-2}=g\prod_{i=0}^{n-1}\frac{(gf_{2i-1}+bf_{2i-2})(bf_{2i-1}+df_{2i-2})(cf_{2i}+ef_{2i-1})(ef_{2i}+af_{2i-1})}{(gf_{2i}+bf_{2i-1})(bf_{2i}+df_{2i-1})(cf_{2i+1}+ef_{2i})(ef_{2i+1}+af_{2i})},$$

$$t_{4n-1}=e\prod_{i=0}^{n-1}\frac{(gf_{2i}+bf_{2i-1})(bf_{2i}+df_{2i-1})(cf_{2i+1}+ef_{2i})(ef_{2i-1}+af_{2i-2})}{(gf_{2i+1}+bf_{2i})(bf_{2i+1}+df_{2i})(cf_{2i+2}+ef_{2i+1})(ef_{2i}+af_{2i-1})},$$

$$t_{4n}=d\prod_{i=0}^{n-1}\frac{(gf_{2i+1}+bf_{2i})(bf_{2i+1}+df_{2i})(cf_{2i}+ef_{2i-1})(ef_{2i}+af_{2i-1})}{(gf_{2i+2}+bf_{2i+1})(bf_{2i+2}+df_{2i+1})(cf_{2i+1}+ef_{2i})(ef_{2i+1}+af_{2i})},$$

$$t_{4n+1}=\frac{ag}{(g+b)}\prod_{i=0}^{n-1}\frac{(gf_{2i+2}+bf_{2i+1})(bf_{2i}+df_{2i-1})(cf_{2i+1}+ef_{2i})(ef_{2i+1}+af_{2i})}{(gf_{2i+3}+bf_{2i+2})(bf_{2i+1}+df_{2i})(cf_{2i+2}+ef_{2i+1})(ef_{2i+2}+af_{2i+1})},$$

where the Fibonacci sequence$\{f_m\}_{m=-2}^{\infty}=\{1,0,1,1,2,3,5,8,13,...\}$.
Proof. Clearly, for $n=0$, the solutions hold. Now let $n>0$ and claim that the solutions are satisfied for $n-1$. That is,

$$s_{4n-6}=c\prod_{i=0}^{n-2}\frac{(gf_{2i}+bf_{2i-1})(bf_{2i}+df_{2i-1})(cf_{2i-1}+ef_{2i-2})(ef_{2i-1}+af_{2i-2})}{(gf_{2i+1}+bf_{2i})(bf_{2i+1}+df_{2i})(cf_{2i}+ef_{2i-1})(ef_{2i}+af_{2i-1})},$$

$$s_{4n-5}=b\prod_{i=0}^{n-2}\frac{(gf_{2i+1}+bf_{2i})(bf_{2i-1}+df_{2i-2})(cf_{2i}+ef_{2i-1})(ef_{2i}+af_{2i-1})}{(gf_{2i+2}+bf_{2i+1})(bf_{2i}+df_{2i-1})(cf_{2i+1}+ef_{2i})(ef_{2i+1}+af_{2i})},$$

$$s_{4n-4}=a\prod_{i=0}^{n-2}\frac{(gf_{2i}+bf_{2i-1})(bf_{2i}+df_{2i-1})(cf_{2i+1}+ef_{2i})(ef_{2i+1}+af_{2i})}{(gf_{2i+1}+bf_{2i})(bf_{2i+1}+df_{2i})(cf_{2i+2}+ef_{2i+1})(ef_{2i+2}+af_{2i+1})},$$

$$s_{4n-3}=\frac{cd}{(c+e)}\prod_{i=0}^{n-2}\frac{(gf_{2i+1}+bf_{2i})(bf_{2i+1}+df_{2i})(cf_{2i+2}+ef_{2i+1})(ef_{2i}+af_{2i-1})}{(gf_{2i+2}+bf_{2i+1})(bf_{2i+2}+df_{2i+1})(cf_{2i+3}+ef_{2i+2})(ef_{2i+1}+af_{2i})},$$

$$t_{4n-6}=g\prod_{i=0}^{n-2}\frac{(gf_{2i-1}+bf_{2i-2})(bf_{2i-1}+df_{2i-2})(cf_{2i}+ef_{2i-1})(ef_{2i}+af_{2i-1})}{(gf_{2i}+bf_{2i-1})(bf_{2i}+df_{2i-1})(cf_{2i+1}+ef_{2i})(ef_{2i+1}+af_{2i})},$$

$$t_{4n-5}=e\prod_{i=0}^{n-2}\frac{(gf_{2i}+bf_{2i-1})(bf_{2i}+df_{2i-1})(cf_{2i+1}+ef_{2i})(ef_{2i-1}+af_{2i-2})}{(gf_{2i+1}+bf_{2i})(bf_{2i+1}+df_{2i})(cf_{2i+2}+ef_{2i+1})(ef_{2i}+af_{2i-1})},$$

$$t_{4n-4}=d\prod_{i=0}^{n-2}\frac{(gf_{2i+1}+bf_{2i})(bf_{2i+1}+df_{2i})(cf_{2i}+ef_{2i-1})(ef_{2i}+af_{2i-1})}{(gf_{2i+2}+bf_{2i+1})(bf_{2i+2}+df_{2i+1})(cf_{2i+1}+ef_{2i})(ef_{2i+1}+af_{2i})},$$

$$t_{4n-3}=\frac{ag}{(g+b)}\prod_{i=0}^{n-2}\frac{(gf_{2i+2}+bf_{2i+1})(bf_{2i}+df_{2i-1})(cf_{2i+1}+ef_{2i})(ef_{2i+1}+af_{2i})}{(gf_{2i+3}+bf_{2i+2})(bf_{2i+1}+df_{2i})(cf_{2i+2}+ef_{2i+1})(ef_{2i+2}+af_{2i+1})}.$$

From system (2.1), we have

$$s_{4n-2}=\frac{t_{4n-3}s_{4n-5}}{s_{4n-5}+t_{4n-4}}$$

$$=\frac{ag}{(g+b)}\frac{\prod_{i=0}^{n-2}\frac{(gf_{2i+2}+bf_{2i+1})(bf_{2i}+df_{2i-1})(cf_{2i+2}+ef_{2i})(ef_{2i+1}+af_{2i})}{(gf_{2i+3}+bf_{2i+2})(bf_{2i+1}+df_{2i})(cf_{2i+1}+ef_{2i+1})(ef_{2i+2}+af_{2i+1})}b\prod_{i=0}^{n-2}\frac{(gf_{2i+1}+bf_{2i})(bf_{2i-1}+df_{2i-2})(cf_{2i}+ef_{2i-1})(ef_{2i}+af_{2i-1})}{(gf_{2i+2}+bf_{2i+1})(bf_{2i}+df_{2i-1})(cf_{2i+1}+ef_{2i})(ef_{2i+1}+af_{2i})}}{b\prod_{i=0}^{n-2}\frac{(gf_{2i+1}+bf_{2i})(bf_{2i-1}+df_{2i-2})(cf_{2i}+ef_{2i-1})(ef_{2i}+af_{2i-1})}{(gf_{2i+2}+bf_{2i+1})(bf_{2i}+df_{2i-1})(cf_{2i+1}+ef_{2i})(ef_{2i+1}+af_{2i})}+d\prod_{i=0}^{n-2}\frac{(gf_{2i+1}+bf_{2i})(bf_{2i+1}+df_{2i})(cf_{2i}+ef_{2i-1})(ef_{2i}+af_{2i-1})}{(gf_{2i+2}+bf_{2i+1})(bf_{2i+2}+df_{2i+1})(cf_{2i+1}+ef_{2i})(ef_{2i+1}+af_{2i})}}$$

$$=\frac{abg}{(g+b)}\prod_{i=0}^{n-2}\frac{\frac{(gf_{2i+2}+bf_{2i+1})(bf_{2i}+df_{2i-1})(cf_{2i+1}+ef_{2i})(ef_{2i+1}+af_{2i})}{(gf_{2i+3}+bf_{2i+2})(bf_{2i+1}+df_{2i})(cf_{2i+2}+ef_{2i+1})(ef_{2i+2}+af_{2i+1})}\frac{(gf_{2i+1}+bf_{2i})(bf_{2i-1}+df_{2i-2})(cf_{2i}+ef_{2i-1})(ef_{2i}+af_{2i-1})}{(gf_{2i+2}+bf_{2i+1})(bf_{2i}+df_{2i-1})(cf_{2i+1}+ef_{2i})(ef_{2i+1}+af_{2i})}}{b\frac{(gf_{2i+1}+bf_{2i})(bf_{2i-1}+df_{2i-2})(cf_{2i}+ef_{2i-1})(ef_{2i}+af_{2i-1})}{(gf_{2i+2}+bf_{2i+1})(bf_{2i}+df_{2i-1})(cf_{2i+1}+ef_{2i})(ef_{2i+1}+af_{2i})}+d\frac{(gf_{2i+1}+bf_{2i})(bf_{2i+1}+df_{2i})(cf_{2i}+ef_{2i-1})(ef_{2i}+af_{2i-1})}{(gf_{2i+2}+bf_{2i+1})(bf_{2i+2}+df_{2i+1})(cf_{2i+1}+ef_{2i})(ef_{2i+1}+af_{2i})}}$$

$$=\frac{abg}{(g+b)}\prod_{i=0}^{n-2}\frac{\frac{(gf_{2i+1}+bf_{2i})(bf_{2i-1}+df_{2i-2})(cf_{2i}+ef_{2i-1})(ef_{2i}+af_{2i-1})}{(gf_{2i+3}+bf_{2i+2})(bf_{2i+1}+df_{2i})(cf_{2i+2}+ef_{2i+1})(ef_{2i+2}+af_{2i+1})}}{\frac{(gf_{2i+1}+bf_{2i})(cf_{2i}+ef_{2i-1})(ef_{2i}+af_{2i-1})}{(gf_{2i+2}+bf_{2i+1})(cf_{2i+1}+ef_{2i})(ef_{2i+1}+af_{2i})}\left(b\frac{(bf_{2i-1}+df_{2i-2})}{(bf_{2i}+df_{2i-1})}+d\frac{(bf_{2i+1}+df_{2i})}{(bf_{2i+2}+df_{2i+1})}\right)}$$

$$=\frac{abg}{(g+b)}\prod_{i=0}^{n-2}\frac{\frac{(bf_{2i-1}+df_{2i-2})}{(gf_{2i+3}+bf_{2i+2})(bf_{2i+1}+df_{2i})(cf_{2i+2}+ef_{2i+1})(ef_{2i+2}+af_{2i+1})}}{\frac{1}{(gf_{2i+2}+bf_{2i+1})(cf_{2i+1}+ef_{2i})(ef_{2i+1}+af_{2i})}\left(b\frac{(bf_{2i-1}+df_{2i-2})}{(bf_{2i}+df_{2i-1})}+d\frac{(bf_{2i+1}+df_{2i})}{(bf_{2i+2}+df_{2i+1})}\right)}$$

$$=\frac{abg}{(g+b)}\prod_{i=0}^{n-2}\frac{\frac{(gf_{2i+2}+bf_{2i+1})(bf_{2i-1}+df_{2i-2})(cf_{2i+1}+ef_{2i})(ef_{2i+1}+af_{2i})}{(gf_{2i+3}+bf_{2i+2})(bf_{2i+1}+df_{2i})(cf_{2i+2}+ef_{2i+1})(ef_{2i+2}+af_{2i+1})}}{b\frac{(bf_{2i-1}+df_{2i-2})}{(bf_{2i}+df_{2i-1})}+d\frac{(bf_{2i+1}+df_{2i})}{(bf_{2i+2}+df_{2i+1})}}$$

$$=\frac{abg}{(g+b)}\frac{\prod_{i=0}^{n-2}\frac{(gf_{2i+2}+bf_{2i+1})(cf_{2i+1}+ef_{2i})(ef_{2i+1}+af_{2i})}{(gf_{2i+3}+bf_{2i+2})(cf_{2i+2}+ef_{2i+1})(ef_{2i+2}+af_{2i+1})}\prod_{i=0}^{n-2}\frac{(bf_{2i-1}+df_{2i-2})}{(bf_{2i+1}+df_{2i})}}{b\prod_{i=0}^{n-2}\frac{(bf_{2i-1}+df_{2i-2})}{(bf_{2i}+df_{2i-1})}+d\prod_{i=0}^{n-2}\frac{(bf_{2i+1}+df_{2i})}{(bf_{2i+2}+df_{2i+1})}}$$

$$=\frac{abg}{(g+b)}\frac{\prod_{i'=1}^{n-1}\frac{(gf_{2i'}+bf_{2i'-1})(cf_{2i'-1}+ef_{2i'-2})(ef_{2i'-1}+af_{2i'-2})}{(gf_{2i'+1}+bf_{2i'})(cf_{2i'}+ef_{2i'-1})(ef_{2i'}+af_{2i'-1})}\frac{d}{(bf_{2n-3}+df_{2n-4})}}{b\prod_{i=0}^{n-2}\frac{(bf_{2i-1}+df_{2i-2})}{(bf_{2i}+df_{2i-1})}+d\prod_{i'=1}^{n-1}\frac{(bf_{2i'-1}+df_{2i'-2})}{(bf_{2i'}+df_{2i'-1})}}$$

$$=\frac{abg}{(g+b)}\frac{\prod_{i'=1}^{n-1}\frac{(gf_{2i'}+bf_{2i'-1})(cf_{2i'-1}+ef_{2i'-2})(ef_{2i'-1}+af_{2i'-2})}{(gf_{2i'+1}+bf_{2i'})(cf_{2i'}+ef_{2i'-1})(ef_{2i'}+af_{2i'-1})}\frac{d}{(bf_{2n-3}+df_{2n-4})}}{b\prod_{i=0}^{n-2}\frac{(bf_{2i-1}+df_{2i-2})}{(bf_{2i}+df_{2i-1})}+b\prod_{i'=0}^{n-1}\frac{(bf_{2i'-1}+df_{2i'-2})}{(bf_{2i'}+df_{2i'-1})}}$$

$$=\frac{dabg}{(g+b)(bf_{2n-3}+df_{2n-4})}\frac{\prod_{i'=1}^{n-1}\frac{(gf_{2i'}+bf_{2i'-1})(cf_{2i'-1}+ef_{2i'-2})(ef_{2i'-1}+af_{2i'-2})}{(gf_{2i'+1}+bf_{2i'})(cf_{2i'}+ef_{2i'-1})(ef_{2i'}+af_{2i'-1})}}{b\prod_{i=0}^{n-1}\frac{(bf_{2i-1}+df_{2i-2})}{(bf_{2i}+df_{2i-1})}\left(\frac{(bf_{2n-2}+df_{2n-3})}{(bf_{2n-3}+df_{2n-4})}+1\right)}$$

$$= \frac{dabg}{(g+b)(bf_{2n-3}+df_{2n-4})} \frac{\prod_{i'=1}^{n-1} \frac{(gf_{2i'}+bf_{2i'-1})(cf_{2i'-1}+ef_{2i'-2})(ef_{2i'-1}+af_{2i'-2})}{(gf_{2i'+1}+bf_{2i'})(cf_{2i'}+ef_{2i'-1})(ef_{2i'}+af_{2i'-1})}}{b\prod_{i=0}^{n-1} \frac{(bf_{2i-1}+df_{2i-2})}{(bf_{2i}+df_{2i-1})} \frac{(b(f_{2n-3}+f_{2n-2})+d(f_{2n-4}+f_{2n-3}))}{(bf_{2n-3}+df_{2n-4})}}$$

$$= \frac{dag}{(g+b)} \frac{\prod_{i'=1}^{n-1} \frac{(gf_{2i'}+bf_{2i'-1})(cf_{2i'-1}+ef_{2i'-2})(ef_{2i'-1}+af_{2i'-2})}{(gf_{2i'+1}+bf_{2i'})(cf_{2i'}+ef_{2i'-1})(ef_{2i'}+af_{2i'-1})}}{\prod_{i=0}^{n-1} \frac{(bf_{2i-1}+df_{2i-2})}{(bf_{2i}+df_{2i-1})}(b(f_{2n-3}+f_{2n-2})+d(f_{2n-4}+f_{2n-3}))}$$

$$= \frac{dag}{(g+b)} \frac{\prod_{i'=1}^{n-1} \frac{(gf_{2i'}+bf_{2i'-1})(cf_{2i'-1}+ef_{2i'-2})(ef_{2i'-1}+af_{2i'-2})}{(gf_{2i'+1}+bf_{2i'})(cf_{2i'}+ef_{2i'-1})(ef_{2i'}+af_{2i'-1})}}{\prod_{i=0}^{n-1} \frac{(bf_{2i-1}+df_{2i-2})}{(bf_{2i}+df_{2i-1})}(bf_{2n-1}+df_{2n-2})}$$

$$= dc \frac{\prod_{i'=0}^{n-1} \frac{(gf_{2i'}+bf_{2i'-1})(cf_{2i'-1}+ef_{2i'-2})(ef_{2i'-1}+af_{2i'-2})}{(gf_{2i'+1}+bf_{2i'})(cf_{2i'}+ef_{2i'-1})(ef_{2i'}+af_{2i'-1})}}{\prod_{i=0}^{n-1} \frac{(bf_{2i-1}+df_{2i-2})}{(bf_{2i}+df_{2i-1})}(bf_{2n-1}+df_{2n-2})}$$

$$= \frac{dc}{(bf_{2n-1}+df_{2n-2})} \prod_{i'=0}^{n-1} \frac{(gf_{2i'}+bf_{2i'-1})(cf_{2i'-1}+ef_{2i'-2})(ef_{2i'-1}+af_{2i'-2})}{(gf_{2i'+1}+bf_{2i'})(cf_{2i'}+ef_{2i'-1})(ef_{2i'}+af_{2i'-1})} \prod_{i=0}^{n-1} \frac{(bf_{2i}+df_{2i-1})}{(bf_{2i-1}+df_{2i-2})}$$

$$= \frac{dc}{(bf_{2n-1}+df_{2n-2})} \prod_{i=0}^{n-1} \frac{(gf_{2i}+bf_{2i-1})(bf_{2i}+df_{2i-1})(cf_{2i-1}+ef_{2i-2})(ef_{2i-1}+af_{2i-2})}{(gf_{2i+1}+bf_{2i})(cf_{2i}+ef_{2i-1})(ef_{2i}+af_{2i-1})} \frac{1}{\prod_{i=0}^{n-1}(bf_{2i-1}+df_{2i-2})}$$

$$= c \prod_{i=0}^{n-1} \frac{(gf_{2i}+bf_{2i-1})(bf_{2i}+df_{2i-1})(cf_{2i-1}+ef_{2i-2})(ef_{2i-1}+af_{2i-2})}{(gf_{2i+1}+bf_{2i})(cf_{2i}+ef_{2i-1})(ef_{2i}+af_{2i-1})} \frac{1}{\prod_{i=1}^{n}(bf_{2i-1}+df_{2i-2})}$$

$$= c \prod_{i=0}^{n-1} \frac{(gf_{2i}+bf_{2i-1})(bf_{2i}+df_{2i-1})(cf_{2i-1}+ef_{2i-2})(ef_{2i-1}+af_{2i-2})}{(gf_{2i+1}+bf_{2i})(cf_{2i}+ef_{2i-1})(ef_{2i}+af_{2i-1})} \frac{1}{\prod_{i=0}^{n-1}(bf_{2i+1}+df_{2i})}$$

$$= c \prod_{i=0}^{n-1} \frac{(gf_{2i}+bf_{2i-1})(bf_{2i}+df_{2i-1})(cf_{2i-1}+ef_{2i-2})(ef_{2i-1}+af_{2i-2})}{(gf_{2i+1}+bf_{2i})(bf_{2i+1}+df_{2i})(cf_{2i}+ef_{2i-1})(ef_{2i}+af_{2i-1})}.$$

Other formulas can be obtain by similar way. Thus, the proof is complete.

2.2. *On the system:* $S_{n+1} = \frac{T_n S_{n-2}}{S_{n-2}+T_{n-1}}, \quad T_{n+1} = \frac{S_n T_{n-2}}{T_{n-2}-S_{n-1}}$

This part studies the shape of the solutions of the following system:

$$s_{n+1} = \frac{t_n s_{n-2}}{s_{n-2}+t_{n-1}}, \quad t_{n+1} = \frac{s_n t_{n-2}}{t_{n-2}-s_{n-1}}, \tag{2.2}$$

where $s_{-2}, s_{-1}, s_0, t_{-2}, t_{-1}, t_0$ are positive real numbers.

Theorem 2. Let $\{s_n, t_n\}_{n=-2}^{\infty}$ be a solution of system (2.2) and suppose that $s_{-2} = c, s_{-1} = b, s_0 = a, t_{-2} = g, t_{-1} = e, t_0 = d$. Then for $n = 0, 1, ...,$

$$s_{4n-2} = -c \prod_{i=0}^{n-1} \frac{(gf_{i+1} - bf_{i-1})(bf_{i-2} + df_{i-1})(cf_{i-1} + ef_i)(ef_{i-1} - af_{i-3})}{(gf_i - bf_{i-2})(bf_i + df_{i+1})(cf_{i-2} + ef_{i-1})(ef_{i+1} - af_{i-1})},$$

$$s_{4n-1} = b \prod_{i=0}^{n-1} \frac{(gf_i - bf_{i-2})(bf_{i-1} + df_i)(cf_{i-2} + ef_{i-1})(ef_{i+1} - af_{i-1})}{(gf_{i+2} - bf_i)(bf_{i-2} + df_{i-1})(cf_i + ef_{i+1})(ef_i - af_{i-2})},$$

$$s_{4n} = a \prod_{i=0}^{n-1} \frac{(gf_{i+1} - bf_{i-1})(bf_{i-2} + df_{i-1})(cf_i + ef_{i+1})(ef_{i-2} - af_{i-2})}{(gf_i - bf_{i-2})(bf_i + df_{i+1})(cf_{i-1} + ef_i)(ef_{i+2} - af_i)},$$

$$s_{4n+1} = \frac{cd}{(c+e)} \prod_{i=0}^{n-1} \frac{(gf_i - bf_{i-2})(bf_i + df_{i+1})(cf_{i-1} + ef_i)(ef_{i+1} - af_{i-1})}{(gf_{i+1} - bf_i)(bf_{i-1} + df_i)(cf_{i+1} + ef_{i+2})(ef_i - af_{i-2})},$$

$$t_{4n-2} = -g \prod_{i=0}^{n-1} \frac{(gf_{i-1} - bf_{i-3})(bf_{i-1} + df_i)(cf_{i-2} + ef_{i-1})(ef_{i+1} - af_{i-1})}{(gf_{i+1} - bf_{i-1})(bf_{i-2} + df_{i-1})(cf_i + ef_{i+1})(ef_i - af_{i-2})},$$

$$t_{4n-1} = -e \prod_{i=0}^{n-1} \frac{(gf_{i+1} - bf_{i-1})(bf_{i-2} + df_{i-1})(cf_i + ef_{i+1})(ef_{i-1} - af_{i-3})}{(gf_i - bf_{i-2})(bf_i + df_{i+1})(cf_{i-1} + ef_i)(ef_{i+1} - af_{i-1})},$$

$$t_{4n} = d \prod_{i=0}^{n-1} \frac{(gf_i - bf_{i-2})(bf_i + df_{i+1})(cf_{i-2} + ef_{i-1})(ef_{i+1} - af_{i-1})}{(gf_{i+2} - bf_i)(bf_{i-1} + df_i)(cf_i + ef_{i+1})(ef_i - af_{i-2})},$$

$$t_{4n+1} = \frac{ag}{(g-b)} \prod_{i=0}^{n-1} \frac{(gf_{i+2} - bf_i)(bf_{i-2} + df_{i-1})(cf_i + ef_{i+1})(ef_i - af_{i-2})}{(gf_{i+1} - bf_{i-1})(bf_i + df_{i+1})(cf_{i-1} + ef_i)(ef_{i+2} - af_i)},$$

where $\{f_m\}_{m=-3}^{\infty} = \{1, 1, 0, 1, 1, 2, 3, 5, 8, 13, ...\}$.

Proof. For $n = 0$, the result holds. Now let $n > 0$, assuming that the results are true for $n - 1$. That is,

$$s_{4n-6} = -c \prod_{i=0}^{n-2} \frac{(gf_{i+1} - bf_{i-1})(bf_{i-2} + df_{i-1})(cf_{i-1} + ef_i)(ef_{i-1} - af_{i-3})}{(gf_i - bf_{i-2})(bf_i + df_{i+1})(cf_{i-2} + ef_{i-1})(ef_{i+1} - af_{i-1})},$$

$$s_{4n-5} = b \prod_{i=0}^{n-2} \frac{(gf_i - bf_{i-2})(bf_{i-1} + df_i)(cf_{i-2} + ef_{i-1})(ef_{i+1} - af_{i-1})}{(gf_{i+2} - bf_i)(bf_{i-2} + df_{i-1})(cf_i + ef_{i+1})(ef_i - af_{i-2})},$$

$$s_{4n-4} = a\prod_{i=0}^{n-2}\frac{(gf_{i+1}-bf_{i-1})(bf_{i-2}+df_{i-1})(cf_i+ef_{i+1})(ef_{i-2}-af_{i-2})}{(gf_i-bf_{i-2})(bf_i+df_{i+1})(cf_{i-1}+ef_i)(ef_{i+2}-af_i)},$$

$$s_{4n-3} = \frac{cd}{(c+e)}\prod_{i=0}^{n-2}\frac{(gf_i-bf_{i-2})(bf_i+df_{i+1})(cf_{i-1}+ef_i)(ef_{i+1}-af_{i-1})}{(gf_{i+1}-bf_i)(bf_{i-1}+df_i)(cf_{i+1}+ef_{i+2})(ef_i-af_{i-2})},$$

$$t_{4n-6} = -g\prod_{i=0}^{n-2}\frac{(gf_{i-1}-bf_{i-3})(bf_{i-1}+df_i)(cf_{i-2}+ef_{i-1})(ef_{i+1}-af_{i-1})}{(gf_{i+1}-bf_{i-1})(bf_{i-2}+df_{i-1})(cf_i+ef_{i+1})(ef_i-af_{i-2})},$$

$$t_{4n-5} = -e\prod_{i=0}^{n-2}\frac{(gf_{i+1}-bf_{i-1})(bf_{i-2}+df_{i-1})(cf_i+ef_{i+1})(ef_{i-1}-af_{i-3})}{(gf_i-bf_{i-2})(bf_i+df_{i+1})(cf_{i-1}+ef_i)(ef_{i+1}-af_{i-1})},$$

$$t_{4n-4} = d\prod_{i=0}^{n-2}\frac{(gf_i-bf_{i-2})(bf_i+df_{i+1})(cf_{i-2}+ef_{i-1})(ef_{i+1}-af_{i-1})}{(gf_{i+2}-bf_i)(bf_{i-1}+df_i)(cf_i+ef_{i+1})(ef_i-af_{i-2})},$$

$$t_{4n-3} = \frac{ag}{(g-b)}\prod_{i=0}^{n-2}\frac{(gf_{i+2}-bf_i)(bf_{i-2}+df_{i-1})(cf_i+ef_{i+1})(ef_i-af_{i-2})}{(gf_{i+1}-bf_{i-1})(bf_i+df_{i+1})(cf_{i-1}+ef_i)(ef_{i+2}-af_i)}.$$

Next, from system (2.2) we have

$$s_{4n-2} = \frac{t_{4n-3}s_{4n-5}}{s_{4n-5}+t_{4n-4}}$$

$$= \frac{ag}{(g-b)}\frac{\prod_{i=0}^{n-2}\frac{(gf_{i+2}-bf_i)(bf_{i-2}+df_{i-1})(cf_i+ef_{i+1})(ef_i-af_{i-2})}{(gf_{i+1}-bf_{i-1})(bf_i+df_{i+1})(cf_{i-1}+ef_i)(ef_{i+2}-af_i)}b\prod_{i=0}^{n-2}\frac{(gf_i-bf_{i-2})(bf_{i-1}+df_i)(cf_{i-2}+ef_{i-1})(ef_{i+1}-af_{i-1})}{(gf_{i+2}-bf_i)(bf_{i-2}+df_{i-1})(cf_i+ef_{i+1})(ef_i-af_{i-2})}}{b\prod_{i=0}^{n-2}\frac{(gf_i-bf_{i-2})(bf_{i-1}+df_i)(cf_{i-2}+ef_{i-1})(ef_{i+1}-af_{i-1})}{(gf_{i+2}-bf_i)(bf_{i-2}+df_{i-1})(cf_i+ef_{i+1})(ef_i-af_{i-2})}+d\prod_{i=0}^{n-2}\frac{(gf_i-bf_{i-2})(bf_i+df_{i+1})(cf_{i-2}+ef_{i-1})(ef_{i+1}-af_{i-1})}{(gf_{i+2}-bf_i)(bf_{i-1}+df_i)(cf_i+ef_{i+1})(ef_i-af_{i-2})}}$$

$$= \frac{abg}{(g-b)}\prod_{i=0}^{n-2}\frac{\frac{(gf_{i+2}-bf_i)(bf_{i-2}+df_{i-1})(cf_i+ef_{i+1})(ef_i-af_{i-2})}{(gf_{i+1}-bf_{i-1})(bf_i+df_{i+1})(cf_{i-1}+ef_i)(ef_{i+2}-af_i)}\frac{(gf_i-bf_{i-2})(bf_{i-1}+df_i)(cf_{i-2}+ef_{i-1})(ef_{i+1}-af_{i-1})}{(gf_{i+2}-bf_i)(bf_{i-2}+df_{i-1})(cf_i+ef_{i+1})(ef_i-af_{i-2})}}{b\frac{(gf_i-bf_{i-2})(bf_{i-1}+df_i)(cf_{i-2}+ef_{i-1})(ef_{i+1}-af_{i-1})}{(gf_{i+2}-bf_i)(bf_{i-2}+df_{i-1})(cf_i+ef_{i+1})(ef_i-af_{i-2})}+d\frac{(gf_i-bf_{i-2})(bf_i+df_{i+1})(cf_{i-2}+ef_{i-1})(ef_{i+1}-af_{i-1})}{(gf_{i+2}-bf_i)(bf_{i-1}+df_i)(cf_i+ef_{i+1})(ef_i-af_{i-2})}}$$

$$= \frac{abg}{(g-b)}\prod_{i=0}^{n-2}\frac{\frac{(gf_i-bf_{i-2})(bf_{i-1}+df_i)(cf_{i-2}+ef_{i-1})(ef_{i+1}-af_{i-1})}{(gf_{i+1}-bf_{i-1})(bf_i+df_{i+1})(cf_{i-1}+ef_i)(ef_{i+2}-af_i)}}{\frac{(gf_i-bf_{i-2})(cf_{i-2}+ef_{i-1})(ef_{i+1}-af_{i-1})}{(gf_{i+2}-bf_i)(cf_i+ef_{i+1})(ef_i-af_{i-2})}\left(b\frac{(bf_{i-1}+df_i)}{(bf_{i-2}+df_{i-1})}+d\frac{(bf_i+df_{i+1})}{(bf_{i-1}+df_i)}\right)}$$

$$= \frac{abg}{(g-b)}\prod_{i=0}^{n-2}\frac{\frac{(bf_{i-1}+df_i)}{(gf_{i+1}-bf_{i-1})(bf_i+df_{i+1})(cf_{i-1}+ef_i)(ef_{i+2}-af_i)}}{\frac{1}{(gf_{i+2}-bf_i)(cf_i+ef_{i+1})(ef_i-af_{i-2})}\left(b\frac{(bf_{i-1}+df_i)}{(bf_{i-2}+df_{i-1})}+d\frac{(bf_i+df_{i+1})}{(bf_{i-1}+df_i)}\right)}$$

$$= \frac{abg}{(g-b)}\prod_{i=0}^{n-2}\frac{\frac{(gf_{i+2}-bf_i)(bf_{i-1}+df_i)(cf_i+ef_{i+1})(ef_i-af_{i-2})}{(gf_{i+1}-bf_{i-1})(bf_i+df_{i+1})(cf_{i-1}+ef_i)(ef_{i+2}-af_i)}}{b\frac{(bf_{i-1}+df_i)}{(bf_{i-2}+df_{i-1})}+d\frac{(bf_i+df_{i+1})}{(bf_{i-1}+df_i)}}$$

$$= \frac{abg}{(g-b)} \frac{\prod_{i=0}^{n-2} \frac{(gf_{i+2}-bf_i)(cf_i+ef_{i+1})(ef_i-af_{i-2})}{(gf_{i+1}-bf_{i-1})(cf_{i-1}+ef_i)(ef_{i+2}-af_i)} \prod_{i=0}^{n-2} \frac{(bf_{i-1}+df_i)}{(bf_i+df_{i+1})}}{b\prod_{i=0}^{n-2} \frac{(bf_{i-1}+df_i)}{(bf_{i-2}+df_{i-1})} + d\prod_{i=0}^{n-2} \frac{(bf_i+df_{i+1})}{(bf_{i-1}+df_i)}}$$

$$= \frac{abg}{(g-b)} \frac{\prod_{i'=1}^{n-1} \frac{(gf_{i'+1}-bf_{i'-1})(cf_{i'-1}+ef_{i'})(ef_{i'-1}-af_{i'-3})}{(gf_{i'}-bf_{i'-2})(cf_{i'-2}+ef_{i'-1})(ef_{i'+1}-af_{i'-1})} \frac{d}{(bf_{n-3}+df_{n-2})}}{b\prod_{i=0}^{n-2} \frac{(bf_{i-1}+df_i)}{(bf_{i-2}+df_{i-1})} + d\prod_{i'=1}^{n-1} \frac{(bf_{i'-1}+df_{i'})}{(bf_{i'-2}+df_{i'-1})}}$$

$$= \frac{abg}{(g-b)} \frac{\prod_{i'=1}^{n-1} \frac{(gf_{i'+1}-bf_{i'-1})(cf_{i'-1}+ef_{i'})(ef_{i'-1}-af_{i'-3})}{(gf_{i'}-bf_{i'-2})(cf_{i'-2}+ef_{i'-1})(ef_{i'+1}-af_{i'-1})} \frac{d}{(bf_{n-3}+df_{n-2})}}{b\prod_{i=0}^{n-2} \frac{(bf_{i-1}+df_i)}{(bf_{i-2}+df_{i-1})} + b\prod_{i'=0}^{n-1} \frac{(bf_{i'-1}+df_{i'})}{(bf_{i'-2}+df_{i'-1})}}$$

$$= \frac{dabg}{(g-b)(bf_{n-3}+df_{n-2})} \frac{\prod_{i'=1}^{n-1} \frac{(gf_{i'+1}-bf_{i'-1})(cf_{i'-1}+ef_{i'})(ef_{i'-1}-af_{i'-3})}{(gf_{i'}-bf_{i'-2})(cf_{i'-2}+ef_{i'-1})(ef_{i'+1}-af_{i'-1})}}{b\prod_{i=0}^{n-1} \frac{(bf_{i-1}+df_i)}{(bf_{i-2}+df_{i-1})}(\frac{(bf_{n-2}+df_{n-1})}{(bf_{n-3}+df_{n-2})}+1)}$$

$$= \frac{dabg}{(g-b)(bf_{n-3}+df_{n-2})} \frac{\prod_{i'=1}^{n-1} \frac{(gf_{i'+1}-bf_{i'-1})(cf_{i'-1}+ef_{i'})(ef_{i'-1}-af_{i'-3})}{(gf_{i'}-bf_{i'-2})(cf_{i'-2}+ef_{i'-1})(ef_{i'+1}-af_{i'-1})}}{b\prod_{i=0}^{n-1} \frac{(bf_{i-1}+df_i)}{(bf_{i-2}+df_{i-1})} \frac{(b(f_{n-3}+f_{n-2})+d(f_{n-2}+f_{n-1}))}{(bf_{n-3}+df_{n-2})}}$$

$$= \frac{dag}{(g-b)} \frac{\prod_{i'=1}^{n-1} \frac{(gf_{i'+1}-bf_{i'-1})(cf_{i'-1}+ef_{i'})(ef_{i'-1}-af_{i'-3})}{(gf_{i'}-bf_{i'-2})(cf_{i'-2}+ef_{i'-1})(ef_{i'+1}-af_{i'-1})}}{\prod_{i=0}^{n-1} \frac{(bf_{i-1}+df_i)}{(bf_{i-2}+df_{i-1})}(b(f_{n-3}+f_{n-2})+d(f_{n-2}+f_{n-1}))}$$

$$= \frac{dag}{(g-b)} \frac{\prod_{i'=1}^{n-1} \frac{(gf_{i'+1}-bf_{i'-1})(cf_{i'-1}+ef_{i'})(ef_{i'-1}-af_{i'-3})}{(gf_{i'}-bf_{i'-2})(cf_{i'-2}+ef_{i'-1})(ef_{i'+1}-af_{i'-1})}}{\prod_{i=0}^{n-1} \frac{(bf_{i-1}+df_i)}{(bf_{i-2}+df_{i-1})}(bf_{n-1}+df_n)}$$

$$= -dc\frac{\prod_{i'=0}^{n-1} \frac{(gf_{i'+1}-bf_{i'-1})(cf_{i'-1}+ef_{i'})(ef_{i'-1}-af_{i'-3})}{(gf_{i'}-bf_{i'-2})(cf_{i'-2}+ef_{i'-1})(ef_{i'+1}-af_{i'-1})}}{\prod_{i=0}^{n-1} \frac{(bf_{i-1}+df_i)}{(bf_{i-2}+df_{i-1})}(bf_{n-1}+df_n)}$$

$$= \frac{-dc}{(bf_{n-1}+df_n)} \prod_{i'=0}^{n-1} \frac{(gf_{i'+1}-bf_{i'-1})(cf_{i'-1}+ef_{i'})(ef_{i'-1}-af_{i'-3})}{(gf_{i'}-bf_{i'-2})(cf_{i'-2}+ef_{i'-1})(ef_{i'+1}-af_{i'-1})} \prod_{i=0}^{n-1} \frac{(bf_{i-2}+df_{i-1})}{(bf_{i-1}+df_i)}$$

$$= \frac{-dc}{(bf_{n-1}+df_n)} \prod_{i=0}^{n-1} \frac{(gf_{i+1}-bf_{i-1})(bf_{i-2}+df_{i-1})(cf_{i-1}+ef_i)(ef_{i-1}-af_{i-3})}{(gf_i-bf_{i-2})(cf_{i-2}+ef_{i-1})(ef_{i+1}-af_{i-1})} \frac{1}{\prod_{i=0}^{n-1}(bf_{i-1}+df_i)}$$

$$= -c \prod_{i=0}^{n-1} \frac{(gf_{i+1}-bf_{i-1})(bf_{i-2}+df_{i-1})(cf_{i-1}+ef_i)(ef_{i-1}-af_{i-3})}{(gf_i-bf_{i-2})(cf_{i-2}+ef_{i-1})(ef_{i+1}-af_{i-1})} \frac{1}{\prod_{i=1}^{n}(bf_{i-1}+df_i)}$$

$$= -c \prod_{i=0}^{n-1} \frac{(gf_{i+1}-bf_{i-1})(bf_{i-2}+df_{i-1})(cf_{i-1}+ef_i)(ef_{i-1}-af_{i-3})}{(gf_i-bf_{i-2})(cf_{i-2}+ef_{i-1})(ef_{i+1}-af_{i-1})} \frac{1}{\prod_{i=0}^{n-1}(bf_i+df_{i+1})}$$

$$= -c \prod_{i=0}^{n-1} \frac{(gf_{i+1}-bf_{i-1})(bf_{i-2}+df_{i-1})(cf_{i-1}+ef_i)(ef_{i-1}-af_{i-3})}{(gf_i-bf_{i-2})(bf_i+df_{i+1})(cf_{i-2}+ef_{i-1})(ef_{i+1}-af_{i-1})}.$$

Other formulas can be proved by identical way. Thus, this completes our proof.

2.3. On the system: $S_{n+1} = \dfrac{T_n S_{n-2}}{S_{n-2}+T_{n-1}}, \quad T_{n+1} = \dfrac{S_n T_{n-2}}{-T_{n-2}+S_{n-1}}$

This section aims to present a fundamental theorem that states the existence of periodic twelve solutions of the following system:

$$s_{n+1} = \frac{t_n s_{n-2}}{s_{n-2}+t_{n-1}}, \quad t_{n+1} = \frac{s_n t_{n-2}}{-t_{n-2}+s_{n-1}}, \tag{2.3}$$

with positive real numbers initial conditions $s_{-2}, s_{-1}, s_0, t_{-2}, t_{-1}, t_0$.

Theorem 3. Let $\{s_n, t_n\}_{n=-2}^{\infty}$ be a solution of system (2.3). Then the solutions of this system have a periodic solutions with period twelve and for $n = 0, 1, ...,$

$$
\begin{cases}
s_{12n-2} = c, & t_{12n-2} = g \\
s_{12n-1} = b, & t_{12n-1} = e \\
s_{12n} = a, & t_{12n} = d \\
s_{12n+1} = \dfrac{cd}{c+e}, & t_{12n+1} = \dfrac{ag}{b-g} \\
s_{12n+2} = \dfrac{agb}{(b-g)(b+d)}, & t_{12n+2} = \dfrac{cde}{(c+e)(a-e)} \\
s_{12n+3} = \dfrac{cde(b-g)}{b(c+e)(a-e)}, & t_{12n+3} = -\dfrac{agb(c+e)}{e(b-g)(b+d)} \\
s_{12n+4} = -\dfrac{gb(c+e)(a-e)}{e(b-g)(b+d)}, & t_{12n+4} = -\dfrac{ce(b-g)(b+d)}{b(c+e)(a-e)} \\
s_{12n+5} = \dfrac{e^2(b-g)(b+d)}{b(c+e)(a-e)}, & t_{12n+5} = \dfrac{bd(c+e)(a-e)}{e(b-g)(b+d)} \\
s_{12n+6} = -\dfrac{bd(c+e)(a-e)}{e(b-g)(b+d)}, & t_{12n+6} = -\dfrac{ae(b-g)(b+d)}{b(c+e)(a-e)} \\
s_{12n+7} = \dfrac{eag(b+d)}{b(c+e)(a-e)}, & t_{12n+7} = \dfrac{bcd(a-e)}{e(b-g)(b+d)} \\
s_{12n+8} = -\dfrac{bcd}{(b-g)(b+d)}, & t_{12n+8} = -\dfrac{eag}{(c+e)(a-e)} \\
s_{12n+9} = -\dfrac{ag}{(a-e)}, & t_{12n+9} = \dfrac{cd}{b+d}
\end{cases}
$$

where $s_{-2} = c, s_{-1} = b, s_0 = a, t_{-2} = g, t_{-1} = e, t_0 = d$.

Proof. Obviously solutions true if $n = 0$. Now, let $n > 0$ and assume that our solutions are satisfied for $n - 1$. That is;

$$\begin{cases} s_{12n-14} = c, & t_{12n-14} = g \\ s_{12n-13} = b, & t_{12n-13} = e \\ s_{12n-12} = a, & t_{12n-12} = d \\ s_{12n-11} = \dfrac{cd}{c+e}, & t_{12n-11} = \dfrac{ag}{b-g} \\ s_{12n-10} = \dfrac{agb}{(b-g)(b+d)}, & t_{12n-10} = \dfrac{cde}{(c+e)(a-e)} \\ s_{12n-9} = \dfrac{cde(b-g)}{b(c+e)(a-e)}, & t_{12n-9} = -\dfrac{agb(c+e)}{e(b-g)(b+d)} \\ s_{12n-8} = -\dfrac{gb(c+e)(a-e)}{e(b-g)(b+d)}, & t_{12n-8} = -\dfrac{ce(b-g)(b+d)}{b(c+e)(a-e)} \\ s_{12n-7} = \dfrac{e^2(b-g)(b+d)}{b(c+e)(a-e)}, & t_{12n-7} = \dfrac{bd(c+e)(a-e)}{e(b-g)(b+d)} \\ s_{12n-6} = -\dfrac{bd(c+e)(a-e)}{e(b-g)(b+d)}, & t_{12n-6} = -\dfrac{ae(b-g)(b+d)}{b(c+e)(a-e)} \\ s_{12n-5} = \dfrac{eag(b+d)}{b(c+e)(a-e)}, & t_{12n-5} = \dfrac{bcd(a-e)}{e(b-g)(b+d)} \\ s_{12n-4} = -\dfrac{bcd}{(b-g)(b+d)}, & t_{12n-4} = -\dfrac{eag}{(c+e)(a-e)} \\ s_{12n-3} = -\dfrac{ag}{(a-e)}, & t_{12n-3} = \dfrac{cd}{b+d} \end{cases} .$$

Next, system (2.3) follows that

$$\begin{aligned} s_{12n-2} &= \frac{t_{12n-3}s_{12n-5}}{s_{12n-5}+t_{12n-4}} \\ &= \frac{\dfrac{cd}{(b+d)}\dfrac{aeg(b+d)}{b(c+e)(a-e)}}{\dfrac{aeg(b+d)}{b(c+e)(a-e)} - \dfrac{aeg}{(c+e)(a-e)}} \end{aligned}$$

$$= \frac{\frac{acdeg}{b(c+e)(a-e)}}{\frac{aeg(b+d)}{b(c+e)(a-e)} - \frac{abeg}{b(c+e)(a-e)}}$$

$$= \frac{\frac{acdeg}{b(c+e)(a-e)}}{\frac{adeg}{b(c+e)(a-e)}}$$

$$= \frac{acdeg}{adeg}$$

$$= c.$$

By similar technique, we can show the other relations. Hence, the is end our proof.

2.4. *On the system:* $S_{n+1} = \frac{T_n S_{n-2}}{S_{n-2} + T_{n-1}}, \quad T_{n+1} = -\frac{S_n T_{n-2}}{T_{n-2} + S_{n-1}}$

Here, our principal task is to formulate the solutions of the following system of difference equations:

$$s_{n+1} = \frac{t_n s_{n-2}}{s_{n-2} + t_{n-1}}, \quad t_{n+1} = -\frac{s_n t_{n-2}}{t_{n-2} + s_{n-1}}, \tag{2.4}$$

where $s_{-2}, s_{-1}, s_0, t_{-2}, t_{-1}, t_0$ are defined as positive real numbers.

Theorem 4. Let $\{s_n, t_n\}_{n=-2}^{\infty}$ be a solution of system (2.4) and let $s_{-2} = c, s_{-1} = b, s_0 = a, t_{-2} = g, t_{-1} = e, t_0 = d$. Then for $n = 0, 1, ...,$

$$s_{4n-2} = -c \prod_{i=0}^{n-1} \frac{(gf_{i-2} + bf_{i-1})(bf_{i+1} + df_{i-1})(cf_{i-1} + ef_{i-3})(ef_{i-1} + af_i)}{(gf_i + bf_{i+1})(bf_i + df_{i-2})(cf_{i+1} + ef_{i-1})(ef_{i-2} + af_{i-1})},$$

$$s_{4n-1} = -b \prod_{i=0}^{n-1} \frac{(gf_i + bf_{i+1})(bf_{i-1} + df_{i-3})(cf_{i+1} + ef_{i-1})(ef_{i-2} + af_{i-1})}{(gf_{i-1} + bf_i)(bf_{i+1} + df_{i-1})(cf_i + ef_{i-2})(ef_i + af_{i+1})},$$

$$s_{4n} = a \prod_{i=0}^{n-1} \frac{(gf_{i-2} + bf_{i-1})(bf_{i+1} + df_{i-1})(cf_i + ef_{i-2})(ef_i + af_{i+1})}{(gf_i + bf_{i+1})(bf_i + df_{i-2})(cf_{i+2} + ef_i)(ef_{i-1} + af_i)},$$

$$s_{4n+1} = \frac{cd}{(c+e)} \prod_{i=0}^{n-1} \frac{(gf_i + bf_{i+1})(bf_i + df_{i-2})(cf_{i+2} + ef_i)(ef_{i-2} + af_{i-1})}{(gf_{i-1} + bf_i)(bf_{i+2} + df_i)(cf_{i+1} + ef_{i-1})(ef_i + af_{i+1})},$$

$$t_{4n-2} = -g \prod_{i=0}^{n-1} \frac{(gf_{i-1} + bf_i)(bf_{i-1} + df_{i-3})(cf_{i+1} + ef_{i-1})(ef_{i-2} + af_{i-1})}{(gf_{i-2} + bf_{i-1})(bf_{i+1} + df_{i-1})(cf_i + ef_{i-2})(ef_i + af_{i+1})},$$

$$t_{4n-1} = e \prod_{i=0}^{n-1} \frac{(gf_{i-2} + bf_{i-1})(bf_{i+1} + df_{i-1})(cf_i + ef_{i-2})(ef_{i-1} + af_i)}{(gf_i + bf_{i+1})(bf_i + df_{i-2})(cf_{i+2} + ef_i)(ef_{i-2} + af_{i-1})},$$

$$t_{4n} = d \prod_{i=0}^{n-1} \frac{(gf_i + bf_{i+1})(bf_i + df_{i-2})(cf_{i+1} + ef_{i-1})(ef_{i-2} + af_{i-1})}{(gf_{i-1} + bf_i)(bf_{i+2} + df_i)(cf_i + ef_{i-2})(ef_i + af_{i+1})},$$

$$t_{4n+1} = -\frac{ag}{(g+b)} \prod_{i=0}^{n-1} \frac{(gf_{i-1} + bf_i)(bf_{i+1} + df_{i-1})(cf_i + ef_{i-2})(ef_i + af_{i+1})}{(gf_{i+1} + bf_{i+2})(bf_i + df_{i-2})(cf_{i+2} + ef_i)(ef_{i-1} + af_i)},$$

where $\{f_m\}_{m=-3}^{\infty} = \{1, 1, 0, 1, 1, 2, 3, 5, 8, 13, ...\}$.
Proof. For $n = 0$, the relations true. Next, assume that $n > 0$ and our solutions are true for $n - 1$. That is,

$$s_{4n-6} = -c \prod_{i=0}^{n-2} \frac{(gf_{i-2} + bf_{i-1})(bf_{i+1} + df_{i-1})(cf_{i-1} + ef_{i-3})(ef_{i-1} + af_i)}{(gf_i + bf_{i+1})(bf_i + df_{i-2})(cf_{i+1} + ef_{i-1})(ef_{i-2} + af_{i-1})},$$

$$s_{4n-5} = -b \prod_{i=0}^{n-2} \frac{(gf_i + bf_{i+1})(bf_{i-1} + df_{i-3})(cf_{i+1} + ef_{i-1})(ef_{i-2} + af_{i-1})}{(gf_{i-1} + bf_i)(bf_{i+1} + df_{i-1})(cf_i + ef_{i-2})(ef_i + af_{i+1})},$$

$$s_{4n-4} = a \prod_{i=0}^{n-2} \frac{(gf_{i-2} + bf_{i-1})(bf_{i+1} + df_{i-1})(cf_i + ef_{i-2})(ef_i + af_{i+1})}{(gf_i + bf_{i+1})(bf_i + df_{i-2})(cf_{i+2} + ef_i)(ef_{i-1} + af_i)},$$

$$s_{4n-3} = \frac{cd}{(c+e)} \prod_{i=0}^{n-2} \frac{(gf_i + bf_{i+1})(bf_i + df_{i-2})(cf_{i+2} + ef_i)(ef_{i-2} + af_{i-1})}{(gf_{i-1} + bf_i)(bf_{i+2} + df_i)(cf_{i+1} + ef_{i-1})(ef_i + af_{i+1})},$$

$$t_{4n-6} = -g \prod_{i=0}^{n-2} \frac{(gf_{i-1} + bf_i)(bf_{i-1} + df_{i-3})(cf_{i+1} + ef_{i-1})(ef_{i-2} + af_{i-1})}{(gf_{i-2} + bf_{i-1})(bf_{i+1} + df_{i-1})(cf_i + ef_{i-2})(ef_i + af_{i+1})},$$

$$t_{4n-5} = e \prod_{i=0}^{n-2} \frac{(gf_{i-2} + bf_{i-1})(bf_{i+1} + df_{i-1})(cf_i + ef_{i-2})(ef_{i-1} + af_i)}{(gf_i + bf_{i+1})(bf_i + df_{i-2})(cf_{i+2} + ef_i)(ef_{i-2} + af_{i-1})},$$

$$t_{4n-4} = d \prod_{i=0}^{n-2} \frac{(gf_i + bf_{i+1})(bf_i + df_{i-2})(cf_{i+1} + ef_{i-1})(ef_{i-2} + af_{i-1})}{(gf_{i-1} + bf_i)(bf_{i+2} + df_i)(cf_i + ef_{i-2})(ef_i + af_{i+1})},$$

$$t_{4n-3} = -\frac{ag}{(g+b)} \prod_{i=0}^{n-2} \frac{(gf_{i-1} + bf_i)(bf_{i+1} + df_{i-1})(cf_i + ef_{i-2})(ef_i + af_{i+1})}{(gf_{i+1} + bf_{i+2})(bf_i + df_{i-2})(cf_{i+2} + ef_i)(ef_{i-1} + af_i)}.$$

Now, from system (2.4) we have

$$s_{4n-2} = \frac{t_{4n-3}s_{4n-5}}{s_{4n-5} + t_{4n-4}}$$

$$= \frac{ag}{(g+b)} \frac{\prod_{i=0}^{n-2} \frac{(gf_{i-1}+bf_i)(bf_{i+1}+df_{i-1})(cf_i+ef_{i-2})(ef_i+af_{i+1})}{(gf_{i+1}+bf_{i+2})(bf_i+df_{i-2})(cf_{i+2}+ef_i)(ef_{i-1}+af_i)} b \prod_{i=0}^{n-2} \frac{(gf_i+bf_{i+1})(bf_{i-1}+df_{i-3})(cf_{i+1}+ef_{i-1})(ef_{i-2}+af_{i-1})}{(gf_{i-1}+bf_i)(bf_{i+1}+df_{i-1})(cf_i+ef_{i-2})(ef_i+af_{i+1})}}{-b\prod_{i=0}^{n-2} \frac{(gf_i+bf_{i+1})(bf_{i-1}+df_{i-3})(cf_{i+1}+ef_{i-1})(ef_{i-2}+af_{i-1})}{(gf_{i-1}+bf_i)(bf_{i+1}+df_{i-1})(cf_i+ef_{i-2})(ef_i+af_{i+1})} + d\prod_{i=0}^{n-2} \frac{(gf_i+bf_{i+1})(bf_i+df_{i-2})(cf_{i+1}+ef_{i-1})(ef_{i-2}+af_{i-1})}{(gf_{i-1}+bf_i)(bf_{i+2}+df_i)(cf_i+ef_{i-2})(ef_i+af_{i+1})}}$$

$$= \frac{abg}{(g+b)} \prod_{i=0}^{n-2} \frac{\frac{(gf_{i-1}+bf_i)(bf_{i+1}+df_{i-1})(cf_i+ef_{i-2})(ef_i+af_{i+1})}{(gf_{i+1}+bf_{i+2})(bf_i+df_{i-2})(cf_{i+2}+ef_i)(ef_{i-1}+af_i)} \frac{(gf_i+bf_{i+1})(bf_{i-1}+df_{i-3})(cf_{i+1}+ef_{i-1})(ef_{i-2}+af_{i-1})}{(gf_{i-1}+bf_i)(bf_{i+1}+df_{i-1})(cf_i+ef_{i-2})(ef_i+af_{i+1})}}{-b\frac{(gf_i+bf_{i+1})(bf_{i-1}+df_{i-3})(cf_{i+1}+ef_{i-1})(ef_{i-2}+af_{i-1})}{(gf_{i-1}+bf_i)(bf_{i+1}+df_{i-1})(cf_i+ef_{i-2})(ef_i+af_{i+1})} + d\frac{(gf_i+bf_{i+1})(bf_i+df_{i-2})(cf_{i+1}+ef_{i-1})(ef_{i-2}+af_{i-1})}{(gf_{i-1}+bf_i)(bf_{i+2}+df_i)(cf_i+ef_{i-2})(ef_i+af_{i+1})}}$$

$$= \frac{abg}{(g+b)} \prod_{i=0}^{n-2} \frac{\frac{(gf_i+bf_{i+1})(bf_{i-1}+df_{i-3})(cf_{i+1}+ef_{i-1})(ef_{i-2}+af_{i-1})}{(gf_{i+1}+bf_{i+2})(bf_i+df_{i-2})(cf_{i+2}+ef_i)(ef_{i-1}+af_i)}}{\frac{(gf_i+bf_{i+1})(cf_{i+1}+ef_{i-1})(ef_{i-2}+af_{i-1})}{(gf_{i-1}+bf_i)(cf_i+ef_{i-2})(ef_i+af_{i+1})}\Big(-b\frac{(bf_{i-1}+df_{i-3})}{(bf_{i+1}+df_{i-1})} + d\frac{(bf_i+df_{i-2})}{(bf_{i+2}+df_i)}\Big)}$$

$$= \frac{abg}{(g+b)} \prod_{i=0}^{n-2} \frac{\frac{(bf_{i-1}+df_{i-3})}{(gf_{i+1}+bf_{i+2})(bf_i+df_{i-2})(cf_{i+2}+ef_i)(ef_{i-1}+af_i)}}{\frac{1}{(gf_{i-1}+bf_i)(cf_i+ef_{i-2})(ef_i+af_{i+1})}\Big(-b\frac{(bf_{i-1}+df_{i-3})}{(bf_{i+1}+df_{i-1})} + d\frac{(bf_i+df_{i-2})}{(bf_{i+2}+df_i)}\Big)}$$

$$= \frac{abg}{(g+b)} \prod_{i=0}^{n-2} \frac{\frac{(gf_{i-1}+bf_i)(bf_{i-1}+df_{i-3})(cf_i+ef_{i-2})(ef_i+af_{i+1})}{(gf_{i+1}+bf_{i+2})(bf_i+df_{i-2})(cf_{i+2}+ef_i)(ef_{i-1}+af_i)}}{-b\frac{(bf_{i-1}+df_{i-3})}{(bf_{i+1}+df_{i-1})} + d\frac{(bf_i+df_{i-2})}{(bf_{i+2}+df_i)}}$$

$$= \frac{abg}{(g+b)} \frac{\prod_{i=0}^{n-2} \frac{(gf_{i-1}+bf_i)(cf_i+ef_{i-2})(ef_i+af_{i+1})}{(gf_{i+1}+bf_{i+2})(cf_{i+2}+ef_i)(ef_{i-1}+af_i)} \prod_{i=0}^{n-2} \frac{(bf_{i-1}+df_{i-3})}{(bf_i+df_{i-2})}}{-b\prod_{i=0}^{n-2} \frac{(bf_{i-1}+df_{i-3})}{(bf_{i+1}+df_{i-1})} + d\prod_{i=0}^{n-2} \frac{(bf_i+df_{i-2})}{(bf_{i+2}+df_i)}}$$

$$= \frac{abg}{(g+b)} \frac{\prod_{i'=1}^{n-1} \frac{(gf_{i'-2}+bf_{i'-1})(cf_{i'-1}+ef_{i'-3})(ef_{i'-1}+af_{i'})}{(gf_{i'}+bf_{i'+1})(cf_{i'+1}+ef_{i'-1})(ef_{i'-2}+af_{i'-1})} \frac{d}{(bf_{n-2}+df_{n-4})}}{-b\prod_{i=0}^{n-2} \frac{(bf_{i-1}+df_{i-3})}{(bf_{i+1}+df_{i-1})} + d\prod_{i'=1}^{n-1} \frac{(bf_{i'-1}+df_{i'-3})}{(bf_{i'+1}+df_{i'-1})}}$$

$$= \frac{abg}{(g+b)} \frac{\prod_{i'=1}^{n-1} \frac{(gf_{i'-2}+bf_{i'-1})(cf_{i'-1}+ef_{i'-3})(ef_{i'-1}+af_{i'})}{(gf_{i'}+bf_{i'+1})(cf_{i'+1}+ef_{i'-1})(ef_{i'-2}+af_{i'-1})} \frac{d}{(bf_{n-2}+df_{n-4})}}{-b\prod_{i=0}^{n-2} \frac{(bf_{i-1}+df_{i-3})}{(bf_{i+1}+df_{i-1})} + b\prod_{i'=0}^{n-1} \frac{(bf_{i'-1}+df_{i'-3})}{(bf_{i'+1}+df_{i'-1})}}$$

$$= \frac{dabg}{(g+b)(bf_{n-2}+df_{n-4})} \frac{\prod_{i'=1}^{n-1} \frac{(gf_{i'-2}+bf_{i'-1})(cf_{i'-1}+ef_{i'-3})(ef_{i'-1}+af_{i'})}{(gf_{i'}+bf_{i'+1})(cf_{i'+1}+ef_{i'-1})(ef_{i'-2}+af_{i'-1})}}{-b\prod_{i=0}^{n-1} \frac{(bf_{i-1}+df_{i-3})}{(bf_{i+1}+df_{i-1})}\Big(\frac{(bf_{n-1}+df_{n-3})}{(bf_{n-2}+df_{n-4})} - 1\Big)}$$

$$= \frac{dabg}{(g+b)(bf_{n-2}+df_{n-4})} \frac{\prod_{i'=1}^{n-1} \frac{(gf_{i'-2}+bf_{i'-1})(cf_{i'-1}+ef_{i'-3})(ef_{i'-1}+af_{i'})}{(gf_{i'}+bf_{i'+1})(cf_{i'+1}+ef_{i'-1})(ef_{i'-2}+af_{i'-1})}}{-b\prod_{i=0}^{n-1} \frac{(bf_{i-1}+df_{i-3})}{(bf_{i+1}+df_{i-1})} \frac{(b(f_{n-1}-f_{n-2})+d(f_{n-3}-f_{n-4}))}{(bf_{n-2}+df_{n-4})}}$$

$$= \frac{dag}{(g+b)} \frac{\prod_{i'=1}^{n-1} \frac{(gf_{i'-2}+bf_{i'-1})(cf_{i'-1}+ef_{i'-3})(ef_{i'-1}+af_{i'})}{(gf_{i'}+bf_{i'+1})(cf_{i'+1}+ef_{i'-1})(ef_{i'-2}+af_{i'-1})}}{-\prod_{i=0}^{n-1} \frac{(bf_{i-1}+df_{i-3})}{(bf_{i+1}+df_{i-1})}((b(f_{n-1}-f_{n-2})+d(f_{n-3}-f_{n-4}))}$$

$$= \frac{dag}{(g+b)} \frac{\prod_{i'=1}^{n-1} \frac{(gf_{i'-2}+bf_{i'-1})(cf_{i'-1}+ef_{i'-3})(ef_{i'-1}+af_{i'})}{(gf_{i'}+bf_{i'+1})(cf_{i'+1}+ef_{i'-1})(ef_{i'-2}+af_{i'-1})}}{-\prod_{i=0}^{n-1} \frac{(bf_{i-1}+df_{i-3})}{(bf_{i+1}+df_{i-1})}(bf_{n-1}+df_{n-3})}$$

$$= dc \frac{\prod_{i'=0}^{n-1} \frac{(gf_{i'-2}+bf_{i'-1})(cf_{i'-1}+ef_{i'-3})(ef_{i'-1}+af_{i'})}{(gf_{i'}+bf_{i'+1})(cf_{i'+1}+ef_{i'-1})(ef_{i'-2}+af_{i'-1})}}{-\prod_{i=0}^{n-1} \frac{(bf_{i-1}+df_{i-3})}{(bf_{i+1}+df_{i-1})}(bf_{n-1}+df_{n-3})}$$

$$= \frac{-dc}{(bf_{n-1}+df_{n-3})} \prod_{i'=0}^{n-1} \frac{(gf_{i'-2}+bf_{i'-1})(cf_{i'-1}+ef_{i'-3})(ef_{i'-1}+af_{i'})}{(gf_{i'}+bf_{i'+1})(cf_{i'+1}+ef_{i'-1})(ef_{i'-2}+af_{i'-1})} \prod_{i=0}^{n-1} \frac{(bf_{i+1}+df_{i-1})}{(bf_{i-1}+df_{i-3})}$$

$$= \frac{-dc}{(bf_{n-1}+df_{n-3})} \prod_{i=0}^{n-1} \frac{(gf_{i-2}+bf_{i-1})(bf_{i+1}+df_{i-1})(cf_{i-1}+ef_{i-3})(ef_{i-1}+af_{i})}{(gf_{i}+bf_{i+1})(cf_{i+1}+ef_{i-1})(ef_{i-2}+af_{i-1})} \frac{1}{\prod_{i=0}^{n-1}(bf_{i-1}+df_{i-3})}$$

$$= -c \prod_{i=0}^{n-1} \frac{(gf_{i-2}+bf_{i-1})(bf_{i+1}+df_{i-1})(cf_{i-1}+ef_{i-3})(ef_{i-1}+af_{i})}{(gf_{i}+bf_{i+1})(cf_{i+1}+ef_{i-1})(ef_{i-2}+af_{i-1})} \frac{1}{\prod_{i=1}^{n}(bf_{i-1}+df_{i-3})}$$

$$= -c \prod_{i=0}^{n-1} \frac{(gf_{i-2}+bf_{i-1})(bf_{i+1}+df_{i-1})(cf_{i-1}+ef_{i-3})(ef_{i-1}+af_{i})}{(gf_{i}+bf_{i+1})(cf_{i+1}+ef_{i-1})(ef_{i-2}+af_{i-1})} \frac{1}{\prod_{i=0}^{n-1}(bf_{i}+df_{i-2})}$$

$$= -c \prod_{i=0}^{n-1} \frac{(gf_{i-2}+bf_{i-1})(bf_{i+1}+df_{i-1})(cf_{i-1}+ef_{i-3})(ef_{i-1}+af_{i})}{(gf_{i}+bf_{i+1})(bf_{i}+df_{i-2})(cf_{i+1}+ef_{i-1})(ef_{i-2}+af_{i-1})}.$$

Similarly, we can prove the others. Thus, the proof is complete.

3. Numerical simulation results

For numerical solutions purpose, we present some examples to confirm our theoretical results graphically of each system that we considered in this paper.

Example 1. This example plots the solution of the system (2.1) with the initial conditions $s_{-2} = 15, s_{-1} = -8, s_0 = 12, t_{-2} = -10, t_{-1} = 14, t_0 = -5$ (see Figure 1).

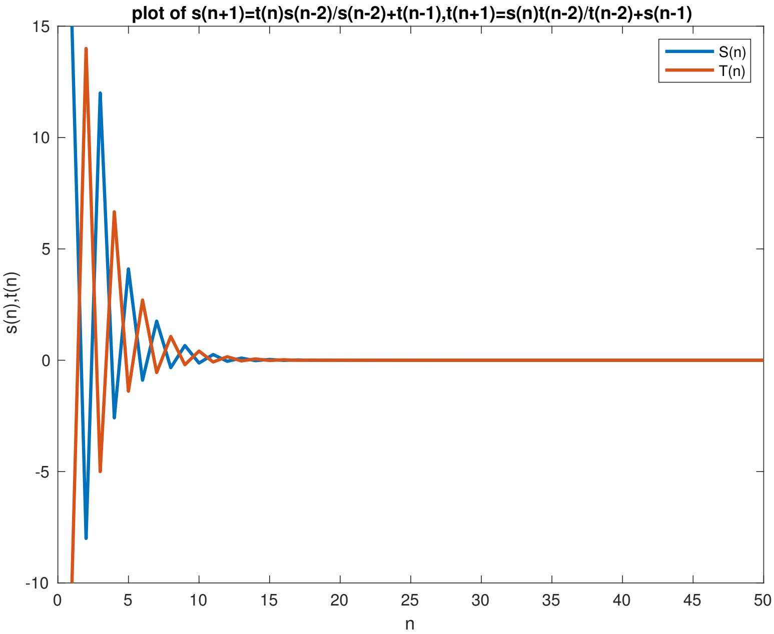

Figure 1. Shows the behavior for system (2.1) with the initial values $s_{-2} = 15, s_{-1} = -8, s_0 = 12, t_{-2} = -10, t_{-1} = 14, t_0 = -5$.

Example 2. Let us consider the behavior's solution of the system (2.2) under $s_{-2} = -5, s_{-1} = 9, s_0 = 1, t_{-2} = 8, t_{-1} = 2, t_0 = 1$ (see Figure 2).

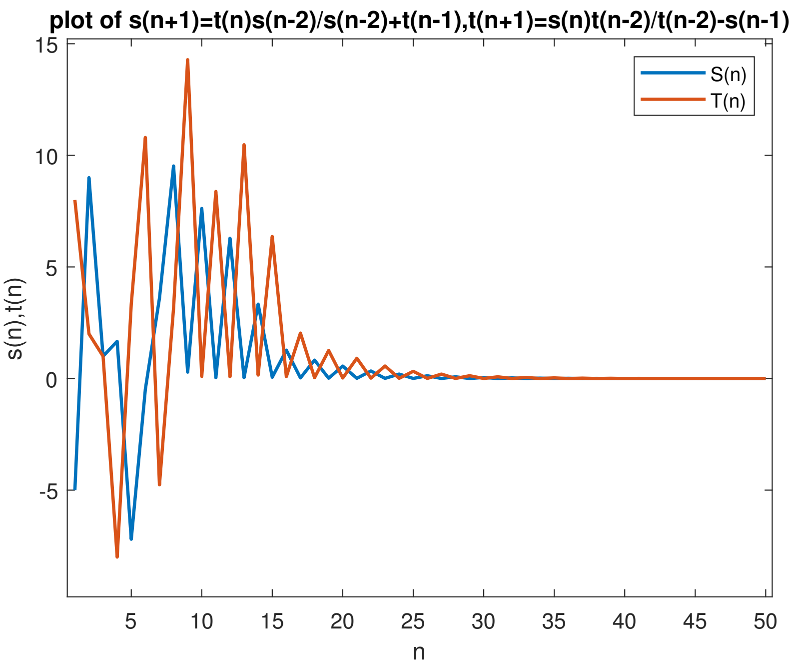

Figure 2. Plot of solution of system (2.2) under the initial values $s_{-2} = -5, s_{-1} = 9, s_0 = 1, t_{-2} = 8, t_{-1} = 2, t_0 = 1$.

Example 3. Figure 3 sketch's the periodicity of the solution of system (2.3) with $s_{-2} = 1, s_{-1} = 2, s_0 = 3, t_{-2} = 3, t_{-1} = 2, t_0 = 1$.

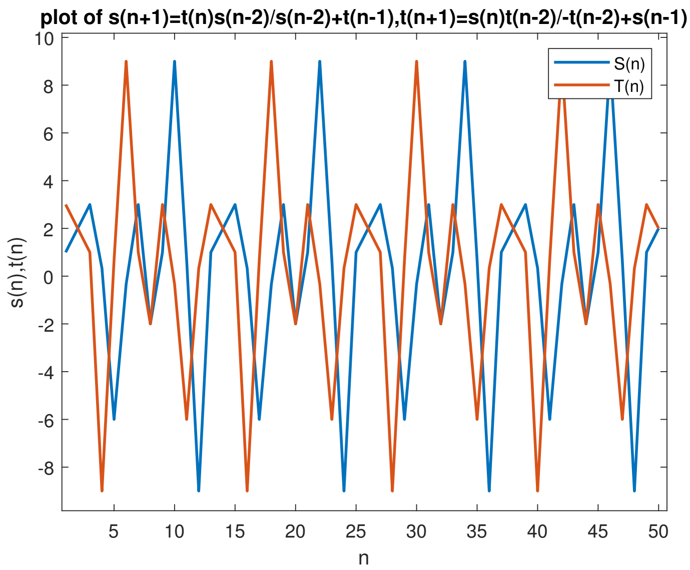

Figure 3. Expresses the solution of system (2.3) when the initial values $s_{-2} = 1, s_{-1} = 2, s_0 = 3, t_{-2} = 3, t_{-1} = 2, t_0 = 1$.

Example 4. For this system, we can show the behavior of solution with initial values $s_{-2} = 1, s_{-1} = 10, s_0 = -3, t_{-2} = 8, t_{-1} = 4, t_0 = -2$ (see Figure 4).

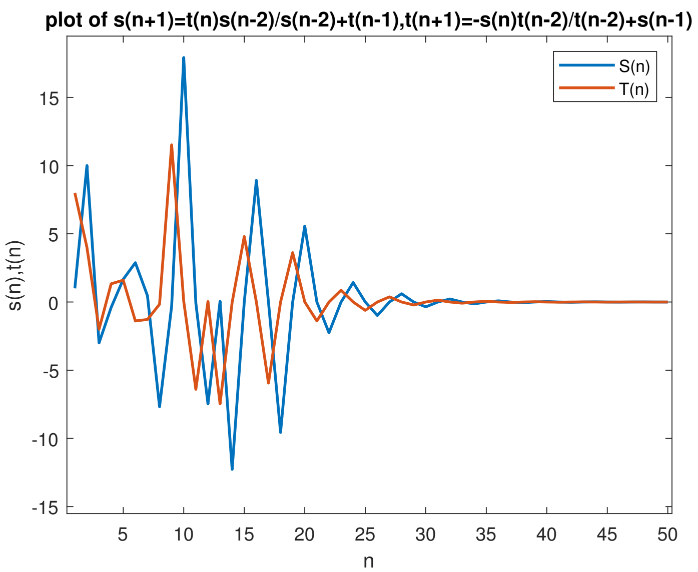

Figure 4. Illustrates the conduct of the solution of system (2.4) with $s_{-2} = 1, s_{-1} = 10, s_0 = -3, t_{-2} = 8, t_{-1} = 4, t_0 = -2$.

4. Conclusions

In this paper, we have discovered the specific form of the exact solutions of model of each systems. In particular, we devised in section (2.1) a specific form of all possible solutions of system $S_{n+1} = \frac{T_n S_{n-2}}{S_{n-2}+T_{n-1}}, \quad T_{n+1} = \frac{S_n T_{n-2}}{T_{n-2}+S_{n-1}}$ with Fibonacci sequence $\{f_m\}_{m=-2}^{\infty} = \{1,0,1,1,2,3,5,8,13,...\}$. Also, we found an exact solutions' forms of the system $S_{n+1} = \frac{T_n S_{n-2}}{S_{n-2}+T_{n-1}}, \quad T_{n+1} = \frac{S_n T_{n-2}}{T_{n-2}-S_{n-1}}$ which we described and proved each iteration in Theorem (2.2). In section (2.3), we created a fundamental theorem (2.3) that states the existence of periodic twelve solutions of system $S_{n+1} = \frac{T_n S_{n-2}}{S_{n-2}+T_{n-1}}, \quad T_{n+1} = \frac{S_n T_{n-2}}{-T_{n-2}+S_{n-1}}$. We have also explored existence of shape and periodic of the solutions of system $S_{n+1} = \frac{T_n S_{n-2}}{S_{n-2}+T_{n-1}}, \quad T_{n+1} = -\frac{S_n T_{n-2}}{T_{n-2}+S_{n-1}}$ in section (2.4). Finally, in section (3) we confirmed our theoretical results in the previous sections by carried out numerical simulation using MATLAB programm.

Conflict of interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

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