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Research article

Bi-Lie n-derivations on triangular rings

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Abstract: The purpose of this article is to prove that every bi-Lie n-derivation of certain triangular rings is the sum of an inner biderivation, an extremal biderivation and an additive central mapping vanishing at $(n-1)^{th}$-commutators for both components, using the notion of maximal left ring of quotients. As a consequence, we characterize the decomposition structure of bi-Lie n-derivations on upper triangular matrix rings.

Keywords: triangular rings; upper triangular matrix rings; extremal biderivation; maximal left ring of quotients; bi-Lie n-derivation
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1. Introduction

Let $\mathcal{A}$ be an associative unital ring and let $\mathcal{Z}(\mathcal{A})$ be the center of R. A ring $\mathcal{A}$ is said to be n-torsion free if $nx = 0$ implies $x = 0$ for some positive integer $n \in \mathfrak{N}$ and arbitrary $x \in \mathcal{A}$. A biadditive mapping $\varphi : \mathcal{A}\times\mathcal{A} \to \mathcal{A}$ is a (Lie) biderivation if it is a (Lie) derivation with respect to both components, that is

$$\varphi(xz, y) = \varphi(x, y)z + x\varphi(z, y) \text{ and } \varphi(x, yz) = \varphi(x, y)z + y\varphi(x, z),$$
$$\varphi([x, z], y) = [\varphi(x, y), z] + [x, \varphi(z, y)] \text{ and } \varphi(x, [y, z]) = [\varphi(x, y), z] + [y, \varphi(x, z)],$$

for all $x, y, z \in \mathcal{A}$. If the algebra $\mathcal{A}$ is noncommutative, then the mapping $\varphi(x, y) = \lambda[x, y]$ for all $x, y \in \mathcal{A}$ and some $\lambda \in \mathcal{Z}(\mathcal{A})$ is called an inner biderivation. A biadditive mapping $\varphi : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is said to be an extremal biderivation if it is of the form $\varphi(x, y) = [x, [y, a]]$ for all $x, y \in \mathcal{A}$ and some $a \notin \mathcal{Z}(\mathcal{A})$ such that $[[\mathcal{A}, \mathcal{A}], a] = 0$. The biderivations play a significant role in the theory of functional identities of algebras or rings, which was first introduced by Maksa [1, 2]. After that, the structures of biderivations or Lie biderivations on various algebras or rings were studied by authors, forming a series of interesting and systematic results, see [3–6]. The kind of problem belongs to the extension of

Herstein's Lie-type mapping research program proposed by Herstein in the AMS Hour Talk of 1961, see [7].

The objective of this paper is to investigate bi-Lie n-derivations on triangular rings $\mathcal{T}$. For this purpose, we recall some necessary basic concepts. Assuming that $n \geq 2$ is a positive integer, we introduce a series of multi-variable polynomials in the following way:

$$\begin{aligned} P_1(x_1) &= x_1, \\ P_2(x_1, x_2) &= [x_1, x_2], \\ &\cdots\cdots \\ P_n(x_1, x_2, \cdots, x_{n-1}, x_n) &= [P_{n-1}(x_1, x_2, \cdots, x_{n-1}), x_n], \end{aligned}$$

where the symbol $[x, y]$ is equal to $xy - yx$ for all $x, y \in \mathcal{A}$ and $P_n(x_1, x_2, \cdots, x_{n-1}, x_n)$ is called $(n-1)^{th}$-commutator.

An additive mapping $L : \mathcal{T} \to \mathcal{T}$ is called a Lie n-derivation if

$$L(P_n(x_1, x_2, \cdots, x_{n-1}, x_n)) = \sum_{k=1}^{n} P_n(x_1, x_2, \cdots, L(x_k), \cdots, x_{n-1}, x_n)$$

for all $x_1, x_2, \cdots, x_{n-1}, x_n \in \mathcal{T}$. On this basis, we introduce the definition of bi-Lie n-derivation. A bi-additive mapping $\varphi : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ is called a bi-Lie n-derivation if it is a Lie n-derivation with respect to both components, that is,

$$\varphi(P_n(x_1, x_2, \cdots, x_{n-1}, x_n), y) = \sum_{k=1}^{n} P_n(x_1, x_2, \cdots, \varphi(x_k, y), \cdots, x_{n-1}, x_n),$$
$$\text{and } \varphi(y, P_n(x_1, x_2, \cdots, x_{n-1}, x_n)) = \sum_{k=1}^{n} P_n(x_1, x_2, \cdots, \varphi(y, x_k), \cdots, x_{n-1}, x_n),$$

for all $x_1, x_2, \cdots, x_{n-1}, x_n, y \in \mathcal{T}$. In recent years, many authors have studied the structure of (Lie) biderivations on triangular rings and their related algebras, see [8–12]. The research related to the mapping of Lie (Jordan)-type biderivation on triangular algebra can be roughly divided into two directions. One is to use faithful bimodule structure and elementary methods to research, such as the biderivation studied by Benkovic [11] and Wang [13] respectively, and the Lie (Jordan) biderivation studied by the first author and his collaborators [12, 14]; the other is to use the structure of the maximal left ring of the quotient ring, which is the structure defined by Utumi in 1957 and also called an Utumi left quotient ring, such that the biderivation on the triangular ring studied by Eremita [10], and the Jordan biderivation on the triangular ring studied by Liu and his collaborators [6]. Therefore, under the Herstein's Lie-type mapping research program, a natural question is proposed: How to characterize the structural form of bi-Lie n-derivation on a triangular ring? This problem leads us to study the structure of bi-Lie n-derivations on triangular rings.

Inspired by Eremita [10] and Liu [6], we use the structure of Utumi left quotient ring to study the decomposition form of bi-Lie n-derivation over triangular rings (see Theorem 3.1) in this article. Meanwhile, we shall make use of Theorem 3.1 to upper triangular matrix rings, attaining the structures of bi-Lie n-derivations.

2. Triangular rings

This part is introduced to define triangular ring. Let $\mathcal{T}$ be a ring with unity I and idempotents e and f satisfying a relation $e + f = I$. A unital ring $\mathcal{T}$ is called a triangular ring, if $e\mathcal{T}f$ is a faithful right $e\mathcal{T}e$-module and also left $f\mathcal{T}f$-module and $f\mathcal{T}e = 0$. Therefore, the triangular ring has the following decomposition form:

$$\mathcal{T} = e\mathcal{T}e + e\mathcal{T}f + f\mathcal{T}f.$$

There are many examples of meeting the structural form of the triangular ring, such as upper triangular matrix rings and nest algebras, see [15]. The center of $\mathcal{T}$ is

$$\mathcal{Z}(\mathcal{T}) = \{a + b \in e\mathcal{T}e \oplus f\mathcal{T}f | \, aexf = exfb, \ \forall \ x \in \mathcal{T})\}.$$

In this paper, we will study the decomposition form of bi-Lie n-derivation over triangular rings by using the structure of quotient rings. Next, we will introduce some necessary knowledge points about Utumi left quotient rings. Let A be a unital ring. In 1956, the concept of the maximal left ring of quotients (also called Utumi left ring of quotients) was introduced by Utumi [16], which is recorded as $Q_{ml}(A)$. Its corresponding center is called the extended centroid of A and recorded as $C(A)$. According to [15, 17], the center $C(\mathcal{T})$ of $\mathcal{T}$ is

$$C(\mathcal{T}) = \{q = a + b \in eQ_{ml}(\mathcal{T})e \oplus fQ_{ml}(\mathcal{T})f | \, qexf = exfq, \ \forall \ x \in \mathcal{T})\}.$$

It is easy to verify that the map $\tau : C(\mathcal{T})e \to C(\mathcal{T})f$ is a ring isomorphism such that $\lambda e{\cdot}exf = exf{\cdot}\tau(\lambda e)$ for all $x \in \mathcal{T}$ and $\lambda \in C(\mathcal{T})$.

Let K, L be subsets of $Q_{ml}(\mathcal{T})$. Set

$$C(K, L) = \{q \in K | \, qx = xq, \ \forall \ x \in L)\}.$$

On behalf of [17, Proposition 2.5], we have $C(\mathcal{T}) = C(Q_{ml}(\mathcal{T}), R)$.

With the help of [3, 5, 13, 15, 17] and above notations, we have listed many important conclusions that are needed in the text. Since these conclusions have been proved, we have only listed them without giving their proofs.

Proposition 2.1. *[3, 5, 13, 15, 17] Let $\mathcal{T}$ be a unital ring. The maximal left ring of quotients $Q_{ml}(\mathcal{T})$ satisfies the following properties:*

(1) $\mathcal{T}$ is a subring of the Utumi left quotient ring $Q_{ml}(\mathcal{T})$ with the same I;
(2) For any dense left ideal $\mathcal{U}$ of $\mathcal{T}$ and a left $\mathcal{T}$-module homomorphism $\varrho : \mathcal{U} \to \mathcal{T}$, there exists $q \in Q_{ml}(\mathcal{T})$ such that ϱ be of the form $\varrho(x) = xq$ for $x \in \mathcal{U}$;
(3) $\mathcal{Z}(\mathcal{T}) \subseteq C(\mathcal{T})$. Furthermore, $\mathcal{Z}(\mathcal{T})e \subseteq C(\mathcal{T})e$ and $\mathcal{Z}(\mathcal{T})f \subseteq C(\mathcal{T})f$.

3. Bi-Lie n-derivations

This part is the mainbody of this paper. We mainly use the properties of Utumi left quotient rings to study the structure of bi-Lie n-derivations on triangular rings. This method is simple and very efficient.

Remark 3.1. *Let $\mathcal{T} = e\mathcal{T}e + e\mathcal{T}f + f\mathcal{T}f$ be a triangular ring. For any $x \in \mathcal{T}$ and for any integer $n \geq 2$, we have*

$$P_n(x, e, \cdots, e) = (-1)^{n-1}exf \text{ and}$$
$$P_n(x, f, \cdots, f) = exf.$$

In particular, $[x, e] = -exf$ and $[x, f] = exf$ for $n = 2$.

Lemma 3.1. *Let $\phi : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ be a bi-Lie n-derivation on $\mathcal{T}$, then ϕ has the following properties:*

(1) $\phi(0, x) = \phi(x, 0) = 0$ for all $x \in \mathcal{T}$;
(2) $\phi(I, x) \in C(\mathcal{T})$ and $\phi(x, I) \in C(\mathcal{T})$ for all $x \in \mathcal{T}$;
(3) $e\phi(e, e)f = -e\phi(f, e)f = -e\phi(e, f)f = e\phi(f, f)f$.

Proof. (1) Suppose that ϕ is a Lie n-derivation in first component, then for any $x \in \mathcal{T}$,

$$\phi(0, x) = \phi(P_n(I, \cdots, I), x) = \sum_{i=1}^{n} P_n(I, \cdots, I, \phi(I, x), I, \cdots, I) = 0.$$

Similarly, $\phi(x, 0) = 0$ for all $x \in \mathcal{T}$.

(2) Since ϕ is a Lie n-derivation in first component, it follows

$$\begin{aligned} 0 =& \phi(0, x) = \phi(P_n(I, y, f, \cdots, f), x) \\ =& P_n(\phi(I, x), y, f, \cdots, f) + P_n(I, \phi(y, x), f, \cdots, f) \\ &+ \sum_{i=3}^{n} P_n(I, y, f, \cdots, f, \phi(f, x), f, \cdots, f) \\ =& e[\phi(I, x), y]f \end{aligned}$$

for arbitrary $x, y \in \mathcal{T}$, we get $e[\phi(I, x), y]f = 0$.

Let's prove the conclusion: $\phi(I, x) \in \mathcal{Z}(\mathcal{T}) \subseteq C(\mathcal{T})$ for all $x \in \mathcal{T}$. Since $P_n(I, y, ezf, f, \cdots, f) = 0$, we have

$$\begin{aligned} 0 =& \phi(0, x) = \phi(P_n(I, y, ezf, f, \cdots, f), x) \\ =& P_n(\phi(I, x), y, ezf, \cdots, f) + P_n(I, \phi(y, x), ezf, \cdots, f) \\ &+ P_n(I, y, \phi(ezf, x), \cdots, f) + \sum_{i=4}^{n} P_n(I, y, f, \cdots, f, \phi(f, x), f, \cdots, f) \\ =& P_n(\phi(I, x), y, ezf, \cdots, f) \\ =& e[\phi(I, x), y]ezf - ezf[\phi(I, x), y]f, \end{aligned}$$

and then

$$e[\phi(I, x), y]ezf - ezf[\phi(I, x), y]f = 0.$$

With the help of the relation $e[\phi(I, x), y]f = 0$, we obtain

$$[\phi(I, x), y] = e[\phi(I, x), y]e + f[\phi(I, x), y]f \in \mathcal{Z}(\mathcal{T}) \subseteq C(\mathcal{T})$$

for all $x, y \in \mathcal{T}$. Since the element y is arbitrary, let $y = e$ in above formula, we have $e\phi(I, x)f = 0$; on the other hand, taking $y = ezf$ in above formula, we arrive at $e\phi(I, x)ezf - ezf\phi(I, x)f \in \mathcal{Z}(\mathcal{T})$, and then $e\phi(I, x)ezf - ezf\phi(I, x)f = 0$, we have $e\phi(I, x)e + f\phi(I, x)f \in \mathcal{Z}(\mathcal{T}) \subseteq C(\mathcal{T})$ for all $x \in \mathcal{T}$.

Likewise, we have $\phi(x, I) \in \mathcal{C}(\mathcal{T})$ for all $x \in \mathcal{T}$.

(3) According to the conclusion (2), we arrive at

$$e\phi(I, x)f = 0 = e\phi(x, I)f. \tag{3.1}$$

We assume $x = e$ and substitute the equality $e + f = I$ for I in (3.1), we may write

$$e\phi(e, e)f = -e\phi(f, e)f = -e\phi(e, f)f. \tag{3.2}$$

An argument similar to the above equations shows that

$$e\phi(f, f)f = -e\phi(f, e)f = -e\phi(e, f)f. \tag{3.3}$$

Combining (3.2) with (3.3), thus, we conclude that

$$e\phi(e, e)f = -e\phi(f, e)f = -e\phi(e, f)f = e\phi(f, f)f. \tag{3.4}$$

Theorem 3.1. *Let $\mathcal{T}$ be a $(n-1)^{th}$-torsion free triangular ring with a nontrivial idempotent e. Assume that $\phi : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ is a bi-Lie n-derivation and $\mathcal{C}(fQ_{ml}(\mathcal{T})f, f\mathcal{T}f) = \mathcal{C}(\mathcal{T})f$ and either $e\mathcal{T}\mathcal{C}(\mathcal{T})e$ or $f\mathcal{T}\mathcal{C}(\mathcal{T})f$ does not contain nonzero central ideals. Then it has the form $\phi = \kappa + \delta + \gamma$, where $\delta : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ is an inner biderivation, $\kappa : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ is an extremal biderivation and γ is a bilinear central map vanishing at $(n-1)^{th}$-commutators.*

In order to give a more concise proof, we review an interesting conclusion (see Lemma 3.2) coming from [11, Remark 4.5.]. On this basis, we transform the bi-Lie n-derivation in Theorem 3.1 into another simpler bi-Lie n-derivation, see Lemma 3.3. For this purpose, we will prove that the biadditive mapping $\kappa : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ defined in Lemma 3.3 is a bi-Lie n-derivation of triangular rings $\mathcal{T}$ for the case $m_0 = e\phi(e, e)f$ appears in Lemma 3.2.

Lemma 3.2. *A triangular algebra $\mathcal{T} = Tri(A, M, B)$ has a nonzero extremal biderivation if and only if there exists $0 \neq m_0 \in M$ such that $[A, A]m_0 = 0 = m_0[B, B]$.*

Lemma 3.3. *Let $\phi : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ be a bi-Lie n-derivation on $\mathcal{T}$. If $\phi(e, e) \neq 0$, then $\phi = \kappa + \psi$, where $\kappa(x, y) = [x, [y, \phi(e, e)]]$ is an extremal biderivation and ψ is a bi-Lie n-derivation satisfying $\psi(e, e) \in \mathcal{C}(\mathcal{T})$.*

Proof. Since ϕ is a Lie n-derivation of $\mathcal{T}$ for the second component, we arrive at

$$\begin{aligned}\phi(e, exf) &= \phi(e, P_n(exf, f, \cdots, f)) \\ &= P_n(\phi(e, exf), f, \cdots, f) + \sum_{i=2}^{n} P_n(exf, f, \cdots, f, \phi(e, f), f, \cdots, f) \\ &= e\phi(e, exf)f + (n-1)(exf\phi(e, f) - \phi(e, f)exf).\end{aligned}$$

Multiplying by e from left and f from right, and since $\mathcal{T}$ is $(n-1)^{th}$-torsion free, we get $e\phi(e, f)exf = exf\phi(e, f)f$ for all $x \in \mathcal{T}$.

Replacing element exf with equation $exf = P_n(e, exf, f, \cdots, f)$ in bi-Lie n-derivation $\phi(e, exf)$, we can get

$$\begin{aligned}\phi(e, exf) =&\phi(e, P_n(e, exf, f, \cdots, f))\\ =&P_n(\phi(e, e), exf, f, \cdots, f) + P_n(e, \phi(e, exf), f, \cdots, f)\\ &+ \sum_{i=3}^{n} P_n(e, exf, f, \cdots, f, \phi(e, f), f, \cdots, f)\\ =&\phi(e, e)exf - exf\phi(e, e) + e\phi(e, exf)f + (n-2)(exf\phi(e, f) - \phi(e, f)exf)\\ =&\phi(e, e)exf - exf\phi(e, e) + e\phi(e, exf)f.\end{aligned}$$

It follows from above equation that $e\phi(e, e)exf = exf\phi(e, e)f$ for all $x \in \mathcal{T}$. Therefore, we have $e\phi(e, e)e + f\phi(e, e)f \in C(\mathcal{T})$.

Assume that $\phi(e, e) \neq 0$ and let us prove the map $\kappa(x, y) = [x, [y, \phi(e, e)]]$ is an extremal biderivation of $\mathcal{T}$. Note that

$$\begin{aligned}\kappa(e, e) &= [e, [e, \phi(e, e)]]\\ &= [e, [e, e\phi(e, e)e + e\phi(e, e)f + f\phi(e, e)f]]\\ &= e\phi(e, e)f,\end{aligned}$$

then $\phi(e, e) - \kappa(e, e) = e\phi(e, e)e + f\phi(e, e)f \in C(\mathcal{T})$.

In the following part, we prove that the element $e\phi(e, e)f$ satisfies conditions

$$e\phi(e, e)f[b, b'] = 0 \text{ and } [a, a']e\phi(e, e)f = 0.$$

In fact, for any $a, a' \in e\mathcal{T}e$, we have

$$\begin{aligned}&\phi(P_n(e, a, f, \cdots, f), P_n(e, a', f, \cdots, f))\\ =&P_n(\phi(P_n(e, a, f, \cdots, f), e), a', f, \cdots, f) + P_n(e, \phi(P_n(e, a, f, \cdots, f), a'), f, \cdots, f)\\ =&-a'\phi(P_n(e, a, f, \cdots, f), e)f + e\phi(P_n(e, a, f, \cdots, f), a')f\\ =&a'a\phi(e, e)f - a'\phi(a, e)f - a\phi(e, a')f + e\phi(a, a')f.\end{aligned} \tag{3.5}$$

On the other hand,

$$\begin{aligned}&\phi(P_n(e, a, f, \cdots, f), P_n(e, a', f, \cdots, f))\\ =&P_n(\phi(e, P_n(e, a', f, \cdots, f)), a, f, \cdots, f) + P_n(e, \phi(a, P_n(e, a', f, \cdots, f)), f, \cdots, f)\\ =&-a\phi(e, P_n(e, a', f, \cdots, f))f + e\phi(a, P_n(e, a', f, \cdots, f))f\\ =&aa'\phi(e, e)f - a\phi(e, a')f - a'\phi(a, e)f + e\phi(a, a')f.\end{aligned} \tag{3.6}$$

Considering (3.5) and (3.6) together, we get that

$$[a, a']e\phi(e, e)f = 0.$$

Through similar calculation process, it can be obtained that

$$e\phi(e, e)f[b, b'] = 0.$$

Therefore, according to Lemma 3.2 and [11, Remark 4.4], we obtain that the biadditive mapping $\kappa(x, y) = [x, [y, \phi(e, e)]]$ is an extremal biderivation of $\mathcal{T}$ and also is a biderivation. And then $\kappa(x, y) = [x, [y, \phi(e, e)]]$ is a Lie biderivation. We know that every Lie derivation is a Lie n-derivation of triangular rings, so we know that $\kappa(x, y) = [x, [y, \phi(e, e)]]$ is a bi-Lie n-derivation of triangular rings. Set $\phi - \kappa = \psi$. it is easy to check that ψ is a bi-Lie n-derivation satisfying $\psi(e, e) \in C(\mathcal{T})$. The proof of the lemma is now complete.

Before proceeding further study, let us remake a note on the rationality of this method.

Remark 3.2. *Owing to Lemma 3.3, we may always subtract an extremal biderivation* $\kappa(x, y) = [x, [y, \phi(e, e)]]$ *from bi-Lie n-derivation* ϕ *on* $\mathcal{T}$ *in Theorem 3.1. Therefore, we consider only those bi-Lie n-derivations* ψ *which satisfies* $e\psi(e, e)f = 0$. *Further in view of Lemma 3.1, we see that*

$$e\psi(e, e)f = -e\psi(f, e)f = -e\psi(e, f)f = e\psi(f, f)f = 0.$$

Lemma 3.4. *Let* $\psi : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ *be a bi-Lie n-derivation on* $\mathcal{T}$. *Then* ψ *satisfies*

(1) $\psi(exf, y) \in e\mathcal{T}f$;
(2) $\psi(x, eyf) \in e\mathcal{T}f$

for all $x, y \in \mathcal{T}$.

Proof. For any $x, y \in \mathcal{T}$,

$$\begin{aligned}\psi(exf, y) &= \psi(P_n(exf, f, \cdots, f), y)\\ &= P_n(\psi(exf, y), f, \cdots, f) + \sum_{i=2}^{n} P_n(exf, f, \cdots, f, \psi(f, y), f, \cdots, f)\\ &= e\psi(exf, y)f + (n-1)(exf\psi(f, y) - \psi(f, y)exf).\end{aligned} \tag{3.7}$$

According to above relation (3.7), multiplying e and f on both sides of the above identity respectively, we find that $e\psi(exf, y)e = 0 = f\psi(exf, y)f$. Hence, we have $\psi(exf, y) \in e\mathcal{T}f$. Using similar methods, we can show that $\psi(x, eyf) \in e\mathcal{T}f$.

Lemma 3.5. *Let* $\psi : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ *be a bi-Lie n-derivation on* $\mathcal{T}$. *Then* ψ *satisfies*

(1) $\psi(exe, eyf) = -\psi(eyf, exe) = \lambda exeyf$;
(2) $\psi(exf, fyf) = -\psi(fyf, exf) = \lambda exfyf$

for all $x, y \in \mathcal{T}$.

Proof. For any $x, y \in \mathcal{T}$, thanks to the equation $eyf = P_n(eyf, f, \cdots, f)$, we have

$$\begin{aligned}\psi(exe, eyf) &= \psi(exe, P_n(eyf, f, \cdots, f))\\ &= P_n(\psi(exe, eyf), f, \cdots, f) + \sum_{i=2}^{n} P_n(eyf, f, \cdots, f, \psi(exe, f), f, \cdots, f)\\ &= e\psi(exe, eyf)f + (n-1)(eyf\psi(exe, f) - \psi(exe, f)eyf).\end{aligned} \tag{3.8}$$

Multiplying by e from left and f from right, and since $\mathcal{T}$ is $(n-1)^{th}$-torsion free, we obtain

$$[eyf, e\psi(exe, f)e + f\psi(exe, f)f] = 0, \tag{3.9}$$

thus we arrive that

$$\psi(exe, eyf) = e\psi(exe, eyf)f \in e\mathcal{T}f.$$

On the other hand, it follows from (3.9) that

$$\begin{aligned}\psi(exe, eyf) =&\psi(exe, P_n(e, eyf, f, \cdots, f))\\ =&P_n(\psi(exe, e), eyf, f, \cdots, f) + P_n(e, \psi(exe, eyf), f, \cdots, f)\\ &+ \sum_{i=3}^{n} P_n(e, eyf, f, \cdots, f, \psi(exe, f), f, \cdots, f)\\ =&\psi(exe, e)eyf - eyf\psi(exe, e) + e\psi(exe, eyf)f.\end{aligned}$$

It is easy to conclude $[e\psi(exe, e)e + f\psi(exe, e)f, eyf] = 0$ and $\psi(exe, eyf) = e\psi(exe, eyf)f$. Through similar calculations, we can see $\psi(eyf, exe) = e\psi(eyf, exe)f \in e\mathcal{T}f$ and $[e\psi(e, exe)e + f\psi(e, exe)f, eyf] = 0$. Define a map $\mathfrak{f} : e\mathcal{T} \to \mathcal{T}$ by $\mathfrak{f}(x) = \psi(e, exf)$ for all $x \in e\mathcal{T}$, then by Eq (3.9), we get that

$$\begin{aligned}\mathfrak{f}(rx) =&\psi(e, P_n(ere, exf, f, \cdots, f))\\ =&P_n(\psi(e, ere), exf, f, \cdots, f) + P_n(ere, \psi(e, exf), f, \cdots, f)\\ &+ \sum_{i=3}^{n} P_n(ere, exf, f, \cdots, f, \psi(e, f), f, \cdots, f)\\ =&\psi(e, ere)exf - exf\psi(e, ere) + ere\psi(e, exf)f + (n-2)(erexf\psi(e, f) - \psi(e, f)erexf)\\ =&ere\psi(e, exf) = r\mathfrak{f}(x)\end{aligned}$$

for all $x \in e\mathcal{T}$, $r \in \mathcal{T}$. This implies that $\mathfrak{f}$ is a left $\mathcal{T}$-module homomorphism. On account of conclusion (3) in Proposition 2.1, there exists $q \in \mathcal{Q}_{ml}(\mathcal{T})$ such that $\mathfrak{f}(x) = xq$ for all $x \in e\mathcal{T}$. In particular, $\mathfrak{f}(e) = eq = 0$. This implies that $q = fq$. Thus, $\mathfrak{f}(x) = xfqf$ for all $x \in e\mathcal{T}$. For any $r \in f\mathcal{T}f$, we have

$$\begin{aligned}\mathfrak{f}(xr) =&\psi(e, P_n(exf, frf, f, \cdots, f))\\ =&P_n(\psi(e, exf), frf, f, \cdots, f) + P_n(exf, \psi(e, frf), f, \cdots, f)\\ &+ \sum_{i=3}^{n} P_n(exf, frf, f, \cdots, f, \psi(e, f), f, \cdots, f)\\ =&\psi(e, exf)frf + exf\psi(e, frf)f - e\psi(e, frf)exf + (n-2)(exfrf\psi(e, f) - \psi(e, f)exfrf)\\ =&\psi(e, exf)frf = \mathfrak{f}(x)r,\end{aligned}$$

which leads to $xfrfqf = xfqfrf$ for all $x \in e\mathcal{T}$. Then we get $e\mathcal{T}(frfqf - fqfrf) = 0$ for all $r \in \mathcal{T}$. Using conclusion (3) in Proposition 2.1, we see that $frfqf = fqfrf$ for all $r \in f\mathcal{T}f$. Then $fqf \in \mathcal{C}(f\mathcal{Q}_{ml}(\mathcal{T})f, f\mathcal{T}f)$ and hence $fqf \in \mathcal{C}(\mathcal{T})f$. Setting $\lambda = \tau^{-1}(fqf)$ and owing to the conclusion (3) in Proposition 2.1, we have $\lambda exf = xfqf$ for all $x \in e\mathcal{T}$. Thus $\psi(e, exf) = \lambda exf$ for all $x \in e\mathcal{T}$. Therefore,

$$\begin{aligned}0 =& \psi(P_n(e, exe, f, \cdots, f), eyf)\\ =& P_n(\psi(e, eyf), exe, f, \cdots, f) + P_n(e, \psi(exe, eyf), f, \cdots, f)\\ =& -exe\psi(e, eyf)f + e\psi(exe, eyf)f\end{aligned}$$

for all $x, y \in \mathcal{T}$. It follows from above equation that $\psi(exe, eyf) = exe\psi(e, eyf)f = \lambda exeyf$. Similarly, there exists $\mu \in C(\mathcal{T})e$, such that $\psi(exf, e) = \mu exf$ for all $x \in \mathcal{T}$.

Next we will prove that $\psi(e, exf) = \lambda exf = -\psi(exf, e)$ for all $x \in \mathcal{T}$. It is sufficient to prove that $\lambda + \mu = 0$. For any $x, y, z \in \mathcal{T}$, we have

$$\begin{aligned}
&\psi(P_n(eze, e, f, \cdots, f), P_n(eye, exf, f, \cdots, f))\\
=&P_n(\psi(P_n(eze, e, f, \cdots, f), eye), exf, f, \cdots, f) + P_n(eye, \psi(P_n(eze, e, f, \cdots, f), exf), f, \cdots, f)\\
&+ \sum_{i=3}^{n} P_n(eye, exf, f, \cdots, f, \psi(P_n(eze, e, f, \cdots, f), f), f, \cdots, f)\\
=&\psi(P_n(eze, e, f, \cdots, f), eye)exf - exf\psi(P_n(eze, e, f, \cdots, f), eye)\\
&+ eye\psi(P_n(eze, e, f, \cdots, f), exf)f\\
&+ (n-2)(eyexf\psi(P_n(eze, e, f, \cdots, f), f) - \psi(P_n(eze, e, f, \cdots, f), f)eyexf)\\
=&eyeze\psi(e, exf)f - eye\psi(eze, exf)f.
\end{aligned} \tag{3.10}$$

On the other hand,

$$\begin{aligned}
&\psi(P_n(eze, e, f, \cdots, f), P_n(eye, exf, f, \cdots, f))\\
=&P_n(\psi(eze, P_n(eye, exf, f, \cdots, f)), e, f, \cdots, f) + P_n(eze, \psi(e, P_n(eye, exf, f, \cdots, f)), f, \cdots, f)\\
=&-e\psi(eze, P_n(eye, exf, f, \cdots, f))f + eze\psi(e, P_n(eye, exf, f, \cdots, f))f\\
=&exf\psi(eze, eye) - \psi(eze, eye)exf - eye\psi(eze, exf)f + ezeye\psi(e, exf)f.
\end{aligned} \tag{3.11}$$

Considering (3.10) and (3.11) together, we get that

$$-[eye, eze]\psi(e, exf) = [\psi(eze, eye), exf] \tag{3.12}$$

for all $x, y, z \in \mathcal{T}$. Similarly, we obtain

$$\begin{aligned}
&\psi(P_n(eze, exf, f, \cdots, f), P_n(eye, e, f, \cdots, f))\\
=&P_n(\psi(P_n(eze, exf, f, \cdots, f), eye), e, f, \cdots, f)\\
&+ P_n(eye, \psi(P_n(eze, exf, f, \cdots, f), e), f, \cdots, f)\\
=&-e\psi(P_n(eze, exf, f, \cdots, f), eye)f + eye\psi(P_n(eze, exf, f, \cdots, f), e)f\\
=&exf\psi(eze, eye) - \psi(eze, eye)exf - eze\psi(exf, eye)f + eyeze\psi(exf, e)f.
\end{aligned} \tag{3.13}$$

On the other hand,

$$\begin{aligned}
&\psi(P_n(eze, exf, f, \cdots, f), P_n(eye, e, f, \cdots, f))\\
=&P_n(\psi(eze, P_n(eye, e, f, \cdots, f)), exf, f, \cdots, f)\\
&+ P_n(eze, \psi(exf, P_n(eye, e, f, \cdots, f)), f, \cdots, f)\\
&+ \sum_{i=3}^{n} P_n(eze, exf, f, \cdots, f, \psi(f, P_n(eye, e, f, \cdots, f)), f, \cdots, f)\\
=&\psi(eze, P_n(eye, e, f, \cdots, f))exf - exf\psi(eze, P_n(eye, e, f, \cdots, f))\\
&+ eze\psi(exf, P_n(eye, e, f, \cdots, f))f\\
&+ (n-2)(ezexf\psi(f, P_n(eye, e, f, \cdots, f) - \psi(f, P_n(eye, e, f, \cdots, f))ezexf)\\
=&ezeye\psi(exf, e)f - eze\psi(exf, eye)f.
\end{aligned} \tag{3.14}$$

Considering (3.13) and (3.14) together, we get that

$$-[eye, eze]\psi(exf, e) = [\psi(eze, eye), -exf]. \tag{3.15}$$

With the help of the preceding two equations (3.12) and (3.15), we get

$$[eye, eze](\psi(e, exf) + \psi(exf, e)) = 0,\ i.e.,\ (\lambda + \mu)[eye, eze]exf = 0$$

for all $x, y, z \in \mathcal{T}$. By [17, Proposition 2.6], we conclude that $(\lambda + \mu)[e\mathcal{T}e, e\mathcal{T}e] = 0$. This leads to $[(\lambda+\mu)e\mathcal{T}C(\mathcal{T})e, e\mathcal{T}C(\mathcal{T})e] = 0$. Then $(\lambda+\mu)e\mathcal{T}C(\mathcal{T})e$ is the central ideal of $e\mathcal{T}C(\mathcal{T})e$. Without loss of generality assume that $e\mathcal{T}C(\mathcal{T})e$ does not contain nonzero central ideals. Hence $\mu = -\lambda$. Further, we have

$$\begin{aligned}
0 &= \psi(eyf, P_n(e, exe, f, \cdots, f)) \\
&= P_n(\psi(eyf, e), exe, f, \cdots, f) + P_n(e, \psi(eyf, exe), f, \cdots, f) \\
&= -exe\psi(eyf, f)f + e\psi(eyf, exe)f,
\end{aligned}$$

and hence $e\psi(eyf, exe)f = exe\psi(eyf, e)f$ for all $x, y \in \mathcal{T}$. Therefore, $\psi(eyf, exe) = -\lambda exeyf$. Therefore, conclusion (1) is valid. The second conclusion can be obtained by a similar method.

Lemma 3.6. *Let $\psi : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ be a bi-Lie n-derivation on $\mathcal{T}$. Then ψ satisfies*

(1) $\psi(exe, fyf) = e\psi(exe, fyf)e + f\psi(exe, fyf)f \in C(\mathcal{T})$;
(2) $\psi(fxf, eye) = e\psi(fyf, exe)e + f\psi(fyf, exe)f \in C(\mathcal{T})$

for all $x, y \in \mathcal{T}$.

Proof. For any $x, y \in \mathcal{T}$,

$$\begin{aligned}
0 &= \psi(P_n(e, exe, f, \cdots, f), fyf) \\
&= P_n(\psi(e, fyf), exe, f, \cdots, f) + P_n(e, \psi(exe, fyf), f, \cdots, f) \\
&= -exe\psi(e, fyf)f + e\psi(exe, fyf)f.
\end{aligned}$$

Thus, $e\psi(exe, fyf)f = exe\psi(e, fyf)f$. Using the property that the second component is Lie n-derivation, we can get

$$\begin{aligned}
0 &= \psi(e, P_n(e, fyf, f, \cdots, f)) \\
&= P_n(\psi(e, e), fyf, f, \cdots, f) + P_n(e, \psi(e, fyf), f, \cdots, f) \\
&= e\psi(e, e)fyf + e\psi(e, fyf)f,
\end{aligned}$$

which leads to $e\psi(e, fyf)f = -e\psi(e, e)fyf$ and hence $e\psi(exe, fyf)f = -exe\psi(e, e)fyf = 0$. In view of Lemma 3.4, we have

$$\begin{aligned}
\psi(exf, fyf) &= \psi(P_n(exf, f, \cdots, f), fyf) \\
&= P_n(\psi(exf, fyf), f, \cdots, f) + \sum_{i=2}^{n} P_n(exf, f, \cdots, f, \psi(f, fyf), f, \cdots, f) \\
&= e\psi(exf, fyf)f + (n-1)(exf\psi(f, fyf) - \psi(f, fyf)exf).
\end{aligned}$$

Multiplying by e from left and f from right, since $\mathcal{T}$ is $(n-1)^{th}$-torsion free, we conclude that $[exf, e\psi(f, fyf)e + f\psi(f, fyf)f] = 0$. Thus, by the same methods, we also have

$$\begin{aligned}\psi(exf, fyf) =& \psi(P_n(e, exf, f, \cdots, f), fyf)\\ =& P_n(\psi(e, fyf), exf, f, \cdots, f) + P_n(e, \psi(exf, fyf), f, \cdots, f)\\ &+ \sum_{i=3}^{n} P_n(e, exf, f, \cdots, f, \psi(f, fyf), f, \cdots, f)\\ =& \psi(e, fyf)exf - exf\psi(e, fyf) + e\psi(exf, fyf)f\\ &+ (n-2)(exf\psi(f, fyf) - \psi(f, fyf)exf)\\ =& \psi(e, fyf)exf - exf\psi(e, fyf) + e\psi(exf, fyf)f.\end{aligned}$$

It is easy to conclude that $[e\psi(e, fyf)e + f\psi(e, fyf)f, exf] = 0$.

In view of Lemmas 3.4 and 3.5, we firstly apply the properties of Lie n-derivation to the second component, and then perform Lie n-derivation operations on the first component, we can get

$$\begin{aligned}&\psi(P_n(exe, e, f, \cdots, f), P_n(fyf, ezf, f, \cdots, f))\\ =&P_n(\psi(P_n(exe, e, f, \cdots, f), fyf), ezf, f, \cdots, f)\\ &+ P_n(fyf, \psi(P_n(exe, e, f, \cdots, f), ezf), f, \cdots, f)\\ &+ \sum_{i=3}^{n} P_n(fyf, ezf, f, \cdots, f, \psi(P_n(exe, e, f, \cdots, f), f), f, \cdots, f)\\ =&\psi(P_n(exe, e, f, \cdots, f), fyf)ezf - ezf\psi(P_n(exe, e, f, \cdots, f), fyf)\\ &- e\psi(P_n(exe, e, f, \cdots, f), ezf)fyf\\ &+ (n-2)(\psi(P_n(exe, e, f, \cdots, f), f)ezfyf - ezfyf\psi(P_n(exe, e, f, \cdots, f), f))\\ =&e\psi(exe, ezf)fyf - exe\psi(e, ezf)fyf.\end{aligned} \tag{3.16}$$

On the other hand, we adjust the order in which the first component and the second component use the Lie n-derivation property, and we can get that

$$\begin{aligned}&\psi(P_n(exe, e, f, \cdots, f), P_n(fyf, ezf, f, \cdots, f))\\ =&P_n(\psi(exe, P_n(fyf, ezf, f, \cdots, f)), e, f, \cdots, f)\\ &+ P_n(exe, \psi(e, P_n(fyf, ezf, f, \cdots, f)), f, \cdots, f)\\ =&-e\psi(exe, P_n(fyf, ezf, f, \cdots, f))f + exe\psi(e, P_n(fyf, ezf, f, \cdots, f))f\\ =&ezf\psi(exe, fyf)f - e\psi(exe, fyf)ezf + e\psi(exe, ezf)fyf\\ &+ exe\psi(e, fyf)ezf - exezf\psi(e, fyf)f - exe\psi(e, ezf)fyf.\end{aligned} \tag{3.17}$$

Considering (3.16) and (3.17) together, we get that

$$[e\psi(exe, fyf)e + f\psi(exe, fyf)f, ezf] = [[exe, ezf], \psi(e, fyf)] = 0,$$

and hence $e\psi(exe, fyf)e + f\psi(exe, fyf)f \in \mathcal{C}(\mathcal{T})$ for all $x, y \in \mathcal{T}$. Similarly, we can conclude that (2) is true.

Lemma 3.7. *Let $\psi : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ be a bi-Lie n-derivation on $\mathcal{T}$. Then ψ satisfies*

(1) $\psi(exe, eye) = \tau^{-1}(f\psi(exe, eye)f) + \lambda[exe, eye] + f\psi(exe, eye)f$, where $f\psi(exe, eye)f \in C(\mathcal{T})f$;

(2) $\psi(fxf, fyf) = e\psi(fxf, fyf)e+\tau(e\psi(fxf, fyf)e)+\tau(\lambda)[fxf, fyf]$, where $e\psi(fxf, fyf)e \in C(\mathcal{T})e$

for all $x, y \in \mathcal{T}$.

Proof. For any $x, y \in \mathcal{T}$, by Lemma 3.6 we find that

$$\begin{aligned} 0 &= \psi(exe, P_n(eye, f, \cdots, f)) \\ &= P_n(\psi(exe, eye), f, \cdots, f) + P_n(eye, \psi(exe, f), f, \cdots, f) \\ &= e\psi(exe, eye)f, \end{aligned}$$

hence, $e\psi(exe, eye)f = 0$. At the same time, we see that

$$\begin{aligned} &\psi(P_n(exe, e, f, \cdots, f), P_n(eze, eyf, f, \cdots, f)) \\ =&P_n(\psi(P_n(exe, e, f, \cdots, f), eze), eyf, f, \cdots, f) \\ &+ P_n(eze, \psi(P_n(exe, e, f, \cdots, f), eyf), f, \cdots, f) \\ &+ \sum_{i=3}^{n} P_n(eze, eyf, f, \cdots, f, \psi(P_n(exe, e, f, \cdots, f), f), f, \cdots, f) \\ =&\psi(P_n(exe, e, f, \cdots, f), eze)eyf - eyf\psi(P_n(exe, e, f, \cdots, f), eze) \\ &+ eze\psi(P_n(exe, e, f, \cdots, f), eyf)f \\ &+ (n-2)(ezeyf\psi(P_n(exe, e, f, \cdots, f), f) - \psi(P_n(exe, e, f, \cdots, f), f)ezeyf) \\ =& - eze\psi(exe, eyf)f + ezexe\psi(e, eyf)f. \end{aligned} \tag{3.18}$$

On the other hand,

$$\begin{aligned} &\psi(P_n(exe, e, f, \cdots, f), P_n(eze, eyf, f, \cdots, f)) \\ =&P_n(\psi(exe, P_n(eze, eyf, f, \cdots, f)), e, f, \cdots, f) \\ &+ P_n(exe, \psi(e, P_n(eze, eyf, f, \cdots, f)), f, \cdots, f) \\ =& - e\psi(exe, P_n(eze, eyf, f, \cdots, f))f + exe\psi(e, P_n(eze, eyf, f, \cdots, f))f \\ =&eyf\psi(exe, eze) - \psi(exe, eze)eyf - eze\psi(exe, eyf)f + exeze\psi(e, eyf)f. \end{aligned} \tag{3.19}$$

Taking advantage of (3.18) and (3.19), we arrive at

$$e\psi(exe, eze)eyf - eyf\psi(exe, eze)f = [exe, eze]\psi(e, eyf) \text{ and}$$
$$e\psi(exe, eze)eyf - \tau^{-1}(f\psi(exe, eze)f)eyf = \lambda[exe, eze]eyf.$$

Making use of the conclusion (3) in Proposition 2.1, we can conclude that $e\psi(exe, eze)e = \tau^{-1}(f\psi(exe, eze)f) + \lambda[exe, eze]$.

Since ψ is a Lie n-derivation for the second component, we find

$$\begin{aligned} 0 &= \psi(exe, P_n(eye, fzf, ezf, f, \cdots, f)) \\ &= P_n(\psi(exe, eye), fzf, ezf, f, \cdots, f) + P_n(eye, \psi(exe, fzf), ezf, f, \cdots, f) \\ &= ezf[fzf, f\psi(exe, eye)f], \end{aligned}$$

and hence $f\psi(exe, eye)f \in C(\mathcal{T})f$ for all $x, y \in \mathcal{T}$. Similarly, we can prove the other part.

Lemma 3.8. *Let $\psi : \mathcal{T}\times\mathcal{T} \to \mathcal{T}$ be a bi-Lie n-derivation on $\mathcal{T}$. Then $\psi(exf, eyf) = 0$ for all $x, y \in \mathcal{T}$.*

Proof. For any $x, y \in \mathcal{T}$,

$$\begin{aligned}\psi(exf, eyf) &= \psi(exf, P_n(eyf, f, \cdots, f))\\ &= P_n(\psi(exf, eyf), f, \cdots, f) + \sum_{i=2}^{n} P_n(eyf, f, \cdots, f, \psi(exf, f), f, \cdots, f)\\ &= e\psi(exf, eyf)f + (n-1)(eyf\psi(exf, f) - \psi(exf, f)eyf)\\ &= e\psi(exf, eyf)f \in e\mathcal{T}f.\end{aligned}$$

Let's fix element $y \in \mathcal{T}$, we define a map $\mathfrak{g} : e\mathcal{T} \to \mathcal{T}$ by $\mathfrak{g}_y(x) = \psi(exf, eyf)$ for all $x \in e\mathcal{T}$. Then by Lemma 3.8, we get

$$\begin{aligned}\mathfrak{g}_y(rx) =&\psi(P_n(ere, exf, f, \cdots, f), eyf)\\ =&P_n(\psi(ere, eyf), exf, f, \cdots, f) + P_n(ere, \psi(exf, eyf), f, \cdots, f)\\ &+ \sum_{i=3}^{n} P_n(ere, exf, f, \cdots, f, \psi(f, eyf), f, \cdots, f)\\ =&ere\psi(exf, eyf)f = r\mathfrak{g}_y(x)\end{aligned}$$

for all $x \in e\mathcal{T}$, $r \in \mathcal{T}$, and hence $\mathfrak{g}_y$ is a left $\mathcal{T}$-module homomorphism. By conclusions (2) and (3) in Proposition 2.1, there exists $q_y \in \mathcal{Q}_{ml}(\mathcal{T})$ such that $\mathfrak{g}_y(x) = xq_y$ for all $x \in e\mathcal{T}$. Clearly, $eq_y = \mathfrak{g}_y(e) = 0$. So $q_y = fq_yf$ implies that $\mathfrak{g}_y(x) = xfq_yf$ for all $x \in e\mathcal{T}$. And then we also have

$$\begin{aligned}\mathfrak{g}_y(xr) =&\psi(P_n(exf, frf, f, \cdots, f), eyf)\\ =&P_n(\psi(exf, eyf), frf, f, \cdots, f) + P_n(exf, \psi(frf, eyf), f, \cdots, f)\\ &+ \sum_{i=3}^{n} P_n(exf, frf, f, \cdots, f, \psi(f, eyf), f, \cdots, f)\\ =&e\psi(exf, eyf)frf = \mathfrak{g}_y(x)r,\end{aligned}$$

and hence $xfrfq_yf = xfq_yfrf$ for all $x \in e\mathcal{T}$, $r \in \mathcal{T}$. Then $e\mathcal{T}(frfq_yf - fq_yfrf) = 0$ for all $r \in \mathcal{T}$. In view of conclusion (3) in Proposition 2.1, we get $frfq_yf = fq_yfrf$ for all $r \in \mathcal{T}$. Consequently, by the assumption of Theorem 3.1, we have $fq_yf \in \mathcal{C}(\mathcal{T})f$. Now, for any $x, y, x', y' \in \mathcal{T}$, by Lemma 3.6, we have

$$\begin{aligned}&\psi(P_n(ex'f, exe, f, \cdots, f), P_n(ey'f, eye, f, \cdots, f))\\ =&P_n(\psi(P_n(ex'f, exe, f, \cdots, f), ey'f), eye, f, \cdots, f)\\ &+ P_n(ey'f, \psi(P_n(ex'f, exe, f, \cdots, f), eye), f, \cdots, f)\\ &+ \sum_{i=3}^{n} P_n(ey'f, eye, f, \cdots, f, \psi(P_n(ex'f, exe, f, \cdots, f), f), f, \cdots, f)\\ =&-eye\psi(P_n(ex'f, exe, f, \cdots, f), ey'f)f\\ &+ ey'f\psi(P_n(ex'f, exe, f, \cdots, f), eye) - \psi(P_n(ex'f, exe, f, \cdots, f), eye)ey'f\\ &+ (n-2)(\psi(P_n(ex'f, exe, f, \cdots, f), f)eyey'f - eyey'f\psi(P_n(ex'f, exe, f, \cdots, f), f))\\ =&eyexe\psi(ex'f, ey'f).\end{aligned} \tag{3.20}$$

On the other hand,

$$
\begin{aligned}
&\psi(P_n(ex'f, exe, f, \cdots, f), P_n(ey'f, eye, f, \cdots, f))\\
=&P_n(\psi(ex'f, P_n(ey'f, eye, f, \cdots, f)), exe, f, \cdots, f)\\
&+ P_n(ex'f, \psi(exe, P_n(ey'f, eye, f, \cdots, f)), f, \cdots, f)\\
&+ \sum_{i=3}^{n} P_n(ex'f, exe, f, \cdots, f, \psi(f, P_n(ey'f, eye, f, \cdots, f), f, \cdots, f)\\
=&-exe\psi(ex'f, P_n(ey'f, eye, f, \cdots, f))f\\
&+ ex'f\psi(exe, P_n(ey'f, eye, f, \cdots, f)) - \psi(exe, P_n(ey'f, eye, f, \cdots, f))ex'f\\
&+ (n-2)(\psi(f, P_n(ey'f, eye, f, \cdots, f)exex'f - exex'f\psi(f, P_n(ey'f, eye, f, \cdots, f))\\
=&exeye\psi(ex'f, ey'f)f.
\end{aligned} \tag{3.21}
$$

Taking advantage of (3.20) and (3.21) together, we get that

$$
\begin{aligned}
0 &= [\psi(ex'f, ey'f), [exe, eye]]\\
&= [exe, eye]ex'fq_{y'}f\\
&= \tau^{-1}(fq_{y'}f)[exe, eye]ex'f
\end{aligned}
$$

for all $x, y, x', y' \in \mathcal{T}$. By conclusion (3) in Proposition 2.1, we have $\tau^{-1}(fq_{y'}f)[e\mathcal{T}e, e\mathcal{T}e] = 0$, this implies that

$$[\tau^{-1}(fq_{y'}f)e\mathcal{T}C(\mathcal{T})e, e\mathcal{T}C(\mathcal{T})e] = 0.$$

It follows from above equation that $\tau^{-1}(fq_{y'}f)e\mathcal{T}C(\mathcal{T})e$ is a central ideal of $e\mathcal{T}C(\mathcal{T})e$. Assume without loss of generality that $e\mathcal{T}C(\mathcal{T})e$ does not contain nonzero central ideals. Then $\tau^{-1}(fq_{y'}f) = 0$, which leads to $fq_{y'}f = 0$ for all $y' \in \mathcal{T}$. So we conclude that $\psi(exf, eyf) = exfq_yf = 0$ for all $x, y \in \mathcal{T}$. This lemma is proved.

Remark 3.3. *Let us define two bilinear maps $\gamma : \mathcal{T} \times \mathcal{T} \to C(\mathcal{T})$ and $\delta : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ by*

$$
\begin{aligned}
\gamma(x, y) =&f\psi(exe, eye + fyf)f + \tau^{-1}(f\psi(exe, eye + fyf)f)\\
&+ e\psi(fxf, eye + fyf)e + \tau(e\psi(fxf, eye + fyf)e),\\
\delta(x, y) =&\psi(x, y) - \gamma(x, y).
\end{aligned}
$$

Clearly, $\gamma(x, y) \in C(\mathcal{T})$ and $\gamma(P_n(x_1, x_2, \cdots, x_n), P_n(y_1, y_2, \cdots, y_n)) = 0$ for all $x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n \in \mathcal{T}$. Also, it is easy to verify that δ is a bi-Lie n-derivation.

On the basis of Remark 3.3 and Lemmas 3.1–3.8, it follows that:

Lemma 3.9. *For any $x, y \in \mathcal{T}$, with notations as above, we have*

(1) $\delta(e, e) = \delta(e, f) = \delta(f, e) = \delta(f, f) = 0$;
(2) $\delta(exe, fyf) = 0 = \delta(fyf, exe)$;
(3) $\delta(exe, eyf) = \lambda exeyf = -\delta(eyf, exe)$ *and* $\delta(exf, fyf) = \lambda exfyf = -\delta(fyf, exf)$;
(4) $\delta(exe, eye) = \lambda[exe, eye]$ *and* $\delta(fxf, fyf) = \tau(\lambda)[fxf, fyf]$;
(5) $\delta(exf, eyf) = 0$.

Lemma 3.10. *With notations as above, we have that δ is an inner biderivation.*

Proof. Let $\alpha = \lambda + \eta(\lambda) \in \mathcal{C}(\mathcal{T})$. Since δ is bilinear, it follows that

$$\begin{aligned}\delta(x, y) =&\delta(exe + exf + fxf, eye + eyf + fyf)\\ =&\delta(exe, eye) + \delta(exe, eyf) + \delta(exe, fyf)\\ &+ \delta(exf, eye) + \delta(exf, eyf) + \delta(exf, fyf)\\ &+ \delta(fxf, eye) + \delta(fxf, eyf) + \delta(fxf, fyf)\\ =&\lambda[exe, eye] + \lambda exeyf - \lambda exeyf + \lambda exfyf - \lambda exfyf + \eta(\lambda)[fxf, fyf]\\ =&\alpha[x, y]\end{aligned}$$

for all $x, y \in \mathcal{T}$. Hence δ is an inner biderivation.

Proof of Theorem 3.1. Now in view of Remark 3.2, we have $\phi - \kappa = \psi$, where $\kappa(x, y) = [x, [y, \phi(e, e)]]$. By Remark 3.3 and Lemmas 3.9, 3.10, we see every bi-Lie n-derivation ψ can be written as sum of inner biderivation and a bilinear central mapping vanishing at n-commutators on $\mathcal{T}$. Therefore, $\psi = \kappa + \delta + \gamma$, where $\kappa(x, y) = [x, [y, \phi(e, e)]]$ is an extremal biderivation, $\delta : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ is an inner biderivation and $\gamma : \mathcal{T} \times \mathcal{T} \to \mathcal{C}(\mathcal{T})$ is a biadditive central mapping which vanishes at $(n-1)^{th}$-commutators on $\mathcal{T}$.

As a consequence of Theorem 3.1, we have:

Corollary 3.1. *Let $T_m(R)$ be a upper triangular matrix ring with $m \geq 3$, where R be an unital ring. If a bi-additive mapping $\varphi : T_m(R) \times T_m(R) \to T_m(R)$ be a bi-Lie n-derivation of $T_m(R)$.Then it has the form $\phi = \zeta + \delta + \gamma$, where $\delta : T_m(R) \times T_m(R) \to T_m(R)$ is an inner biderivation, $\zeta : T_m(R) \times T_m(R) \to T_m(R)$ is an extremal biderivation and γ is a bilinear central map vanishing at commutators.*

Proof. With the help of [13, Corollary 2.1], it can be seen that upper triangular algebra $T_m(R)$ $(m \geq 3)$ coincides with the conditions of Theorem 3.1, so this corollary holds.

When $n = 2$, we have the following corollary.

Corollary 3.2. *Let $\mathcal{T}$ be a triangular ring and e is the nontrivial idempotent of it. Assume that $\phi : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ is a bi-Lie derivation and $\mathcal{C}(fQ_{ml}(\mathcal{T})f, f\mathcal{T}f) = \mathcal{C}(\mathcal{T})f$ and either $e\mathcal{T}\mathcal{C}(\mathcal{T})e$ or $f\mathcal{T}\mathcal{C}(\mathcal{T})f$ does not contain nonzero central ideals. Then it has the form $\phi = \zeta + \delta + \gamma$, where $\delta : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ is an inner biderivation, $\zeta : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ is an extremal biderivation and γ is a bilinear central map vanishing at commutators.*

4. Conclusions

The purpose of this article is to prove that every bi-Lie n-derivation of certain triangular rings is the sum of an inner biderivation, an extremal biderivation and an additive central mapping vanishing at $(n-1)^{th}$-commutators for both components, using the notion of maximal left ring of quotients.

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Conflict of interest

The authors declare no conflicts of interest.

References

1. G. Maksa, On the trace of symmetric biderivations, *C. R. Math. Rep. Acad. Sci. Canada.*, **9** (1987), 303–308.
2. G. Maksa, A remark on symmetric biadditive functions having nonnegative diagonalization, *Glasnik Math.*, **15** (1980), 279–282.
3. Y. Wang, Biderivations of triangular rings, *Linear Multilinear Algebra*, **64** (2016), 1952–1959. https://doi.org/10.1080/03081087.2015.1127887
4. N. M. Ghosseiri, On biderivations of upper triangular matrix rings, *Linear Algebra Appl.*, **438** (2013), 250–260. https://doi.org/10.1016/j.laa.2012.07.039
5. Y. Wang, On functional identities of degree 2 and centralizing maps in triangular rings, *Oper. Matrices*, **10** (2016), 485–499. https://doi.org/10.7153/oam-10-28
6. L. Liu, M. Y. Liu, On Jordan biderivations of triangular rings, *Oper. Matrices*, **15** (2021), 1417–1426. https://doi.org/10.7153/oam-2021-15-88
7. I. N. Herstein, Lie and Jordan structures in simple associative rings, *Bull. Amer. Math. Soc.*, **67** (1961), 517–531. https://doi.org/10.1090/S0002-9904-1961-10666-6
8. Y. N. Ding, J. K. Li, Characterizations of Lie n-derivations of unital algebras with nontrivial idempotents, *Filomat*, **32** (2018), 4731–4754. https://doi.org/10.2298/FIL1813731D
9. D. Benkovič, Generalized Lie n-derivations of triangular algebras, *Commun. Algebra*, **47** (2019), 5294–5302. https://doi.org/10.1080/00927872.2019.1617875
10. D. Eremita, Biderivations of triangular rings revisited, *Bull. Malays. Math. Soc.*, **40** (2017), 505–527. https://doi.org/10.1007/s40840-017-0451-6
11. D. Benkovič, Biderivations of triangular algebras, *Linear Algebra Appl.*, **431** (2009), 1587–1602. https://doi.org/10.1016/j.laa.2009.05.029
12. X. F. Liang, D. D. Ren, F. Wei, Lie biderivations of triangular algebras, arXiv:2002.12498.
13. Y. Wang, Functional identities of degree 2 in arbitrary triangular rings, *Linear Algebra Appl.*, **479** (2015), 171–184. https://doi.org/10.1016/j.laa.2015.04.018
14. D. D. Ren, X. F. Liang, Jordan biderivations on triangular algebras, *Adv. Math. (China)*, **51** (2022), 299–312.
15. D. Eremita, Functional identities of degree 2 in triangular rings, *Linear Algebra Appl.*, **438** (2013), 584–597. https://doi.org/10.1016/j.laa.2012.07.028
16. Y. Utumi, On quotient rings, *Osaka J. Math.*, **8** (1956), 1–18.
17. D. Eremita, Functional identities of degree 2 in triangular rings revisited, *Linear Multilinear Algebra*, **63** (2015), 534–553. https://doi.org/10.1080/03081087.2013.877012