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Research article

A kind of even order Bernoulli-type operator with bivariate Shepard

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Abstract: It is known that an efficient method for interpolation of very large scattered data sets is the method of Shepard. Unfortunately, it reproduces only the constants. In this paper, we first generalize an expansion in bivariate even order Bernoulli polynomials for real functions possessing a sufficient number of derivatives. Finally, by combining the known Shepard operator with the even order Bernoulli bivariate operator, we construct a kind of new approximated operator satisfying the higher order polynomial reproducibility. We study this combined operator and give some error bounds in terms of the modulus of continuity of high order and also with Peano's theorem. Numerical comparisons show that this new technique provides the higher degree of accuracy. Furthermore, the advantage of our method is that the algorithm is very simple and easy to implement.

Keywords: Shepard operator; Bernoulli operator; even order Bernoulli operator; error estimations
Mathematics Subject Classification: 41A05, 65D05, 65D15

1. Introduction

Suppose that f is a real-valued function defined on a domain $D \subset \mathbb{R}^2$ and $Z \subset D$, $(x_i, y_i) \in Z, i = 0, 1, \ldots, N$. The classical Shepard operator (first introduced in [1]) is defined by

$$(S_0 f)(x, y) = \sum_{i=0}^{N} A_{i,\mu}(x, y) f(x_i, y_i)$$

where

$$A_{i,\mu}(x,y) = \frac{\prod\limits_{\substack{j=0\\ j\neq i}}^{N} d_j^{\mu}(x,y)}{\sum\limits_{k=0}^{N}\prod\limits_{\substack{j=0\\ j\neq k}}^{N} d_j^{\mu}(x,y)} \tag{1.1}$$

with $d_j(x,y) = ((x-x_j)^2 + (y-y_j)^2)^{\frac{1}{2}}$ and $\mu \in \mathbb{R}_+$.

It follows that

$$\sum_{i=0}^{N} A_{i,\mu}(x,y) = 1 \tag{1.2}$$

A basic characteristic of an approximation operator is its degree of exactness, usually abbreviated by "dex".

The basic properties of S_0 are expressed as follows:
(1)$(S_0 f)(x_i, y_i) = f(x_i, y_i), i = 0, 1, \ldots, N$,
(2)$\text{dex}(S_0) = 0$.

Due to its small degree of exactness we are interested in extending the Shepard operator S_0 by combining it with some other operators. Several improved operators have been constructed to increase the degree of exactness of the classical Shepard operator: Taylor [2–4], Lagrange [3,5], Hermite [3,6], Birkhoff [3, 7], Bernoulli [8, 9], Lidstone [10], least square approximations [11–14] and splines [3]. The Shepard method can also refer to recent developments on the subject [15–21]. Recently, many other works have been studied on this multiquadric operator, see for example [22–29] and the methods are successfully applied in other scientific disciplines [30–34].

In the present paper, we first generalize an expansion in bivariate even order Bernoulli polynomials for real functions having a sufficient number of derivatives. To obtain an new operator with higher accuracy and better reproduction qualities, we combine the classical Shepard operator with the even order Bernoulli bivariate operator: the generalized Taylor polynomial.

The remainder of this paper is organized as follows. In section 2, we introduce the generalized Taylor polynomial of degree $(2m, 2n)$ and give new results on the error of approximation that will be used later in the paper. In section 3, we apply previous results to derive a kind of even order Bernoulli-type operator with bivariate Shepard, and give some error bounds in terms of the modulus of continuity of high order and also with Peano's theorem. In section 4, numerical examples are shown to compare the approximation capacity of the new operators with other existing methods and give some numerical examples. In section 5, we give the conclusions.

2. Some remarks about the generalized Taylor polynomial

2.1. Preliminaries

The Bernoulli polynomials are quite important in many branches of mathematics. A considerable amount of literature exists about the Bernoulli polynomials, i.e., the polynomials of the sequence defined recursively by means of the following relations [35]:

$$\begin{cases} B_0(x) = 1, \\ B_n'(x) = nB_{n-1}(x), & n \geq 1, \\ \int_0^1 B_n(x)dx = 0, & n \geq 1, \end{cases}$$

where $B_n(x)$ express the n-th degree Bernoulli polynomial. the values of $B_n(x)$ at $x = 0$ are known as Bernoulli numbers and are denoted by B_n. For functions in the class $C^m([a, b])$,$a, b \in \mathbb{R}$,$a < b$,$h = b-a$, we have the following relation [8]:

$$\begin{aligned} f(x) =& f(a) + \sum_{k=1}^{m} \frac{B_k\left(\frac{x-a}{h}\right) - B_k}{k!} h^{k-1}\left(f^{(k-1)}(b) - f^{(k-1)}(a)\right) \\ &+ \frac{h^{m-1}}{m!} \int_a^b f^{(m)}(t)\left(B_m\left(\frac{b-t}{h}\right) - B_m\left(\frac{(x-t)-[x-t]}{h}\right)\right)dt. \end{aligned}$$

2.2. *Construction of the expansion in the even order Bernoulli bivariate polynomial*

Let us recall the polynomial sequence defined recursively by means of the following relations [36]:

$$\begin{cases} v_0(x) = 1, \\ v_k'(x) = \int_0^x v_{k-1}(t)dt, & k \geq 1, \\ \int_0^1 v_k(x)dx = 0, & k \geq 1. \end{cases} \tag{2.1}$$

The polynomial sequence (2.1) is related to Bernoulli polynomials of even degree as follows

Proposition 1 (see [36])**.** *For each $k \geq 1$*

$$v_k(x) = \frac{2^{2k}}{(2k)!} B_{2k}\left(\frac{1+x}{2}\right)$$

At the same time, [37] provided the following well-known properties of Bernoulli polynomials:

$$\begin{cases} B_n\left(\frac{1}{2}\right) = -(1 - 2^{1-n})B_n(0), & n \geq 1, \\ B_n = B_n(0) = B_n(1), & n \geq 2, \\ B_{2n+1}(0) = 0, & n \geq 1. \end{cases}$$

For functions in the class $C^{2m+1}([a, b])$, $a, b \in \mathbb{R}$, $a < b$, $h = b - a$, we have the similar univariate even order Bernoulli interpolation formula [36]

$$f(x) = EB_m[f; a, b; h](x) + R_m[f; a, b; h](x), \quad x \in [a, b] \tag{2.2}$$

where the polynomial approximant is defined by

$$\begin{aligned} EB_m[f; a, b; h](x) =& f(a) + \sum_{j=1}^{m} h^{2j-1}\left(f^{(2j-1)}(b)\left(v_j\left(\frac{x-a}{h}\right) - v_j(0)\right)\right. \\ &\left. - f^{(2j-1)}(a)\left(v_j\left(\frac{b-x}{h}\right) - v_j(1)\right)\right) \end{aligned} \tag{2.3}$$

and the remainder term is

$$R_m[f;a,b;h](x) = h^{2m}\int_a^b f^{(2m+1)}(t)K_m\left(\frac{x-a}{h},\frac{t-a}{h}\right)dt \tag{2.4}$$

with $K_m(x,t)$ defined by the following relations

$$K_m(x,t) = \begin{cases} -\sum_{j=1}^{n}\left(v_j(x)-v_j(0)\right)\frac{(1-t)^{2m-2j+1}}{(2m-2j+1)!}, & x \le t,\\ \frac{t^{2m}}{(2m)!}+\sum_{j=1}^{m}\left(v_j(1-x)-v_j(1)\right)\frac{t^{2m-2j+1}}{(2m-2j+1)!}, & t \le x.\end{cases}$$

Remark 1. *For $f \in C^{2m+1}([a,b])$ we have*

$$\begin{aligned} EB_m[f](x) &:= EB_m[f;a,b;h](x)\\ &= f(a)+\sum_{j=1}^{m} h^{2j-1}\left(f^{(2j-1)}(b)V_j\left(\frac{x-a}{h}\right)-f^{(2j-1)}(a)\Lambda_j\left(\frac{b-x}{h}\right)\right),\\ V_j\left(\frac{x-a}{h}\right) &= v_j\left(\frac{x-a}{h}\right)-v_j(0),\\ \Lambda_j\left(\frac{b-x}{h}\right) &= v_j\left(\frac{b-x}{h}\right)-v_j(1)\end{aligned}$$

and denote by $T_{2m}[f;a](x)$ the well-known Taylor polynomial of degree $2m$ centered at a. It is easy to find that

$$\lim_{h\to 0} EB_m[f;a,b;h](x) = T_{2m}[f;a](x).$$

For this reason we call the polynomial $EB_m[f;a,b;h]$ the generalized Taylor polynomial for f of degree $2m$ in $[a,b]$.

Let $X = [a,b]\times[c,d]$ be a rectangular domain in the plane $\mathbb{R}^2$. We denote by $C^{(2m+1,2n+1)}(X)$ the space of functions $f : X \to \mathbb{R}$ possessing continuous partial derivatives

$$f^{(i,j)}(x,y) = \frac{\partial^{i+j}}{\partial x^i \partial y^j} f(x,y), \quad (x,y)\in X$$

for all (i, j), $0 \le i \le 2m+1$, $0 \le j \le 2n+1$.

We set the mth power of the argument

$$(\cdot)^m = \begin{cases}(\cdot)^m, & m \ge 0\\ 0, & m < 0\end{cases},$$

$h = b-a, k = d-c$ and obtain the following Theorem.

Theorem 1. *Let $f \in C^{(2m+1,2n+1)}(X)$, $m,n \ge 1$. Then at each $(x,y)\in X$ the following identity holds:*

$$
\begin{aligned}
f(x,y) = f(a,c) &+ \sum_{i=1}^{m} h^{2i-1}\left(f^{(2i-1,0)}(b,c)V_i\left(\frac{x-a}{h}\right) - f^{(2i-1,0)}(a,c)\Lambda_i\left(\frac{b-x}{h}\right)\right) \\
&+ \sum_{j=1}^{n} k^{2j-1}\left(f^{(0,2j-1)}(a,d)V_j\left(\frac{y-c}{k}\right) - f^{(0,2j-1)}(a,c)\Lambda_j\left(\frac{d-y}{k}\right)\right) \\
&+ \sum_{i=1}^{m}\sum_{j=1}^{n} h^{2i-1}k^{2j-1}\left(f^{(2i-1,2j-1)}(b,d)V_i\left(\frac{x-a}{h}\right)V_j\left(\frac{y-c}{k}\right)\right. \\
&- f^{(2i-1,2j-1)}(b,c)V_i\left(\frac{x-a}{h}\right)\Lambda_j\left(\frac{d-y}{k}\right) - f^{(2i-1,2j-1)}(a,d)\Lambda_i\left(\frac{b-x}{h}\right)V_j\left(\frac{y-c}{k}\right) \\
&\left.+f^{(2i-1,2j-1)}(a,c)\Lambda_i\left(\frac{b-x}{h}\right)\Lambda_j\left(\frac{d-y}{k}\right)\right) + R_{m,n}[f;a,b;c,d;h,k](x,y)
\end{aligned} \tag{2.5}
$$

with the remainder

$$
\begin{aligned}
&R_{m,n}[f;a,b;c,d;h,k](x,y) \\
&= \sum_{j<2n+1}\int_a^b f^{(2m+1,j)}(s,c)H^x_{m,j}(x,y,s)ds + \sum_{i<2m+1}\int_c^d f^{(i,2n+1)}(a,t)H^y_{i,n}(x,y,t)dt \\
&\quad + \int_a^b\int_c^d f^{(2m+1,2n+1)}(s,t)H^{x,y}_{m,n}(x,y,s,t)dsdt
\end{aligned} \tag{2.6}
$$

where $H^x_{m,j}(x,y,s)$, $H^y_{i,n}(x,y,t)$, and $H^{x,y}_{m,n}(x,y,s,t)$ are the Peano's kernels.

Proof. By applying the Peano's Theorem for bidimensional case [38] and by using the relation (2.5), we obtain the following equalities:

$$
\begin{aligned}
H^x_{m,j}(x,y,s) &= R_{m,n}\left[\frac{(x-s)_+^{2m}}{(2m)!}\frac{(y-c)^j}{j!};a,b;c,d;h,k\right](x,y), \\
H^y_{i,n}(x,y,t) &= R_{m,n}\left[\frac{(x-a)^i}{i!}\frac{(y-t)_+^{2n}}{(2n)!};a,b;c,d;h,k\right](x,y), \\
H^{x,y}_{m,n}(x,y,t) &= R_{m,n}\left[\frac{(x-s)_+^{2m}}{(2m)!}\frac{(y-t)_+^{2n}}{(2n)!};a,b;c,d;h,k\right](x,y)
\end{aligned}
$$

where

$$
z_+ = \begin{cases} z, & z > 0, \\ 0, & z \le 0. \end{cases}
$$

Furthermore,

$$
\begin{aligned}
H^x_{m,j}(x,y,s) = & \frac{(x-s)_+^{2m}}{(2m)!}\frac{(y-c)^j}{j!} - \Bigg[\frac{(x-s)_+^{2m}}{(2m)!}\frac{(y-c)^j}{j!}\Big|_{\substack{x=a\\y=c}} \\
& + \sum_{p=1}^{m} h^{2p-1}\Bigg(\left(\frac{(x-s)_+^{2m}}{(2m)!}\frac{(y-c)^j}{j!}\right)^{(2p-1,0)}\Big|_{\substack{x=b\\y=c}} V_p\left(\frac{x-a}{h}\right) \\
& - \left(\frac{(x-s)_+^{2m}}{(2m)!}\frac{(y-c)^j}{j!}\right)^{(2p-1,0)}\Big|_{\substack{x=a\\y=c}} \Lambda_p\left(\frac{b-x}{h}\right)\Bigg) \\
& + \sum_{q=1}^{m} k^{2q-1}\Bigg(\left(\frac{(x-s)_+^{2m}}{(2m)!}\frac{(y-c)^j}{j!}\right)^{(0,2q-1)}\Big|_{\substack{x=a\\y=d}} V_q\left(\frac{y-c}{k}\right) \\
& - \left(\frac{(x-s)_+^{2m}}{(2m)!}\frac{(y-c)^j}{j!}\right)^{(0,2q-1)}\Big|_{\substack{x=a\\y=c}} \Lambda_q\left(\frac{d-y}{k}\right)\Bigg) + \sum_{p=1}^{m}\sum_{q=1}^{n} h^{2p-1}k^{2q-1} \\
& \times \Bigg(\left(\frac{(x-s)_+^{2m}}{(2m)!}\frac{(y-c)^j}{j!}\right)^{(2p-1,2q-1)}\Big|_{\substack{x=b\\y=d}} V_p\left(\frac{x-a}{h}\right) V_q\left(\frac{y-c}{k}\right) \\
& - \left(\frac{(x-s)_+^{2m}}{(2m)!}\frac{(y-c)^j}{j!}\right)^{(2p-1,2q-1)}\Big|_{\substack{x=b\\y=c}} V_p\left(\frac{x-a}{h}\right) \Lambda_q\left(\frac{d-y}{k}\right) \\
& - \left(\frac{(x-s)_+^{2m}}{(2m)!}\frac{(y-c)^j}{j!}\right)^{(2p-1,2q-1)}\Big|_{\substack{x=a\\y=d}} \Lambda_p\left(\frac{b-x}{h}\right) V_q\left(\frac{y-c}{k}\right) \\
& + \left(\frac{(x-s)_+^{2m}}{(2m)!}\frac{(y-c)^j}{j!}\right)^{(2p-1,2q-1)}\Big|_{\substack{x=a\\y=c}} \Lambda_p\left(\frac{b-x}{h}\right) \Lambda_q\left(\frac{d-y}{k}\right)\Bigg)\Bigg]
\end{aligned}
$$

with

$$
\begin{aligned}
& \frac{(x-s)_+^{2m}}{(2m)!}\frac{(y-c)^j}{j!}\Big|_{\substack{x=a\\y=c}} = 0, \\
& \left(\frac{(x-s)_+^{2m}}{(2m)!}\frac{(y-c)^j}{j!}\right)^{(2p-1,0)}\Big|_{\substack{x=b\\y=c}} = \begin{cases} \frac{(b-s)^{2m-2p+1}}{(2m-2p+1)!}, & j=0 \\ 0, & j=1,2,\ldots,2n \end{cases}, \\
& \left(\frac{(x-s)_+^{2m}}{(2m)!}\frac{(y-c)^j}{j!}\right)^{(2p-1,0)}\Big|_{\substack{x=a\\y=c}} = 0, \\
& \left(\frac{(x-s)_+^{2m}}{(2m)!}\frac{(y-c)^j}{j!}\right)^{(0,2q-1)}\Big|_{\substack{x=a\\y=d}} = 0, \\
& \left(\frac{(x-s)_+^{2m}}{(2m)!}\frac{(y-c)^j}{j!}\right)^{(0,2q-1)}\Big|_{\substack{x=a\\y=c}} = 0, \\
& \left(\frac{(x-s)_+^{2m}}{(2m)!}\frac{(y-c)^j}{j!}\right)^{(2p-1,2q-1)}\Big|_{\substack{x=b\\y=d}} = \frac{(b-s)^{2m-2p+1}k^{j-2q+1}}{(2m-2p+1)!(j-2q+1)!},
\end{aligned}
$$

$$\left(\frac{(x-s)_+^{2m}}{(2m)!}\frac{(y-c)^j}{j!}\right)^{(2p-1,2q-1)}\Bigg|_{\substack{x=b\\y=c}} = \begin{cases}\dfrac{(b-s)^{2m-2p+1}}{(2m-2p+1)!(j-2q+1)!}, & j-2q+1=0\\ 0, & j-2q+1\neq 0\end{cases},$$

$$\left(\frac{(x-s)_+^{2m}}{(2m)!}\frac{(y-c)^j}{j!}\right)^{(2p-1,2q-1)}\Bigg|_{\substack{x=a\\y=d}} = 0,$$

$$\left(\frac{(x-s)_+^{2m}}{(2m)!}\frac{(y-c)^j}{j!}\right)^{(2p-1,2q-1)}\Bigg|_{\substack{x=a\\y=c}} = 0,$$

so, we obtain

$$\begin{aligned}&H_{m,j}^x(x,y,s)\\ &= \frac{(x-s)_+^{2m}}{(2m)!}\frac{(y-c)^j}{j!} - \sum_{p=1}^m \frac{(b-s)^{2m-2p+1}h^{2p-1}}{(2m-2p+1)!}\cdot\lambda_1\cdot V_p\left(\frac{x-a}{h}\right) - \sum_{p=1}^m\sum_{q=1}^n \frac{(b-s)^{2m-2p+1}h^{2p-1}k^{2q-1}}{(2m-2p+1)!(j-2q+1)!}\\ &\times\left(k^{j-2q+1}\cdot V_p\left(\frac{x-a}{h}\right)V_q\left(\frac{y-c}{k}\right) - \kappa_1\cdot V_p\left(\frac{x-a}{h}\right)\Lambda_q\left(\frac{d-y}{k}\right)\right)\end{aligned}$$

where

$$\lambda_1 = \begin{cases}1, & j=0\\ 0, & j=1,2,\ldots,2n\end{cases}, \kappa_1 = \begin{cases}1, & j-2q+1=0\\ 0, & j-2q+1\neq 0\end{cases}.$$

The remaining kernels $H_{i,n}^y(x,y,t)$ and $H_{m,n}^{x,y}(x,y,s,t)$ may be obtained by the analogous arguments as follows:

$$\begin{aligned}&H_{i,n}^y(x,y,t)\\ &= \frac{(x-a)^i}{i!}\frac{(y-t)_+^{2n}}{(2n)!} - \sum_{q=1}^n \frac{(d-t)^{2n-2q+1}k^{2q-1}}{(2n-2q+1)!}\cdot\lambda_2\cdot V_q\left(\frac{y-c}{k}\right) - \sum_{p=1}^m\sum_{q=1}^n \frac{(d-t)^{2n-2q+1}h^{2p-1}k^{2q-1}}{(2n-2q+1)!(i-2p+1)!}\\ &\times\left(h^{i-2p+1}\cdot V_p\left(\frac{x-a}{h}\right)V_q\left(\frac{y-c}{k}\right) - \kappa_2\cdot\Lambda_p\left(\frac{b-x}{h}\right)V_q\left(\frac{y-c}{k}\right)\right)\end{aligned}$$

with

$$\lambda_2 = \begin{cases}1, & i=0\\ 0, & i=1,2,\ldots,2m\end{cases}, \kappa_2 = \begin{cases}1, & i-2p+1=0\\ 0, & i-2p+1\neq 0\end{cases}$$

and

$$\begin{aligned}H_{m,n}^{x,y}(x,y,s,t) =& \frac{(x-s)_+^{2m}}{(2m)!}\frac{(y-t)_+^{2n}}{(2n)!} - \sum_{i=1}^m\sum_{j=1}^n \frac{(b-s)^{2m-2p+1}(d-t)^{2n-2q+1}}{(2m-2p+1)!(2n-2q+1)!}\\ &\times h^{2p-1}k^{2q-1}\cdot V_p\left(\frac{x-a}{h}\right)V_q\left(\frac{y-c}{k}\right).\end{aligned}$$

□

Remark 2 (The generalized Taylor polynomial)**.** *For* $f \in C^{(2m+1,2n+1)}(X)$ *we denote by*

$EB_{m,n}[f](x,y) := EB_{m,n}[f;a,b;c,d;h,k](x,y)$ the polynomial of degree $(2m,2n)$ of the variables x,y

$$\begin{aligned}
EB_{m,n}[f](x,y) =& f(a,c) + \sum_{i=1}^{m} h^{2i-1}\left(f^{(2i-1,0)}(b,c)V_i\left(\frac{x-a}{h}\right) - f^{(2i-1,0)}(a,c)\Lambda_i\left(\frac{b-x}{h}\right)\right)\\
&+ \sum_{j=1}^{n} k^{2j-1}\left(f^{(0,2j-1)}(a,d)V_j\left(\frac{y-c}{k}\right) - f^{(0,2j-1)}(a,c)\Lambda_j\left(\frac{d-y}{k}\right)\right)\\
&+ \sum_{i=1}^{m}\sum_{j=1}^{n} h^{2i-1}k^{2j-1}\left(f^{(2i-1,2j-1)}(b,d)V_i\left(\frac{x-a}{h}\right)V_j\left(\frac{y-c}{k}\right)\right.\\
&- f^{(2i-1,2j-1)}(b,c)V_i\left(\frac{x-a}{h}\right)\Lambda_j\left(\frac{d-y}{k}\right) - f^{(2i-1,2j-1)}(a,d)\Lambda_i\left(\frac{b-x}{h}\right)V_j\left(\frac{y-c}{k}\right)\\
&\left. + f^{(2i-1,2j-1)}(a,c)\Lambda_i\left(\frac{b-x}{h}\right)\Lambda_j\left(\frac{d-y}{k}\right)\right)
\end{aligned} \tag{2.7}$$

and by $T_{2m,2n}[f;a;c](x,y)$ the well-known Taylor polynomial of degree $(2m,2n)$ of the variables x,y [39]

$$T_{2m,2n}[f;a,c](x,y) = \sum_{i=0}^{2m}\sum_{j=0}^{2n} \frac{(x-a)^i(y-c)^j}{i!j!} f^{(i,j)}(a,c).$$

Then

$$\lim_{h,k\to 0} EB_{m,n}[f;a,b;c,d;h,k](x,y) = T_{2m,2n}[f;a,c](x,y).$$

Remark 3 (Interpolation problem)**.** *the polynomial approximant $EB_{m,n}[f;a,b;c,d;h,k](x,y)$ satisfies the interpolation conditions as follows:*

$$EB_{m,n}[f](a,c) = f(a,c), \tag{2.8}$$

$$\begin{aligned}
&EB_{m,n}[f]^{(2i-1,0)}(a,c) = f^{(2i-1,0)}(a,c), 1 \le i \le m,\\
&EB_{m,n}[f]^{(0,2j-1)}(a,c) = f^{(0,2j-1)}(a,c), 1 \le j \le n,\\
&EB_{m,n}[f]^{(2i-1,2j-1)}(a,c) = f^{(2i-1,2j-1)}(a,c), 1 \le i \le m, 1 \le j \le n.
\end{aligned} \tag{2.9}$$

Remark 4 (Degree of exactness of the polynomial approximant (2.7))**.** *We denote by $\mathbb{P}^{(2m,2n)}$,$m,n \ge 1$ the space of polynomials $P(x,y)$ of degree (s,t) with $s \le 2m, t \le 2n$. We can easily prove that the polynomial approximant* (2.7) *has the degree of exactness* $(2m,2n)$*, i.e.*

$$R_{m,n}[P;a,b;c,d;h,k](x,y) = 0, \quad \forall P \in \mathbb{P}^{(2m,2n)}.$$

3. A kind of even order Bernoulli-type operator with bivariate Shepard

To increase the approximation capability of the expansion in bivariate even order Bernoulli polynomials, we study a kind of improved Shepard scheme by combining the Shepard operator with the even order Bernoulli bivariate operator.

Let us consider a function $f \in C^{(2m+1,2n+1)}(X)$, $X = [a,b]\times[c,d]$ and $N+1$ distinct points $(x_i,y_i) \in X$, $i = 0,1,\ldots,N$; we also set $h_i = x_{i+1} - x_i$, $k_i = y_{i+1} - y_i$, $i = 0,1,\ldots,N$ considering a fictive node $(x_{N+1},y_{N+1}) = (x_{N-1},y_{N-1})$.

Definition 1. *For each fixed $\mu > 0$ and $m, n = 1, 2, \dots$ a kind of even order Bernoulli-type operator with bivariate Shepard is introduced by*

$$S_{EB_{m,n}}[f](x, y) \;=\; \sum_{i=0}^{N} A_{i,\mu}(x, y) EB_{m,n}[f; x_i, x_{i+1}; y_i, y_{i+1}; h_i, k_i](x, y), (x, y) \in X \tag{3.1}$$

where $A_{i,\mu}(x, y)$ is the weight functions in barycentric form defined in (1.1) *and $EB^i_{m,n}[f](x, y) := EB_{m,n}[f; x_i, x_{i+1}; y_i, y_{i+1}; h_i, k_i](x, y)$, $i = 0, 1, \dots, N$ denote the even order Bernoulli bivariate interpolation operators constructed in* (2.7).

The following theorems can be easily checked.

Theorem 2. *The operator $S_{EB_{m,n}}[\cdot]$ is an interpolation operator in (x_i, y_i), $i = 0, 1, \dots, N$.*

Proof. The result can be obtained from the following well-known property

$$A_{i,\mu}(x_k, y_k) = \delta_{i,k}, \quad i, k = 0, 1, \dots, N. \tag{3.2}$$

□

Theorem 3. *The degree of exactness of the operator $S_{EB_{m,n}}[\cdot]$ is $(2m, 2n)$.*

Proof. The result can be obtained from the relation (1.2) since degree of exactness of the operator $EB^i_{m,n}[\cdot]$ is $(2m, 2n)$, $i = 0, 1, \dots, N$. □

Theorem 4. *For $f \in C^{(2m+1,2n+1)}(X)$ the operator $S_{EB_{m,n}}[f]$ also has the interpolation properties as follows:*

$$S_{EB_{m,n}}[f]^{(p,q)}(x_r, y_r) = f^{(p,q)}(x_r, y_r), r = 0, 1, \dots, N$$

where $1 \le p + q$ and $\max\{p + q | 0 \le p \le 2\alpha - 1, 1 \le \alpha \le m, 0 \le q \le 2\beta - 1, 1 \le \beta \le n\} < \mu$.

Proof. For $1 \le p + q$ and $\max\{p + q | 0 \le p \le 2\alpha - 1, 1 \le \alpha \le m, 0 \le q \le 2\beta - 1, 1 \le \beta \le n\} < \mu$ it is not difficult to see that

$$A^{(p,q)}_{i,\mu}(x_k, y_k) = 0, \quad i, k = 0, 1, \dots, N. \tag{3.3}$$

By applying the Leibniz rule and by using the relations (2.9), (3.2), (3.3) we get

$$\begin{aligned}
S_{EB_{m,n}}[f]^{(p,q)}(x_r, y_r) &= \sum_{i=0}^{N} \frac{\partial^q}{\partial y^q} \left(\frac{\partial^p}{\partial x^p} \left(A_{i,\mu}(x, y) EB^i_{m,n}[f](x, y) \right) \right) \Big|_{\substack{x=x_r \\ y=y_r}} \\
&= \sum_{i=0}^{N} \frac{\partial^q}{\partial y^q} \left(\sum_{l_1=0}^{p} \binom{p}{l_1} A^{(p-l_1,0)}_{i,\mu}(x, y) EB^i_{m,n}[f]^{(l_1,0)}(x, y) \right) \Big|_{\substack{x=x_r \\ y=y_r}} \\
&= \sum_{i=0}^{N} \left(\sum_{l_1=0}^{p} \binom{p}{l_1} \frac{\partial^q}{\partial y^q} \left(A^{(p-l_1,0)}_{i,\mu}(x, y) EB^i_{m,n}[f]^{(l_1,0)}(x, y) \right) \right) \Big|_{\substack{x=x_r \\ y=y_r}} \\
&= \sum_{i=0}^{N} \sum_{l_2=0}^{n} \sum_{l_1=0}^{m} \binom{q}{l_2} \binom{p}{l_1} A^{(p-l_1,q-l_2)}_{i,\mu}(x_r, y_r) EB^i_{m,n}[f]^{(l_1,l_2)}(x_r, y_r) \\
&= EB^r_{m,n}[f]^{(p,q)}(x_r, y_r) \\
&= f^{(p,q)}(x_r, y_r).
\end{aligned}$$

□

Next, Based on the mesh length, we have an estimation of the approximation error by using the modulus of smoothness of order k. Let us recall the following theorem. Some detailed definition is introduced in [40].

Theorem 5 (see for example [40], Th.7.3, p.225)**.** *Given a quasi-interpolation operator Q of order r, for each $f \in C[a, b]$, it follows the following estimation:*

$$\|f - Qf\|_\infty \leq C_r \omega_r(f; \delta)_\infty$$

where C_r is a constant and δ is defined by

$$\delta = \max_{0 \leq i \leq N} |x_{i+1} - x_i|.$$

Given functions $f \in C^{2m+1}[a, b]$ and $g \in C^{2n+1}[c, d]$ and corresponding even order Bernoulli polynomials $EB_m f$ and $EB_n g (m, n \geq 1)$ introduced by (2.3). Since the operators EB_m and EB_n are quasi-interpolation operators of order $2m + 1$ and $2n + 1$ respectively (see the definitions in p.144-146 of [40]), from Theorem 5 it follows the following estimates

$$\begin{aligned} \|f - EB_m[f]\|_\infty &\leq C_{2m+1} \omega_{2m+1}(f; \delta_1)_\infty, \\ \|f - EB_n[g]\|_\infty &\leq C_{2n+1} \omega_{2n+1}(f; \delta_2)_\infty \end{aligned} \tag{3.4}$$

where $\omega_k(f; t)_\infty$ denotes the k-th modulus of smoothness of a function f(see [41]) having

$$\begin{aligned} \delta_1 &= \max_{0 \leq i \leq N} |x_{i+1} - x_i|, \\ \delta_2 &= \max_{0 \leq j \leq N} |y_{j+1} - y_j| \end{aligned} \tag{3.5}$$

and C_{2m+1}, C_{2n+1} are some constants.

By applying the modulus of smoothness of high order, we give an estimation of the error.

Theorem 6. *Let $f \in C^{(2m+1,2n+1)}(X)$, $X = [a, b] \times [c, d]$ then*

$$\begin{aligned} \|f - S_{EB_{m,n}}[f]\|_\infty \leq & C_{2m+1} \max_{y \in [c,d]} \omega_{2m+1}(f(\cdot, y); \delta_1)_\infty \\ & + C_{2n+1} \max_{x \in [a,b]} \omega_{2n+1} f(x, \cdot); \delta_{2\infty} \\ & + C_{2m+1} \max_{y \in [c,d]} \omega_{2m+1}((f - EB_n[f])(\cdot, y); \delta_1)_\infty \end{aligned}$$

where δ_1, δ_2 are expressed in (3.5) and C_{2m+1}, C_{2n+1} are constants.

Proof. In terms of the relation (1.2), we have

$$\begin{aligned} f(x, y) - S_{EB_{m,n}}[f](x, y) &= f(x, y) - \sum_{i=0}^{N} A_{i,\mu}(x, y) EB^i_{m,n}[f](x, y) \\ &= \sum_{i=0}^{N} A_{i,\mu}(x, y) \left[f(x, y) - EB^i_{m,n}[f](x, y) \right]. \end{aligned}$$

We know from the relation (2.7)

$$EB_{m,n}[f](x,y) = EB_n\left[EB_m[f]\right](x,y) = EB_m\left[EB_n[f]\right](x,y),$$

so

$$\begin{aligned} f(x,y) - EB_{m,n}[f](x,y) =& f(x,y) - EB_m[f](x,y) + f(x,y) - EB_n[f](x,y) \\ &+ EB_m[f - EB_n[f]](x,y) - (f - EB_n[f])(x,y). \end{aligned}$$

We have

$$\begin{aligned} f(x,y) - S_{EB_{m,n}}[f](x,y) =& \sum_{i=0}^{N} A_{i,\mu}(x,y)(f(x,y) - EB_m^i[f](x,y)) \\ &+ \sum_{i=0}^{N} A_{i,\mu}(x,y)(f(x,y) - EB_n^i[f](x,y)) \\ &+ \sum_{i=0}^{N} A_{i,\mu}(x,y)\left(EB_m^i[f - EB_n^i[f]](x,y) - (f - EB_n^i[f])(x,y)\right) \end{aligned} \tag{3.6}$$

with $EB_m^i[f] = EB_m[f; x_i, x_{i+1}; h_i]$ and $EB_n^i[f] = EB_n[f; y_i, y_{i+1}; k_i]$. It follows from the relation (3.6)

$$\begin{aligned} |f(x,y) - S_{EB_{m,n}}[f](x,y)| \leq& \max_{y\in[c,d]} \|f(\cdot,y) - EB_m[f](\cdot,y)\|_\infty \sum_{i=0}^{N} A_{i,\mu}(x,y) \\ &+ \max_{x\in[a,b]} \|f(x,\cdot) - EB_n[f](x,\cdot)\|_\infty \sum_{i=0}^{N} A_{i,\mu}(x,y) \\ &+ \max_{y\in[c,d]} \|(f - EB_n[f])(\cdot,y) - EB_m[f - EB_n[f]](\cdot,y)\|_\infty \sum_{i=0}^{N} A_{i,\mu}(x,y). \end{aligned}$$

By using the relations (1.2) and (3.4), we can finish the proof. □

Furthermore,by applying the Peano's theorem we give the following integral representations of the error.

Theorem 7. *If $f \in C^{(2m+1,2n+1)}(X)$ and $X = [a,b] \times [c,d]$, then for the remainder term*

$$R_{EB_{m,n}}[f](x,y) = f(x,y) - S_{EB_{m,n}}[f](x,y) \tag{3.7}$$

we have

$$\begin{aligned} R_{EB_{m,n}}[f](x,y) =& \sum_{j<2n+1} \int_a^b f^{(2m+1,j)}(s,c) K_{m,j}^x(x,y,s)ds \\ &+ \sum_{i<2m+1} \int_c^d f^{(i,2n+1)}(a,t) K_{i,n}^y(x,y,t)dt \\ &+ \int_a^b \int_c^d f^{(2m+1,2n+1)}(s,t) K_{m,n}^{x,y}(x,y,s,t)dsdt \end{aligned}$$

where $K_{m,j}^x(x,y,s)$, $K_{i,n}^y(x,y,t)$, and $K_{m,n}^{x,y}(x,y,s,t)$ are the Peano's kernels.

Proof. By applying the Peano's Theorem for bidimensional case [38] and using the relation (2.5), we give the following equalities:

$$K^x_{m,j}(x,y,s) = R_{EB_{m,n}}\left[\frac{(x-s)_+^{2m}}{(2m)!}\frac{(y-c)^j}{j!}\right](x,y),$$
$$K^y_{i,n}(x,y,t) = R_{EB_{m,n}}\left[\frac{(x-a)^i}{i!}\frac{(y-t)_+^{2n}}{(2n)!}\right](x,y),$$
$$K^{x,y}_{m,n}(x,y,t) = R_{EB_{m,n}}\left[\frac{(x-s)_+^{2m}}{(2m)!}\frac{(y-t)_+^{2n}}{(2n)!}\right](x,y),$$

such that

$$K^x_{m,j}(x,y,s) = \frac{(x-s)_+^{2m}}{(2m)!}\frac{(y-c)^j}{j!} - \sum_{i=0}^{N} A_{i,\mu}(x,y) EB_{m,n}\left[\frac{(x-s)_+^{2m}}{(2m)!}\frac{(y-c)^j}{j!}; x_i, x_{i+1}; y_i, y_{i+1}; h_i, k_i\right](x,y)$$

with

$$\begin{aligned}
&EB_{m,n}\left[\frac{(x-s)_+^{2m}}{(2m)!}\frac{(y-c)^j}{j!}; x_i, x_{i+1}; y_i, y_{i+1}; h_i, k_i\right](x,y)\\
&= \frac{(x_i-s)_+^{2m}}{(2m)!}\frac{(y_i-c)^j}{j!} + \sum_{p=1}^{m} h_i^{2p-1}\left(\left(\frac{(x-s)_+^{2m}}{(2m)!}\frac{(y-c)^j}{j!}\right)^{(2p-1,0)}\Big|_{\substack{x=x_{i+1}\\ y=y_i}} V_p\left(\frac{x-x_i}{h_i}\right)\right.\\
&\left. - \left(\frac{(x-s)_+^{2m}}{(2m)!}\frac{(y-c)^j}{j!}\right)^{(2p-1,0)}\Big|_{\substack{x=x_i\\ y=y_i}} \Lambda_p\left(\frac{x_{i+1}-x}{h_i}\right)\right)\\
&+ \sum_{q=1}^{m} k_i^{2q-1}\left(\left(\frac{(x-s)_+^{2m}}{(2m)!}\frac{(y-c)^j}{j!}\right)^{(0,2q-1)}\Big|_{\substack{x=x_i\\ y=y_{i+1}}} V_q\left(\frac{y-y_i}{k_i}\right)\right.\\
&\left. - \left(\frac{(x-s)_+^{2m}}{(2m)!}\frac{(y-c)^j}{j!}\right)^{(0,2q-1)}\Big|_{\substack{x=x_i\\ y=y_i}} \Lambda_q\left(\frac{y_{i+1}-y}{k_i}\right)\right) + \sum_{p=1}^{m}\sum_{q=1}^{n} h_i^{2p-1}k_i^{2q-1}\\
&\times\left(\left(\frac{(x-s)_+^{2m}}{(2m)!}\frac{(y-c)^j}{j!}\right)^{(2p-1,2q-1)}\Big|_{\substack{x=x_{i+1}\\ y=y_{i+1}}} V_p\left(\frac{x-x_i}{h_i}\right) V_q\left(\frac{y-y_i}{k_i}\right)\right.\\
&- \left(\frac{(x-s)_+^{2m}}{(2m)!}\frac{(y-c)^j}{j!}\right)^{(2p-1,2q-1)}\Big|_{\substack{x=x_{i+1}\\ y=y_i}} V_p\left(\frac{x-x_i}{h_i}\right) \Lambda_q\left(\frac{y_{i+1}-y}{k_i}\right)\\
&- \left(\frac{(x-s)_+^{2m}}{(2m)!}\frac{(y-c)^j}{j!}\right)^{(2p-1,2q-1)}\Big|_{\substack{x=x_i\\ y=y_{i+1}}} \Lambda_p\left(\frac{x_{i+1}-x}{h_i}\right) V_q\left(\frac{y-y_i}{k_i}\right)\\
&\left. + \left(\frac{(x-s)_+^{2m}}{(2m)!}\frac{(y-c)^j}{j!}\right)^{(2p-1,2q-1)}\Big|_{\substack{x=x_i\\ y=y_i}} \Lambda_p\left(\frac{x_{i+1}-x}{h_i}\right) \Lambda_q\left(\frac{y_{i+1}-y}{k_i}\right)\right)
\end{aligned}$$

where

$$\left(\frac{(x-s)_+^{2m}}{(2m)!}\frac{(y-c)^j}{j!}\right)^{(2p-1,0)}\Big|_{\substack{x=x_{i+1}\\y=y_i}}=\frac{(x_{i+1}-s)^{2m-2p+1}}{(2m-2p+1)!}\frac{(y_i-c)^j}{j!},$$

$$\left(\frac{(x-s)_+^{2m}}{(2m)!}\frac{(y-c)^j}{j!}\right)^{(2p-1,0)}\Big|_{\substack{x=x_i\\y=y_i}}=\frac{(x_i-s)^{2m-2p+1}}{(2m-2p+1)!}\frac{(y_i-c)^j}{j!},$$

$$\left(\frac{(x-s)_+^{2m}}{(2m)!}\frac{(y-c)^j}{j!}\right)^{(0,2q-1)}\Big|_{\substack{x=x_i\\y=y_{i+1}}}=\frac{(x_i-s)^{2m}}{(2m)!}\frac{(y_{i+1}-c)^{j-2q+1}}{(j-2q+1)!},$$

$$\left(\frac{(x-s)_+^{2m}}{(2m)!}\frac{(y-c)^j}{j!}\right)^{(0,2q-1)}\Big|_{\substack{x=x_i\\y=y_i}}=\frac{(x_i-s)^{2m}}{(2m)!}\frac{(y_i-c)^{j-2q+1}}{(j-2q+1)!},$$

$$\left(\frac{(x-s)_+^{2m}}{(2m)!}\frac{(y-c)^j}{j!}\right)^{(2p-1,2q-1)}\Big|_{\substack{x=x_{i+1}\\y=y_{i+1}}}=\frac{(x_{i+1}-s)^{2m-2p+1}}{(2m-2p+1)!}\frac{(y_{i+1}-c)^{j-2q+1}}{(j-2q+1)!},$$

$$\left(\frac{(x-s)_+^{2m}}{(2m)!}\frac{(y-c)^j}{j!}\right)^{(2p-1,2q-1)}\Big|_{\substack{x=x_{i+1}\\y=y_i}}=\frac{(x_{i+1}-s)^{2m-2p+1}}{(2m-2p+1)!}\frac{(y_i-c)^{j-2q+1}}{(j-2q+1)!},$$

$$\left(\frac{(x-s)_+^{2m}}{(2m)!}\frac{(y-c)^j}{j!}\right)^{(2p-1,2q-1)}\Big|_{\substack{x=x_i\\y=y_{i+1}}}=\frac{(x_i-s)^{2m-2p+1}}{(2m-2p+1)!}\frac{(y_{i+1}-c)^{j-2q+1}}{(j-2q+1)!},$$

$$\left(\frac{(x-s)_+^{2m}}{(2m)!}\frac{(y-c)^j}{j!}\right)^{(2p-1,2q-1)}\Big|_{\substack{x=x_i\\y=y_i}}=\frac{(x_i-s)^{2m-2p+1}}{(2m-2p+1)!}\frac{(y_i-c)^{j-2q+1}}{(j-2q+1)!}.$$

The rest Peano's kernels $K_{i,n}^{y}(x,y,t)$ and $K_{m,n}^{x,y}(x,y,s,t)$ are obtain in terms of the same manner. □

4. Numerical experiments

To test the bivariate even order Bernoulli-type Shepard operator, we consider the following test functions (see, e.g., [42]) on the computational domain $[0,1]\times[0,1]$:

$$\text{Gentle}\quad f_1(x,y)=\frac{\exp[-\frac{81}{16}((x-0.5)^2+(y-0.5)^2)]}{3},\tag{4.1}$$

$$\text{Sphere}\quad f_2(x,y)=\frac{\sqrt{64-81((x-0.5)^2+(y-0.5)^2)}}{9}-0.5,\tag{4.2}$$

$$\text{Saddle}\quad f_3(x,y)=\frac{1.25+\cos(5.4y)}{6+6(3x-1)^2},\tag{4.3}$$

$$\text{Steep}\quad f_4(x,y)=\frac{\exp[-\frac{81}{4}((x-0.5)^2+(y-0.5)^2)]}{3}.\tag{4.4}$$

Table 1. Gentle.

	$S_{EB_{m,n}}f_1$		$S_{B_{m,n}}f_1$		Sf_1	
$(\mu$, m, n)	$\varepsilon_{\max}$	$\varepsilon_{\text{mean}}$	$\varepsilon_{\max}$	$\varepsilon_{\text{mean}}$	$\varepsilon_{\max}$	$\varepsilon_{\text{mean}}$
(2,1,1)	0.046116	0.023056	0.044358	0.027047	0.068958	0.020117
(2,1,2)	0.045824	0.011934	0.056505	0.028519	0.068958	0.020117
(2,2,1)	0.045824	0.011934	0.056505	0.028519	0.068958	0.020117
(2,2,2)	0.056211	0.017020	0.048871	0.023836	0.068958	0.020117
(3,1,1)	0.011739	0.002895	0.013299	0.004323	0.026690	0.008430
(3,1,2)	0.008405	0.001943	0.016112	0.004427	0.026690	0.008430
(3,2,1)	0.008405	0.001943	0.016112	0.004427	0.026690	0.008430
(3,2,2)	0.012096	0.003204	0.013120	0.003056	0.026690	0.008430
(4,1,1)	0.004504	0.000600	0.007000	0.001245	0.020594	0.008050
(4,1,2)	0.003556	0.000652	0.007054	0.001137	0.020594	0.008050
(4,2,1)	0.003556	0.000652	0.007054	0.001137	0.020594	0.008050
(4,2,2)	0.002998	0.000561	0.004870	0.000696	0.020594	0.008050

For each function $f_i, i = 1, 2, 3, 4$, we will compare the numerical results of our new operator $\mathcal{S}_{EB_{m,n}}$ with other bivariate Bernoulli-type Shepard operator $S_{B_{m,n}}$ (see [9]) and known Shepard operator Sf (see [1]).

Table 2. Sphere.

	$S_{EB_{m,n}}f_2$		$S_{B_{m,n}}f_2$		Sf_2	
$(\mu$, m, n)	$\varepsilon_{\max}$	$\varepsilon_{\text{mean}}$	$\varepsilon_{\max}$	$\varepsilon_{\text{mean}}$	$\varepsilon_{\max}$	$\varepsilon_{\text{mean}}$
(2,1,1)	0.018906	0.007241	0.065107	0.033133	0.047419	0.016148
(2,1,2)	0.079777	0.013153	0.038972	0.016720	0.047419	0.016148
(2,2,1)	0.079777	0.013153	0.038972	0.016720	0.047419	0.016148
(2,2,2)	0.650462	0.051623	0.020006	0.007866	0.047419	0.016148
(3,1,1)	0.002141	0.000878	0.011961	0.005532	0.022392	0.007249
(3,1,2)	0.010344	0.001833	0.008402	0.003471	0.022392	0.007249
(3,2,1)	0.010344	0.001833	0.008402	0.003471	0.022392	0.007249
(3,2,2)	0.091497	0.006739	0.002393	0.001009	0.022392	0.007249
(4,1,1)	0.001080	0.000161	0.005216	0.001646	0.025498	0.007724
(4,1,2)	0.001562	0.000292	0.003304	0.001211	0.025498	0.007724
(4,2,1)	0.001562	0.000292	0.003304	0.001211	0.025498	0.007724
(4,2,2)	0.012993	0.000792	0.000831	0.000191	0.025498	0.007724

We use uniform grids of 15 × 15, 15 × 10, 10 × 15, 10 × 10, 10 × 6, 6 × 10, and 6 × 6 nodes for the operators $S_{B_{11}}$, $S_{B_{12}}$, $S_{B_{21}}$, $S_{B_{22}}$($S_{EB_{11}}$,S), $S_{EB_{1,2}}$, $S_{EB_{2,1}}$, $S_{EB_{2,2}}$ with $\mu = 2, 3, 4$, respectively. In order to estimate the errors as accurate as possible, we compute the approximating functions at the points $(\frac{i}{23}, \frac{j}{23}), (i = 1, 2, \cdots, 22; j = 1, 2, \cdots, 22)$. Tables 1–4 display mean and max errors for the different approximation operators above. The numerical results show that the approximating powers of the even order bivariate Bernoulli-type operator $S_{EB_{m,n}}$ are comparable with that of the bivariate Shepard-Bernoulli operator $S_{B_{m,n}}$.

Table 3. Saddle.

	$S_{EB_{m,n}}f_3$		$S_{B_{m,n}}f_3$		Sf_3	
$(\mu, \text{m}, \text{n})$	$\varepsilon_{\max}$	$\varepsilon_{\text{mean}}$	$\varepsilon_{\max}$	$\varepsilon_{\text{mean}}$	$\varepsilon_{\max}$	$\varepsilon_{\text{mean}}$
(2,1,1)	0.034160	0.014168	0.034948	0.008515	0.068958	0.020529
(2,1,2)	0.105085	0.018923	0.034727	0.013261	0.068958	0.020529
(2,2,1)	0.043400	0.017487	0.039529	0.015048	0.068958	0.020529
(2,2,2)	0.090092	0.019846	0.036666	0.014922	0.068958	0.020529
(3,1,1)	0.012499	0.002427	0.010136	0.002009	0.037655	0.008294
(3,1,2)	0.022955	0.003218	0.016492	0.002547	0.037655	0.008294
(3,2,1)	0.017262	0.003633	0.012480	0.003170	0.037655	0.008294
(3,2,2)	0.019105	0.002659	0.014021	0.002697	0.037655	0.008294
(4,1,1)	0.007125	0.000779	0.010698	0.001231	0.027259	0.007390
(4,1,2)	0.007411	0.000734	0.009745	0.001026	0.027259	0.007390
(4,2,1)	0.005174	0.000937	0.008533	0.001394	0.027259	0.007390
(4,2,2)	0.003347	0.000480	0.007805	0.000911	0.027259	0.007390

Table 4. Steep.

	$S_{EB_{m,n}}f_4$		$S_{B_{m,n}}f_4$		Sf_4	
$(\mu, \text{m}, \text{n})$	$\varepsilon_{\max}$	$\varepsilon_{\text{mean}}$	$\varepsilon_{\max}$	$\varepsilon_{\text{mean}}$	$\varepsilon_{\max}$	$\varepsilon_{\text{mean}}$
(2,1,1)	0.082872	0.030905	0.035551	0.015396	0.138247	0.020464
(2,1,2)	0.098944	0.027763	0.068106	0.022786	0.138247	0.020464
(2,2,1)	0.098944	0.027763	0.068106	0.022786	0.138247	0.020464
(2,2,2)	0.098123	0.023376	0.089691	0.030825	0.138247	0.020464
(3,1,1)	0.025288	0.003583	0.020997	0.002300	0.067153	0.009210
(3,1,2)	0.036910	0.004416	0.029179	0.003196	0.067153	0.009210
(3,2,1)	0.036910	0.004416	0.029179	0.003196	0.067153	0.009210
(3,2,2)	0.037544	0.004200	0.032058	0.003690	0.067153	0.009210
(4,1,1)	0.013497	0.001071	0.022890	0.001703	0.041699	0.007456
(4,1,2)	0.018735	0.001348	0.020749	0.001658	0.041699	0.007456
(4,2,1)	0.018735	0.001348	0.020749	0.001658	0.041699	0.007456
(4,2,2)	0.015110	0.001014	0.018654	0.001458	0.041699	0.007456

5. Conclusions

In this paper, a kind of bivariate even order Bernoulli-type Shepard operator is constructed by combining the known Shepard operator with the generalized Taylor polynomial as the expansion in the bivariate even order Bernoulli polynomials. A result on the some error bounds of the new operator is given. Numerical tests show that the operator offers a higher of accuracy. Furthermore, the associated algorithm is easily implemented.

In our future work, we plan to apply it to solve partial differential equations, and good results may be obtained. Moreover, we could construct stochastic quasi-interpolation operator with even order Bernoulli Polynomials.

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Conflict of interest

The author declares that he has no conflict of interest.

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