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Research article

Error estimations of a weak Galerkin finite element method for a linear system of $\ell \geq 2$ coupled singularly perturbed reaction-diffusion equations in the energy and balanced norms

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Abstract: This paper introduces a weak Galerkin finite element method for a system of $\ell \geq 2$ coupled singularly perturbed reaction-diffusion problems. The proposed method is independent of parameter and uses piecewise discontinuous polynomials on interior of each element and constant on the boundary of each element. By the Schur complement technique, the interior unknowns can be locally efficiently eliminated from the resulting linear system, and the degrees of freedom of the proposed method are comparable with the classical FEM. It has been reported that the energy norm is not adequate for singularly perturbed reaction-diffusion problems since it can not efficiently reflect the behaviour of the boundary layer parts when the diffusion coefficient is very small. For the first time, the error estimates in the balanced norm has been presented for a system of coupled singularly perturbed problems when each equation has different parameter. Optimal and uniform error estimates have been established in the energy and balanced norm on an uniform Shishkin mesh. Finally, we carry out various numerical experiments to verify the theoretical findings.

Keywords: weak Galerkin finite element method; discrete gradient operator; boundary layers; linear system of singularly reaction-diffusion problems; energy and balanced norm
Mathematics Subject Classification: 35J50, 65N15, 65N30

1. Introduction

We consider the following system of $\ell \geq 2$ coupled singularly perturbed reaction-diffusion boundary value problems. Find $\mathbf{u} \in (C^2(0,1) \cap C[0,1])^\ell$ such that

$$\begin{cases} \mathcal{L}\mathbf{u} = -\mathcal{E}\mathbf{u}'' + \mathbf{A}\mathbf{u} = \mathbf{g} \text{ in } \Omega = (0,1), \\ \mathbf{u}(0) = \mathbf{u_0}, \quad \mathbf{u}(\mathbf{1}) = \mathbf{u_1}, \end{cases} \tag{1.1}$$

where $\mathcal{E} = \operatorname{diag}(\varepsilon_1^2, \ldots, \varepsilon_\ell^2)$ with small perturbation parameters $0 < \varepsilon_i << 1$, $i = 1, 2 \ldots, \ell$, the vector-valued function $\mathbf{g} = (g_1, g_2, \ldots, g_\ell)^T$ and the reaction coefficients matrix $\mathbf{A} = (a_{kl})_{k,l=1}^\ell$ are twice continuously differentiable on $[0,1]$ and the constants $\mathbf{u_0}$ and $\mathbf{u_1}$ are given. The exact solution of (1.1) is the vector $\mathbf{u} = (u_1, u_2, \ldots, u_\ell)^T$. We assume that $\mathbf{A} : [0,1] \to \mathbb{R}^{(\ell,\ell)}$ and the vector valued function $\mathbf{g} : [0,1] \to \mathbb{R}^\ell$ are independent of the perturbation parameters $\mathcal{E}$ and the reaction matrix $\mathbf{A}$ is strongly diagonally dominant with

$$\sum_{\substack{j=1 \\ j \neq i}}^{\ell} \left\| \frac{a_{ij}}{a_{ii}} \right\|_\infty < 1, \quad \text{for} \quad i = 1, 2, \ldots, \ell. \tag{1.2}$$

Then the condition (1.2) implies that $\mathbf{A}$ is an M-matrix and its inverse is positive definite and bounded in the maximum norm (see e.g., [1]). Under these assumptions, the problem (1.1) has a unique solution $\mathbf{u} = (u_1, \ldots, u_\ell)^T \in (C^2(0,1) \cap C[0,1])^\ell$. In this paper, without loss of generality we will assume the most general case

$$0 < \varepsilon_1 \leq \varepsilon_2 \leq \cdots \leq \varepsilon_\ell \ll 1, \tag{1.3}$$

which always can be done, if necessary, by renumbering of the equations in the system.

It has been well-known that standard numerical methods including finite difference (FD) methods and finite element methods (FEM) are inefficient and inaccurate when applied to singularly perturbed problems (SPPs) on uniform meshes. The solutions of SPPs have boundary or/and interior layers which are very thin regions in which the solution or its derivative change abruptly. The width of the layers depends on the perturbation parameter and these layers are not resolved unless a large number of mesh points are used which is computationally expensive. As a remedy, fitted mesh methods based on layer-adapted meshes have been proposed and studied during recent years for solving boundary layer problems. The construction of these meshes require a priori knowledge on the bounds for the solution and its derivative. These meshes are finer in the part of boundary layers and coarser in the outside of region of the boundary layers. The well-known layer-adapted meshes are piecewise uniform Shishkin meshes [2] and Bakhvalov-type meshes [3]. We refer the readers to the books [1, 2, 4] and references therein for more details.

Unlike the non-coupled SPPs, the boundary layer behaviour of the solution to each equation in the system can be dramatically different and complicated. Each solution in the coupled system may have a sublayer corresponding to each of the perturbation parameter in the domain if the perturbation parameter in each equation has a different magnitude. This renders the construction of numerical methods very subtle. Shiskin [5] considered coupled system of two reaction diffusion equations on an infinite strip and he proved that the finite difference method is a robust method and has the rate of convergence $O(N^{-1/4})$ on piecewise uniform meshes when the perturbation parameters are small and different each other. Later, it is shown that the method has higher order convergence in [6–9] using piecewise uniform Shishkin mesh. The finite element method has been developed and analyzed for SPPs in the papers [7, 10, 11]. Recently, the numerical solution of system of reaction diffusion problems have been presented in [12–14] and reference therein.

Although there has been increasing interest in the numerical solution of coupled system of two singularly perturbed differential equations, few articles discuss the numerical solution of coupled system of more than two singularly perturbed differential equations. Kellogg et al. [15] considered a system of $\ell \geq 2$ reaction-diffusion equations in two dimension with the same perturbation parameter for each equation in the system. A system of $\ell \geq 2$ reaction-diffusion equations each of which has a different perturbation parameter has been studied and analyzed in one dimension in [9]. In general, the reaction coefficient matrix $\mathbf{A}$ is assumed to be diagonally dominant with positive diagonal and nonpositive off-diagonal elements in most of the papers. However, this condition is weakened in [9]. Usually, error analyses in FEMs or DG methods have been analyzed in the energy norms derived from corresponding variational formulations. Unfortunately, these norms are too weak to capture the boundary layers of SPPs of reaction-diffusion type, see, e.g., [14, 16–18]. Up to now, the only work on the balanced error estimates of finite element method for a system of $\ell \geq 2$ coupled singularly perturbed reaction-diffusion two-point boundary value problems is the paper of Lin and Stynes [14]. They have proved that the classical FEM using quadratic C^1 splines is order of $O(N^{-1} \ln N)$ in the balanced norm provided that each perturbation parameter is equal to the same small number. A new FEM is presented for SPPs of reaction-diffusion type in the weighted and balanced norm in [19]. The convergence analysis of the classical FEM in a balanced norm on Bakhvalov-type rectangular meshes has been studied in [20]. The analysis is much more involved and complicated when each parameter in the system is different. The error analysis in the balanced norm, to the best of the authors' knowledge, is not studied in the literature when the perturbation parameters are different. In this paper, to fill this gap, we derive the error estimates of a weak Galerkin method for a system of $\ell \geq 2$ coupled SPPs of reaction-diffusion type in the energy and balanced norms when each equation in the system has a different parameter. The classical FEM on the balanced norm using C^0-elements is open [21].

Wang and Ye [22] first introduced the weak Galerkin finite element method (WG-FEM) and presented for the second order elliptic equation. The key in the WG finite element scheme is to use the weak functions and weak derivatives on the completely discontinuous piecewise polynomials spaces. Since then, many papers have been devoted to WG finite element methods including the implementation results in [23], parabolic problems in [24], the Maxwell equations [25], the Stokes equations [26], the Helmholtz equations with high wave numbers in [27] and the multi-term time fractional diffusion equations in [28]. In [29], a discrete gradient and divergence operators have been introduced for convection-dominated problems. A uniformly convergent weak Galerkin finite element method on Shishkin mesh for convection-diffusion problem in one dimension has been presented in [30]. Uniform convergence of the WG-FEM on Shishkin mesh for SPPs of convection-dominated type has been studied in 2D in [31] and in 1D in [32] and singularly perturbed reaction-convection-diffusion problems with two parameters has been analyzed in [33]. Uniform convergence of a weak Galerkin method on Bakhvalov-type mesh for singularly perturbed convection-diffusion problem has been analyzed in [34, 35] and nonlinear singularly perturbed reaction-diffusion problems in [36]. Supercloseness in an energy norm of a WG-FEM on a

Bakhvalov-type mesh for a singularly perturbed two-point boundary value problem has been demonstrated in [37] and superconvergence results in [38]. The WG-FEM for two coupled system of SPPs of reaction-diffusion type has been presented in the energy norm in [39]. We wish to study a robust WG-FEM for the coupled systems of SPPs of reaction-diffusion type. Thus, the main aim of this paper is to construct a uniformly convergent WG-FEM for the problem (1.1).

The paper is organized as follows. In Section 2, we present and study a decomposition of the exact solution and a uniform Shishkin mesh. We introduce the WG-FEM in Section 3. Stability properties of the proposed method have been demonstrated in Section 4. Error analysis in the energy and balanced norm is presented in Sections 5 and 6, respectively. In Section 7, the numerical results are conducted to confirm the theory in the previous sections. Finally. conclusion is given in Section 8.

In this work, by C we mean a generic constant independent of N and the perturbation parameters $\varepsilon_i,\ i = 1, \dots, \ell$ which may not be the same at each occurrence. Constants with subscript such as C_c are fixed numbers and also do not depend on $\varepsilon_i,\ i = 1, \dots, \ell$, and the mesh parameter N.

2. Preliminaries

In this section, we first give a decomposition of the analytical solution of the linear system (1.1). Then we will derive the bounds for the solution and its derivatives. Next, a piecewise-uniform Shishkin mesh is constructed. Sobolev spaces with the related norms and some basic notations are introduced at the end of this section.

2.1. Properties of the solution

The solution of the system (1.1) can be decomposed as $\mathbf{u} = \mathbf{R} + \mathbf{L}$ where $\mathbf{R}$ is the regular solution part and $\mathbf{L}$ is the layer parts. In light of (1.2), there is a constant $\rho \in (0, 1)$ such that

$$\sum_{\substack{j=1\\ j\neq i}}^{\ell} \left\| \frac{a_{ij}}{a_{ii}} \right\|_\infty < \rho, \quad \text{for} \quad i = 1, 2, \dots, \ell. \tag{2.1}$$

Define $\alpha = \alpha(\rho)$ by

$$\alpha^2 := (1 - \rho) \min_{i=1,\dots,\ell} \min_{0\le x\le 1} a_{ii}(x).$$

For the future reference, we set

$$\mathcal{B}_\mu^\alpha(x) := e^{-\alpha x/\mu} + e^{-\alpha(1-x)/\mu},$$

and define $\ell \times \ell$ matrix $\Gamma = (c_{ij})$ by

$$c_{ii} = 1, \text{ and } c_{ij} = -\|\frac{a_{ij}}{a_{ii}}\|_\infty \text{ for } i \neq j. \tag{2.2}$$

We assume that the matrix Γ is inverse monotone, that is, Γ^{-1} exists and

$$\Gamma^{-1} \geq 0. \tag{2.3}$$

We first provide the stability of the solution of (1.1) from [9].

Lemma 2.1. *Assume that* $\mathbf{u}$ *is the solution of (1.1) and the reaction coefficient matrix* $\mathbf{A}$ *has strictly positive diagonal elements* $a_{ii} > 0$ *for* $i = 1, 2, \ldots, \ell$. *Let the matrix* Γ *be inverse monotone. Then the solution to each equation in the system has the following bounds*

$$|u_i(x)| \leq \sum_{j=1}^{\ell} (\Gamma^{-1})_{ij} \max\left\{\left\|\frac{g_j}{a_{jj}}\right\|, |u_{0,i}|, |u_{1,i}|\right\}, \quad i = 1, \ldots, \ell.$$

Proof. We refer the reader to [9] for the detailed proof. □

The following theorem states that the coercivity of $\mathbf{A}$ and inverse monotone property (2.3) of the matrix Γ are related.

Theorem 2.2. *[40] Assume that the reaction coefficient matrix* $\mathbf{A}$ *has strictly positive diagonal elements* $a_{ii} > 0$ *for* $i = 1, 2, \ldots, \ell$ *and the matrix* Γ *is inverse monotone. Then, there is a constant diagonal matrix* $\mathbf{D}$ *with positive elements and a positive constant* β *such that*

$$\boldsymbol{v}^T \boldsymbol{D} \boldsymbol{A} \boldsymbol{v} \geq \beta \boldsymbol{v}^T \boldsymbol{v}, \quad \forall \boldsymbol{v} \in \mathbb{R}^{\ell}, \quad x \in [0, 1].$$

Remark 2.1.

(1) If the matrix $\mathbf{A}$ *has the property (1.2), then the matrix* Γ *is a strongly diagonally dominant* L_0 *matrix which implies that the matix* Γ *is inverse monotone.*

(2) If $\mathbf{A}$ *and* $\mathbf{g}$ *are twice continuously differentiable, then the above stability result guarantees the existence of a unique solution* $\boldsymbol{u} \in C^4[0, 1]^{\ell}$.

(3) The reaction matrix is assumed to be strongly diagonally dominant with positive diagonal elements and nonpositive off-diagonal elements in most of existence papers on coupled system of SPPs with the exception [9, 41]. This assumption implies that the operator $\mathcal{L}$ *is inverse monotone and satisfies the maximum principle which is a useful tool in finite difference method. In this paper, the assumptions on* $\mathbf{A}$ *are weakened and we consider problems in a more general setting.*

(4) Since the form of system (1.1) and the matrix Γ *do not change when a constant positive diagonal matrix is applied on the left, Theorem 2.2 implies that we can assume, without loss of generality, the reaction matrix* $\mathbf{A}$ *is coercive if it has positive diagonal elements. That means that there exists* $\eta > 0$ *such that*

$$\boldsymbol{v}^T \boldsymbol{A} \boldsymbol{v} \geq \eta \boldsymbol{v}^T \boldsymbol{v}, \quad \forall \boldsymbol{v} \in \mathbb{R}^{\ell}. \tag{2.4}$$

We have to consider the solution decomposition consisting of smooth and layer components because of the boundary layers. Thus, we will use the following decomposition of $\mathbf{u}$ in the forthcoming analysis.

$$\mathbf{u} = \mathbf{R} + \mathbf{L}_L + \mathbf{L}_R,$$

where $\mathbf{R}$ is the smooth part, $\mathbf{L}_L$ and $\mathbf{L}_R$ are the boundary layer parts, and satisfy the following boundary value problems, respectively

$$\mathcal{L}\mathbf{R} = \mathbf{g} \quad \text{on} \quad \Omega \quad \text{and} \quad \mathbf{R}(0) = \mathbf{A}^{-1}(0)\mathbf{g}(0), \quad \mathbf{R}(1) = \mathbf{A}^{-1}(1)\mathbf{g}(1), \tag{2.5}$$

$$\mathcal{L}\mathbf{L}_L = \mathbf{0} \quad \text{on} \quad \Omega \quad \text{and} \quad \mathbf{L}_L(0) = \mathbf{u}_0 - \mathbf{R}(0), \quad \mathbf{L}_L(1) = 0., \tag{2.6}$$

$$\mathcal{L}\mathbf{L}_R = \mathbf{0} \quad \text{on} \quad \Omega \quad \text{and} \quad \mathbf{L}_R(0) = 0, \quad \mathbf{L}_R(1) = \mathbf{u}_1 - \mathbf{R}(1). \tag{2.7}$$

Here, the existence of the inverse matrix $\mathbf{A}^{-1}$ is guaranteed by the condition (1.2).

Theorem 2.3. *Assume that* $\boldsymbol{A}$ *and* $\mathbf{g}$ *are twice continuously differentiable. Then the solution* $\boldsymbol{u}$ *of the system (1.1) can be decomposed as* $\boldsymbol{u} = \boldsymbol{R} + \boldsymbol{L}_L + \boldsymbol{L}_R$*, where* $\boldsymbol{R}$ *and* $\mathbf{L} = \mathbf{L}_L + \mathbf{L}_R$ *satisfy*

$$|R_i^{(k)}(x)| \leq C, \qquad \textit{for} \quad k = 0, 1, \ldots, 4, \quad i = 1, \ldots, \ell \tag{2.8}$$

$$|L_i^{(k)}(x)| \leq C \sum_{m=i}^{\ell} \varepsilon_m^{-k} \mathcal{B}_{\varepsilon_m}^{\alpha}(x), \qquad \textit{for} \quad k = 0, 1, 2, \quad i = 1, \ldots, \ell \tag{2.9}$$

$$|L_i^{(k)}(x)| \leq C \varepsilon_i^{2-k} \sum_{m=1}^{\ell} \varepsilon_m^{-2} \mathcal{B}_{\varepsilon_m}^{\alpha}(x), \qquad \textit{for} \quad k = 3, 4, \quad i = 1, \ldots, \ell \tag{2.10}$$

Proof. A detailed proof can be found in [42]. □

2.2. *Uniform Shishkin mesh*

Let N be an integer divisible by $2(\ell + 1)$. We define the transition points

$$\lambda_{\ell+1} = \frac{1}{2}, \quad \lambda_s = \min\left\{\frac{s\lambda_{s+1}}{s+1}, \frac{\sigma \varepsilon_s}{\alpha} \ln N\right\}, \quad s = \ell, \ldots, 1, \quad \text{and} \quad \lambda_0 = 0,$$

where σ is a user-chosen constant with $\sigma = O(1)$. In general, this parameter is chosen as $\sigma \geq k + 1$ where k is the order of polynomials in the approximation space. Then we divide each of the intervals $\Omega_s := [\lambda_s, \lambda_{s+1}]$ and ${}_s\Omega := [1 - \lambda_{s+1}, 1 - \lambda_s]$, $s = 0, \ldots, \ell$ into $\dfrac{N}{2(\ell+1)}$ subintervals of equal mesh size

$$H_s = H_{2\ell+1-s} = \frac{2(\ell + 1)(\lambda_{s+1} - \lambda_s)}{N}, \quad s = 0, \ldots, \ell.$$

An example of a piecewise-uniform Shishkin mesh with $N = 32$ elements for a system of $\ell = 3$ reaction-diffusion equations is shown in Figure 1.

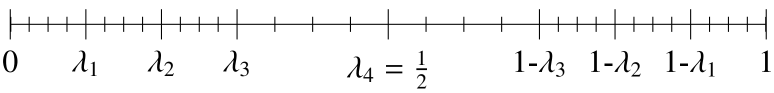


Figure 1. Piecewise-uniform Shishkin mesh for $\ell = 3$ reaction-diffusion equations.

We next define the nodes recursively as

$$x_0 = 0, \quad x_n = x_{n-1} + h_n \quad \text{for } n = 1, \ldots, N, \text{ where}$$
$$h_n = \begin{cases} H_0, & n = 1, \ldots, \dfrac{N}{2(\ell+1)}, \\ H_1, & n = \dfrac{N}{2(\ell+1)} + 1, \ldots, \dfrac{N}{(\ell+1)}, \\ \vdots & \vdots \\ H_{2\ell+1}, & n = \dfrac{N(2\ell+1)}{2(\ell+1)}, \ldots, N. \end{cases}$$

We denote the mesh and a partition of the domain Ω by $I_n = [x_{n-1}, x_n], \quad n = 1, \ldots, N$ and $\mathcal{T}_N = \{I_n : n = 1, \ldots, N\}$, respectively. For $I_n \in \mathcal{T}_N$, the outward unit normal $\mathbf{n}_{I_n}$ on I_n is defined as $\mathbf{n}_{I_n}(x_n) = 1$ and $\mathbf{n}_{I_n}(x_{n-1}) = -1$; for simplicity, we use $\mathbf{n}$ instead of $\mathbf{n}_{I_n}$.

We use the following basic notations in the sequel. By $L^2(\Omega)$, we denote the space of square integrable functions on Ω with the norm $\|u\|^2_{L^2(\Omega)} = \int_\Omega u^2(x)\,dx$ and sometimes, we will use the abbreviation $\|\cdot\| = \|u\|^2_{L^2(\Omega)}$. The standard Sobolev space is denoted by $H^k(\Omega)$ with the norm $\|\cdot\|_{k,\Omega}$ and semi-norm $|\cdot|_{k,\Omega}$ given as

$$\|u\|^2_{k,\Omega} = \sum_{j=0}^{k} \|u^{(j)}\|^2_{L^2(\Omega)}, \quad |u|^2_{k,\Omega} = \|u^{(k)}\|^2_{L^2(\Omega)}.$$

We define the norm for a vector-valued function $\mathbf{u}$ as

$$\|\mathbf{u}\|^2_{k,\Omega} = \sum_{i=1}^{\ell} \|u_i\|^2_{k,\Omega}.$$

For each interval I_n, the broken Sobolev space is defined by

$$H^k_N(\Omega) = \{u \in L^2(\Omega) : u|_{I_n} \in H^k(I_n), \quad \forall I_n \in \mathcal{T}_N\},$$

and the corresponding norm and semi-norm

$$\|\mathbf{u}\|^2_{H^k_N(\Omega)} = \sum_{n=1}^{N}\sum_{i=1}^{\ell} \|u_i\|^2_{k,I_n}, \quad |\mathbf{u}|^2_{H^k_N(\Omega)} = \sum_{n=1}^{N}\sum_{i=1}^{\ell} |u_i|^2_{k,I_n}.$$

For the future reference we use the following notations

$$\begin{aligned}
(u, v) &= \sum_{I_n \in \mathcal{T}_N} (u, v)_{I_n} = \sum_{I_n \in \mathcal{T}_N} \int_{I_n} u(x)v(x)\,dx, \\
\langle u, v\rangle &= \sum_{I_n \in \mathcal{T}_N} \langle u, v\rangle_{\partial I_n} = \sum_{I_n \in \mathcal{T}_N} \Big(u(x_n)v(x_n) + u(x_{n-1})v(x_{n-1})\Big), \\
\|u\|^2 &= \sum_{n=1}^{N} \|u\|^2_{I_n} = \sum_{n=1}^{N} (u, u)_{I_n}.
\end{aligned}$$

3. The WG-FEM method

This section is devoted to introduce novel concepts such as weak functions and weak derivatives from which we define our method for the problem (1.1). For the rest of the paper, we denote by $\mathbb{P}_k(I_n)$ the set of polynomials defined on I_n with degree at most k. The space of weak functions $\mathcal{W}(I_n)$ on I_n is defined by

$$\mathcal{W}(I_n) = \{u = \{u_0, u_b\} : u_0 \in L^2(I_n), v_b \in L^\infty(\partial I_n)\}.$$

Here, a weak function $u = \{u_0, u_b\}$ has two components and the first component u_0 represents the value of u in (x_{n-1}, x_n) and u_b is interpreted as the value of u on $\partial I_n = \{x_{n-1}, x_n\}$. From now on, we assume that $k = 2$ unless otherwise mentioned.

Let $S_N(I_n)$ be a local weak Galerkin (WG) finite element space given by

$$S_N(I_n) = \{u = \{u_0, u_b\} : u_0|_{I_n} \in \mathbb{P}_k(I_n), u_b|_{\partial I_n} \in \mathbb{P}_0(\partial I_n) \quad \forall I_n \in \mathcal{T}_N\}, \tag{3.1}$$

where $\mathbb{P}_0(\partial I_n)$ stands for constant polynomials on ∂I_n. We remark that the results can be extended to $\mathbb{P}_k$ elements when $k > 2$. However, in this case some additional compatibility conditions of the data will be required in order to have (2.8).

Next, we define a global WG finite element space S_N that comprises of weak functions $u = \{u_0, u_b\}$ such that $u_0|_{I_n} \in \mathbb{P}_k(I_n)$ and $u_b|_{x_n}$ is the constant for $n = 0, \ldots, N$.

Let S_N^0 be the subspace of S_N with zero boundary conditions, that is,

$$S_N^0 = \{u = \{u_0, u_b\} : u \in S_N, u_b(0) = u_b(1) = 0\}. \tag{3.2}$$

The weak derivative $d_{w,I_n}u \in \mathbb{P}_{k-1}(I_n)$ of a function $u \in S_N(I_n)$ is defined to be the solution of the following equation

$$(d_{w,I_n}u, v)_{I_n} = -(u_0, v')_{I_n} + \langle u_b, \mathbf{n}v\rangle_{\partial I_n}, \qquad \forall v \in \mathbb{P}_{k-1}(I_n), \tag{3.3}$$

where

$$(w, z)_{I_n} = \int_{I_n} w(x)z(x)\, dx,$$

and

$$\langle w, z\mathbf{n}\rangle_{\partial I_n} = w(x_n)z(x_n) - w(x_{n-1})z(x_{n-1}).$$

The discrete weak derivative $d_w u$ of the weak function $u = \{u_0, u_b\}$ on the finite element space S_N is defined by

$$(d_w u)|_{I_n} = d_{w,I_n}\left(u|_{I_n}\right), \quad \forall u \in S_N.$$

Our WG-FEM scheme for the system of singularly perturbed reaction-diffusion problems (1.1) is given as follows.

Algorithm 1 The weak Galerkin scheme for the linear system of singularly perturbed diffusion-reaction problem.

The WG-FEM for the problem (1.1) is to find $\mathbf{u}_N = (u_1^N, \ldots, u_\ell^N) \in [S_N^0]^\ell$ which solves the following:

$$a(\mathbf{u}_N, \mathbf{v}_N) = L(\mathbf{v}_N), \quad \forall \mathbf{v}_N = (v_1^N, \ldots, v_\ell^N) \in [S_N^0]^\ell. \tag{3.4}$$

Here, the bilinear and the linear forms are defined by, for any $u_i^N = \{u_{i0}, u_{ib}\}$,

$$a(\mathbf{u}_N, \mathbf{v}_N) = \sum_{i=1}^{\ell} \varepsilon_i^2 (d_w u_i^N, d_w v_i^N) + \sum_{i=1}^{\ell}\sum_{j=1}^{\ell} (a_{ij}u_{j0}, v_{i0}) + \sum_{i=1}^{\ell} s(u_i^N, v_i^N), \tag{3.5}$$

$$s(u_i^N, v_i^N) = \sum_{n=1}^{N} \langle \varrho_n(u_{i0} - u_{ib}), v_{i0} - v_{ib}\rangle_{\partial I_n},$$

$$L(\mathbf{v}_N) = \sum_{i=1}^{\ell} (g_i, v_{i0}),$$

where $\varrho_n \geq 0, n = 1, \ldots N$ is the penalization parameter associated with the node x_n defined as follows:

$$\varrho_n = \begin{cases} 1, & \text{for} \quad I_n \subset \Omega_\lambda = [\lambda_\ell, 1 - \lambda_\ell], \\ \dfrac{N}{\ln N}, & \text{for} \quad I_n \subset \Omega \setminus \Omega_\lambda. \end{cases} \tag{3.6}$$

Choosing a penalty parameter in stabilized numerical methods is an important issue in uniform convergence estimates. Usually, the penalization parameter will depend on the perturbation parameters. For example, $\varrho_n = \varepsilon_\ell^2 h_n^{-1}$ is taken in the WG finite element schemes [22, 26, 29]. However, a uniform convergence rate can not be attained for a penalization constant depending on the perturbation parameters.

4. Stability of the method

In the following analysis, we will recall the following multiplicative trace inequality and the inverse inequality.

$$\|v\|^2_{L^2(\partial I_n)} \leq C(h_n^{-1}\|v\|^2_{L^2(I_n)} + \|v\|_{L^2(I_n)}\|v'\|_{L^2(I_n)}), \quad \forall v \in H^1(I_n), \tag{4.1}$$

$$\|v_N\|_{L^p(\partial I_n)} \leq C h_n^{-1/p}\|v_N\|_{L^p(I_n)}, \quad \forall 1 \leq p \leq \infty, \quad \forall v_N \in \mathbb{P}_k(I_n). \tag{4.2}$$

We introduce the $\mathcal{E}$-weighted energy norm $|||\cdot|||$ in $[S_N^0]^\ell$ as follows: for $\mathbf{v} = (v_1^N, \ldots, v_\ell^N)^T = (\{v_{10}, v_{1b}\}, \ldots, \{v_{\ell 0}, v_{\ell b}\})^T \in [S_N^0]^\ell$,

$$|||\mathbf{v}|||^2 = \sum_{i=1}^{\ell} \varepsilon_i^2 \|d_w v_i^N\|^2 + \eta \sum_{i=1}^{\ell} \|v_{i0}\|^2 + \sum_{i=1}^{\ell} s(v_i^N, v_i^N), \tag{4.3}$$

where η is the coercivity constant of $\mathbf{A}$.

We also introduce the discrete H^1 energy-like norm $|||\cdot|||_\varepsilon$ in $S_N^\ell + H^1(\Omega)^\ell$ defined as

$$|||\mathbf{v}|||_\varepsilon^2 = \sum_{i=1}^{\ell} \varepsilon_i^2 \|v_{i0}'\|^2 + \eta \sum_{i=1}^{\ell} \|v_{i0}\|^2 + \sum_{i=1}^{\ell} s(v_i^N, v_i^N), \tag{4.4}$$

where v_{i0}' is the ordinary derivative of a functions $v_{i0}(x)$.

We show that the norms $|||\cdot|||$ and $|||\cdot|||_\varepsilon$ defined by (4.3) and (4.4), respectively are equivalent in the weak Galerkin finite element space $[S_N^0]^\ell$ in the next lemma.

Lemma 4.1. *Let $\boldsymbol{v}_N \in [S_N^0]^\ell$. Then there are two positive constant C_l and C_s such that*

$$C_l|||\boldsymbol{v}_N||| \leq |||\boldsymbol{v}_N|||_\varepsilon \leq C_s|||\boldsymbol{v}_N|||. \tag{4.5}$$

Proof. For any $v_i^N = \{v_{i0}, v_{ib}\} \in S_N^0$, by the definition of weak derivative (3.3) and integration by parts we arrive at

$$(d_w v_i^N, w)_{I_n} = (v_{i0}', w)_{I_n} + \langle v_{ib} - v_{i0}, w\mathbf{n}\rangle_{\partial I_n}, \quad \forall w \in \mathbb{P}_{k-1}(I_n). \tag{4.6}$$

Choosing $w = d_w v_i^N$ in the above Eq (4.6) yields

$$\|d_w v_i^N\|^2_{I_n} = (v_{i0}', d_w v_i^N)_{I_n} + \langle v_{ib} - v_{i0}, d_w v_i^N \mathbf{n}\rangle_{\partial I_n}.$$

Summing up the above equation over all $I_n \in \mathcal{T}_N$, and using the inverse inequality (4.2), we obtain

$$\|d_w v_i^N\|^2 \le C(\|v_{i0}'\|^2 + \sum_{n=1}^{N} h_n^{-1}\|v_{ib} - v_{i0}\|_{\partial I_n}^2)^{1/2}\|d_w v_i^N\|.$$

Therefore, we have

$$\|d_w v_i^N\|^2 \le C(\|v_{i0}'\|^2 + \sum_{n=1}^{N} h_n^{-1}\|v_{ib} - v_{i0}\|_{\partial I_n}^2). \tag{4.7}$$

From the penalty parameter (3.6), we have

$$\frac{\varepsilon_i^2 h_n^{-1}}{\varrho_n} \le \frac{\varepsilon_i h_n^{-1}}{\varrho_n} \le C, \quad \text{for } n = 1, \ldots, N. \tag{4.8}$$

To see this, the minimal possible $h_n = H_0 = \dfrac{2(\ell+1)\lambda_1}{N}$ implies that $\dfrac{\varepsilon_1 h_n^{-1}}{\varrho_n} = \dfrac{\alpha}{2(\ell+1)\sigma} =: C$. Hence using (4.8), we obtain

$$\sum_{n=1}^{N} \varepsilon_i^2 h_n^{-1}\|v_{ib} - v_{i0}\|_{\partial I_n}^2 = \sum_{n=1}^{N} \frac{\varepsilon_i^2 h_n^{-1}}{\varrho_n}\varrho_n\|v_{ib} - v_{i0}\|_{\partial I_n}^2 \le Cs(v_i^N, v_i^N),$$

which together with (4.7) implies that

$$\varepsilon_i^2\|d_w v_i^N\|^2 \le 2\Big(\varepsilon_i^2\|v_{i0}'\|^2 + s(v_i^N, v_i^N)\Big). \tag{4.9}$$

On the other hand, taking $w = v_{i0}'$ in the Eq (4.6) yields

$$\|v_{i0}'\|_{I_n}^2 = (v_{i0}', d_w v_i^N)_{I_n} - \langle v_{ib} - v_{i0}, v_{i0}'\mathbf{n}\rangle_{\partial I_n}.$$

Summing up the above equation over all $I_n \in \mathcal{T}_N$, using the inverse inequality (4.2), we have

$$\|v_{i0}'\|^2 \le C(\|d_w v_i^N\|^2 + \sum_{n=1}^{N} h_n^{-1}\|v_{ib} - v_{i0}\|_{\partial I_n}^2)^{1/2}\|v_{i0}'\|.$$

Therefore, we have

$$\|v_{i0}'\|^2 \le C(\|d_w v_i^N\|^2 + \sum_{n=1}^{N} h_n^{-1}\|v_{ib} - v_{i0}\|_{\partial I_n}^2). \tag{4.10}$$

With the help of (4.8), we result in

$$\varepsilon_i^2\|v_{i0}'\|^2 \le C\Big(\varepsilon_i^2\|d_w v_i^N\|^2 + s(v_i^N, v_i^N)\Big). \tag{4.11}$$

We obtain the desired result (4.5) in view of the inequalities (4.9) and (4.11) and the definition of the norms $|||\cdot||$ and $|||\cdot|||_\varepsilon$. Thus we complete the proof. □

We next show that the coercivity of the bilinear form $a(\cdot,\cdot)$ on $[S_N^0]^\ell$ in the energy norm $|||\cdot|||$ defined by (4.3).

Lemma 4.2. *Let $\boldsymbol{v}_N \in [S_N^0]^\ell$. Then there holds*

$$a(\boldsymbol{v}_N, \boldsymbol{v}_N) \geq |||\boldsymbol{v}_N|||^2. \tag{4.12}$$

Proof. Using the coercivity (2.4) of the reaction matrix $\mathbf{A}$, we have

$$\begin{aligned} a(\mathbf{v}_N, \mathbf{v}_N) &= \sum_{i=1}^{\ell} \varepsilon_i^2 \|d_w v_i^N\|^2 + \sum_{i=1}^{\ell}\sum_{j=1}^{\ell} (a_{ij} v_{j0}, v_{i0}) + \sum_{i=1}^{\ell} s(v_i^N, v_i^N) \\ &\geq \sum_{i=1}^{\ell} \varepsilon_i^2 \|d_w v_i^N\|^2 + \eta \sum_{i=1}^{\ell} \|v_{i0}\|^2 + \sum_{i=1}^{\ell} s(v_i^N, v_i^N) \\ &= |||\mathbf{v}_N|||^2. \end{aligned}$$

The proof is completed. ◻

In light of Lemma 4.2, we deduce that

$$|||\mathbf{u}_N||| \leq \|\mathbf{g}\|,$$

which in turn implies the problem (3.4) has a unique solution. The existence follows from the uniqueness.

As a result of Lemma 4.1 and Lemma 4.2, we conclude that the bilinear form $a(\cdot,\cdot)$ is also coercive in the energy like norm $|||\cdot|||_\varepsilon$ defined by (4.4).

Lemma 4.3. *Let $\boldsymbol{v}_N, \boldsymbol{w}_N \in [S_N^0]^\ell$. Then there exist positive constants C_c and C_e such that*

$$a(\boldsymbol{v}_N, \boldsymbol{w}_N) \leq C_c |||\boldsymbol{v}_N|||_\varepsilon |||\boldsymbol{w}_N|||_\varepsilon, \tag{4.13}$$

$$a(\boldsymbol{v}_N, \boldsymbol{v}_N) \geq C_e |||\boldsymbol{v}_N|||_\varepsilon^2. \tag{4.14}$$

5. Error analysis in the energy norm

In this section, we study the error analysis of the proposed numerical scheme applied to the problem (1.1) in the energy norm associated with the bilinear form. We will show that the WG-FEM solution converges uniformly in the energy norm with respect to the perturbation parameters. For the uniform convergence analysis on Shishkin mesh, we will use a special interpolation operator given in [11]. On each interval I_n, we introduce the set of $k+1$ nodal functional N_ℓ defined as follows: for any $v \in C(I_n)$

$$\begin{aligned} N_0(v) &= v(x_{n-1}), \quad N_k(v) = v(x_n), \\ N_m(v) &= \frac{1}{h_n^m} \int_{x_{n-1}}^{x_n} (x - x_{n-1})^{m-1} v(x)\, dx, \quad m = 1, \ldots, k-1. \end{aligned}$$

A local interpolation $\mathcal{I} : H^1(I_n) \to \mathbb{P}_k(I_n)$ is now defined by

$$N_m(\mathcal{I}v - v) = 0, \quad m = 0, 1, \ldots, k. \tag{5.1}$$

The local interpolation operator $\mathcal{I}$ can used for constructing a continuous global interpolation.

Since $\mathcal{I}v|_{I_n}$ is continuous on I_n and is in the $H^1(I_n)$ space, we denote $\mathcal{I}v|_{\partial I_n}$ by $\mathcal{I}v|_{I_n}$, for simplicity. Form this fact we observe that for any $v \in H^1(I_n)$ we have

$$d_w(\mathcal{I}v) = (\mathcal{I}v)'. \tag{5.2}$$

Lemma 5.1. *[11] For any $w \in H^{k+1}(I_n), I_n \in \mathcal{T}_N$, the interpolation $\mathcal{I}w$ defined by (5.1) has the following estimates:*

$$|w - \mathcal{I}w|_{l,I_n} \leq Ch_n^{k+1-l}|w|_{k+1,I_n}, \quad l = 0, 1, \ldots, k+1, \tag{5.3}$$

$$\|w - \mathcal{I}w\|_{L_\infty(I_n)} \leq Ch_n^{k+1}|w|_{k+1,\infty,I_n}, \tag{5.4}$$

where h_n is the length of element I_n and C is independent of h_n, and ε_i, $i = 1, \ldots, \ell$.

Lemma 5.2. *Let $\mathcal{I}\boldsymbol{R}$ and $\mathcal{I}\boldsymbol{L}$ be the interpolations of the regular part $\boldsymbol{R}$ and the layer part $\boldsymbol{L}$ of the solution $\boldsymbol{u} \in H^{k+1}(\Omega)$ on the piecewise-uniform Shishkin mesh, respectively. Assume also that $\varepsilon_\ell \ln N \leq \ell\alpha/(2(\ell+1)\sigma)$ and let $\Omega_\lambda = [\lambda_\ell, 1-\lambda_\ell]$. Then, we have $\mathcal{I}\boldsymbol{u} = \mathcal{I}\boldsymbol{R} + \mathcal{I}\boldsymbol{L}$ and the following interpolation estimates are satisfied for $i = 1, \ldots, \ell$*

$$\|(R_i - \mathcal{I}R_i)^{(l)}\|_{L^2(\Omega)} \leq CN^{l-(k+1)}, \quad l = 0, 1, 2, \tag{5.5}$$

$$\|L_i - \mathcal{I}L_i\|_{L^2(\Omega\setminus\Omega_\lambda)} \leq C\varepsilon^{1/2}(N^{-1}\ln N)^{k+1}, \tag{5.6}$$

$$N^{-1}\|(\mathcal{I}L_i)'\|_{L^2(\Omega_\lambda)} + \|\mathcal{I}L_i\|_{L^2(\Omega_\lambda)} \leq C(\varepsilon^{1/2} + N^{-1/2})N^{-\sigma}, \tag{5.7}$$

$$\|L_i\|_{L^\infty(\Omega_\lambda)} + \varepsilon^{-1/2}\|L_i\|_{L^2(\Omega_\lambda)} \leq CN^{-\sigma}, \tag{5.8}$$

$$\|(L_i)^{(l)}\|_{L^2(\Omega_\lambda)} \leq C\varepsilon^{1/2-l}N^{-\sigma}, \quad l = 1, 2, \tag{5.9}$$

$$\|L_i - \mathcal{I}L_i\|_{L^2(\Omega_\lambda)} \leq C(\varepsilon^{1/2} + N^{-1/2})N^{-\sigma}, \tag{5.10}$$

where $\varepsilon^{1/2} := \varepsilon_1^{1/2} + \ldots, +\varepsilon_\ell^{1/2}$. Furthermore, the following estimates hold true

$$\|(L_i - \mathcal{I}L_i)^{(l)}\|_{L^2(\Omega_\lambda)} \leq C\varepsilon^{1/2-l}N^{-\sigma}, \quad l = 1, 2, \tag{5.11}$$

$$\|(L_i - \mathcal{I}L_i)^{(l)}\|_{L^2(\Omega\setminus\Omega_\lambda)} \leq C\varepsilon^{1/2-l}(N^{-1}\ln N)^{k+1-\ell}, \quad l = 1, 2. \tag{5.12}$$

Proof. The linearity of the interpolation implies that $\mathcal{I}\mathbf{u} = \mathcal{I}(\mathbf{R} + \mathbf{L}) = \mathcal{I}\mathbf{R} + \mathcal{I}\mathbf{L}$. Applying the estimate (5.3), the bounds for the derivatives of regular components R_i of the solution in Lemma 2.1 and using (2.8), we obtain

$$\|(R_i - \mathcal{I}R_i)^{(l)}\| \leq CN^{l-3}|R_i|_{k+1,\Omega} \leq CN^{l-(k+1)}, \quad l = 0, 1, 2, \quad i = 1, \ldots, \ell.$$

This completes the proof of estimates (5.5).

Using the fact that $\mathcal{B}^\alpha_{\varepsilon_i}(x) \leq \mathcal{B}^\alpha_{\varepsilon_\ell}(x)$ for $i = 1, \ldots, \ell$ and $\lambda_\ell = \dfrac{\sigma\varepsilon_\ell}{\alpha}\ln N$, we have

$$\begin{aligned}
\|L_i\|_{L^\infty(\Omega_\lambda)} &\leq C \max_{[\lambda_\ell, 1-\lambda_\ell]} \sum_{m=i}^{\ell} \mathcal{B}^\alpha_{\varepsilon_m}(x) \\
&\leq C \max_{[\lambda_\ell, 1-\lambda_\ell]} \Big(\exp(-\alpha x/\varepsilon_\ell) + \exp(-\alpha(1-x)/\varepsilon_\ell)\Big) \\
&\leq CN^{-\sigma}.
\end{aligned}$$

The L^2- norm estimate of the layer part of the solution on the sub-interval Ω_λ follows from

$$\begin{aligned}\|L_i\|^2_{L^2(\Omega_\lambda)} &\leq C\int_{\lambda_\ell}^{1-\lambda_\ell}(\sum_{m=i}^{\ell}\mathcal{B}^{\alpha}_{\varepsilon_m}(x))^2\,dx\\ &\leq C\sum_{m=i}^{\ell}\int_{\lambda_\ell}^{1-\lambda_\ell}\Big(\exp(-2\alpha x/\varepsilon_m)+\exp(-2\alpha(1-x)/\varepsilon_m)\Big)\,dx\\ &\leq C\varepsilon N^{-2\sigma}.\end{aligned}$$

Hence, from the above inequalities we have

$$\|L_i\|_{L^\infty(\Omega_\lambda)}+\varepsilon^{-1/2}\|L_i\|_{L^2(\Omega_\lambda)}\leq CN^{-\sigma}.$$

Thus, we complete the proof of the estimate (5.8).

We also have

$$\begin{aligned}\|(L_i)^{(l)}\|^2_{L^2(\Omega_\lambda)} &\leq C\int_{\lambda_\ell}^{1-\lambda_\ell}\Big((\sum_{m=i}^{\ell}\mathcal{B}^{\alpha}_{\varepsilon_m}(x))^{(l)}\Big)^2\,dx\\ &\leq C\sum_{m=i}^{\ell}\varepsilon_m^{-2l}\int_{\lambda_\ell}^{1-\lambda_\ell}\Big(\exp(-2\alpha x/\varepsilon_m)+\exp(-2\alpha(1-x)/\varepsilon_m)\Big)\,dx\\ &\leq C\varepsilon^{1-2l}N^{-2\sigma}.\end{aligned}$$

This proves the estimate (5.9).

Due to (5.3) of Lemma 5.1 and the bounds for derivatives (2.9), we obtain at once

$$\begin{aligned}&\|L_i-\mathcal{I}L_i\|^2_{L^2(\Omega\setminus\Omega_\lambda)}=\sum_{I_n\subset\Omega\setminus\Omega_\lambda}\|L_i-\mathcal{I}L_i\|^2_{L^2(I_n)}\leq\sum_{I_n\subset\Omega\setminus\Omega_\lambda}h_n^{2(k+1)}\|L_i^{(k+1)}\|^2_{L^2(I_n)}\\ \leq& C\sum_{s=0}^{\ell-1}H_s^{2(k+1)}\varepsilon_i^{-2}\Big(\int_{\lambda_s}^{\lambda_{s+1}}\Big(\sum_{m=1}^{\ell}\varepsilon_m^{-2}\mathcal{B}^{\alpha}_{\varepsilon_m}(x)\Big)^2\,dx+\int_{1-\lambda_{\ell-s}}^{1-\lambda_{\ell-1-s}}\Big(\sum_{m=1}^{\ell}\varepsilon_m^{-2}\mathcal{B}^{\alpha}_{\varepsilon_m}(x)\Big)^2\,dx\Big)\\ \leq& C\sum_{m=1}^{\ell}\sum_{s=0}^{\ell-1}\Big[\frac{2(\ell+1)(\lambda_{s+1}-\lambda_s)}{N}\Big]^{2(k+1)}\varepsilon_m^{-2(k+1)}\varepsilon_m\leq C\varepsilon(N^{-1}\ln N)^{2(k+1)}.\end{aligned}$$

Thus, the estimate (5.6) is proved.

For the proof of (5.7) we follow [11]. An inverse estimate yields that

$$N^{-1}\|(\mathcal{I}L_i)'\|_{L^2(\Omega_\lambda)}\leq C\|\mathcal{I}L_i\|_{L^2(\Omega_\lambda)}.$$

We will derive a bound for $\|\mathcal{I}L_i\|_{L^2(\Omega_\lambda)}$. For the interval $I_n=(x_{n-1},x_n)$, we have the estimate for the local nodal functional $N_m(L_i)$ as

$$|N_m(L_i)|\leq C\sum_{p=i}^{\ell}\Big(\exp(-\alpha x_{n-1}/\varepsilon_p)+\exp(-\alpha(1-x_n)/\varepsilon_p)\Big).$$

The local representation

$$\mathcal{I}L_i|_{I_n} = \sum_{m=0}^{k} N_m(L_i)\phi_m$$

implies that

$$\begin{aligned}\|\mathcal{I}L_i\|^2_{L^2(I_n)} &\le \sum_{m=0}^{k} |N_m(L_i)|^2 \|\phi_m\|^2_{L^2(I_n)} \\ &\le CN^{-1}\sum_{p=i}^{\ell}\Big(\exp(-2\alpha x_{n-1}/\varepsilon_p)+\exp(-2\alpha(1-x_n)/\varepsilon_p)\Big),\end{aligned} \tag{5.13}$$

where we use the fact $\|\phi_m\|_{L^2(I_n)} \le CN^{-1}$. Summing up over all $I_n \subset \Omega_\lambda$ yields that

$$\sum_{n=\frac{\ell N}{2(\ell+1)}+1}^{\frac{(\ell+2)N}{2(\ell+1)}} \|\mathcal{I}L_i\|^2_{L^2(I_n)} \le CN^{-1}\sum_{n=\frac{\ell N}{2(\ell+1)}+1}^{\frac{(\ell+2)N}{2(\ell+1)}}\sum_{p=i}^{\ell}\Big(\exp(-2\alpha x_{n-1}/\varepsilon_p)+\exp(-2\alpha(1-x_n)/\varepsilon_p)\Big).$$

Since the mesh size on Ω_λ is $H_\ell = H_{\ell+1}$, the term in the parenthesis on the right hand side of the above inequality can be written as

$$\begin{aligned}&\exp(-2\alpha x_{n-1}/\varepsilon_p)+\exp(-2\alpha(1-x_n)/\varepsilon_p)\\ =&\exp((-2\alpha x_{n-1}+2\alpha x_n-2\alpha x_n)/\varepsilon_p)+\exp((-2\alpha(1-x_n)+2\alpha x_{n-1}-2\alpha x_{n-1})/\varepsilon_p)\\ \le&\exp(2H_\ell\alpha/\varepsilon_p)\Big(\exp(-2\alpha x/\varepsilon_p)+\exp(-2\alpha(1-x)/\varepsilon_p)\Big) \quad \text{for} \quad x_{n-1}<x<x_n.\end{aligned}$$

Integrating the above inequality on $I_n \subset \Omega_\lambda$ and using the fact that $H_\ell = O(N^{-1})$, we have

$$\begin{aligned}&N^{-1}\Big(\exp(-2\alpha x_{n-1}/\varepsilon_p)+\exp(-2\alpha(1-x_n)/\varepsilon_p)\Big)\\ \le&\exp(2H_\ell\alpha/\varepsilon_p)\int_{x_{n-1}}^{x_n}\Big(\exp(-2\alpha x/\varepsilon_p)+\exp(-2\alpha(1-x)/\varepsilon_p)\Big)dx.\end{aligned}$$

Summing up the above inequality for $n = \frac{\ell N}{2(\ell+1)}+1,\dots,\frac{(\ell+2)N}{2(\ell+1)}-1$ leads to

$$N^{-1}\sum_{n=\frac{\ell N}{2(\ell+1)}+1}^{\frac{(\ell+2)N}{2(\ell+1)}-1}\sum_{p=i}^{\ell}\Big(\exp(-2\alpha x_{n-1}/\varepsilon_p)+\exp(-2\alpha(1-x_n)/\varepsilon_p)\Big)\le C\varepsilon N^{-2\sigma}.$$

It remains to bound on the last interval $(x_{\frac{(\ell+2)N}{2(\ell+1)}-1}, x_{\frac{(\ell+2)N}{2(\ell+1)}})$. From the inequality (5.13), we have

$$\begin{aligned}\|\mathcal{I}L_i\|^2_{L^2\left(I_{\frac{(\ell+2)N}{2(\ell+1)}}\right)} &\le N^{-1}\sum_{p=i}^{\ell}\Big(\exp(-2\alpha x_{\frac{(\ell+2)N}{2(\ell+1)}-1}/\varepsilon_p)+\exp(-2\alpha(1-x_{\frac{(\ell+2)N}{2(\ell+1)}})/\varepsilon_p)\Big)\\ &\le CN^{-(1+2\sigma)}.\end{aligned}$$

These two last estimates give the desired estimate. Thus the estimate (5.7) is proved.

From (5.7) and (5.8), we get

$$\|L_i - \mathcal{I}L_i\|_{L^2(\Omega_\lambda)} \le \|L_i\|_{L^2(\Omega_\lambda)} + \|\mathcal{I}L_i\|_{L^2(\Omega_\lambda)} \le C(\varepsilon^{1/2} + N^{-1/2})N^{-\sigma},$$

which completes the proof of (5.10).

Using the triangle inequality and (5.7) and (5.9), we have

$$\|(L_i - \mathcal{I}L_i)'\|_{L^2(\Omega_\lambda)} \le \|L_i'\|_{L^2(\Omega_\lambda)} + \|(\mathcal{I}L_i)'\|_{L^2(\Omega_\lambda)} \le C\varepsilon^{-1/2}N^{-\sigma}.$$

Similarly, using the inverse estimate, we get

$$\begin{aligned}\|(L_i - \mathcal{I}L_i)''\|_{L^2(\Omega_\lambda)} &\le \|L_i''\|_{L^2(\Omega_\lambda)} + CN\|(\mathcal{I}L_i)'\|_{L^2(\Omega_\lambda)}\\ &\le C\varepsilon^{-3/2}[1 + (\varepsilon N)^{3/2} + (\varepsilon N)^2]N^{-(k+1)}\\ &\le C\varepsilon^{-3/2}N^{-\sigma}.\end{aligned}$$

Hence, we complete the proof of (5.11).

By (5.3) and (2.10), we have for $l = 1, 2$,

$$\begin{aligned}&\|(L_i - \mathcal{I}L_i)^{(l)}\|^2_{L^2(\Omega\backslash\Omega_\lambda)}\\ =&\sum_{I_n\subset\Omega\backslash\Omega_\lambda} \|(L_i - \mathcal{I}L_i)^{(l)}\|^2_{L^2(I_n)} \le \sum_{I_n\subset\Omega\backslash\Omega_\lambda} Ch_n^{2(k+1-l)}\|L_i^{(k+1)}\|^2_{L^2(I_n)}\\ \le& C\sum_{s=0}^{\ell-1} H_s^{2(k+1-l)}\varepsilon_i^{-2}\Big(\int_{\lambda_s}^{\lambda_{s+1}}\Big(\sum_{m=1}^{\ell}\varepsilon_m^{-2}\mathcal{B}^\alpha_{\varepsilon_m}(x)\Big)^2 dx + \int_{1-\lambda_{\ell-s}}^{1-\lambda_{\ell-1-s}}\Big(\sum_{m=1}^{\ell}\varepsilon_m^{-2}\mathcal{B}^\alpha_{\varepsilon_m}(x)\Big)^2 dx\Big)\\ \le& C\sum_{m=1}^{\ell}\sum_{s=0}^{\ell-1}\Big[\frac{2(\ell+1)(\lambda_{s+1}-\lambda_s)}{N}\Big]^{2(k+1-l)}\varepsilon_m^{-2(k+1)}\varepsilon_m \le C\varepsilon^{1-2l}(N^{-1}\ln N)^{2(k+1-l)},\end{aligned}$$

which shows (5.12). Thus we complete the proof.

□

The exact solution of problem (1.1) does not satisfy the WG-FEM scheme (3.4) and hence the WG-FEM lacks of consistency. Consequently, inconsistency leads to loss of the classical Galerkin orthogonality. As a result, we follow different techniques from the ones used in the standard finite element procedure to derive the error estimates.

Now Strang's second lemma provides a quasi-optimal bound for $|||\mathbf{u} - \mathbf{u}_N|||_\varepsilon$.

Theorem 5.3. *Let $\boldsymbol{u}$ and $\boldsymbol{u}_N$ be the solutions of problems (1.1) and (3.4) respectively. Then there exists a positive constant C independent of N and ε_i such that*

$$|||\boldsymbol{u} - \boldsymbol{u}_N|||_\varepsilon \le C\Big(\inf_{\boldsymbol{v}_N\in[S_N^0]^\ell}|||\boldsymbol{u} - \boldsymbol{v}_N|||_\varepsilon + \sup_{\boldsymbol{w}_N\in[S_N^0]^\ell}\frac{|a(\boldsymbol{u}, \boldsymbol{w}_N) - L(\boldsymbol{w}_N)|}{|||\boldsymbol{w}_N|||_\varepsilon}\Big), \tag{5.14}$$

where $a(\cdot,\cdot)$ is the bilinear form given by (3.5).

First, we will establish some error equations which will be needed in the error analysis below.

Lemma 5.4. *Let $\boldsymbol{u} = (u_1, \ldots, u_\ell)$ be the solution of the problem (1.1). Then for any $\boldsymbol{v}_N = (v_1^N, \ldots, v_\ell^N) = (\{v_{10}, v_{1b}\}, \ldots, \{v_{\ell 0}, v_{\ell b}\}) \in [S_N^0]^\ell$, we have*

$$-\varepsilon_i^2\Big(u_i'', v_{i0}\Big) = \varepsilon_i^2\Big(d_w(\mathcal{I}u_i), d_w v_i^N\Big) - T_1(u_i, v_i^N), \quad i = 1, \ldots, \ell, \tag{5.15}$$

$$\sum_{i=1}^{\ell}\sum_{j=1}^{\ell}(a_{ij}u_j, v_{i0}) = \sum_{i=1}^{\ell}\sum_{j=1}^{\ell}(a_{ij}\mathcal{I}u_j, v_{i0}) - T_2(\boldsymbol{u}, \boldsymbol{v}_N), \tag{5.16}$$

where

$$T_1(u_i, v_i^N) = \varepsilon_i^2\langle (u_i - \mathcal{I}u_i)', (v_{i0} - v_{ib})\boldsymbol{n}\rangle, \tag{5.17}$$

$$T_2(\boldsymbol{u}, \boldsymbol{v}_N) = \sum_{i=1}^{\ell}\sum_{j=1}^{\ell}(a_{ij}(\mathcal{I}u_j - u_j), v_{i0}). \tag{5.18}$$

Proof. For any $\mathbf{v}_N \in [S_N^0]^\ell$, using the commutative property (5.2) of the interpolation operator we have

$$\Big(d_w(\mathcal{I}u_i), d_w v_i^N\Big)_{I_n} = \Big((\mathcal{I}u_i)', d_w v_i^N\Big)_{I_n}, \quad \forall I_n \in \mathcal{T}_N. \tag{5.19}$$

Using the definition of the weak derivative (3.3) and integration by parts, we have

$$\begin{aligned}\Big(d_w v_i^N, (\mathcal{I}u_i)'\Big)_{I_n} &= -\Big(v_{i0}, (\mathcal{I}u_i)''\Big)_{I_n} + \langle v_{ib}, \mathbf{n}(\mathcal{I}u_i)'\rangle_{\partial I_n} \\ &= \Big(v_{i0}', (\mathcal{I}u_i)'\Big)_{I_n} - \langle v_{i0} - v_{ib}, \mathbf{n}(\mathcal{I}u_i)'\rangle_{\partial I_n}.\end{aligned} \tag{5.20}$$

From the definition of the interpolation and integration by parts, we obtain

$$\Big((u_i - \mathcal{I}u_i)', v_{i0}'\Big)_{I_n} = -\Big(u_i - \mathcal{I}u_i, v_{i0}''\Big)_{I_n} + \langle u_i - \mathcal{I}u_i, \mathbf{n}v_{i0}'\rangle_{\partial I_n} = 0,$$

which implies that

$$\Big((\mathcal{I}u_i)', v_{i0}'\Big)_{I_n} = \Big(u_i', v_{i0}'\Big)_{I_n}. \tag{5.21}$$

We infer from the Eqs (5.19)–(5.21) that

$$\Big(d_w(\mathcal{I}u_i), d_w v_i^N\Big)_{I_n} = \Big(u_i', v_{i0}'\Big)_{I_n} - \langle v_{i0} - v_{ib}, \mathbf{n}(\mathcal{I}u_i)'\rangle_{\partial I_n}. \tag{5.22}$$

Summing up the Eq (5.22) over all interval $I_n \in \mathcal{T}_N$, we find

$$\Big(d_w(\mathcal{I}u_i), d_w v_i^N\Big) = \Big(u_i', v_{i0}'\Big) - \langle v_{i0} - v_{ib}, \mathbf{n}(\mathcal{I}u_i)'\rangle. \tag{5.23}$$

Using integration by parts, one can show that

$$-\Big(u_i'', v_{i0}\Big)_{I_n} = \Big(u_i', v_{i0}'\Big)_{I_n} - \langle u_i', \mathbf{n}v_{i0}\rangle_{\partial I_n}.$$

Summing up the above equation over all interval $I_n \in \mathcal{T}_N$, we get

$$\Big(u_i', v_{i0}'\Big) = -\Big(u_i'', v_{i0}\Big) + \langle u_i', \mathbf{n}(v_{i0} - v_{ib})\rangle, \tag{5.24}$$

where we used the fact that $\langle u_i', \mathbf{n}v_{ib}\rangle = 0$. Finally, by plugging the Eq (5.24) into (5.23), we arrive at the desired result (5.15).

Lastly, the Eq (5.18) clearly holds. We complete the proof.

□

The following lemma will be useful in the error analysis.

Lemma 5.5. *Assume that $\boldsymbol{u} = (u_1, \ldots, u_\ell)$, with $u_i \in H^{k+1}(\Omega)$ is the solution of the problem (1.1). Then we have the following estimate*

$$\sum_{I_n \subset \Omega} \|\theta_i'\|^2_{L^2(\partial I_n)} \leq \begin{cases} C\varepsilon_i^{-2}(N^{-1}\ln N)^{2k-1}, & I_n \subset \Omega \setminus \Omega_\lambda, \\ C\varepsilon_i^{-2}N^{-2(k+1)}, & I_n \subset \Omega_\lambda, \end{cases}$$

where $\theta_i = u_i - \mathcal{I}u_i$ for $i = 1, \ldots, \ell$.

Proof. From the trace inequality (4.1), we can write

$$\|\theta_i'\|^2_{L^2(\partial I_n)} \leq C(h_n^{-1}\|\theta_i'\|^2_{L^2(I_n)} + \|\theta_i'\|_{L^2(I_n)}\|\theta_i''\|_{L^2(I_n)}).$$

It remains to estimate $\|\theta_i'\|_{L^2(I_n)}$ and $\|\theta_i''\|_{L^2(I_n)}$, individually. From the estimate (5.5), one has

$$\begin{aligned} \|(R_i - \mathcal{I}R_i)'\|_{L^2(\Omega)} &\leq CN^{-k}, \quad i = 1, 2, \ldots, \ell, \\ \|(R_i - \mathcal{I}R_i)''\|_{L^2(\Omega)} &\leq CN^{1-k}, \quad i = 1, 2, \ldots, \ell. \end{aligned} \tag{5.25}$$

With the help of the estimate (5.11) and (5.12) one can show that

$$\begin{aligned} \|(L_i - \mathcal{I}L_i)'\|_{L^2(\Omega_\lambda)} &\leq C\varepsilon_i^{-1/2}N^{-\sigma}, \quad i = 1, 2, \ldots, \ell, \\ \|(L_i - \mathcal{I}L_i)''\|_{L^2(\Omega_\lambda)} &\leq C\varepsilon_i^{-3/2}N^{-\sigma}, \quad i = 1, 2, \ldots, \ell, \\ \|(L_i - \mathcal{I}L_i)'\|_{L^2(\Omega\setminus\Omega_\lambda)} &\leq C\varepsilon_i^{-1/2}(N^{-1}\ln N)^{k}, \quad i = 1, 2, \ldots, \ell, \\ \|(L_i - \mathcal{I}L_i)''\|_{L^2(\Omega\setminus\Omega_\lambda)} &\leq C\varepsilon_i^{-3/2}(N^{-1}\ln N)^{k-1}, \quad i = 1, 2, \ldots, \ell. \end{aligned} \tag{5.26}$$

With the help of the above estimates, the fact that $\sigma \geq k + 1$, and the triangle inequality, one can conclude that

$$\sum_{I_n \subset \Omega} \|\theta_i'\|_{L^2(I_n)} \leq \begin{cases} C\varepsilon_i^{-1/2}N^{-k}(\varepsilon_i^{1/2} + \ln^k N), & I_n \subset \Omega \setminus \Omega_\lambda, \\ C\varepsilon_i^{-1/2}N^{-k}(\varepsilon_i^{1/2} + N^{-1}), & I_n \subset \Omega_\lambda, \end{cases} \tag{5.27}$$

and

$$\sum_{I_n \subset \Omega} \|\theta_i''\|_{L^2(I_n)} \leq \begin{cases} C\varepsilon_i^{-3/2}N^{1-k}(\varepsilon_i^{3/2} + \ln^{k-1} N), & I_n \subset \Omega \setminus \Omega_\lambda, \\ C\varepsilon_i^{-3/2}N^{1-k}(\varepsilon_i^{3/2} + N^{-2}), & I_n \subset \Omega_\lambda. \end{cases}$$

The desired result follows from combining the above estimates and the mesh size h_n. Thus, we complete the proof. □

Lemma 5.6. *Assume that $u_i \in H^{k+1}(\Omega)$ and the penalization parameter ϱ_n is given by (3.6). If $\sigma \geq k+1$, then we have*

$$T(\boldsymbol{u}, \boldsymbol{v}_N) \leq C(\varepsilon^{1/2}(N^{-1}\ln N)^k + N^{-(k+1)})|||\boldsymbol{v}_N|||_\varepsilon, \tag{5.28}$$

where $T(\boldsymbol{u}, \boldsymbol{v}_N) = \sum_{i=1}^{\ell} T_1(u_i, v_i^N) + T_2(\boldsymbol{u}, \boldsymbol{v}_N)$ and C is independent of N and ε_i, $i = 1, \ldots, \ell$.

Proof. It follows from Cauchy-Schwarz inequality, Lemma 5.5 and the penalization parameter (3.6) that

$$\begin{aligned}
|T_1(u_i, v_i^N)| &\leq \sum_{n=1}^{N} \varepsilon_i^2 |\langle (u_i - \mathcal{I}u_i)', v_{i0} - v_{ib}\rangle_{\partial I_n}| \\
&\leq \sum_{n=1}^{N} \varepsilon_i^2 \|(u_i - \mathcal{I}u_i)'\|_{L^2(\partial I_n)} \|v_{i0} - v_{ib}\|_{L^2(\partial I_n)} \\
&\leq \left\{ \sum_{n=1}^{N} \frac{\varepsilon_i^3}{\varrho_n} \|(u_i - \mathcal{I}u_i)'\|^2_{L^2(\partial I_n)} \right\}^{1/2} \left\{ \sum_{n=1}^{N} \varrho_n \|v_{i0} - v_{ib}\|^2_{L^2(\partial I_n)} \right\}^{1/2} \\
&\leq \Big\{ \sum_{I_n \in \Omega \setminus \Omega_\lambda} \frac{\varepsilon_i^3}{\varrho_n} \|(u_i - \mathcal{I}u_i)'\|^2_{L^2(\partial I_n)} \\
&\quad + \sum_{I_n \in \Omega_\lambda} \frac{\varepsilon_i^3}{\varrho_n} \|(u_i - \mathcal{I}u_i)'\|^2_{L^2(\partial I_n)} \Big\}^{1/2} s^{1/2}(v_i^N, v_i^N) \\
&\leq C\varepsilon^{1/2} (N^{-1} \ln N)^k s^{1/2}(v_i^N, v_i^N).
\end{aligned}$$

As a result

$$|T_1(\mathbf{u}, \mathbf{v}_N)| \leq \sum_{i=1}^{\ell} T_1(u_i, v_i^N) \leq C\varepsilon^{1/2} (N^{-1} \ln N)^k |||\mathbf{v}_N|||_\varepsilon. \tag{5.29}$$

We next bound the term $T_2(\mathbf{u}, \mathbf{v}_N)$. We need to estimate $\|u_i - \mathcal{I}u_i\|$, $i = 1, \ldots, \ell$. Using the estimates (5.5)–(5.8) of Lemma 5.2 and Cauchy-Schwarz inequality, taking $\sigma \geq k + 1$, we get

$$\begin{aligned}
\|u_i - \mathcal{I}u_i\|_{L^2(\Omega)} &\leq \|R_i - \mathcal{I}R_i\|_{L^2(\Omega)} + \|L_i - \mathcal{I}L_i\|_{L^2(\Omega \setminus \Omega_\lambda)} \\
&\quad + \|L_i\|_{L^2(\Omega_\lambda)} + \|\mathcal{I}L_i\|_{L^2(\Omega_\lambda)} \\
&\leq CN^{-(k+1)} [1 + \varepsilon_\ell^{1/2} (\ln N)^{k+1} \\
&\quad + \varepsilon_\ell^{1/2} N^{-(\sigma-3)} + N^{-(\sigma-5/2)}] \\
&\leq CN^{-(k+1)} (1 + \varepsilon_\ell^{1/2} (\ln N)^{k+1}) \\
&\leq CN^{-(k+1)}.
\end{aligned} \tag{5.30}$$

The above estimate (5.30) and Cauchy-Schwarz inequality yield the following bound

$$\sum_{i=1}^{\ell} \sum_{j=1}^{\ell} (a_{ij}(\mathcal{I}u_j - u_j), v_{i0}) \leq C \sum_{i=1}^{\ell} \sum_{j=1}^{\ell} \|u_j - \mathcal{I}u_j\| \|v_{i0}\| \leq CN^{-(k+1)} \|v_{i0}\|.$$

From the above estimate, we have

$$|T_2(\mathbf{u}, \mathbf{v}_N)| \leq CN^{-(k+1)} |||\mathbf{v}_N|||_\varepsilon. \tag{5.31}$$

From the estimates (5.29) and (5.31), we have the desired result. Thus we complete the proof. □

Theorem 5.7. *Let $\boldsymbol{u} = (u_1, \dots, u_\ell) = \boldsymbol{R} + \boldsymbol{L}$ with $u_i \in H^{k+1}(\Omega)$ be the solution of the problem (1.1) and assume that the conditions of Lemma 5.2 hold with $\sigma \geq k+1$. Then, the following estimates hold true:*

$$|||\boldsymbol{R} - \mathcal{I}\boldsymbol{R}|||_\varepsilon \leq CN^{-(k+1)} \text{ and } |||\boldsymbol{L} - \mathcal{I}\boldsymbol{L}|||_\varepsilon \leq C(\varepsilon_\ell^{1/2}(N^{-1}\ln N)^k + N^{-(k+1)}), \tag{5.32}$$

where C is independent of N and $\varepsilon_i, i = 1, \dots, \ell$.

Proof. Since $\theta_i^R := R_i - \mathcal{I}R_i$ and $\theta_i^L := L_i - \mathcal{I}L_i$ are continuous on Ω, we get $s(\theta_i^R, \theta_i^R) = s(\theta_i^L, \theta_i^L) = 0$ for $i = 1, \dots, \ell$. Then we have

$$|||R_i - \mathcal{I}R_i|||_\varepsilon^2 = \sum_{i=1}^{\ell} \varepsilon_i^2 \|(\theta_i^R)'\|^2 + \eta \sum_{i=1}^{\ell} \|\theta_i^R\|^2, \tag{5.33}$$

$$|||L_i - \mathcal{I}L_i|||_\varepsilon^2 = \sum_{i=1}^{\ell} \varepsilon_i^2 \|(\theta_i^L)'\|^2 + \eta \sum_{i=1}^{\ell} \|\theta_i^L\|^2. \tag{5.34}$$

In the light of the interpolation errors (5.5) and (2.8), we obtain for $i = 1, \dots, \ell$

$$\begin{aligned} \varepsilon_i^2 \|(\theta_i^R)'\|^2 &\leq \varepsilon_i^2 (N^{-k}|R_i|_{3,\Omega})^2 \leq C\varepsilon_i^2 N^{-2k}, \\ \|\theta_i^R\|^2 &\leq CN^{-2(k+1)}, \end{aligned}$$

which together with (5.33) yields

$$|||\mathbf{R} - \mathcal{I}\mathbf{R}|||_\varepsilon \leq CN^{-(k+1)}.$$

Using the inequalities (5.11), (5.12) and (5.30), we have

$$\begin{aligned} \sum_{i=1}^{\ell} \varepsilon_i^2 \|(\theta_i^L)'\|^2 &\leq \sum_{i=1}^{\ell} \varepsilon_i^2 \Big(\|(\theta_i^L)'\|^2_{L^2(\Omega\setminus\Omega_\lambda)} + \|(\theta_i^L)'\|^2_{L^2(\Omega_\lambda)} \Big) \leq C\varepsilon((N^{-1}\ln N)^{2k} + N^{-2\sigma}), \\ \|\theta_i^L\|^2 &\leq N^{-2(k+1)}, \end{aligned}$$

which together with (5.34) gives the desired result

$$|||\mathbf{L} - \mathcal{I}\mathbf{L}|||_\varepsilon \leq C(\varepsilon^{1/2}(N^{-1}\ln N)^k + N^{-(k+1)}),$$

where we have used $\sigma \geq k+1$. The proof is completed. □

We next estimate the consistency error $\sup_{\mathbf{w}_N \in [S_N^0]^\ell} \dfrac{|a(\mathbf{u}, \mathbf{w}_N) - L(\mathbf{w}_N)|}{|||\mathbf{w}_N|||_\varepsilon}$.

Lemma 5.8. *Assume that $\boldsymbol{u} = (u_1, \dots, u_\ell), u_i \in H^{k+1}(\Omega), i = 1, \dots, \ell$ is the solution of (1.1). If $\sigma \geq k+1$, then the following estimate holds true:*

$$\sup_{\boldsymbol{w}_N \in [S_N^0]^\ell} \frac{|a(\boldsymbol{u}, \boldsymbol{w}_N) - L(\boldsymbol{w}_N)|}{|||\boldsymbol{w}_N|||_\varepsilon} \leq C(\varepsilon^{1/2}(N^{-1}\ln N)^k + N^{-(k+1)}),$$

where C is independent of N, and $\varepsilon_i, \ i = 1, \dots, \ell$.

Proof. Using the definition of bilinear form (3.4), the fact that $s(u_i, w_i^N) = s(u_i - \mathcal{I}u_i, w_i^N) = 0$ for $i = 1, \dots, \ell$ and Lemma 5.4, we have

$$\begin{aligned} E_{\mathbf{u}}(\mathbf{w}_N) :=& a(\mathbf{u}, \mathbf{w}_N) - L(\mathbf{w}_N) = a(\mathcal{I}\mathbf{u}, \mathbf{w}_N) + a(\mathbf{u} - \mathcal{I}\mathbf{u}, \mathbf{w}_N) - L(\mathbf{w}_N) \\ =& \sum_{i=1}^{\ell} \Big(-\varepsilon^2 u_i'' + \sum_{j=1}^{\ell} a_{ij} u_j - g_i, w_{i0} \Big) + T(\mathbf{u}, \mathbf{w}_N) + a(\mathbf{u} - \mathcal{I}\mathbf{u}, \mathbf{w}_N) \\ =& T(\mathbf{u}, \mathbf{w}_N) + a(\mathbf{u} - \mathcal{I}\mathbf{u}, \mathbf{w}_N), \end{aligned}$$

where $T(\mathbf{u}, \mathbf{w}_N) = T_1(\mathbf{u}, \mathbf{w}_N) + T_2(\mathbf{u}, \mathbf{w}_N)$ and $T_1(\mathbf{u}, \mathbf{w}_N) = \sum_{i=1}^{\ell} T_1(u_i, w_i^N)$ with $T_1(u_i, w_i^N)$ and $T_2(\mathbf{u}, \mathbf{w}_N)$ are given in (5.17) and (5.18), respectively. By Lemma 5.6, if $\sigma \geq k+1$, the first term on the right side of the above equation can be estimated as

$$T(\mathbf{u}, \mathbf{w}_N) \leq C(\varepsilon^{1/2}(N^{-1}\ln N)^k + N^{-(k+1)})|||\mathbf{w}_N|||_{\varepsilon}. \tag{5.35}$$

For the second term, we use the continuity property (4.14) of bilinear form $a(\cdot, \cdot)$ and again the fact that $s(u_i - \mathcal{I}u_i, w_i^N) = 0, i = 1, \dots, \ell$ and we obtain

$$\begin{aligned} a(\mathbf{u} - \mathcal{I}\mathbf{u}, \mathbf{w}_N) \leq& C_c |||\mathbf{u} - \mathcal{I}\mathbf{u}|||_{\varepsilon} |||\mathbf{w}_N|||_{\varepsilon} \\ =& C_c \sum_{i=1}^{\ell} \Big(\varepsilon_i^2 \|(u_i - \mathcal{I}u_i)'\|^2 + \eta \|u_i - \mathcal{I}u_i\|^2 \Big)^{1/2} |||\mathbf{w}_N|||_{\varepsilon} \\ \leq& C \sum_{i=1}^{\ell} \Big(\varepsilon_i^2 \|(R_i - \mathcal{I}R_i)'\|^2 + \varepsilon_i^2 \|(L_i - \mathcal{I}L_i)'\|^2_{L^2(\Omega \setminus \Omega_\lambda)} \\ &+ \varepsilon_i^2 \|(L_i - \mathcal{I}L_i)'\|^2_{L^2(\Omega_\lambda)} + \eta \|u_i - \mathcal{I}u_i\|^2 \Big)^{1/2} |||\mathbf{w}_N|||_{\varepsilon}. \end{aligned}$$

Appealing the estimates (5.5), (5.11), (5.12), (5.30) and using (2.8), if $\sigma \geq k+1$ we obtain

$$\begin{aligned} a(\mathbf{u} - \mathcal{I}\mathbf{u}, \mathbf{w}_N) \leq& C\Big(\varepsilon^2 N^{-2k} + \varepsilon^2 \varepsilon^{-1} (N^{-1}\ln N)^{2k} + \varepsilon^2 \varepsilon^{-1} N^{-2(k+1)} + N^{-2(k+1)} \Big)^{1/2} |||\mathbf{w}_N|||_{\varepsilon} \\ \leq& C(\varepsilon^{1/2}(N^{-1}\ln N)^k + N^{-(k+1)})|||\mathbf{w}_N|||_{\varepsilon}. \end{aligned} \tag{5.36}$$

From (5.35) and (5.36), we arrive at

$$\sup_{\mathbf{w}_N \in [S_N^0]^{\ell}} \frac{|E_{\mathbf{u}}(\mathbf{w}_N)|}{|||\mathbf{w}_N|||_{\varepsilon}} \leq C(\varepsilon^{1/2}(N^{-1}\ln N)^k + N^{-(k+1)}),$$

which is the desired result. We complete the proof. □

Theorem 5.9. *Assume that* $\boldsymbol{u} = (u_1, \dots, u_\ell), u_i \in H^{k+1}(\Omega), i = 1, \dots, \ell$ *is the exact solution and* $\boldsymbol{u}_N \in [S_N^0]^{\ell}$ *is the WG-FEM solution given by (3.4) on the uniform Shishkin mesh for the problem (1.1), respectively. If* $\sigma \geq k+1$*, then we have the following estimate*

$$|||\boldsymbol{u} - \boldsymbol{u}_N|||_{\varepsilon} \leq C(\varepsilon^{1/2}(N^{-1}\ln N)^k + N^{-(k+1)}),$$

where C is independent of N and $\varepsilon_i, i = 1, \dots, \ell$*.*

Proof. Using Theorem 5.7, if $\sigma \geq k+1$ we have

$$|||\mathbf{R} - \mathcal{I}\mathbf{R}|||_\varepsilon \leq CN^{-(k+1)}$$
$$|||\mathbf{L} - \mathcal{I}\mathbf{L}|||_\varepsilon \leq C(\varepsilon^{1/2}(N^{-1}\ln N)^k + N^{-(k+1)}).$$

Hence, we obtain

$$|||\mathbf{u} - \mathcal{I}\mathbf{u}|||_\varepsilon \leq |||\mathbf{R} - \mathcal{I}\mathbf{R}|||_\varepsilon + |||\mathbf{L} - \mathcal{I}\mathbf{L}|||_\varepsilon \leq C(\varepsilon^{1/2}(N^{-1}\ln N)^k + N^{-(k+1)}).$$

Note that $\mathcal{I}\mathbf{R}$ and $\mathcal{I}\mathbf{L}$ are both in $[S_N^0]^\ell$, the set of piecewise discontinuous polynomials of degree at most k, and that $\mathcal{I}\mathbf{R}+\mathcal{I}\mathbf{L} \in [S_N^0]^\ell$. Take $\mathbf{v}_N = (\mathcal{I}R_1+\mathcal{I}L_1, \ldots, \mathcal{I}R_\ell+\mathcal{I}L_\ell) \in S_N^\ell$. Invoking Theorem 5.3 and Lemma 5.8, the desired result follows.

□

6. Balanced norm estimates

As stated in the introduction, error estimates in the corresponding energy norm of FEMs not adequate. The reason arises from the fact that the energy norm of the boundary layer functions $\exp(\frac{-x}{\varepsilon_\ell})$ and $\exp(\frac{-(1-x)}{\varepsilon_\ell})$ are of order $\mathcal{O}(\varepsilon_\ell^{1/2})$. Therefore, the error estimates in the energy norm is not much strong than the L^2-norm if $\varepsilon_\ell \ll 1$. A stronger norm obtained by scaling of the coefficient of the H^1-seminorm captures correctly the boundary layers. This norm is called the balanced norm defined as follows. For $\mathbf{v} = (v_1^N, \ldots, v_\ell^N)^T = (\{v_{10}, v_{1b}\}, \ldots, \{v_{\ell 0}, v_{\ell b}\})^T \in [S_N^0]^\ell$,

$$\|\mathbf{v}\|_b^2 = \sum_{i=1}^{\ell} \varepsilon_i \|d_w v_i^N\|^2 + \eta \sum_{i=1}^{\ell} \|v_i^N\|^2 + s^b(\mathbf{v}_N, \mathbf{v}_N), \tag{6.1}$$

where $s^b(\mathbf{u}_N, \mathbf{v}_N)$ is given by

$$s^b(\mathbf{u}_N, \mathbf{v}_N) = \sum_{i=1}^{\ell} \langle \varrho_n^b(u_{i0} - u_{ib}), v_{i0} - v_{ib} \rangle. \tag{6.2}$$

Here, the penalization parameter ϱ_n^b is now defined as

$$\varrho_n^b = \begin{cases} \varepsilon, & \text{on } \Omega_\lambda, \\ \dfrac{\varepsilon N}{\ln N}, & \text{on } \Omega \setminus \Omega_\lambda, \end{cases} \tag{6.3}$$

where $\varepsilon = \sum_{i=1}^{\ell} \varepsilon_i$.

We note that the error bound $N^{-(k+1)}$ independent of $\varepsilon^{1/2}$ in Theorem 5.9 comes from the estimate of the L^2-norm of $\mathbf{u} - \mathcal{I}\mathbf{u}$ in the energy norm error estimates. These terms can be handled by replacing the special interpolation operator $\mathcal{I}$ defined by (5.1) with a projection operator $Q_h : H^1(I_n) \to S_N$ defined as follows.

Let $P_h : L^2(I_n) \to \mathbb{P}_k(I_n)$ be the local weighted L^2-projection restricted to interval I_n defined by

$$(\sum_{i=1}^{\ell} a_{ii}(P_h u_i - u_i), v)_{I_n} = 0, \quad \forall v \in \mathbb{P}_k(I_n), \quad n = 1, 2, \ldots, N. \tag{6.4}$$

This weighted L^2-projection is well-defined because we assume that the diagonal elements are positive and the reaction coefficient matrix is strongly diagonally dominant matrix. With the aid of the Bramble-Hilbert lemma, one can show that for $i = 1, \ldots, \ell$

$$\|u_i - P_h u_i\|_{L^2(I_n)} + h_n \|(u_i - P_h u_i)'\|_{L^2(I_n)} \le C h_n^{s+1} |u_i|_{s+1,I_n}, \quad 0 \le s \le k. \tag{6.5}$$

We introduce the projection operator $Q_h : H^1(I_n) \to S_N$ such that

$$Q_h u_i|_{I_n} = \{Q_0 u_i, Q_b u_i\} = \{P_h u_i, \{u_i(x_{n-1}), u_i(x_n)\}\}, \quad n = 1, 2, \ldots, N. \tag{6.6}$$

Clearly, $Q_h u_i \in S_N^0$ if $u_i \in H_0^1(I_n)$ for $i = 1, \ldots, \ell$. By (6.5), we have

$$\|Q_0 u_i - u_i\|_{L^2(I_n)} \le C h_n^{s+1} |u_i|_{s+1,I_n}, \quad 0 \le s \le k, \quad i = 1, \ldots, \ell. \tag{6.7}$$

The following trace and inverse inequalities will be used in the forthcoming analysis [43]. For any function $\phi \in H(I_n)$, we have

$$\|\phi\|^2_{L^2(\partial I_n)} \le C(h_n^{-1}\|\phi\|^2_{L^2(I_n)} + h_n\|\phi'\|^2_{L^2(I_n)}), \tag{6.8}$$

$$\|v_N'\|_{L^2(\partial I_n)} \le C h_n^{-1} \|v_N\|_{L^2(I_n)}, \quad \forall v_N \in \mathbb{P}_k(I_n). \tag{6.9}$$

We would like to derive similar estimates as Lemma 5.2 for the projection operator Q_0 which is essentially the generalized L^2-projection. The following lemma will serve this purpose.

Lemma 6.1. *Assume that the conclusions of Lemma 5.2 hold. Then we have the following error estimates for the operator Q_0 on the uniform Shishkin mesh.*

$$\|u_i - Q_0 u_i\|_{L^\infty(\Omega)} \le C\|u_i - \mathcal{I} u_i\|_{L^\infty(\Omega)}, \tag{6.10}$$

$$\sum_{I_n \subset \Omega_\lambda} \|u_i - Q_0 u_i\|^2_{L^2(I_n)} \le C N^{-2k-3} \tag{6.11}$$

$$\sum_{I_n \subset \Omega\setminus\Omega_\lambda} \|u_i - Q_0 u_i\|^2_{L^2(I_n)} \le C\varepsilon (N^{-1}\ln N)^{2(k+1)}, \tag{6.12}$$

$$\sum_{I_n \subset \Omega_\lambda} \|(u_i - Q_0 u_i)'\|^2_{L^2(I_n)} \le C\varepsilon^{-1/2} N^{-2k}, \tag{6.13}$$

$$\sum_{I_n \subset \Omega\setminus\Omega_\lambda} \|(u_i - Q_0 u_i)'\|^2_{L^2(I_n)} \le C\varepsilon^{-1} N^{-2k} \ln^{2k+1} N. \tag{6.14}$$

Proof. It is known that the L^2-projection Q_0 is L^∞-stable [44]. Therefore, by the triangle inequality we have

$$\begin{aligned}\|u_i - Q_0 u_i\|_{L^\infty(\Omega)} &\le \|u_i - \mathcal{I} u_i\|_{L^\infty(\Omega)} + \|Q_0(u_i - \mathcal{I} u_i)\|_{L^\infty(\Omega)} \\ &\le C\|u_i - \mathcal{I} u_i\|_{L^\infty(\Omega)},\end{aligned}$$

which proves (6.10). Using this inequality, we get

$$\sum_{I_n \subset \Omega_\lambda} \|u_i - Q_0 u_i\|^2_{L^2(I_n)} \le \sum_{I_n \subset \Omega_\lambda} h_n \|u_i - Q_0 u_i\|^2_{L^\infty(I_n)}$$

$$\leq C \sum_{I_n \subset \Omega_\lambda} h_n \|u_i - \mathcal{I} u_i\|^2_{L^\infty(I_n)}$$
$$\leq CN^{-2k-3},$$

where we used Lemma 5.2 and the fact that $h_n = O(N^{-1})$ in Ω_λ. It follows from (6.7) that

$$\begin{aligned}\sum_{I_n \subset \Omega \setminus \Omega_\lambda} \|Q_0 u_i - u_i\|_{L^2(I_n)} &\leq C \sum_{I_n \subset \Omega \setminus \Omega_\lambda} h_n^{k+1} |u_i|_{k+1, I_n} \\ &\leq C \sum_{I_n \subset \Omega \setminus \Omega_\lambda} h_n^{k+1} (\|R_i^{(k+1)}\|_{L^2(I_n)} + \|L_i^{(k+1)}\|_{L^2(I_n)}).\end{aligned}$$

The first term on the right side of the above inequality can be bounded as

$$\begin{aligned}\sum_{I_n \in \Omega \setminus \Omega_\lambda} h_n^{2(k+1)} \|R_i^{(k+1)}\|^2_{L^2(I_n)} &\leq \sum_{I_n \in \Omega \setminus \Omega_\lambda} h_n^{2(k+1)} \int_{x_{n-1}}^{x_n} |R^{(k+1)}(x)|^2 dx \\ &\leq C \sum_{I_n \in \Omega \setminus \Omega_\lambda} h_n^{2k+3} \leq C\varepsilon (N^{-1} \ln N)^{2k+3}.\end{aligned} \tag{6.15}$$

Next, we estimate the layer parts on $\Omega \setminus \Omega_\lambda$.

$$\begin{aligned}\sum_{I_n \in \Omega \setminus \Omega_\lambda} h_n^{2(k+1)} \|L_i^{(k+1)}\|^2_{L^2(I_n)} \leq & C \sum_{s=0}^{\ell-1} H_s^{2(k+1)} \varepsilon_i^{-2(k-1)} \Big(\int_{\lambda_s}^{\lambda_{s+1}} \Big(\sum_{m=1}^{\ell} \varepsilon_m^{-2} \mathcal{B}^\alpha_{\varepsilon_m}(x) \Big)^2 dx \\ & + \int_{1-\lambda_{\ell-s}}^{1-\lambda_{\ell-1-s}} \Big(\sum_{m=1}^{\ell} \varepsilon_m^{-2} \mathcal{B}^\alpha_{\varepsilon_m}(x) \Big)^2 dx \Big) \\ \leq & C \sum_{m=1}^{\ell} \sum_{s=0}^{\ell-1} \Big[\frac{2(\ell+1)(\lambda_{s+1} - \lambda_s)}{N} \Big]^{2(k+1)} \varepsilon_m^{-2(k+1)} \varepsilon_m \\ \leq & C\varepsilon (N^{-1} \ln N)^{2(k+1)}.\end{aligned} \tag{6.16}$$

From (6.15) and (6.16), we get

$$\sum_{I_n \subset \Omega \setminus \Omega_\lambda} \|Q_0 u_i - u_i\|^2_{L^2(I_n)} \leq C\varepsilon (N^{-1} \ln N)^{2(k+1)},$$

which completes the proof of (6.12).

With the help of an inverse inequality on Ω_λ, we obtain

$$\begin{aligned}\|(\mathcal{I} u_i - Q_0 u_i)'\|_{L^2(I_n)} &\leq CN \|\mathcal{I} u_i - Q_0 u_i\|_{L^2(I_n)} \\ &= CN \Big(\|\mathcal{I} u_i - u_i\|_{L^2(I_n)} + \|u_i - Q_0 u_i\|_{L^2(I_n)} \Big) \\ &\leq CN^{-k},\end{aligned}$$

because $\|\mathcal{I} u_i - u_i\|_{L^2(I_n)}$ and $\|Q_0 u_i - u_i\|_{L^2(I_n)}$ are both of order $O(N^{-(k+1)})$ on Ω_λ. By Lemma 5.2 and the above estimate, we arrive at

$$\sum_{I_n \subset \Omega_\lambda} \|(\mathcal{I} u_i - Q_0 u_i)'\|^2_{L^2(I_n)} \leq CN^{-2k}. \tag{6.17}$$

On the other hand, from Lemma 5.2, we have

$$\varepsilon\|(\mathcal{I}u_i - u_i)'\|^2_{L^2(\Omega_\lambda)} \le \varepsilon\|(R_i - \mathcal{I}R_i)'\|^2_{L^2(\Omega_\lambda)} + \varepsilon\|(L_i - \mathcal{I}L_i)'\|^2_{L^2(\Omega_\lambda)} \le C\varepsilon^{1/2}N^{-2k}. \tag{6.18}$$

Combining (6.17) and (6.18) gives the desired result (6.13). Finally, using an inverse estimate we obtain at once

$$\begin{aligned}\sum_{I_n\subset\Omega\setminus\Omega_\lambda}\|(\mathcal{I}u_i - Q_0u_i)'\|^2_{L^2(I_n)} &\le C|\Omega\setminus\Omega_\lambda|\sum_{I_n\subset\Omega\setminus\Omega_\lambda}\|(\mathcal{I}u_i - Q_0u_i)'\|^2_{L^\infty(I_n)}\\ &\le C\sum_{I_n\subset\Omega\setminus\Omega_\lambda}\frac{|\Omega\setminus\Omega_\lambda|}{h_n^2}\|\mathcal{I}u_i - Q_0u_i\|^2_{L^\infty(I_n)}\\ &\le C\frac{\varepsilon\ln N}{\varepsilon^2(N^{-1}\ln N)^2}(N^{-1}\ln N)^{2(k+1)}\\ &\le C\varepsilon^{-1}N^{-2k}(\ln N)^{2(k+1)},\end{aligned}$$

which proves (6.14). Thus, we complete the proof. □

We derive the following error equations involving the projection Q_h which are similar to ones in Lemma 5.4. To this end, we still need another special projection operator defined as follows. We refer interested readers to [45] for details.

Lemma 6.2. *[45] For $u_i \in H^1(\Omega)$, there is a projection operator $\pi_h u_i \in H^1(0,1)$, restricted on element I_n, $\pi_h u_i \in \mathbb{P}_{k+1}(I_n)$ satisfies*

$$((\pi_h u_i)', q) = (u_i', q)_{I_n}, \quad \forall q \in P_k(I_n), \quad i = 1, 2, \ldots, \ell, \tag{6.19}$$

$$\pi_h u_i(x_n) = u_i(x_n), \quad n = 1, \ldots, N, \quad i = 1, \ldots, \ell, \tag{6.20}$$

$$\|u_i - \pi_h u_i\|_{L^2(I_n)} + h_n\|u_i' - (\pi_h u_i)'\|_{L^2(I_n)} \le Ch_n^{s+1}\|u_i\|_{s+1}, \quad 0 \le s \le k.$$

Lemma 6.3. *Let $\boldsymbol{u} = (u_1, \ldots, u_\ell)$ be the solution of the problem (1.1). Then for any $\boldsymbol{v}_N = (v_1^N, \ldots, v_\ell^N) = (\{v_{10}, v_{1b}\}, \ldots, \{v_{\ell 0}, v_{\ell b}\}) \in [S_N^0]^\ell$, we have*

$$-\varepsilon_i^2\Big(u_i'', v_{i0}\Big) = \varepsilon_i^2\Big(d_w(Q_h u_i), d_w v_i^N\Big) - T_1^b(u_i, v_i^N), \quad i = 1, \ldots, \ell, \tag{6.21}$$

$$\sum_{i=1}^{\ell}\sum_{j=1}^{\ell}(a_{ij}u_j, v_{i0}) = \sum_{i=1}^{\ell}\sum_{j=1}^{\ell}(a_{ij}Q_0u_j, v_{i0}) - T_2^b(\boldsymbol{u}, \boldsymbol{v}_N), \tag{6.22}$$

where

$$T_1^b(\boldsymbol{u}, \boldsymbol{v}_N) = \sum_{i=1}^{\ell}\varepsilon_i^2\langle(u_i - \pi_h u_i)', (v_{i0} - v_{ib})\boldsymbol{n}\rangle, \tag{6.23}$$

$$T_2^b(\boldsymbol{u}, \boldsymbol{v}_N) = \sum_{i=1}^{\ell}\sum_{j=1}^{\ell}(a_{ij}(Q_0u_j - u_j), v_{i0}). \tag{6.24}$$

Proof. Using the definition of the operator Q_h, the weak derivative (3.3), integration by parts and (6.19), we have

$$\begin{aligned}(d_w(Q_hu_i),q)_{I_n} &= -(Q_0u_i,q')_{I_n} + (Q_hu)_nq_n - (Q_hu)_{n-1}q_{n-1}\\ &= -(u_i,q')_{I_n} + u_nq_n - u_{n-1}q_{n-1}\\ &=(u_i',q)_{I_n}\\ &=((\pi_hu_i)',q)_{I_n}, \quad \forall q\in\mathbb{P}_2(I_n), \quad \forall I_n\in\mathcal{T}_N,\end{aligned}$$

where $v_n = v(x_n)$ and $v_{n-1} = v(x_{n-1})$ for a function v. This implies that

$$(d_w(Q_hu_i), d_wv_i^N)_{I_n} = ((\pi_hu_i)', d_wv_i^N)_{I_n}, \quad \forall I_n\in\mathcal{T}_N. \tag{6.25}$$

Following the same procedures as in the energy norm estimates, we prove (6.21). Clearly, we have the Eq (6.24). We complete the proof. □

Lemma 6.4. *Assume that $\boldsymbol{u} = (u_1,\ldots,u_\ell)$, with $u_i\in H^{k+1}(\Omega)$ is the solution of the problem (1.1) and the penalization parameter ϱ_n^b is given by (6.3). Then we have*

$$|T^b(\boldsymbol{u},\boldsymbol{v}_N)| \le C\varepsilon^{1/2}(N^{-1}\ln N)^k|||v_N|||_\varepsilon, \tag{6.26}$$
$$|s^b(Q_hu_i,v_N)| \le C\varepsilon^{1/2}N^{-k}(\ln N)^{k+1/2}|||v_N|||_\varepsilon, \tag{6.27}$$

where C is independent of N and ε_i, $i = 1,\ldots,\ell$, $T^b(\boldsymbol{u},\boldsymbol{v}_N) = T_1^b(\boldsymbol{u},\boldsymbol{v}_N) + T_2^b(\boldsymbol{u},\boldsymbol{v}_N)$, and $s^b(Q_hu_i,v_N)$ is given by (6.2).

Proof. Note that $T_2^b(\mathbf{u},\mathbf{v}_N) = 0$ due to the definition of the projection Q_h. By the inverse estimate (6.9), Lemma 5.5 and Lemma 6.2, we obtain at once

$$\begin{aligned}\sum_{I_n\subset\Omega}\|\xi_i'\|^2_{L^2(\partial I_n)} &\le \sum_{I_n\subset\Omega}\|\theta_i'\|^2_{L^2(\partial I_n)} + \sum_{I_n\subset\Omega}\|(\mathcal{I}u_i-\pi_hu_i)'\|^2_{L^2(\partial I_n)}\\ &\le \sum_{I_n\subset\Omega}\|\theta_i'\|^2_{L^2(\partial I_n)} + C\sum_{I_n\subset\Omega}h_n^{-2}\|\mathcal{I}u_i-\pi_hu_i\|^2_{L^2(I_n)}\\ &\le \begin{cases} C\varepsilon_i^{-2}(N^{-1}\ln N)^{2k-1}, & I_n\subset\Omega\setminus\Omega_\lambda,\\ C\varepsilon_i^{-2}N^{-2(k+1)}, & I_n\subset\Omega_\lambda,\end{cases}\end{aligned}$$

where $\xi_i = u_i - \pi_hu_i$ and $\theta_i = u_i - \mathcal{I}u_i$ for $i = 1,\ldots,\ell$.

İmitating the arguments in the energy norm estimates and using the above fact, one can prove that

$$T^b(\mathbf{u},\mathbf{v}_N) = T_1^b(\mathbf{u},\mathbf{v}_N) \le C\varepsilon^{1/2}(N^{-1}\ln N)^k|||v_N|||_\varepsilon.$$

It follows from Cauchy–Schwarz inequality, the trace inequality (6.8) and Lemma 6.1 that

$$\begin{aligned}|s^b(Q_hu_i,v_N)| &\le \sum_{n=1}^N\varrho_n^b|\langle Q_0u_i - Q_bu_i, v_0-v_b\rangle_{\partial I_n}|\\ &= \sum_{n=1}^N\varrho_n^b|\langle Q_0u_i - u_i, v_0-v_b\rangle_{\partial I_n}|\end{aligned}$$

$$
\begin{aligned}
&\le C\Big(\sum_{n=1}^{N}\varepsilon\varrho_n^b\|u_i-Q_0u_i\|^2_{L^2(\partial I_n)}\Big)^{1/2}\Big(\sum_{n=0}^{N-1}\varepsilon^{-1}\varrho_n^b\|v_0-v_b\|^2_{L^2(\partial I_n)}\Big)^{1/2}\\
&\le C\Big(\sum_{n=1}^{N}\varepsilon\varrho_n^b(h_n^{-1}\|u_i-Q_0u_i\|^2_{L^2(I_n)}+h_n\|(u_i-Q_0u_i)'\|^2_{L^2(I_n)})\Big)^{1/2}|||v_N|||_\varepsilon\\
&\le C\Big[\Big(\sum_{I_n\subset\Omega_\lambda}\varepsilon\varrho_n^b(h_n^{-1}\|u_i-Q_0u_i\|^2_{L^2(I_n)}+h_n\|(u_i-Q_0u_i)'\|^2_{L^2(I_n)})\Big)^{1/2}|||v_N|||_\varepsilon\\
&\quad+\Big(\sum_{I_n\subset\Omega\setminus\Omega_\lambda}\varepsilon\varrho_n^b(h_n^{-1}\|u_i-Q_0u_i\|^2_{L^2(I_n)}+h_n\|(u_i-Q_0u_i)'\|^2_{L^2(I_n)})\Big)^{1/2}\Big]|||v_N|||_\varepsilon\\
&\le C\Big[\Big(\varepsilon^2(NN^{-(2k+3)}+N^{-1}\varepsilon^{-1/2}N^{-2k})\Big)^{1/2}\\
&\quad+\Big(\frac{\varepsilon^2N}{\ln N}(\frac{N}{\varepsilon\ln N}\varepsilon(N^{-1}\ln N)^{2(k+1)}+\frac{\varepsilon\ln N}{N}\varepsilon^{-1}(N^{-2k}\ln^{2k+1}N)\Big)^{1/2}\Big]|||v_N|||_\varepsilon\\
&\le C\varepsilon^{1/2}N^{-k}(\ln N)^{k+1/2}|||v_N|||_\varepsilon.
\end{aligned}
$$

Here, we used the fact that $\varepsilon^{-1}\varrho_n^b=\varrho_n$. Therefore, we complete the proof. □

The main result of this section is the following theorem.

Theorem 6.5. *Assume that $\boldsymbol{u}=(u_1,\dots,u_\ell), u_i\in H^{k+1}(\Omega), i=1,\dots,\ell$ is the exact solution and $\boldsymbol{u}_N=\{u_1^N,\dots,u_\ell^N\}\in[S_N^0]^\ell$ is the WG-FEM solution given by (3.4) on the uniform Shishkin mesh for the problem (1.1), respectively. If $\sigma\ge k+1$, then we have the following improved balanced error estimate*

$$\|\boldsymbol{u}-\boldsymbol{u}_N\|_b\le CN^{-k}(\ln N)^{k+1/2},$$

where C is independent of N and $\varepsilon_i,\ i=1,\dots,\ell$.

Proof. From Lemma 6.1 and Lemma 6.4, we obtain at once

$$
\begin{aligned}
\|\mathbf{u}-Q_h\mathbf{u}\|_b^2&\le C\Big[\sum_{i=1}^{\ell}\varepsilon_i\|(u_i-Q_0u_i)'\|^2+\sum_{i=1}^{\ell}\|u_i-Q_0u_i\|^2\\
&\quad+\sum_{i=1}^{\ell}s(u_i-Q_hu_i,u_i-Q_hu_i)\Big]\\
&=\Big[\sum_{i=1}^{\ell}\varepsilon_i\|(u_i-Q_0u_i)'\|^2+\sum_{i=1}^{\ell}\|u_i-Q_0u_i\|^2\\
&\quad+\sum_{i=1}^{\ell}s^b(Q_hu_i,Q_hu_i)\Big]\\
&\le C\Big[\varepsilon\varepsilon^{-1/2}N^{-2k}+\varepsilon\varepsilon^{-1}N^{-2k}\ln^{2k+1}N+\varepsilon(N^{-1}\ln N)^{2(k+1)}\\
&\quad+N^{-(2k+3)}+\varepsilon N^{-2k}(\ln N)^{2k+1}\Big]\\
&\le CN^{-2k}(\ln N)^{2k+1},
\end{aligned}
\tag{6.28}
$$

where we have used that $s^b(u_i, u_i) = 0$. Imitating the analyses in the energy norm estimates and using again Lemma 6.4, we have

$$\begin{aligned}|||\mathbf{u}_N - Q_h\mathbf{u}|||_\varepsilon^2 \leq & a(\mathbf{u}_N - Q_h\mathbf{u}, \mathbf{u}_N - Q_h\mathbf{u})\\ \leq & \sum_{i=1}^{\ell}\Big(T_1^b(u_i^N - Q_hu_i, u_i^N - Q_hu_i) + s^b(u_i^N - Q_hu_i, u_i^N - Q_hu_i)\Big)\\ \leq & \varepsilon^{1/2}N^{-k}(\ln N)^{k+1/2}|||\mathbf{u}_N - Q_h\mathbf{u}|||_\varepsilon.\end{aligned}$$

Therefore, we obtain

$$\|\mathbf{u}_N - Q_h\mathbf{u}\|_b^2 \leq C\varepsilon^{-1}|||\mathbf{u}_N - Q_h\mathbf{u}|||_\varepsilon^2 \leq CN^{-2k}(\ln N)^{2k+1}.$$

Next, using the triangle inequality and combining the above estimate with (6.28) yield

$$\|\mathbf{u} - \mathbf{u}_N\|_b \leq CN^{-k}(\ln N)^{k+1/2}.$$

The proof is now completed. □

7. Numerical experiment

We present various numerical experiments to show the performance of the WG-FEM in this section. All the integration was calculated by using 5-point Gauss-Legendre quadrature integral formula.

Example 7.1. *Consider the following coupled system of reaction-diffusion problem with constant coefficients*

$$\begin{cases} -\mathcal{E}\boldsymbol{u}'' + \boldsymbol{A}\boldsymbol{u} = \boldsymbol{g} \quad in \quad \Omega = (0,1),\\ \boldsymbol{u}(0) = \boldsymbol{0}, \quad \boldsymbol{u}(1) = \boldsymbol{0},\end{cases} \tag{7.1}$$

where $\mathcal{E} = diag(\varepsilon_1^2, \varepsilon_2^2)$ *with* $0 < \varepsilon_1 \leq \varepsilon_2 << 1$, $\quad \mathbf{g} = (g_1, g_2)^T$, $\quad \boldsymbol{A} = \begin{bmatrix} 2 & -1\\ -1 & 2\end{bmatrix}$ *and* g_1, g_2 *are chosen such that*

$$\begin{aligned} u^1(x) &= \frac{e^{-x/\varepsilon_1} + e^{-(1-x)/\varepsilon_1}}{1 + e^{-1/\varepsilon_1}} + \frac{e^{-x/\varepsilon_2} + e^{-(1-x)/\varepsilon_2}}{1 + e^{-1/\varepsilon_2}} - 2,\\ u^2(x) &= \frac{e^{-x/\varepsilon_2} + e^{-(1-x)/\varepsilon_2}}{1 + e^{-1/\varepsilon_2}} - 1,\end{aligned}$$

is the exact solution $\boldsymbol{u}(x) = (u^1(x), u^2(x))$ *of the system of reaction-diffusion problem (7.1). Note that* $R_i(x), i = 1, 2$ *is constant and (2.8) holds. We know that the solution has exponential layers of width* $O(\varepsilon_2|\ln\varepsilon_2|)$ *at* $x = 0$ *and* $x = 1$*, while only* $u^1(x)$ *has an additional sublayer of width* $O(\varepsilon_1|\ln\varepsilon_1|)$*. We take* $\rho > 1/2$*,* $\alpha = 0.99$ *and* $\sigma = 3$ *for this problem.*

We applied the WG-FEM (3.4) for solving the problem (7.1). The numerical errors $\mathbf{e} := \mathbf{u} - \mathbf{u}_N$ are computed in the energy norm by

$$\mathbf{e}_{\varepsilon_1,\varepsilon_2}^N = |||\mathbf{e}|||_\varepsilon^2 = \sum_{i=1}^{2}\varepsilon_i^2\|d_w e_i^N\|^2 + \eta\sum_{i=1}^{2}\|e_{i0}\|^2 + \sum_{i=1}^{2}s(e_i^N, e_i^N),$$

for a fixed $\varepsilon_1, \varepsilon_2$ and N. We report the numerical experiments for the uniform error calculated by

$$\mathbf{e}^N = \max_{\varepsilon_1,\varepsilon_2=1,10^{-1},\dots,10^{-10}} \mathbf{e}^N_{\varepsilon_1,\varepsilon_2}$$

in Table 1. The order of convergence r_ε is computed using mesh levels $(N_1, |||\mathbf{e}^{N_1}|||_\varepsilon)$ and $(N_2, |||\mathbf{e}^{N_2}|||_\varepsilon)$:

$$r_\varepsilon = \frac{\ln(|||\mathbf{e}^{N_1}|||_\varepsilon / |||\mathbf{e}^{N_2}|||_\varepsilon)}{\ln(N_1^{-1}\ln N_1) - \ln(N_2^{-1}\ln N_2)}. \tag{7.2}$$

Table 1 shows that the energy norm error estimates exhibit k-order convergence which agrees perfectly with the theoretical error estimates.

Table 1. History of convergence of the WG-FEM in the energy norm $|||\cdot|||_\varepsilon$ for Example 7.1.

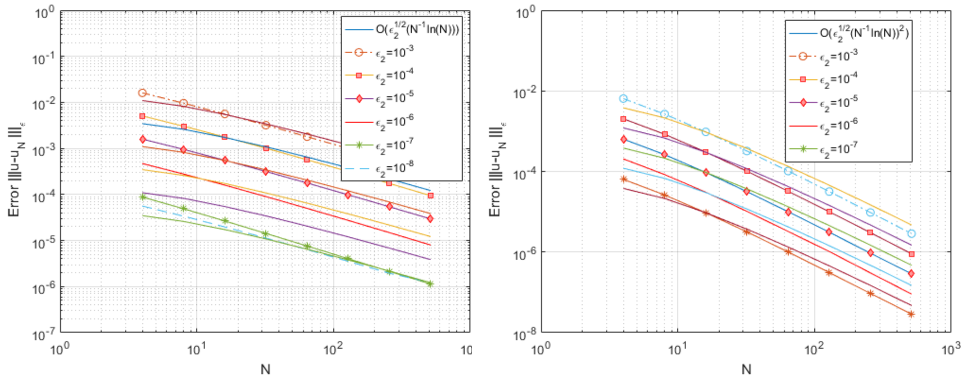
N	$k=1$		$k=2$	
	$\mathbf{e}^N$	r_ε	$\mathbf{e}^N$	r_ε
6	1.1284e-01	-	4.2924e-02	-
12	5.6774e-02	1.46	2.1549e-02	1.46
24	2.8440e-02	1.35	9.0168e-03	1.70
48	1.4228e-02	1.28	3.2876e-03	1.87
96	7.1152e-03	1.23	1.1018e-03	1.95
192	3.5577e-03	1.20	3.5170e-04	1.98
384	1.7888e-03	1.17	1.0885e-04	1.99
768	9.4867e-04	1.10	3.4639e-05	1.99

In order to pay attention to the dependency of the energy norm on the parameters, we compute the energy norm estimates for a fixed ε_1 and different values of ε_2. For instance, we first fixed $\varepsilon_1 = 10^{-10}$ and take different values of $\varepsilon_2 = 10^{-4}, \dots, 10^{-9}$. The results are presented in Table 2, Table 4, Figures 2a and 2b. These results verify that the method is robust on the uniform Shishkin mesh and the order of convergence is of $O(\varepsilon^{1/2}(N^{-1}\ln N)^k)$, where $\varepsilon^{1/2} = \varepsilon_1^{1/2} + \varepsilon_2^{1/2}$ using the linear $k = 1$ and quadratic $k = 2$ element functions, which is in excellent agreement with the main result of Theorem 5.9. Moreover, we infer from Table 2 that $\frac{|||u-u_N|||_{\varepsilon_j}}{|||u-u_N|||_{\varepsilon_{j+2}}} \approx \sqrt{\frac{10^{-j}}{10^{-(j+2)}}}$ for $\varepsilon_j = \{10^{-10}, 10^{-j}\}, \quad j = 4, \dots, 9$, where $|||u-u_N|||^2_{\varepsilon_j} = (10^{-10})^2\|d_w e_1^N\|^2 + (10^{-j})^2\|d_w e_2^N\|^2 + \eta\sum_{i=1}^2 \|e_{i0}\|^2 + \sum_{i=1}^2 s(e_i^N, e_i^N)$.

This implies that the errors might be affected by a term involving $\sqrt{\varepsilon_2}$. We observe almost linear convergence up to a logarithmic factor using the linear elements and almost quadratic convergence using the quadratic elements if N gets larger. Hence, for larger $N \geq 64$, the rate of convergence is of order $O(\varepsilon_2^{1/2}(N^{-1}\ln N)^k)$ which agrees with the theory indicated by Theorem 5.9. Table 2 shows that the errors and the order of convergence are dominated by the term $N^{-(k+1)}$ when N and ε are smaller. We also observe that if ε_2 decreases for a fixed ε_1, the energy norm error estimates get smaller. These observations suggest that the main result of Theorem 5.9 is sharp.

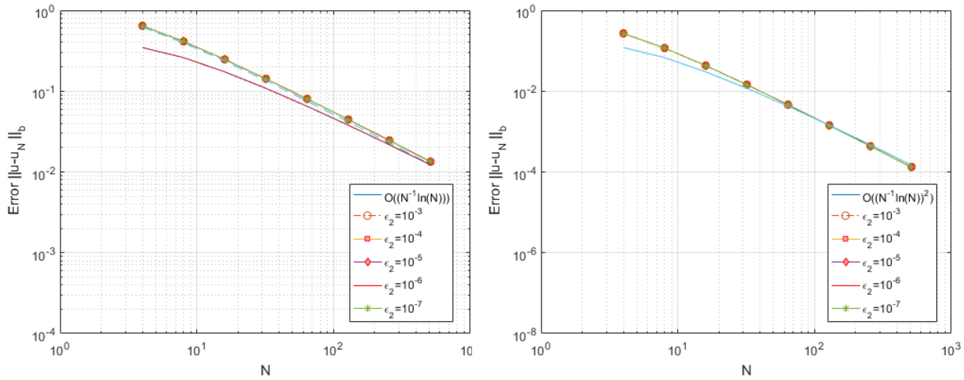
Table 2. Energy norm error estimates and order of convergence with $\varepsilon_1 = 10^{-10}, \varepsilon_2 = 10^{-4}, \ldots, 10^{-9}, k = 1, 2$, for Example 7.1.

$\varepsilon_2/k = 1$	N							
	6	12	24	48	96	192	384	768
10^{-4}	5.2495e-03	3.1587e-03	1.8429e-03	1.0531e-03	5.9237e-04	3.2910e-04	1.8100e-04	9.8729e-05
$r_{10^{-4}}$	-	0.97	0.99	1.00	1.00	1.00	1.00	1.00
10^{-5}	1.6076e-03	9.6196e-04	5.5992e-04	3.1967e-04	1.7976e-04	9.9856e-05	5.4919e-05	2.9956e-05
$r_{10^{-5}}$	-	0.99	1.01	1.00	1.00	1.00	1.00	1.00
10^{-6}	5.0801e-04	3.0395e-04	1.7690e-04	1.0098e-04	5.6778e-05	3.1536e-05	1.7342e-05	9.4732e-06
$r_{10^{-6}}$	-	0.99	1.01	1.00	1.00	1.00	1.00	1.00
10^{-7}	1.5949e-04	9.5331e-05	5.5419e-05	3.1598e-05	1.7744e-05	9.8436e-06	5.4068e-06	2.9644e-06
$r_{10^{-7}}$	-	0.99	1.01	1.01	1.00	1.00	1.00	0.99
10^{-8}	4.6992e-05	2.7802e-05	1.5980e-05	9.0030e-06	4.9943e-06	2.7368e-06	1.4915e-06	9.0148e-07
$r_{10^{-8}}$	-	1.01	1.03	1.03	1.03	1.02	1.02	0.90
10^{-9}	1.5921e-05	9.5379e-06	5.5415e-06	3.1571e-06	1.7716e-06	9.8469e-07	5.4071e-07	2.9678e-07
$r_{10^{-9}}$	-	0.99	1.01	1.01	1.00	1.00	1.00	0.99
$\varepsilon_2/k = 2$								
10^{-4}	2.0323e-03	8.3959e-04	3.0403e-04	1.0161e-04	3.2400e-05	1.0025e-05	3.0394e-06	9.8278e-07
$r_{10^{-4}}$	-	1.73	1.88	1.96	1.99	2.00	2.00	1.99
10^{-5}	6.4261e-04	2.6547e-04	9.6128e-05	3.2126e-05	1.0244e-05	3.1701e-06	9.7357e-07	3.0816e-07
$r_{10^{-5}}$	-	1.73	1.88	1.96	1.99	2.00	2.00	1.99
10^{-6}	2.0300e-04	8.3843e-05	3.0354e-05	1.0142e-05	3.2335e-06	1.0005e-06	3.2525e-07	1.0300e-07
$r_{10^{-6}}$	-	1.73	1.88	1.96	1.99	2.00	2.00	1.99
10^{-7}	6.3520e-05	2.6183e-05	9.4607e-06	3.1550e-06	1.0041e-06	3.1010e-07	9.6874e-07	3.0022e-07
$r_{10^{-7}}$	-	1.73	1.88	1.96	1.99	2.00	2.00	1.99
10^{-8}	1.8097e-05	7.3152e-06	2.5923e-06	8.4824e-07	2.6607e-07	9.0074e-08	2.9335e-08	9.2754e-09
$r_{10^{-8}}$	-	1.77	1.92	2.00	2.02	2.00	2.00	2.00
10^{-9}	2.7387e-06	1.0380e-06	3.5300e-07	1.1388e-07	3.8601e-08	1.2545e-08	4.0893e-09	1.2875e-09
$r_{10^{-9}}$	-	1.74	1.90	2.00	2.02	2.00	2.00	2.00



(a) Energy norm using $\mathbb{P}_1$ element with $\varepsilon_1 = 10^{-8}, \varepsilon_2 = 10^{-3}, \ldots, 10^{-8}$.

(b) Energy norm using $\mathbb{P}_2$ element with $\varepsilon_1 = 10^{-8}, \varepsilon_2 = 10^{-8}, 10^{-3}, \ldots, 10^{-8}$.



(c) Balanced norm using $\mathbb{P}_1$ element with $\varepsilon_1 = 10^{-12}, \varepsilon_2 = 10^{-12}, 10^{-3}, \ldots, 10^{-8}$.

(d) Balanced norm using $\mathbb{P}_2$ element with $\varepsilon_1 = 10^{-8}, \varepsilon_2 = 10^{-12}, 10^{-3}, \ldots, 10^{-8}$.

Figure 2. Convergence curve of the errors in the energy norm for Example 7.1 with fixed $\varepsilon_1 = 10^{-8}$ and varying $\varepsilon_2 = 10^{-3}, \ldots, 10^{-8}$ using linear elements in Figure 2a and quadratic elements in Figure 2b. Figures 2c and 2d depict the error curves in the balanced norm with fixed $\varepsilon_1 = 10^{-12}$ and varying $\varepsilon_2 = 10^{-3}, \ldots, 10^{-8}$ using linear elements in and quadratic elements, respectively.

On the other hand, we compute the numerical errors $\mathbf{e} := \mathbf{u} - \mathbf{u}_N$ with respect to the balanced norm by

$$\mathbf{e}^{N,b}_{\varepsilon_1,\varepsilon_2} = \|\mathbf{e}\|_b = \sum_{i=1}^{2} \varepsilon_i \|d_w e_i^N\|^2 + \eta \sum_{i=1}^{2} \|e_{i0}\|^2 + \sum_{i=1}^{2} s(e_i^N, e_i^N),$$

for a fixed $\varepsilon_1, \varepsilon_2$ and N. We list the uniform balanced error bounds $\mathbf{e}^{N,b}$ calculated as before in Table 3. We also report the numerical results in the balanced norm in Table 4 and we notice that the error estimates in the balanced norm remain almost unchanged as ε_2 decreases for a fixed ε_1 unlike the estimates in the energy norm. This confirms the theory stated in Theorem 6.5. We have plotted the balanced norm error estimates for a fixed ε and varying ε_2 on log-log scale in Figures 2c and 2d for a viewable illustrations. Evidently, the errors stay almost constant while the parameters vary and behave like

$$\|\mathbf{u} - \mathbf{u}_N\|_b \le C(N^{-1} \ln N)^k.$$

This confirms the result of Theorem 6.5 up to a root of $\ln N$.

Table 3. History of convergence of the WG-FEM in the balanced norm $\|\cdot\|_b$ norm for Example 7.1.

N	$k=1$		$k=2$	
	$\mathbf{e}^{N,b}$	r_ε	$\mathbf{e}^{N,b}$	r_ε
6	6.4414e-01	-	2.6892e-01	-
12	4.1317e-01	0.87	1.1638e-01	1.64
24	2.4710e-01	0.95	4.2967e-02	1.85
48	1.4238e-01	0.99	1.4454e-02	1.95
96	8.0297e-02	1.00	4.6177e-03	1.98
192	4.4646e-02	1.00	1.4293e-03	2.00
384	2.4561e-02	1.00	4.4355e-04	1.96
768	1.3400e-02	1.00	1.4159e-04	1.98

Table 4. Balanced norm error estimates and order of convergence with $\varepsilon_1 = 10^{-10}, \varepsilon_2 = 10^{-4}, \ldots, 10^{-9}, k = 1, 2$, for Example 7.1.

$\varepsilon_2/k=1$	N							
	6	12	24	48	96	192	384	768
10^{-4}	6.4390e-01	4.1311e-01	2.4709e-01	1.4237e-01	8.0296e-02	4.4645e-02	2.4561e-02	1.3398e-02
$r_{10^{-4}}$	-	0.87	0.95	0.99	1.00	1.00	1.00	1.00
10^{-5}	6.4387e-01	4.1309e-01	2.4707e-01	1.4236e-01	8.0291e-02	4.4643e-02	2.4559e-02	1.3397e-02
$r_{10^{-5}}$	-	0.87	0.95	0.99	1.00	1.00	1.00	1.00
10^{-6}	6.4366e-01	4.1292e-011	2.4696e-01	1.4229e-01	8.0244e-02	4.4613e-02	2.4542e-02	1.3387e-02
$r_{10^{-6}}$	-	0.87	0.95	0.99	1.00	1.00	1.00	1.00
10^{-7}	6.4316e-01	4.1272e-01	2.4665e-01	1.4251e-01	8.0228e-02	4.4608e-02	2.4536e-02	1.3376e-02
$r_{10^{-7}}$	-	0.87	0.95	0.99	1.00	1.00	1.00	1.00
10^{-8}	6.4319e-01	4.1270e-01	2.4566e-01	1.4251e-01	8.0225e-02	4.4607e-02	2.4528e-02	1.3373e-02
$r_{10^{-8}}$	-	0.87	0.95	0.99	1.00	1.00	1.00	1.00
10^{-9}	6.4317e-01	4.1267e-01	2.4565e-01	1.4247e-01	8.0217e-02	4.4603e-02	2.4524e-02	1.3370e-02
$r_{10^{-9}}$	-	0.87	0.95	0.99	1.00	1.00	1.00	1.00
$\varepsilon_2/k=2$								
10^{-4}	2.6883e-01	1.1637e-01	4.2964e-02	1.4453e-02	4.6176e-03	1.4294e-03	4.3370e-04	1.3180e-04
$r_{10^{-4}}$	-	1.64	1.85	1.95	1.98	1.99	1.99	1.97
10^{-5}	2.6881e-01	1.1636e-01	4.2961e-02	1.4452e-02	4.6172e-03	1.4293e-03	4.3366e-04	1.3175e-04
$r_{10^{-4}}$	-	1.64	1.85	1.95	1.98	1.99	1.99	1.97
10^{-6}	2.6880e-01	1.1634e-01	4.2960e-02	1.4449e-02	4.6170e-03	1.4292e-03	4.3365e-04	1.3173e-04
$r_{10^{-6}}$	-	1.64	1.85	1.95	1.98	1.99	1.99	1.97
10^{-7}	2.6878e-01	1.1632e-01	4.2961e-02	1.4449e-02	4.6168e-03	1.4290e-03	4.3362e-04	1.3171e-04
$r_{10^{-7}}$	-	1.64	1.85	1.95	1.98	1.99	1.99	1.97
10^{-8}	2.6879e-01	1.1631e-01	4.2960e-02	1.4450e-02	4.6167e-03	1.4287e-03	4.3360e-04	1.3170e-04
	-	1.64	1.85	1.95	1.98	1.99	1.99	1.97
10^{-9}	2.6877e-01	1.1630e-01	4.2958e-02	1.4448e-02	4.6166e-03	1.4288e-03	4.3359e-04	1.3170e-04
$r_{10^{-9}}$	-	1.64	1.85	1.95	1.98	1.99	1.99	1.97

Example 7.2. *We next consider the problem (1.1) with variable coefficients*

$$\boldsymbol{A} = \begin{pmatrix} 3 & 1-x & x-1 \\ 2 & 4+x & -1 \\ 2 & 0 & 3 \end{pmatrix} \text{ and } \boldsymbol{g} = \begin{pmatrix} 1 \\ x \\ 1+x^2 \end{pmatrix}.$$

We take $\rho = 3/4$, $\alpha = 0.80$ and $\sigma = 3$. The exact solution is unknown. Hence a finer mesh constructed as below is used for estimating the numerical errors.

We compute the errors $\mathbf{e} = \mathbf{u}_N - \mathbf{u}_{2N}$ where $\mathbf{u}_{2N}$ is our numerical solution computed on a mesh consisting of the initial uniform Shishkin mesh and the midpoints $x_{n+1/2} = \frac{x_n + x_{n+1}}{2}, n = 0, \ldots, N-1$. Therefore we calculate

$$\mathbf{e}^N_{\varepsilon_1,\varepsilon_2,\varepsilon_3} = |||\mathbf{e}|||_\varepsilon = \sum_{i=1}^{3} \varepsilon_i^2 \|d_w e_i^N\|^2 + \eta \sum_{i=1}^{2} \|e_{i0}\|^2 + \sum_{i=1}^{2} s(e_i^N, e_i^N),$$

for a fixed $\varepsilon_1, \varepsilon_2, \varepsilon_3$ and N. The numerical results are listed in Tables 5 and 6 for the uniform errors in the energy and balanced norms, respectively

$$\mathbf{e}^N = \max_{\varepsilon_1,\varepsilon_2,\varepsilon_3 = 1, 10^{-1}, \ldots, 10^{-10}} \mathbf{e}^N_{\varepsilon_1,\varepsilon_2,\varepsilon_3},$$
$$\mathbf{e}^{N,b} = \max_{\varepsilon_1,\varepsilon_2,\varepsilon_3 = 1, 10^{-1}, \ldots, 10^{-10}} \mathbf{e}^{N,b}_{\varepsilon_1,\varepsilon_2,\varepsilon_3},$$

where $\mathbf{e}^{N,b}_{\varepsilon_1,\varepsilon_2,\varepsilon_3}$ is defined as before and the order for convergence is calculated by (7.2). The results clearly suggest that the k-order uniform convergence in the energy norm, which is in good agreement with the main result of Theorem 5.9. The errors in the balanced norm behave like $O(N^{-1} \ln N)^k$ which agrees with the results of Theorem 6.5 up to a square root of $\ln N$. As before, we observe from Table 7 and Figures 3a and 3b that the energy norm estimates depend on $\varepsilon^{1/2} = \varepsilon_1^{1/2} + \varepsilon_2^{1/2} + \varepsilon_3^{1/2}$ and errors change decreasingly as $\varepsilon \to 0$ while the balanced norm estimates do not depend on the parameters and the errors remain almost unchanged as seen from Table 8 and Figures 3c and 3d.

Table 5. History of convergence of the WG-FEM in the energy norm $|||\cdot|||_\varepsilon$ for Example 7.2.

N	$k=1$		$k=2$	
	$\mathbf{e}^N$	r_ε	$\mathbf{e}^N$	r_ε
16	4.1486e-01	-	2.6801e-01	-
32	2.9723e-01	0.71	1.6172e-01	1.07
64	1.9341e-01	0.84	7.8526e-02	1.41
128	1.1715e-01	0.93	3.1321e-02	1.71
256	6.7975e-02	0.97	1.0951e-02	1.88
512	3.8480e-02	0.99	3.5564e-03	1.95
1024	2.1446e-02	0.99	1.1086e-03	1.98

Table 6. History of convergence of the WG-FEM in the balanced norm $\|\cdot\|_b$ for Example 7.2.

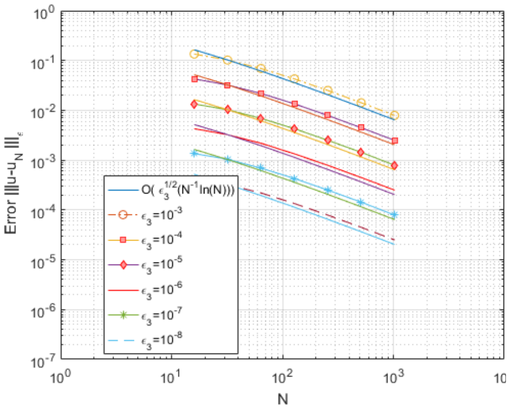
N	$k=1$		$k=2$	
	$\mathbf{e}^{N,b}$	r_ε	$\mathbf{e}^{N,b}$	r_ε
16	3.1691e00	-	2.5247e00	-
32	2.8939e00	0.19	1.8568e00	0.65
64	2.1681e00	0.57	1.0458e00	1.12
128	1.4063e00	0.80	4.6494e-01	1.50
256	8.3916e-01	0.92	1.7412e-01	1.76
512	4.7971e-01	0.97	5.8437e-02	1.90
1024	2.6815e-01	0.99	1.8476e-02	2.00

Table 7. Energy norm error estimates and order of convergence with $\varepsilon_1 = 10^{-10}, \varepsilon_2 = 10^{-3}, \ldots, 10^{-8}, k = 1, 2$, for Example 7.2.

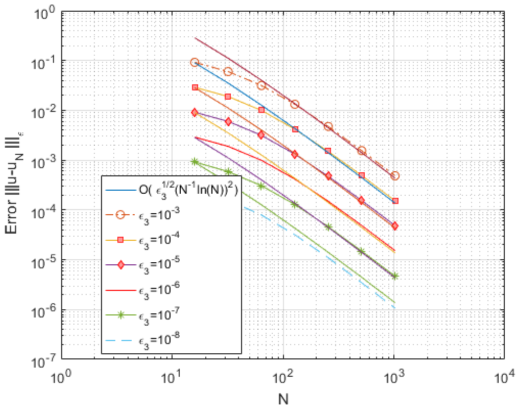
$\varepsilon_2/k=1$	N						
	16	32	64	128	256	512	1024
10^{-3}	1.3469e-01	1.0211e-01	6.9269e-02	4.2854e-02	2.5059e-02	1.4210e-02	7.9163e-03
$r_{10^{-3}}$	-	0.59	0.76	0.89	0.96	0.99	1.00
10^{-4}	4.2826e-02	3.2265e-02	2.1878e-02	1.3535e-02	7.9145e-03	4.4877e-03	2.4997e-03
$r_{10^{-4}}$	-	0.60	0.76	0.89	0.96	0.99	1.00
10^{-5}	1.4555e-02	1.0287e-02	6.9246e-03	4.2798e-03	2.5021e-03	1.4187e-03	7.9017e-04
$r_{10^{-5}}$	-	0.74	0.77	0.89	0.96	0.99	1.00
10^{-6}	7.0511e-03	3.5121e-03	2.2119e-03	1.3538e-03	7.9006e-04	4.4773e-04	2.4931e-04
$r_{10^{-6}}$	-	1.48	0.91	0.91	0.96	0.99	1.00
10^{-7}	5.7885e-03	1.7282e-03	7.6617e-04	4.2944e-04	2.4621e-04	1.3883e-04	7.7093e-05
$r_{10^{-7}}$	-	2.57	1.59	1.07	0.99	1.00	1.00
10^{-8}	5.6470e-03	1.4335e-03	3.9795e-04	1.4327e-04	6.8296e-05	3.6229e-05	1.9525e-05
$r_{10^{-8}}$	-	2.91	2.50	1.90	1.32	1.10	1.02
$\varepsilon_2/k=2$							
10^{-3}	9.1950e-02	6.0047e-02	3.1457e-02	1.3221e-02	4.7423e-03	1.5542e-03	4.8544e-04
$r_{10^{-3}}$	-	0.91	1.27	1.61	1.83	1.94	1.98
10^{-4}	2.9024e-02	1.8958e-02	9.9341e-03	4.1758e-03	1.4979e-03	4.9086e-04	1.5329e-04
$r_{10^{-4}}$	-	0.91	1.27	1.61	1.83	1.94	1.98
10^{-5}	9.1767e-03	5.9932e-03	3.1404e-03	1.3200e-03	4.7348e-04	1.5515e-04	4.8452e-05
$r_{10^{-5}}$	-	0.91	1.27	1.61	1.83	1.94	1.98
10^{-6}	2.9026e-03	1.8921e-03	9.9100e-04	4.1640e-04	1.4930e-04	4.8908e-05	1.5269e-05
$r_{10^{-6}}$	-	0.91	1.27	1.61	1.83	1.94	1.98
10^{-7}	9.2029e-04	5.8861e-04	3.0694e-04	1.2848e-04	4.5896e-05	1.4982e-05	4.6637e-06
$r_{10^{-7}}$	-	0.95	1.27	1.62	1.84	1.95	1.99
10^{-8}	3.0336e-04	1.5840e-04	7.8929e-05	3.1821e-05	1.0972e-05	3.4626e-06	1.0673e-06
$r_{10^{-8}}$	-	1.38	1.37	1.69	1.90	2.00	2.00

Table 8. Balanced norm error estimates and order of convergence with $\varepsilon_1 = 10^{-10}, \varepsilon_2 = 10^{-3}, \ldots, 10^{-8}, k = 1, 2$, for Example 7.2.

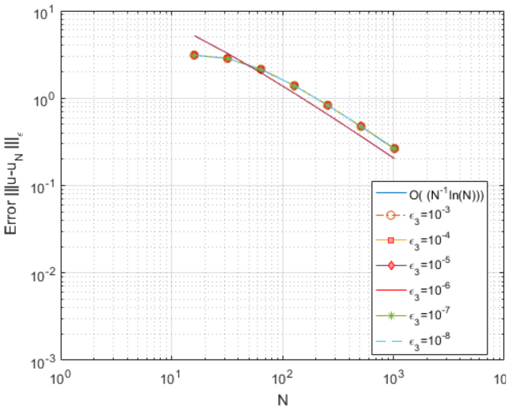
$\varepsilon_2/k = 1$	N						
	16	32	64	128	256	512	1024
10^{-3}	3.1062e00	2.8495e00	2.1418e00	1.3911e00	8.3003e-01	4.7414e-01	2.6477e-01
$r_{10^{-3}}$	-	0.18	0.56	0.80	0.92	0.97	0.99
10^{-4}	3.0996e00	2.8450e00	2.1391e00	1.3895e00	8.2909e-01	4.7356e-01	2.6442e-01
$r_{10^{-4}}$	-	0.18	0.56	0.80	0.92	0.97	0.99
10^{-5}	3.0987e00	2.8445e00	2.1390e00	1.3892e00	8.2907e-01	4.7348e-01	2.6440e-01
$r_{10^{-5}}$	-	0.18	0.56	0.80	0.92	0.97	0.99
10^{-6}	3.0980e00	2.8442e00	2.1387e00	1.3891e00	8.2903e-01	4.7342e-01	2.6436e-01
$r_{10^{-6}}$	-	0.18	0.56	0.80	0.92	0.97	0.99
10^{-7}	3.0978e00	2.8440e00	2.1384e00	1.3890e00	8.2901e-01	4.7340e-01	2.6433e-01
$r_{10^{-7}}$	-	0.18	0.56	0.80	0.92	0.97	0.99
10^{-8}	3.0975e00	2.8438e00	2.1382e00	1.3888e00	8.2888e-01	4.7338e-01	2.6430e-01
$r_{10^{-8}}$	-	0.18	0.56	0.80	0.92	0.97	0.99
$\varepsilon_2/k = 2$							
10^{-3}	2.4792e00	1.8292e00	1.0335e00	4.6054e-01	1.7262e-01	5.7921e-02	1.8266e-02
$r_{10^{-3}}$	-	0.65	1.12	1.50	1.75	1.90	1.96
10^{-4}	2.4790e00	1.8291e00	1.0333e00	4.6052e-01	1.7261e-01	5.7920e-02	1.8263e-02
$r_{10^{-4}}$	-	0.65	1.12	1.50	1.75	1.90	1.96
10^{-5}	2.4785e00	1.8288e00	1.0330e00	4.6050e-01	1.7258e-01	5.7917e-02	1.8260e-02
$r_{10^{-5}}$	-	0.65	1.12	1.50	1.75	1.90	1.96
10^{-6}	2.4776e00	1.8282e00	1.0325e00	4.6047e-01	1.7255e-01	5.7913e-02	1.8256e-02
$r_{10^{-6}}$	-	0.65	1.12	1.50	1.75	1.90	1.96
10^{-7}	2.4765e00	1.8277e00	1.0322e00	4.6043e-01	1.7252e-01	5.7910e-02	1.8252e-02
$r_{10^{-7}}$	-	0.65	1.12	1.50	1.75	1.90	1.96
10^{-8}	2.4754e00	1.8270e00	1.0315e00	4.6036e-01	1.7247e-01	5.7906e-02	1.8245e-02
$r_{10^{-8}}$	-	0.65	1.12	1.50	1.75	1.90	1.96



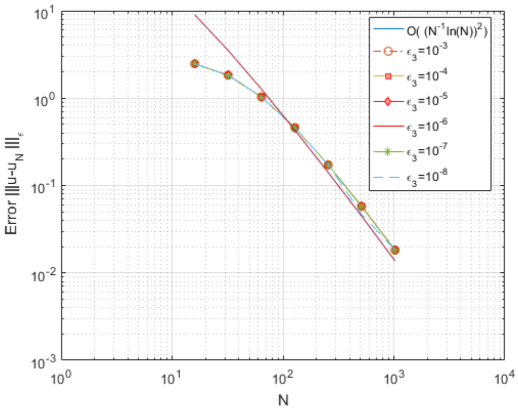
(a) Energy norm using $\mathbb{P}_1$ element with $\varepsilon_1 = 10^{-8}, \varepsilon_2 = 10^{-3}, \ldots, 10^{-8}$.



(b) Energy norm using $\mathbb{P}_2$ element with $\varepsilon_1 = 10^{-8}, \varepsilon_2 = 10^{-8}, 10^{-3}, \ldots, 10^{-8}$.



(c) Balanced norm using $\mathbb{P}_1$ element with $\varepsilon_1 = 10^{-12}, \varepsilon_2 = 10^{-12}, 10^{-3}, \ldots, 10^{-8}$.



(d) Balanced norm using $\mathbb{P}_2$ element with $\varepsilon_1 = 10^{-8}, \varepsilon_2 = 10^{-12}, 10^{-3}, \ldots, 10^{-8}$.

Figure 3. Convergence curve of the errors in the energy norm for Example 7.2 with fixed $\varepsilon_1 = 10^{-10}, \varepsilon_2 = 10^{-8}$ and varying $\varepsilon_3 = 10^{-3}, \ldots, 10^{-8}$ using linear elements in Figure 2a and quadratic elements in Figure 2b. Figures 2c and 2d depict the error curves in the balanced norm with fixed $\varepsilon_1 = 10^{-10}, \varepsilon_2 = 10^{-8}$ and varying $\varepsilon_3 = 10^{-3}, \ldots, 10^{-8}$ using linear elements in and quadratic elements, respectively.

8. Conclusions

In this paper, we studied the WG-FEM for system of SPPs of reaction-diffusion type in which the equations have diffusion parameters of the different magnitudes on a piecewise uniform Shishkin mesh. With the help of a special interpolation operator, we derived optimal and uniform error bounds in the energy and the balanced norms up to a logarithmic factor. The proposed WG-FEM uses the procedure of elimination of the interior unknowns from the discrete linear system and thus the method is comparable with the classical FEM. We will investigate sharper error bounds in balanced norm and extend these results to to high dimensional problem on a tensor product meshes in the future work.

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Conflict of interest

The authors declare that they have no conflict of interest.

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