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Research article

Estimation and prediction for two-parameter Pareto distribution based on progressively double Type-II hybrid censored data

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Abstract: In this paper, a new censoring test plan called progressively double Type-II hybrid censoring scheme is introduced for the first time. Based on this type of censored data, the maximum likelihood estimates of the unknown parameters and reliability for the two-parameter Pareto distribution are obtained. Using the Bayesian method, the Bayesian estimates of the unknown parameters and reliability are obtained under the symmetric and asymmetric loss functions. The failure times of all withdrawn units are predicted using the classical and Bayesian methods, including the predictive values and the prediction intervals. The mean values and mean square errors of the estimators are calculated by Monte-Carlo simulation, and the mean square errors between them are compared, and the results show that all Bayesian estimates are better than the corresponding maximum likelihood estimates. Using a real data set, we compute the Bayesian estimates of the unknown parameters and reliability, and predict the observations of the censored units.

Keywords: two-parameter Pareto distribution; progressively double Type-II hybrid censoring; maximum likelihood estimation; posterior density; Bayesian estimation
Mathematics Subject Classification: 62F10, 62F15

1. Introduction

In the fields of reliability engineering and survival analysis, due to time or other constraints, it is often impossible to obtain the lifetimes of all tested units, so we can use a censored test scheme to obtain censored data. So far, many types of censoring schemes have been produced, and the corresponding statistical inference theories have been developed rapidly. Due to simplicity and applicability, Type-I and Type-II are the two most common and widely used censoring schemes. Type-I censoring is to stop the test when the preset time arrives, while Type-II censoring is to stop the test after a given number of failure data is obtained. A common feature of these two schemes is that the living units cannot be removed during the test. Sometimes it is necessary to withdraw some of the living units to observe their degradation, in this context, the progressive Type-II censoring scheme has been proposed. There are many research results based on the above three types of censored data. For more details, see references [1–8]. In some cases, for some high-reliability and long-life units, if the Type-II censoring scheme is adopted, the test time will be very long, and it will cost more money. In order to improve the test efficiency, another censoring scheme has been proposed, that is, the first-failure censoring scheme, which can save time and test costs. Based on the first-failure censored data, many scholars have studied the statistical inference on the parameters of various distributions, for example, Wu et al. [9,10] obtained maximum likelihood estimates and confidence intervals of the parameters for Gompertz and Burr XII distributions on the basis of the first-failure censored data. Wu and Kus [11] considered the advantages of the above censoring schemes, extended the first-failure censoring scheme, and proposed the progressive first-failure censoring scheme, and proved that this scheme had shorter expected test times than the progressive Type-II censoring scheme. Based on the characteristics of Type-I and Type-II censoring schemes, Epstein [12] first proposed a mixture of Type-I and Type-II censoring schemes, that is, Type-I hybrid censoring scheme. Later, Childs et al. [13] introduced another mixture of Type-I and Type-II censoring schemes, namely, Type-II hybrid censoring scheme. If n units are put into the life test at time zero, they have the following ordered lifetimes: $X_{1:n}, X_{2:n}, ..., X_{n:n}$, and T represents the specified censoring time, r represents the number of failed units determined in advance. The Type-I hybrid censoring test is terminated at a random time $T_1^* = \min(X_{r:n}, T)$, and the Type-II hybrid censoring test is terminated at a random time $T_2^* = \max(X_{r:n}, T)$. The advantage of the Type-I hybrid censoring scheme is that the test time will not exceed T , and the disadvantage is that little failure data may be obtained. The advantage of the Type-II hybrid censoring scheme is that at least r failure data can be obtained, but the disadvantage is that the duration of the test cannot be controlled. For Type-I and Type-II hybrid censoring schemes, living units cannot be removed during the test. Therefore, Kundu et al. [14] extended the hybrid censoring scheme and proposed a progressive Type-I hybrid censoring scheme. At present, in the fields of reliability and survival analysis, hybrid censoring schemes are very popular in analyzing high-reliability life data. Interested readers may refer to references [15,16]. According to the existing censoring schemes, Long and Zhang [17] proposed a mixture of two Type-II censoring schemes, which are called double Type-II hybrid censoring scheme. When the lifetimes of the tested units follow two-parameter Pareto distribution, the statistical inference methods for the unknown parameters and reliability indicators are given based on this type of hybrid censored data. In this paper, we will generalize the double Type-II hybrid censoring scheme, propose a progressively double Type-II hybrid censoring scheme, and discuss the estimation of the unknown parameters and reliability under the two-parameter Pareto model, as well as the prediction of the failure times of evacuated units. The specific test scheme will be introduced in Section 2.

One of the most important problems in life test is to predict future failure times based on observed data. In the early stages of the test, we can predict how expensive the test will be and whether measures need to be taken to adjust the test scheme. So far, many scholars have done a lot of work on prediction, for example, see references [18–22]. In this paper, we will further explore the prediction problems under the progressively double Type-II hybrid censored data.

The rest of the paper is organized as follows: In Section 2, the two-parameter Pareto model and progressively double Type-II hybrid censoring scheme are introduced, and the maximum likelihood estimates of the unknown parameters and reliability are given. The Bayesian estimates of the unknown parameters and reliability are obtained in Section 3. In Section 4, the classical and Bayesian methods are used to predict failure times of the evacuated units. In Section 5, Monte-Carlo simulation is used to verify the goodness of the estimators. A real data set is analyzed in Section 6. We conclude the paper in Section 7.

2. Model description and maximum likelihood estimation

The two-parameter Pareto distribution was originally proposed as an income distribution, mainly used to analyze economic and natural phenomena, and later the distribution was also applied to the fields of reliability and survival analysis. Its cumulative distribution function and probability density function are respectively given as follow

$$F(x;\lambda,\theta)=1-\frac{\lambda^{\theta}}{x^{\theta}},\quad f(x;\lambda,\theta)=\frac{\theta\lambda^{\theta}}{x^{\theta+1}},\quad x\geq\lambda, \tag{2.1}$$

where $\lambda(>0)$ is the scale parameter and $\theta(>0)$ is the shape parameter. If X represents the lifetime of unit, the reliability function is

$$R(x)=\frac{\lambda^{\theta}}{x^{\theta}}. \tag{2.2}$$

In this paper, it is assumed that the lifetimes of the tested units follow the two-parameter Pareto distribution (2.1), and based on progressively double Type-II hybrid censored data, we will discuss the relevant estimation and prediction problems of this distribution.

In the reliability test, progressively double Type-II hybrid censoring is a mixture of two progressive Type-II hybrid censoring schemes. The model is described as follows:

Suppose that n independent and identically distributed units are put into the test, the time t_0 and the positive integers m_1, m_2 are determined in advance, and $m_1 < m_2 \leq n$ are satisfied. When the first unit fails, the failure time is denoted as $X_{1:n}$, and R_1 non-failed units are removed from the remaining $(n-1)$ units. When the second unit fails, the failure time is denoted as $X_{2:n}$, and R_2 non-failed units are removed from the remaining $(n-2-R_1)$ units. By analogy, the failure time of the m_1-th unit is denoted as $X_{m_1:n}$. If $X_{m_1:n}\geq t_0$, the test is terminated at time $X_{m_1:n}$, and all the $R_{m_1}=n-m_1-\sum_{i=1}^{m_1-1}R_i$ units that have not failed are withdrawn from the test, where $0<X_{1:n}\leq X_{2:n}\leq\ldots\leq X_{m_1:n}$ are the order times of failure. If $X_{m_1:n}<t_0$, the test is terminated when m_2 units fail, and the $R_{m_2}=n-m_2-\sum_{i=1}^{m_2-1}R_i$ units that have not failed are withdrawn from the test,

where $0 < X_{1:n} \le X_{2:n} \le ... \le X_{m_2:n}$ are the order times of failure. The number of removed units $R_1, R_2, ..., R_{m_2}$ can be determined in advance.

Using the above test scheme, we can obtain the following two types of censored data:

Case I: $\left(X_{1:n}, R_1, X_{2:n}, R_2, ..., X_{m_1:n}, R_{m_1}\right)$, if $X_{m_1:n} \ge t_0$;

Case II: $\left(X_{1:n}, R_1, X_{2:n}, R_2, ..., X_{m_2:n}, R_{m_2}\right)$, if $X_{m_1:n} < t_0$.

Denote

$$k = \begin{cases} m_1, & \text{Case I} \\ m_2, & \text{Case II} \end{cases}.$$

Then the obtained progressively double Type-II hybrid censored data can be expressed as $\underline{x} = \left(x_{1:n}, R_1, x_{2:n}, R_2, ..., x_{k:n}, R_k\right)$, if $R_1 = R_2 = ... = R_{k-1} = 0,\ R_k = n - k$, $\underline{x}$ is double Type-II hybrid censored data.

Based on the experimental data $\underline{x}$, the likelihood function can be expressed as

$$L(\lambda,\theta) \propto \prod_{i=1}^{k} f(x_{i:n};\lambda,\theta)\left[1 - F(x_{i:n};\lambda,\theta)\right]^{R_i} I(x_{1:n} \ge \lambda)$$

$$\propto \theta^k \lambda^{n\theta} \exp\left\{-\theta\left(\sum_{i=1}^{k} \ln x_{i:n} + \sum_{i=1}^{k} R_i \ln x_{i:n}\right)\right\} I(x_{1:n} \ge \lambda), \tag{2.3}$$

where $I(\cdot)$ is an indicator function.

From (2.3), the maximum likelihood estimates of θ and λ can be obtained as

$$\tilde{\theta} = \frac{k}{\sum_{i=1}^{k} \ln x_{i:n} + \sum_{i=1}^{k} R_i \ln x_{i:n} - n \ln x_{1:n}}, \quad \tilde{\lambda} = x_{1:n}.$$

According to the invariance of the maximum likelihood estimation, the maximum likelihood estimate of the reliability function $R(x)$ is

$$\tilde{R}(x) = (\tilde{\lambda} / x)^{\tilde{\theta}}.$$

When λ is known, the maximum likelihood estimates of θ and $R(x)$ are

$$\hat{\theta}=\frac{k}{\sum_{i=1}^{k}\ln x_{i:n}+\sum_{i=1}^{k}R_i\ln x_{i:n}-n\ln\lambda},\quad \hat{R}(x)=(\lambda/x)^{\hat{\theta}}.$$

3. Bayesian estimation

In this section, when λ is known and θ is unknown, the prior distribution of θ is taken as Gamma distribution, and the Bayesian estimates of θ and reliability function are given. When both λ and θ are unknown, the Bayesian estimates of λ,θ and reliability function are obtained under three types of loss functions.

3.1. Bayesian estimation when λ is known

Since λ is known, we only need to regard θ as a random variable, and the prior distribution of θ can be taken as Gamma distribution, and its probability density function is

$$\pi(\theta)=\frac{b^a}{\Gamma(a)}\theta^{a-1}e^{-b\theta},\quad \theta>0, \tag{3.1}$$

where the hyper-parameters $a>0,b>0$, $\Gamma(\cdot)$ represents the Gamma function.

According to reference [4], in order to ensure the robustness of Bayesian estimation, the value of a should satisfy $0<a<1$, and the value of b should not be too large.
From (2.3) and (3.1), according to the Bayesian formula, the posterior density function of θ can be obtained as

$$\pi(\theta|\underline{x})=\frac{(A+b)^{k+a}}{\Gamma(k+a)}\theta^{k+a-1}e^{-\theta(A+b)},\quad \theta>0, \tag{3.2}$$

where $A=\sum_{i=1}^{k}\ln x_{i:n}+\sum_{i=1}^{k}R_i\ln x_{i:n}-n\ln\lambda$.

In statistical decision theory and Bayesian analysis, the squared error loss function is a symmetric loss function that is often used, and an important advantage is that the Bayesian estimation of the estimated quantity can be easily calculated. The squared error loss is defined as

$$L_1(\phi(\beta),\delta)=\left[\delta-\phi(\beta)\right]^2,$$

where δ is an estimate of $\phi(\beta)$.

Under the squared error loss, the Bayesian estimate of $\phi(\beta)$ is

$$\hat{\phi}_S=E_\phi[\phi(\beta)|data], \tag{3.3}$$

where $E_{\phi}(\cdot)$ denotes posterior expectation with respect to the posterior density of $\phi(\beta)$.

Using (3.2) and (3.3), the Bayesian estimates of θ and $R(x)$ under the squared error loss are given, respectively, by

$$\hat{\theta}_{BS}=\frac{k+a}{A+b},$$

$$\hat{R}_{BS}(x)=\frac{(A+b)^{k+a}}{(A+b+\ln x-\ln\lambda)^{k+a}}.$$

Based on the square error loss, the risks of overestimation and underestimation are the same. In some estimation and prediction problems, overestimation and underestimation will have different estimation risks, so the symmetric loss function may be unreasonable. Therefore, we consider two kinds of asymmetric loss functions, namely, the linear-exponential (LINEX) loss and general entropy loss functions, which are defined as

$$L_2(\phi(\beta),\delta)=e^{c(\delta-\phi(\beta))}-c(\delta-\phi(\beta))-1,(c\neq 0),$$

$$L_3(\phi(\beta),\delta)\propto\left(\frac{\delta}{\phi(\beta)}\right)^{q}-q\ln\left(\frac{\delta}{\phi(\beta)}\right)-1.$$

For the LINEX loss function, when $c<0$, the loss of underestimation is greater than that of overestimation, and when $c>0$, the opposite is true. For the general entropy loss function, when $q<0$, the loss of underestimation is greater than that of overestimation, and the opposite is true when $q>0$.

Under the LINEX loss and general entropy loss, the Bayesian estimates of $\phi(\beta)$ are

$$\hat{\phi}_L=-\frac{1}{c}\ln[E_{\phi}(e^{-c\phi(\beta)}\,|data)],\tag{3.4}$$

$$\hat{\phi}_G=\left\{E_{\phi}[(\phi(\beta))^{-q}\,|data]\right\}^{-1/q}\tag{3.5}$$

Using (3.2) and (3.4), when $c>-(A+b)$, the Bayesian estimates of θ and $R(x)$ under the LINEX loss are given, respectively, by

$$\hat{\theta}_{BL}=\frac{k+a}{c}\ln\left(1+\frac{c}{A+b}\right),$$

$$\hat{R}_{BL}(x)=-\frac{1}{c}\ln\left\{(A+b)^{k+a}\sum_{j=0}^{\infty}\frac{(-c)^j}{j!}\left[A+b+j(\ln x-\ln\lambda)\right]^{-(k+a)}\right\}.$$

Using (3.2) and (3.5), when $q < k + a$, the Bayesian estimates of θ and $R(x)$ under the general entropy loss are given, respectively, by

$$\hat{\theta}_{BG} = \left[\frac{\Gamma(k+a)}{\Gamma(k+a-q)}\right]^{1/q} (A+b)^{-1},$$

$$\hat{R}_{BG}(x) = \left[1 - \frac{q(\ln x - \ln\lambda)}{A+b}\right]^{\frac{k+a}{q}}.$$

In addition, the Bayesian credible interval for θ can be obtained from the posterior density (3.2). Since $2(A+b)\theta \sim \chi^2(2k+2a)$, the $100(1-\gamma)\%$ equal-tail credible interval of θ is (L,U), where $L = \frac{\chi^2_{1-\gamma/2}(2k+2a)}{2(A+b)}$, $U = \frac{\chi^2_{\gamma/2}(2k+2a)}{2(A+b)}$, $\chi^2_\gamma(k)$ is the $100\gamma\%$ right-tail percentile of the chi-squared distribution with k degrees of freedom.

3.2. Bayesian estimation when λ and θ are unknown

In most cases, both λ and θ are unknown. According to Bayesian theories, they are regarded as random variables, and prior distributions need to be given in advance. Here we take the prior distribution of λ as the noninformative prior distribution, namely

$$\pi_1(\lambda) = \frac{1}{\lambda}, \quad \lambda > 0. \tag{3.6}$$

The prior distribution of θ is still taken as Gamma distribution, its probability density function is (3.1), and it is assumed that λ and θ are independent.

Using the prior distributions (3.1) and (3.6), the joint posterior density of (λ,θ) is

$$\pi(\lambda,\theta|\underline{x}) = \frac{L(\lambda,\theta)\pi_1(\lambda)\pi(\theta)}{\int_0^{+\infty}\int_0^{x_{1:n}} L(\lambda,\theta)\pi_1(\lambda)\pi(\theta)d\lambda d\theta}$$

$$= \frac{n(B+b-n\ln x_{1:n})^{k+a-1}}{\Gamma(k+a-1)}\theta^{k+a-1}\lambda^{n\theta-1}e^{-\theta(B+b)}, \tag{3.7}$$

where $B = \sum_{i=1}^{k}\ln x_{i:n} + \sum_{i=1}^{k}R_i \ln x_{i:n}$, $0 < \lambda \le x_{1:n}, 0 < \theta < +\infty$.

Therefore, the posterior distribution of θ is Gamma distribution, and its probability density function is

$$\pi(\theta|\underline{x}) = \int_0^{x_{1:n}} \pi(\lambda,\theta|\underline{x})d\lambda$$

$$= \frac{(B+b-n\ln x_{1:n})^{k+a-1}}{\Gamma(k+a-1)} \theta^{k+a-2} e^{-\theta(B+b-n\ln x_{1:n})}, \theta > 0. \quad (3.8)$$

In addition, the posterior density of λ is

$$\pi(\lambda|\underline{x}) = \int_0^{+\infty} \pi(\lambda,\theta|\underline{x}) d\theta$$

$$= n(k+a-1)\lambda^{-1}(B+b-n\ln x_{1:n})^{k+a-1}(B+b-n\ln\lambda)^{-(k+a)}, 0 < \lambda \le x_{1:n}. \quad (3.9)$$

So under the squared error loss, the Bayesian estimate of θ is

$$\tilde{\theta}_{BS} = \int_0^{+\infty} \theta\pi(\theta|\underline{x}) d\theta = \frac{k+a-1}{B+b-n\ln x_{1:n}}.$$

Under the squared error loss, the Bayesian estimate of λ is

$$\tilde{\lambda}_{BS} = \int_0^{+\infty} \lambda\pi(\lambda|\underline{x}) d\lambda$$

$$= n(k+a-1)(B+b-n\ln x_{1:n})^{k+a-1} \int_0^{x_{1:n}} (B+b-n\ln\lambda)^{-(k+a)} d\lambda. \quad (3.10)$$

Under the squared error loss, the Bayesian estimate of $R(x)$ is

$$\tilde{R}_{BS}(x) = \int_0^{+\infty} \int_0^{x_{1:n}} \lambda^{\theta} x^{-\theta} \pi(\lambda,\theta|\underline{x}) d\lambda d\theta$$

$$= \frac{n}{n+1} \cdot \frac{(B+b-n\ln x_{1:n})^{k+a-1}}{\Gamma(k+a-1)} \int_0^{+\infty} \theta^{k+a-2} \exp\{-\theta[B+b+\ln x-(n+1)\ln x_{1:n}]\} d\theta$$

$$= \frac{n}{n+1} \left[\frac{B+b-n\ln x_{1:n}}{B+b+\ln x-(n+1)\ln x_{1:n}} \right]^{k+a-1}.$$

Using (3.4), (3.8) and (3.9), when $c > -(B+b-n\ln x_{1:n})$, the Bayesian estimates of θ and λ under the LINEX loss are given, respectively, by

$$\tilde{\theta}_{BL} = \frac{k+a-1}{c} \ln\left(1+\frac{c}{B+b-n\ln x_{1:n}}\right),$$

$$\tilde{\lambda}_{BL} = -\frac{1}{c} \ln\left\{ n(k+a-1)(B+b-n\ln x_{1:n})^{k+a-1} \int_0^{x_{1:n}} \lambda^{-1}(B+b-n\ln\lambda)^{-(k+a)} e^{-c\lambda} d\lambda \right\}. \quad (3.11)$$

Because

$$E_{\lambda,\theta}\left[e^{-cR(x)}\left|\underline{x}\right.\right]=E_{\lambda,\theta}\left(e^{-c\lambda^{\theta}x^{-\theta}}\left|\underline{x}\right.\right)$$

$$=n\left(B+b-n\ln x_{1:n}\right)^{k+a-1}\sum_{j=0}^{\infty}\frac{(-c)^{j}}{j!(n+j)}\left[B+b+j\ln x-(n+j)\ln x_{1:n})\right]^{-(k+a-1)}.$$

The Bayesian estimate of $R(x)$ under the LINEX loss is

$$\tilde{R}_{BL}(x)=-\frac{1}{c}\ln\left\{E_{\lambda,\theta}\left[e^{-cR(x)}\left|\underline{x}\right.\right]\right\}$$

$$=-\frac{1}{c}\ln\left\{n\left(B+b-n\ln x_{1:n}\right)^{k+a-1}\sum_{j=0}^{\infty}\frac{(-c)^{j}}{j!(n+j)}\left[B+b+j\ln x-(n+j)\ln x_{1:n})\right]^{-(k+a-1)}\right\}.$$

Using (3.5), (3.8) and (3.9), when $q<k+a-1$, the Bayesian estimates of θ and λ under the general entropy loss are given, respectively, by

$$\tilde{\theta}_{BG}=\left[\frac{\Gamma(k+a-1)}{\Gamma(k+a-q-1)}\right]^{1/q}(B+b-n\ln x_{1:n})^{-1},$$

$$\tilde{\lambda}_{BG}=\left\{n(k+a-1)(B+b-n\ln x_{1:n})^{k+a-1}\int_{0}^{x_{1:n}}\lambda^{-q-1}(B+b-n\ln\lambda)^{-(k+a)}d\lambda\right\}^{-1/q}. \qquad (3.12)$$

Because

$$E_{\lambda,\theta}\left[R^{-q}(x)\left|\underline{x}\right.\right]=E_{\lambda,\theta}\left(\lambda^{-q\theta}x^{q\theta}\left|\underline{x}\right.\right)$$

$$=\frac{n(B+b-n\ln x_{1:n})^{k+a-1}}{(n-q)\left[B+b-q\ln x-(n-q)\ln x_{1:n}\right]^{k+a-1}},$$

the Bayesian estimate of $R(x)$ under the general entropy loss is

$$\tilde{R}_{BG}(x)=\left\{\frac{n(B+b-n\ln x_{1:n})^{k+a-1}}{(n-q)\left[B+b-q\ln x-(n-q)\ln x_{1:n}\right]^{k+a-1}}\right\}^{-1/q}.$$

The approximate values of (3.10)–(3.12) can be obtained by numerical method, so as to obtain the Bayesian estimates of λ.

According to the posterior density (3.9) of λ, its cumulative distribution function can be obtained as

$$F_{\lambda}(y)=\frac{(B+b-n\ln x_{1:n})^{k+a-1}}{(B+b-n\ln y)^{k+a-1}},0<y\le x_{1:n}.$$

Since $F_\lambda(y) \sim U(0,1)$, let $u_1, u_2, ..., u_N$ be mutually independent random numbers from a uniform distribution $U(0,1)$. Using the inverse transform method,

$y_i = \exp\left\{\frac{1}{n}\left[B+b-(B+b-n\ln x_{1:n})u_i^{-1/(k+a-1)}\right]\right\}$, $(i=1,2,...,N)$ are the random numbers from the density function (3.9). So we can obtain

$$\tilde{\lambda}_{BS} \approx \frac{1}{N}\sum_{i=1}^{N} y_i, \quad \tilde{\lambda}_{BL} \approx -\frac{1}{c}\ln\left(\frac{1}{N}\sum_{i=1}^{N} e^{-cy_i}\right), \quad \tilde{\lambda}_{BG} \approx \left(\frac{1}{N}\sum_{i=1}^{N} y_i^{-q}\right)^{-1/q}.$$

Previously, we discussed the point estimation of the unknown parameters, and the Bayesian credible intervals for θ and λ will be given below.
According to the posterior density (3.8) of θ, we can obtain

$$2(B+b-n\ln x_{1:n})\theta \sim \chi^2(2k+2a-2),$$

the $100(1-\gamma)\%$ equal-tail credible interval of θ is (θ_L, θ_U), where

$$\theta_L = \frac{\chi^2_{1-\gamma/2}(2k+2a-2)}{2(B+b-n\ln x_{1:n})}, \quad \theta_U = \frac{\chi^2_{\gamma/2}(2k+2a-2)}{2(B+b-n\ln x_{1:n})}.$$

According to the posterior density (3.9), the $100(1-\gamma)\%$ equal-tail credible interval of λ is (λ_L, λ_U), where λ_L and λ_U should satisfy

$$P(\lambda < \lambda_L) = \int_0^{\lambda_L} \pi(\lambda|\underline{x})d\lambda = \gamma/2, \quad P(\lambda > \lambda_U) = \int_{\lambda_U}^{x_{1:n}} \pi(\lambda|\underline{x})d\lambda = \gamma/2. \tag{3.13}$$

According to (3.13), we can further obtain

$$\lambda_L = \exp\left\{\frac{1}{n}\left[B+b-(B+b-n\ln x_{1:n})(\gamma/2)^{-\frac{1}{k+a-1}}\right]\right\},$$

$$\lambda_U = \exp\left\{\frac{1}{n}\left[B+b-(B+b-n\ln x_{1:n})(1-\gamma/2)^{-\frac{1}{k+a-1}}\right]\right\}.$$

4. Prediction of censored units

In this section, we will give the prediction methods of censored observations Z_{ij}, $j=1,2,...,R_i$, and $i=1,2,...,k$. For the given observation data $\underline{x}$, then the conditional density of Z_{ij} is

$$f(z_{ij}|\underline{x},\lambda,\theta) = j\binom{R_i}{j} f(z_{ij})\left[F(z_{ij})-F(x_{i:n})\right]^{j-1}\left[1-F(z_{ij})\right]^{R_i-j}\left[1-F(x_{i:n})\right]^{-R_i}, \tag{4.1}$$

where $z_{ij} > x_{i:n}$, $j = 1, 2, ..., R_i$.

4.1. Best unbiased predictor

In this subsection, we consider the best unbiased predictor (BUP) to predict censored observation Z_{ij}. A statistic $\hat{z}_{ij}$ is called a BUP of Z_{ij}, if the predictor error $\hat{z}_{ij} - Z_{ij}$ has a mean zero and $Var(\hat{z}_{ij} - Z_{ij})$ is less than or equal to the variance of any unbiased predictor of Z_{ij}. Therefore, the BUP of Z_{ij} can be obtained as $E(Z_{ij} \mid \underline{x})$. In (4.1), if $\frac{1-F(z_{ij})}{1-F(x_{i:n})} = u$, then $u \mid \underline{x} \sim Beta(R_i - j + 1, j)$. If the BUP of Z_{ij} is denoted as $\hat{z}_{BUP}^{(ij)}$, we can obtain

$$\hat{z}_{BUP}^{(ij)} = \int_{x_{i:n}}^{+\infty} z_{ij} f(z_{ij} \mid \underline{x}, \lambda, \theta) dz_{ij}$$

$$= \frac{x_{i:n}}{Beta(R_i - j + 1, j)} \int_0^1 u^{R_i - j - 1/\theta} (1-u)^{j-1} du$$

$$= \frac{x_{i:n}}{Beta\left(R_i - j + 1, \ \ j\right)} \cdot Beta\left(R_i - j - 1/\theta + 1, \ \ j\right). \tag{4.2}$$

The above expression contains the unknown parameter θ, and the desired BUP can be obtained by substituting $\tilde{\theta}$ into (4.2).

4.2. Conditional median predictor

In this subsection, we consider using the conditional median to predict censored observation Z_{ij}. According to the definition of the conditional median, for a given $\underline{x}$, the median of the conditional distribution of Z_{ij} is the conditional median predictor (CMP), denoted as $\hat{z}_{CMP}^{(ij)}$, it needs to satisfy

$$P\left(Z_{ij} \le \hat{z}_{CMP}^{(ij)}\right) = P\left(Z_{ij} \ge \hat{z}_{CMP}^{(ij)}\right).$$

Using the conditional density function (4.1), it can be expressed as

$$\int_{\hat{z}_{CMP}^{(ij)}}^{+\infty} f(z_{ij} \mid \underline{x}, \lambda, \theta) = 0.5. \tag{4.3}$$

Substituting (2.1) into (4.1), the conditional density of Z_{ij} is

$$f(z_{ij} \mid \underline{x}, \lambda, \theta) = j \binom{R_i}{j} \theta \sum_{m=0}^{j-1} (-1)^m \binom{j-1}{m} x_{i:n}^{(R_i + m - j + 1)\theta} z_{ij}^{-(R_i + m - j + 1)\theta - 1}, \quad z_{ij} > x_{i:n}. \tag{4.4}$$

Then (4.3) can be transformed into

$$j\binom{R_i}{j}\sum_{m=0}^{j-1}(-1)^m\binom{j-1}{m}\frac{1}{R_i+m-j+1}\left(\frac{x_{i:n}}{\hat{z}_{CMP}^{(ij)}}\right)^{\theta(R_i+m-j+1)}=0.5\,. \tag{4.5}$$

Therefore, the CMP $\hat{z}_{CMP}^{(ij)}$ of Z_{ij} is the solution of Eq (4.5), and θ is replaced by its maximum likelihood estimate $\tilde{\theta}$.

4.3. Bayesian median predictor

In the previous subsections we used the classical methods to predict Z_{ij}. In this subsection we consider the Bayesian method to predict Z_{ij}. Using the posterior density (3.7) of (λ,θ), the corresponding posterior prediction density is

$$h(z_{ij}|\underline{x})=\int_0^{+\infty}\int_0^{x_{1:n}} f(z_{ij}|\underline{x},\lambda,\theta)\pi(\lambda,\theta|\underline{x})d\lambda d\theta$$

$$=(k+a-1)(B+b-n\ln x_{1:n})^{k+a-1}j\binom{R_i}{j}\sum_{m=0}^{j-1}(-1)^m\binom{j-1}{m}$$

$$\times z_{ij}^{-1}\left[B+b-n\ln x_{1:n}+(R_i+m-j+1)(\ln z_{ij}-\ln x_{i:n})\right]^{-(k+a)}. \tag{4.6}$$

According to (4.6), the Bayesian posterior survival function is

$$S(t|\underline{x})=\int_t^{+\infty}h(z_{ij}|\underline{x})dz_{ij}$$

$$=(B+b-n\ln x_{1:n})^{k+a-1}j\binom{R_i}{j}\sum_{m=0}^{j-1}(-1)^m\binom{j-1}{m}$$

$$\times\frac{1}{R_i+m-j+1}\left[B+b-n\ln x_{1:n}+(R_i+m-j+1)(\ln t-\ln x_{i:n})\right]^{-(k+a)+1},t>x_{i:n}. \tag{4.7}$$

According to the definition of the median, if the Bayesian median predictor (BMP) of Z_{ij} is $\hat{z}_{BMP}^{(ij)}$, it needs to satisfy

$$S(\hat{z}_{BMP}^{(ij)}|\underline{x})=0.5,$$

that is

$$(B+b-n\ln x_{1:n})^{k+a-1}j\binom{R_i}{j}\sum_{m=0}^{j-1}(-1)^m\binom{j-1}{m}$$

$$\times \frac{1}{R_i+m-j+1}\Big[B+b-n\ln x_{1:n}+(R_i+m-j+1)(\ln \hat{z}_{BMP}^{(ij)}-\ln x_{i:n})\Big]^{-(k+a)+1}=0.5.$$

4.4. Prediction intervals

In the previous subsections, we obtained point prediction of Z_{ij} using a variety of methods. Prediction intervals of censored observations will be constructed below, and two types of prediction intervals are obtained in this subsection using the classical method and the Bayesian method, respectively.
According to the conditional density (4.4), the predictive survival function can be obtained as

$$S^*(t|\underline{x},\lambda,\theta)=\int_t^{+\infty} f(z_{ij}|\underline{x},\lambda,\theta)dz_{ij}$$

$$=j\binom{R_i}{j}\sum_{m=0}^{j-1}(-1)^m\binom{j-1}{m}\frac{1}{R_i+m-j+1}\left(\frac{x_{i:n}}{t}\right)^{(R_i+m-j+1)\theta}, \quad t>x_{i:n}.$$

Therefore, the $100(1-\gamma)\%$ classical prediction interval of Z_{ij} is $(\hat{z}_L^{(ij)},\hat{z}_U^{(ij)})$, and the lower bound $\hat{z}_L^{(ij)}$ and the upper bound $\hat{z}_U^{(ij)}$ should satisfy

$$j\binom{R_i}{j}\sum_{m=0}^{j-1}(-1)^m\binom{j-1}{m}\frac{1}{R_i+m-j+1}\left(\frac{x_{i:n}}{\hat{z}_L^{(ij)}}\right)^{(R_i+m-j+1)\theta}=1-\gamma/2,$$

$$j\binom{R_i}{j}\sum_{m=0}^{j-1}(-1)^m\binom{j-1}{m}\frac{1}{R_i+m-j+1}\left(\frac{x_{i:n}}{\hat{z}_U^{(ij)}}\right)^{(R_i+m-j+1)\theta}=\gamma/2,$$

where θ is replaced by its maximum likelihood estimate $\tilde{\theta}$.

According to the Bayesian posterior survival function (4.7), the $100(1-\gamma)\%$ Bayesian prediction interval of Z_{ij} is $(\tilde{z}_L^{(ij)},\tilde{z}_U^{(ij)})$, and $\tilde{z}_L^{(ij)}$ and $\tilde{z}_U^{(ij)}$ should satisfy

$$(B+b-n\ln x_{1:n})^{k+a-1}\, j\binom{R_i}{j}\sum_{m=0}^{j-1}(-1)^m\binom{j-1}{m}$$

$$\times \frac{1}{R_i+m-j+1}\Big[B+b-n\ln x_{1:n}+(R_i+m-j+1)(\ln \tilde{z}_L^{(ij)}-\ln x_{i:n})\Big]^{-(k+a)+1}=1-\gamma/2,$$

$$(B+b-n\ln x_{1:n})^{k+a-1}\, j\binom{R_i}{j}\sum_{m=0}^{j-1}(-1)^m\binom{j-1}{m}$$

$$\times \frac{1}{R_i + m - j + 1}\left[B + b - n\ln x_{1:n} + (R_i + m - j + 1)(\ln \tilde{z}_U^{(ij)} - \ln x_{i:n})\right]^{-(k+a)+1} = \gamma / 2 .$$

5. Simulation study

In this section, we will use Monte-Carlo simulation to verify the properties of the estimators. The simulated samples come from the progressively double Type-II hybrid censoring scheme of the two-parameter Pareto distribution, and $\lambda = 6, \theta = 2$. When the values of $m_1, m_2, (R_1, R_2, ..., R_k)$ and t_0 are given under small sample size $(n = 20)$, medium sample size $(n = 40)$, and large sample size $(n = 60)$, respectively, the progressively double Type-II hybrid censored data with three sample sizes can be obtained. The values of the hyper-parameters are taken as $(a, b) = (1,1)$, and $c = 1, q = 1$ in the loss functions. The values of the estimators are calculated based on the censored data. Each estimator is simulated 10000 times to calculate mean value (MV) and mean square error (MSE), and the formula of MSE is

$$MSE(\hat{\phi}) = \frac{1}{10000}\sum(\hat{\phi} - \phi)^2 ,$$

where $\hat{\phi}$ is an estimate of ϕ.

For convenience, short notation is used to represent different, for example, scheme $(3, 0, 0, 0, 0)$ is denoted as $(3, 0^{*4})$. The MVs and MSEs for all point estimates of λ, θ and $R(x)$ are listed in Tables 1–5. From the values in the Tables, it is easily observed that when n is fixed, the MSEs of all estimators decrease as t_0 increases. Furthermore, when t_0 is fixed, the MSEs of all estimators decrease as n increases. In terms of MSEs, all Bayesian estimates are better than the corresponding maximum likelihood estimates under the same condition, and the differences between them decrease rapidly as n increases. Especially in the case of small sample size, the advantage of Bayesian estimation is more obvious. In general, for three types of Bayesian estimates of θ and $R(x)$, the MSE is minimal under the LINEX loss. For the Bayesian estimation of λ, the MSE is minimal under the general entropy loss. Under the same condition, when λ is unknown, the MSEs of the point estimates of θ and $R(x)$ is larger than the MSEs when λ is known. From the MVs of the point estimates, the maximum likelihood estimates of the unknown parameters θ and λ are greater than their Bayesian estimates. The maximum likelihood estimate of the reliability function is less than three Bayesian estimates.

Table 1. MVs and MSEs of point estimates for θ when λ is known.

n	m_1	m_2	$(R_1, R_2, ..., R_k)$	t_0		$\hat{\theta}$	$\hat{\theta}_{BS}$	$\hat{\theta}_{BL}$	$\hat{\theta}_{BG}$
20	10	14	$(0^{*9},10)$ or $(0^{*9},3,0^{*3},3)$	8	MV	2.13812	1.91591	1.75973	1.74171
					MSE	0.32570	0.20271	0.19925	0.22839
				12	MV	1.88524	1.77095	1.67082	1.65297
					MSE	0.28010	0.19222	0.18354	0.22487

Continued on next page

n	m_1	m_2	$(R_1, R_2, ..., R_k)$	t_0		$\hat{\theta}$	$\hat{\theta}_{BS}$	$\hat{\theta}_{BL}$	$\hat{\theta}_{BG}$
40	20	28	$(0^{*19},20)$ or $(0^{*19},6,0^{*7},6)$	8	MV	2.02012	1.91813	1.83252	1.82686
					MSE	0.21293	0.16274	0.15797	0.17158
				12	MV	2.03174	1.95786	1.89290	1.89014
					MSE	0.13764	0.11101	0.10686	1.87791
60	30	42	$(0^{*29},30)$ or $(0^{*29},8,0^{*11},10)$	8	MV	2.00764	1.94051	1.88055	1.87791
					MSE	0.14173	0.11815	0.11490	0.12226
				12	MV	2.03286	1.98336	1.93827	1.93738
					MSE	0.08793	0.07465	0.07148	0.07494

Table 2. MVs and MSEs of point estimates for $R(x)$ when λ is known $(x = 7.5)$.

n	m_1	m_2	$(R_1, R_2, ..., R_k)$	t_0		$\hat{R}$	$\hat{R}_{BS}$	$\hat{R}_{BL}$	$\hat{R}_{BG}$
20	10	14	$(0^{*9},10)$ or $(0^{*9},3,0^{*3},3)$	8	MV	0.62604	0.66065	0.65742	0.64968
					MSE	0.00689	0.00432	0.00428	0.00448
				12	MV	0.65946	0.67904	0.67543	0.67195
					MSE	0.00454	0.00406	0.00365	0.00379
40	20	28	$(0^{*19},20)$ or $(0^{*19},6,0^{*7},6)$	8	MV	0.64046	0.65712	0.65401	0.65138
					MSE	0.00409	0.00337	0.00336	0.00342
				12	MV	0.63761	0.64990	0.64625	0.64562
					MSE	0.00265	0.00225	0.00224	0.00229
60	30	42	$(0^{*29},30)$ or $(0^{*29},8,0^{*11},10)$	8	MV	0.64111	0.65233	0.64901	0.64838
					MSE	0.00272	0.00239	0.00237	0.00241
				12	MV	0.63667	0.64501	0.64300	0.64207
					MSE	0.00166	0.00146	0.00146	0.00148

Table 3. MVs and MSEs of point estimates for θ when λ is unknown.

n	m_1	m_2	$(R_1, R_2, ..., R_k)$	t_0		$\tilde{\theta}$	$\tilde{\theta}_{BS}$	$\tilde{\theta}_{BL}$	$\tilde{\theta}_{BG}$
20	10	14	$(0^{*9},10)$ or $(0^{*9},3,0^{*3},3)$	8	MV	2.23162	1.90169	1.73313	1.71152
					MSE	0.37844	0.23303	0.22707	0.26414
				12	MV	2.00654	1.74541	1.64163	1.62074
					MSE	0.32407	0.18069	0.21918	0.24375
40	20	28	$(0^{*19},20)$ or $(0^{*19},6,0^{*7},6)$	8	MV	2.11199	1.90141	1.81309	1.80634
					MSE	0.25496	0.16798	0.16572	0.18034
				12	MV	2.09806	1.94752	1.88105	1.87796
					MSE	0.15981	0.11365	0.11056	0.11801
60	30	42	$(0^{*29},30)$ or $(0^{*29},8,0^{*11},10)$	8	MV	2.06281	1.92611	1.86518	1.86191
					MSE	0.15110	0.11631	0.11539	0.12265
				12	MV	2.07338	1.97403	1.92826	1.92703
					MSE	0.09413	0.07295	0.07074	0.07421

Table 4. MVs and MSEs of point estimates for $R(x)$ when λ is unknown $(x = 7.5)$.

n	m_1	m_2	$(R_1, R_2, ..., R_k)$	t_0		$\tilde{R}$	$\tilde{R}_{BS}$	$\tilde{R}_{BL}$	$\tilde{R}_{BG}$
20	10	14	$(0^{*9},10)$ or $(0^{*9},3,0^{*3},3)$	8	MV	0.62867	0.66028	0.65698	0.64910
					MSE	0.00871	0.00441	0.00434	0.00457
				12	MV	0.67303	0.67696	0.67346	0.66931
					MSE	0.00597	0.00431	0.00394	0.00404
40	20	28	$(0^{*19},20)$ or $(0^{*19},6,0^{*7},6)$	8	MV	0.64217	0.65660	0.65311	0.65082
					MSE	0.00451	0.00332	0.00314	0.00337
				12	MV	0.64290	0.64897	0.64582	0.64460
					MSE	0.00288	0.00223	0.00213	0.00227
60	30	42	$(0^{*29},30)$ or $(0^{*29},8,0^{*11},10)$	8	MV	0.64226	0.65203	0.65101	0.64807
					MSE	0.00286	0.00235	0.00221	0.00237
				12	MV	0.63998	0.64431	0.64232	0.64133
					MSE	0.00175	0.00145	0.00138	0.00148

Table 5. MVs and MSEs of point estimates for λ.

n	m_1	m_2	$(R_1, R_2, ..., R_k)$	t_0		$\tilde{\lambda}$	$\tilde{\lambda}_{BS}$	$\tilde{\lambda}_{BL}$	$\tilde{\lambda}_{BG}$
20	10	14	$(0^{*9},10)$ or $(0^{*9},3,0^{*3},3)$	8	MV	6.14564	5.95291	5.93523	5.95753
					MSE	0.04373	0.02135	0.02324	0.02056
				12	MV	6.14564	5.96641	5.94884	5.96968
					MSE	0.04373	0.01873	0.02016	0.01750
40	20	28	$(0^{*19},20)$ or $(0^{*19},6,0^{*7},6)$	8	MV	6.06610	5.96132	5.95652	5.96550
					MSE	0.00822	0.00765	0.00824	0.00758
				12	MV	6.06610	5.97758	5.97013	5.97908
					MSE	0.00822	0.00714	0.00789	0.00702
60	30	42	$(0^{*29},30)$ or $(0^{*29},8,0^{*11},10)$	8	MV	6.04169	5.97864	5.96840	5.98213
					MSE	0.00307	0.00302	0.00324	0.00286
				12	MV	6.04169	5.98151	5.97204	5.98540
					MSE	0.00307	0.00294	0.00311	0.00279

6. Numerical example

Next, we will analyze the example from reference [22], in which the failure data follow Type-II Pareto distribution. After transformation, the failure data from the two-parameter Pareto distribution can be obtained, which are arranged in ascending order as follows: 0.5009, 0.5040, 0.5142, 0.5221, 0.5261, 0.5418, 0.5473, 0.5834, 0.6091, 0.6252, 0.6404, 0.6498, 0.6750, 0.7031, 0.7099, 0.7168, 0.7918, 0.8465, 0.9035, 1.1143.

If we take $m_1 = 10$, $m_2 = 14$, $t_0 = 0.7$, and the number of units progressively removed is $(R_1, R_2, ..., R_k) = (0^{*9}, 3, 0^{*3}, 3)$, then the censored data are obtained as 0.5009, 0.5040, 0.5142, 0.5221, 0.5261, 0.5418, 0.5473, 0.5834, 0.6091, 0.6252, 0.6404, 0.6750, 0.7031, 0.7168.

Using the conclusions in this paper, when a,b,c and q are taken different values, we can calculate the Bayesian estimates of unknown parameters and reliability under three types of loss functions, and the obtained estimates are shown in Tables 6 and 7. According to the test data, the failure observations of the removed units are predicted, including point prediction and interval prediction, and the computational results are shown in Table 8.

Table 6. Bayesian estimates of θ and λ.

(a,b)	$\tilde{\theta}_{BS}$	$\tilde{\lambda}_{BS}$	(c,q)	$\tilde{\theta}_{BL}$	$\tilde{\lambda}_{BL}$	$\tilde{\theta}_{BG}$	$\tilde{\lambda}_{BG}$
(1, 1)	2.89176	0.49175	(1, 2)	2.62876	0.49172	2.78657	0.49145
			(2, -1)	2.42054	0.49167	3.09831	0.49175
			(-1, -2)	3.23918	0.49179	3.19992	0.49185
			(-2, 1)	3.73041	0.49185	2.89176	0.49156
(1, 2)	2.39671	0.48991	(1, 2)	2.21233	0.48985	2.30953	0.48947
			(2, -1)	2.06115	0.48978	2.56790	0.48991
			(-1, -2)	2.62876	0.48997	2.65212	0.49005
			(-2, 1)	2.93397	0.49004	2.39671	0.48962

Table 7. Bayesian estimates of $R(x)$ $(x = 0.6)$.

(a,b)	$\tilde{R}_{BS}$	(c,q)	$\tilde{R}_{BL}$	$\tilde{R}_{BG}$
(1, 1)	0.57045	(1, 2)	0.56709	0.55145
		(2, -1)	0.56372	0.57045
		(-1, -2)	0.57379	0.57629
		(-2, 1)	0.57712	0.55805
(1, 2)	0.62194	(1, 2)	0.61904	0.60696
		(2, -1)	0.61611	0.62194
		(-1, -2)	0.62483	0.62658
		(-2, 1)	0.62768	0.61215

Table 8. Prediction of Z_{ij} $(a = 1, b = 1, \gamma = 0.05)$.

i	j	$\hat{z}_{BUP}^{(ij)}$	$\hat{z}_{CMP}^{(ij)}$	$\hat{z}_{BMP}^{(ij)}$	$(\hat{z}_L^{(ij)}, \hat{z}_U^{(ij)})$	$(\tilde{z}_L^{(ij)}, \tilde{z}_U^{(ij)})$
10	1	0.68813	0.66612	0.67856	(0.62665, 0.87605)	(0.62702, 1.01671)
	2	0.79755	0.75616	0.79946	(0.64239, 1.19508)	(0.64642, 1.63324)
	3	1.09914	0.96407	1.09542	(0.68745, 2.31989)	(0.70087, 4.43453)
14	1	0.78859	0.76372	0.77801	(0.71846, 1.00445)	(0.71889, 1.16586)
	2	0.91441	0.86695	0.91650	(0.73649, 1.37017)	(0.74109, 1.87320)
	3	1.26018	1.10534	1.25543	(0.78815, 2.66005)	(0.80359, 5.08850)

It can be seen from Tables 6 and 7 that under the LINEX loss, the Bayesian estimates of θ and

$R(x)$ are larger when c is negative than when c is positive. Also, under the general entropy loss, the Bayesian estimates of θ and $R(x)$ are larger when q is negative than when q is positive. For different values of a,b,c and q, the Bayesian estimates of λ are very close under three types of loss functions. In Table 8, comparing the point prediction under three types of loss functions, it is found that the CMP is the smallest. The length of the Bayesian prediction intervals of Z_{ij} is greater than that of the classical prediction interval, and the real observations are within the prediction intervals.

7. Conclusions

In this paper, based on the progressively double Type-II hybrid censored data, the statistical inference of the two-parameter Pareto distribution is studied by using the classical and Bayesian methods. The maximum likelihood estimates of the unknown parameter(s) and reliability are obtained when the scale parameter is known and unknown, respectively. In the Bayesian method, we obtain the Bayesian estimates of the unknown parameter(s) and reliability under the squared loss, LINEX loss and general entropy loss, respectively. Since the Bayesian estimation of λ cannot be obtained in an explicit form, a Monte-Carlo simulation method is proposed to obtain its Bayesian estimation, and we also obtain the Bayesian credible intervals of the unknown parameters. The point prediction and interval prediction of the failure observations of the withdrawn units are carried out by the classical and Bayesian methods, and the point prediction includes the best unbiased predictor, the conditional median predictor and the Bayesian median predictor. Simulation results show that, on the basis of MSE, the Bayesian estimation is better than the corresponding maximum likelihood estimation. Based on a real data set, we calculate the Bayesian estimates of the unknown parameters and reliability, and predict the observations of the censored units. We mainly consider the application of the progressively double Type-II hybrid censoring scheme in the two-parameter Pareto model, and this scheme can also be applied to other life distributions.

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Conflict of interest

All authors declare no conflicts of interest.

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