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Research article

Global weak solutions of nonlinear rotation-Camassa-Holm model

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Abstract: A nonlinear rotation-Camassa-Holm equation, physically depicting the motion of equatorial water waves and having the Coriolis effect, is investigated. Using the viscous approximation tool, we obtain an upper bound estimate about the space derivative of the viscous solution and a high order integrable estimate about the time-space variables. Utilizing these two estimates, we prove that there exist $H^1(\mathbb{R})$ global weak solutions to the rotation-Camassa-Holm model.

Keywords: global weak solutions; existence; rotation-Camassa-Holm equation
Mathematics Subject Classification: 35G25, 35L05

1. Preliminary

The objective of this work is to deal with the rotation-Camassa-Holm model (RCH)

$$m_t + Vm_x + 2V_x m + kV_x - \frac{\alpha_0}{\alpha} V_{xxx} + \frac{h_1}{\beta^2} V^2 V_x + \frac{h_2}{\beta^3} V^3 V_x = 0, \tag{1.1}$$

where $m = V - V_{xx}$,

$$k = \sqrt{1+F^2} - F, \quad \beta = \frac{k^2}{1+k^2}, \quad \alpha_0 = \frac{k(k^4+6k^2-1)}{6(k^2+1)^2}, \quad \alpha = \frac{3k^4+8k^2-1}{6(k^2+1)^2}$$
$$h_1 = \frac{-3k(k^2-1)(k^2-2)}{2(1+k^2)^3}, \quad h_2 = \frac{(k^2-2)(k^2-1)^2(8k^2-1)}{2(1+k^2)^5},$$

in which constant F is a parameter to depict the Coriolis effect due to the Earth's rotation. Gui et al. [9] derive the nonlinear RCH equation (1.1) (also see [3, 10]), depicting the motion of the fluid associated with the Coriolis effect.

Recently, many works focus on the study of Eq (1.1). Zhang [23] investigates the well-posedness for Eq (1.1) on the torus in the sense of Hadamard if assuming its initial value in the space H^s with the Sobolev index $s > \frac{3}{2}$, and gives a Cauchy-Kowalevski type proposition for Eq (1.1) under certain conditions. It is shown in Gui et al. [9] that Eq (1.1) has similar dynamical features with those of Camassa-Holm and irrotational Euler equations. The travelling wave solutions are found and classified in [10]. The well-posedness, geometrical analysis and a more general classification of travelling wave solution for Eq (1.1) are carried out in Silva and Freire [17]. Tu et al. [20] investigate the well-posedness of the global conservative solutions to Eq (1.1).

If $F = 0$ (implying $h_1 = h_2 = 0$), namely, the Coriolis effects disappear, Eq (1.1) becomes the standard Camassa-Holm (CH) model [2], which has been investigated by many scholars [1, 6–8, 16]. For some dynamical characteristics of the CH, we refer the reader to the references [11–15, 22].

Motivated by the works made in [4, 21], in which the $H^1(\mathbb{R})$ global weak solution to the CH model is studied without restricting that the initial value obeys the sign condition, we investigate the rotation-Camassa-Holm equation (1.1) and utilize the viscous approximation technique to handle the existence of global weak solution in $H^1(\mathbb{R})$. As the term V_{xxx} appears in Eq (1.1), it yields difficulties to establish estimates of solutions for the viscous approximation of Eq (1.1) (In fact, using a change of coordinates, Silva and Freire [18] eliminate the term V_{xxx} and discuss other dynamical features of Eq (1.1)). The key contribution of this work is that we overcome these difficulties and establish a high order integrable estimate and prove that $\frac{\partial V(t,x)}{\partial x}$ possesses upper bound. These two estimates take key roles in proving the existence of the $H^1(\mathbb{R})$ global weak solution for Eq (1.1) without the sign condition.

This work is structured by the following steps. Definition of the $H^1(\mathbb{R})$ global weak solutions and several Lemmas are given in Section 2. The main conclusion and its proof are presented in Section 3.

2. Lemmas

We rewrite the initial value problem for the RCH equation (1.1)

$$\begin{cases} V_t - V_{txx} + 3VV_x + kV_x + \frac{h_1}{\beta^2}V^2V_x + \frac{h_2}{\beta^3}V^3V_x = 2V_xV_{xx} + (V + \frac{\alpha_0}{\alpha})V_{xxx}, \\ V(0,x) = V_0(x), \quad x \in \mathbb{R}. \end{cases} \tag{2.1}$$

Employing operator $\Lambda^{-2} = (1 - \partial_x^2)^{-1}$ to multiply Eq (1.1), we have

$$\begin{cases} V_t + (V + \frac{\alpha_0}{\alpha})V_x + \frac{\partial A}{\partial x} = 0, \\ V(0,x) = V_0(x), \end{cases} \tag{2.2}$$

where

$$\frac{\partial A}{\partial x} = \Lambda^{-2}\Big[(k - \frac{\alpha_0}{\alpha})V + V^2 + \frac{h_1}{3\beta^2}V^3 + \frac{h_2}{4\beta^3}V^4 + \frac{1}{2}V_x^2\Big]_x.$$

It can be found in [9, 10, 17, 18] that

$$\int_{\mathbb{R}}\Big(V^2 + V_x^2\Big)dx = \int_{\mathbb{R}}\Big(V_0^2 + V_{0x}^2\Big)dx.$$

We cite the definition (see [4, 21]).

Definition 2.1. *The solution $V(t,x) : [0,\infty)\times\mathbb{R} \to \mathbb{R}$ is called as a global weak solution to system (2.1) or (2.2) if*
(1) $V \in C([0,\infty)\times\mathbb{R}) \cap L^\infty\Big([0,\infty); H^1(\mathbb{R})\Big)$;
(2) $\|V(t,.)\|_{H^1(\mathbb{R})} \le \|V_0\|_{H^1(\mathbb{R})}$;
(3) $V = V(t,x)$ obeys (2.2) in the sense of distribution.

Define $\phi(x) = e^{\frac{1}{x^2-1}}$ if $|x| < 1$ and $\phi(x) = 0$ if $|x| \geq 1$. Set $\phi_\varepsilon(x) = \varepsilon^{-\frac{1}{4}}\phi\left(\varepsilon^{-\frac{1}{4}}x\right)$ with $0 < \varepsilon < \frac{1}{4}$. Assume $V_{\varepsilon,0} = \phi_\varepsilon \star V_0$, where $\star$ represents the convolution, we see that $V_{\varepsilon,0} \in C^\infty$ for $V_0(x) \in H^s$, $s > 0$. To discuss global weak solutions for Eq (1.1), we handle the following viscous approximation problem:

$$\begin{cases} \frac{\partial V_\varepsilon}{\partial t} + (V_\varepsilon + \frac{\alpha_0}{\alpha})\frac{\partial V_\varepsilon}{\partial x} + \frac{\partial A_\varepsilon}{\partial x} = \varepsilon\frac{\partial^2 V_\varepsilon}{\partial x^2}, \\ V(0,x) = V_{\varepsilon,0}(x), \end{cases} \tag{2.3}$$

in which

$$A_\varepsilon(t,x) = \Lambda^{-2}\Big[(k - \frac{\alpha_0}{\alpha})V_\varepsilon + V_\varepsilon^2 + \frac{h_1}{3\beta^2}V_\varepsilon^3 + \frac{h_2}{4\beta^3}V_\varepsilon^4 + \frac{1}{2}(\frac{\partial V_\varepsilon}{\partial x})^2\Big].$$

Utilizing (2.3) and denoting $p_\varepsilon(t,x) = \frac{\partial V_\varepsilon(t,x)}{\partial x}$ yield

$$\begin{aligned} & \frac{\partial p_\varepsilon}{\partial t} + (V_\varepsilon + \frac{\alpha_0}{\alpha})\frac{\partial p_\varepsilon}{\partial x} - \varepsilon\frac{\partial^2 p_\varepsilon}{\partial x^2} + \frac{1}{2}p_\varepsilon^2 \\ = \;& (k - \frac{\alpha_0}{\alpha})V_\varepsilon + V_\varepsilon^2 + \frac{h_1}{3\beta^2}V_\varepsilon^3 + \frac{h_2}{4\beta^3}V_\varepsilon^4 - \Lambda^{-2}(V_\varepsilon^2 + (k - \frac{\alpha_0}{\alpha})V_\varepsilon + \frac{h_1}{3\beta^2}V_\varepsilon^3 + \frac{h_2}{4\beta^3}V_\varepsilon^4 + \frac{1}{2}(\frac{\partial V_\varepsilon}{\partial x})^2) \\ = \;& B_\varepsilon(t,x). \end{aligned} \tag{2.4}$$

Simply for writing, let c represent arbitrary positive constants (independent of ε).

Lemma 2.1. *Let $V_0 \in H^1(\mathbb{R})$. For each number $\sigma \geq 2$, system (2.3) has a unique solution $V_\varepsilon \in C([0,\infty); H^\sigma(\mathbb{R}))$ and*

$$\int_{\mathbb{R}}\left(V_\varepsilon^2 + \left(\frac{\partial V_\varepsilon}{\partial x}\right)^2\right)dx + 2\varepsilon\int_0^t\int_{\mathbb{R}}\left[\left(\frac{\partial V_\varepsilon}{\partial x}\right)^2 + \left(\frac{\partial^2 V_\varepsilon}{\partial x^2}\right)^2\right](s,x)dxds = \left\|V_{\varepsilon,0}\right\|^2_{H^1(\mathbb{R})}, \tag{2.5}$$

which has the equivalent expression

$$\|V_\varepsilon(t,.)\|^2_{H^1(\mathbb{R})} + 2\varepsilon\int_0^t\left\|\frac{\partial V_\varepsilon}{\partial x}(s,.)\right\|^2_{H^1(\mathbb{R})}ds = \left\|V_{\varepsilon,0}\right\|^2_{H^1(\mathbb{R})}.$$

Proof. For parameter $\sigma \geq 2$, we acquire $V_{\varepsilon,0} \in C([0,\infty); H^\infty(\mathbb{R}))$. Employing the conclusion in [5] derives that system (2.3) has a unique solution $V_\varepsilon(t,x) \in C([0,\infty); H^\sigma(\mathbb{R}))$. Using (2.3) arises

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}}\left(V_\varepsilon^2 + V_{\varepsilon x}^2\right)dx &= 2\int_{\mathbb{R}}V_\varepsilon(V_{\varepsilon t} - V_{\varepsilon txx})dx \\ &= 2\varepsilon\int_{\mathbb{R}}(V_\varepsilon V_{\varepsilon xx} - V_\varepsilon V_{\varepsilon xxxx})dx \\ &= -2\varepsilon\int_{\mathbb{R}}\left((V_{\varepsilon x})^2 + (V_{\varepsilon xx})^2\right)dx. \end{aligned}$$

Integrating about variable t for both sides of the above identity, we obtain (2.5).

In fact, as $\varepsilon \to 0$, we have

$$\|V_\varepsilon\|_{L^\infty(\mathbb{R})} \leq \|V_\varepsilon\|_{H^1(\mathbb{R})} \leq \|V_{\varepsilon,0}\|_{H^1(\mathbb{R})} \leq \|V_0\|_{H^1(\mathbb{R})}, \text{ and } V_{\varepsilon,0} \to V_0 \text{ in } H^1(\mathbb{R}). \tag{2.6}$$

Lemma 2.2. *If $V_0(x) \in H^1(\mathbb{R})$, for $A_\varepsilon(t,x)$ and $B_\varepsilon(t,x)$, then*

$$\|A_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R})} \le c, \quad \|\frac{\partial A_\varepsilon}{\partial x}(t,\cdot)\|_{L^\infty(\mathbb{R})} \le c, \tag{2.7}$$

$$\|A_\varepsilon(t,\cdot)\|_{L^1(\mathbb{R})} \le c, \quad \|\frac{\partial A_\varepsilon}{\partial x}(t,\cdot)\|_{L^1(\mathbb{R})} \le c, \tag{2.8}$$

$$\|A_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})} \le c, \quad \|\frac{\partial A_\varepsilon}{\partial x}(t,\cdot)\|_{L^2(\mathbb{R})} \le c, \tag{2.9}$$

and

$$\|B_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R})} \le c, \quad \|B_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})} \le c, \tag{2.10}$$

where $c = c(\|V_0\|_{H^1(\mathbb{R})})$.

Proof. For any function $U(x)$ and the operator Λ^{-2}, it holds that

$$\Lambda^{-2}U(x) = \frac{1}{2}\int_{\mathbb{R}} e^{-|x-y|}U(y)dy \ \text{ for } U(x) \in L^r(\mathbb{R}),\ 1 \le r \le \infty, \tag{2.11}$$

and

$$\begin{aligned}\left|\Lambda^{-2}U_x(x)\right| &= \Big|\frac{1}{2}\int_{\mathbb{R}} e^{-|x-y|}\frac{\partial U(y)}{\partial y}dy\Big| \\ &= \Big| -\frac{1}{2}e^{-x}\int_{-\infty}^{x} U(y)dy + \frac{1}{2}e^{x}\int_{x}^{\infty} e^{-y}U(y)dy\Big| \\ &\le \frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-y|}|U(y)|dy. \end{aligned} \tag{2.12}$$

Utilizing (2.6), (2.11), (2.12), the expression of function $A_\varepsilon(t,x)$ and the Tonelli theorem, we have

$$\| \Lambda^{-2}\Big((k-\frac{\alpha_0}{\alpha})V_\varepsilon + V_\varepsilon^2 + \frac{h_1}{3\beta^2}V_\varepsilon^3 + \frac{h_2}{4\beta^3}V_\varepsilon^4 + \frac{1}{2}V_{\varepsilon x}^2\Big) \|_{L^\infty(\mathbb{R})} \le c$$

and

$$\| \Lambda^{-2}\Big((k-\frac{\alpha_0}{\alpha})V_\varepsilon + V_\varepsilon^2 + \frac{h_1}{3\beta^2}V_\varepsilon^3 + \frac{h_2}{4\beta^3}V_\varepsilon^4 + \frac{1}{2}V_{\varepsilon x}^2\Big)_x \|_{L^\infty(\mathbb{R})} \le c,$$

which derive that (2.7) and (2.8) hold. Utilizing (2.7) and (2.8) yields

$$\|A_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}^2 \le \|A_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R})}\|A_\varepsilon(t,\cdot)\|_{L^1(\mathbb{R})} \le c$$

and

$$\|\frac{\partial A_\varepsilon(t,\cdot)}{\partial x}\|_{L^2(\mathbb{R})}^2 \le \|\frac{\partial A_\varepsilon(t,\cdot)}{\partial x}\|_{L^\infty(\mathbb{R})}\|\frac{\partial A_\varepsilon(t,\cdot)}{\partial x}\|_{L^1(\mathbb{R})} \le c,$$

which complete the proof of (2.9). Furthermore, using (2.4) and (2.6), we have

$$\|B_\varepsilon\|_{L^\infty(\mathbb{R})} \le c, \quad \|B_\varepsilon\|_{L^2(\mathbb{R})} \le c,$$

which finishes the proof of (2.10).

Lemma 2.3. *Provided that* $0<\alpha_1<1$, $T>0$, *constants* $a<b$, *then*

$$\int_0^T\int_a^b\left|\frac{\partial V_\varepsilon(t,x)}{\partial x}\right|^{2+\alpha_1}dxdt\le c_1, \tag{2.13}$$

where constant c_1 *depends on* a,b,α_1,T,k *and* $\|V_0\|_{H^1(\mathbb{R})}$.

Proof. We utilize the methods in Xin and Zhang [21] to prove this lemma. Let function $g(x)\in C^\infty(\mathbb{R})$ and satisfy

$$0\le g(x)\le 1,\quad g(x)=\begin{cases}0, & x\in(-\infty,a-1]\cup[b+1,\infty),\\ 1, & x\in[a,b].\end{cases}$$

Define function $f(\eta):=\eta(|\eta|+1)^{\alpha_1}$, $\eta\in\mathbb{R}$. We note that the function f belongs to $C^1(\mathbb{R})$ except $\eta=0$. Here we give the expressions of its first and second derivatives as follows:

$$\begin{aligned}f'(\eta)&=((\alpha_1+1)|\eta|+1)(|\eta|+1)^{\alpha_1-1},\\ f''(\eta)&=\alpha_1\,\mathrm{sign}(\eta)(|\eta|+1)^{\alpha_1-2}((\alpha_1+1)|\eta|+2)\\ &=\alpha_1(\alpha_1+1)\,\mathrm{sign}(\eta)(|\eta|+1)^{\alpha_1-1}+(1-\alpha_1)\alpha_1\,\mathrm{sign}(\eta)(|\eta|+1)^{\alpha_1-2},\end{aligned}$$

from which we have

$$|f(\eta)|\le|\eta|^{\alpha_1+1}+|\eta|,\quad |f'(\eta)|\le(\alpha_1+1)|\eta|+1,\quad |f''(\eta)|\le 2\alpha_1, \tag{2.14}$$

and

$$\begin{aligned}\eta f(\eta)-\frac{1}{2}\eta^2f'(\eta)&=\frac{1-\alpha_1}{2}\eta^2(|\eta|+1)^{\alpha_1}+\frac{\alpha_1}{2}\eta^2(|\eta|+1)^{\alpha_1-1}\\ &\ge\frac{1-\alpha_1}{2}\eta^2(|\eta|+1)^{\alpha_1}.\end{aligned} \tag{2.15}$$

Note that

$$\int_0^T\int_{\mathbb{R}}g(x)f'(p_\varepsilon)p_{\varepsilon t}dxdt=\int_{\mathbb{R}}g(x)dx\int_0^T df(p_\varepsilon)=\int_{\mathbb{R}}\left[f(p_\varepsilon(T,x))-f(p_\varepsilon(0,x))\right]g(x)dx, \tag{2.16}$$

$$\begin{aligned}\int_0^T\int_{\mathbb{R}}g(x)f'(p_\varepsilon)\Big(V_\varepsilon+\frac{\alpha_0}{\alpha}\Big)p_{\varepsilon x}dxdt&=\int_0^T dt\int_{\mathbb{R}}g(x)(V_\varepsilon+\frac{\alpha_0}{\alpha})df(p_\varepsilon)\\ &=-\int_{\mathbb{R}}f(p_\varepsilon)\Big[g'(x)(V_\varepsilon+\frac{\alpha_0}{\alpha})+g(x)p_\varepsilon\Big]dx.\end{aligned} \tag{2.17}$$

Making use of $g(x)f'(p_\varepsilon)$ to multiply (2.4), from (2.16) and (2.17), integrating over $([0,\infty)\times\mathbb{R})$ by parts, we obtain

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}}g(x)p_\varepsilon f(p_\varepsilon)dtdx-\frac{1}{2}\int_0^T\int_{\mathbb{R}}p_\varepsilon^2g(x)f'(p_\varepsilon)dtdx\\ =&\int_{\mathbb{R}}\Big[f(p_\varepsilon(T,x))-f(p_\varepsilon(0,x))\Big]g(x)dx+\int_0^T\int_{\mathbb{R}}(V_\varepsilon+\frac{\alpha_0}{\alpha})g'(x)f(p_\varepsilon)dtdx\end{aligned}$$

$$
\begin{aligned}
&+ \quad \varepsilon \int_0^T \int_{\mathbb{R}} g'(x) f'(p_\varepsilon) \frac{\partial p_\varepsilon}{\partial x} dt dx + \varepsilon \int_0^T \int_{\mathbb{R}} g(x) f''(p_\varepsilon) (\frac{\partial p_\varepsilon}{\partial x})^2 dt dx \\
&- \quad \int_0^T \int_{\mathbb{R}} B_\varepsilon f'(p_\varepsilon) g(x) dt dx. \qquad (2.18)
\end{aligned}
$$

Applying (2.15) yields

$$
\begin{aligned}
& \int_0^T \int_{\mathbb{R}} g(x) p_\varepsilon f(p_\varepsilon) dt dx - \frac{1}{2} \int_0^T \int_{\mathbb{R}} p_\varepsilon^2 g(x) f'(p_\varepsilon) dt dx \\
= & \int_0^T \int_{\mathbb{R}} g(x) \Big(p_\varepsilon f(p_\varepsilon) - \frac{1}{2} p_\varepsilon^2 f'(p_\varepsilon) \Big) dt dx \\
\geq & \frac{(1-\alpha_1)}{2} \int_0^T \int_{\mathbb{R}} g(x) p_\varepsilon^2 (|p_\varepsilon| + 1)^{\alpha_1} dt dx. \qquad (2.19)
\end{aligned}
$$

For $t \geq 0$, using $0 < \alpha_1 < 1$, (2.14) and the Hölder inequality gives rise to

$$
\begin{aligned}
\Big| \int_{\mathbb{R}} g(x) f(p_\varepsilon) dx \Big| \leq & \int_{\mathbb{R}} g(x) (|p_\varepsilon|^{\alpha_1+1} + |p_\varepsilon|) dx \\
\leq & \|g(x)\|_{L^{2/(1-\alpha_1)}(\mathbb{R})} \|p_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}^{\alpha_1+1} + \|g(x)\|_{L^2(\mathbb{R})} \|p_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})} \\
\leq & (b+2-a)^{(1-\alpha_1)/2} \|V_0\|_{H^1(\mathbb{R})}^{\alpha_1+1} + (b+2-a)^{1/2} \|V_0\|_{H^1(\mathbb{R})}, \qquad (2.20)
\end{aligned}
$$

and

$$
\begin{aligned}
& \Big| \int_0^T \int_{\mathbb{R}} V_\varepsilon g'(x) f(p_\varepsilon) dt dx \Big| \\
\leq & \int_0^T \int_{\mathbb{R}} |V_\varepsilon| |g'(x)| (|p_\varepsilon|^{\alpha_1+1} + |p_\varepsilon|) dt dx \\
\leq & \int_0^T \int_{\mathbb{R}} \|V_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R})} |g'(x)| (|p_\varepsilon|^{\alpha_1+1} + |p_\varepsilon|) dt dx \\
\leq & c \int_0^T \Big(\|g'(x)\|_{L^{2/(1-\alpha_1)}(\mathbb{R})} \|p_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}^{\alpha_1+1} + \|g'(x)\|_{L^2(\mathbb{R})} \|p_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})} \Big) dt \\
\leq & c \int_0^T \Big(\|g(x)'\|_{L^{2/(1-\alpha_1)}(\mathbb{R})} \|V_0\|_{L^2(\mathbb{R})}^{\alpha_1+1} + \|g'(x)\|_{L^2(\mathbb{R})} \|V_0\|_{L^2(\mathbb{R})} \Big) dt. \qquad (2.21)
\end{aligned}
$$

Moreover, we have

$$
\varepsilon \int_0^T \int_{\mathbb{R}} \frac{\partial p_\varepsilon}{\partial x} g'(x) f'(p_\varepsilon) dt dx = -\varepsilon \int_0^T \int_{\mathbb{R}} f(p_\varepsilon) g''(x) dt dx. \qquad (2.22)
$$

Utilizing the Hölder inequality and (2.14) leads to

$$
\begin{aligned}
\Big| \varepsilon \int_0^T \int_{\mathbb{R}} g'(x) \frac{\partial p_\varepsilon}{\partial x} f(p_\varepsilon) dt dx \Big| \leq & \varepsilon \int_0^T \int_{\mathbb{R}} |f(p_\varepsilon)| |g''(x)| dt dx \\
\leq & \varepsilon \int_0^T \int_{\mathbb{R}} \Big(|p_\varepsilon|^{\alpha_1+1} + |p_\varepsilon| \Big) | g''(x)| dt dx
\end{aligned}
$$

$$
\begin{aligned}
&\le \varepsilon\int_0^T\left(\| g''\|_{L^{2/(1-\alpha_1)}(\mathbb{R})}\,\|p_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}^{\alpha_1+1}+\|g''\|_{L^2(\mathbb{R})}\,\|p_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}\right)dt\\
&\le \varepsilon T\left(\|g''\|_{L^{2/(1-\alpha_1)}(\mathbb{R})}\,\|V_0\|_{H^1(\mathbb{R})}^{\alpha_1+1}+\|g''\|_{L^2(\mathbb{R})}\,\|V_0\|_{H^1(\mathbb{R})}\right). \qquad (2.23)
\end{aligned}
$$

Using the last part of (2.14), we have

$$
\varepsilon\left|\int_{\Pi_T}\left(\frac{\partial p_\varepsilon}{\partial x}\right)^2 g(x)f''(p_\varepsilon)\,dtdx\right|\le 2\alpha_1\varepsilon\int_{\Pi_T}\left(\frac{\partial p_\varepsilon}{\partial x}\right)^2 dtdx\le \alpha_1\,\|V_0\|_{H^1(\mathbb{R})}^2. \qquad (2.24)
$$

As shown in Lemma 2.2, there exists a constant $c_0>0$ to ensure that

$$
\|B_\varepsilon\|_{L^\infty(\mathbb{R})}\le c_0. \qquad (2.25)
$$

Utilizing the second part in (2.14) arises

$$
\begin{aligned}
&\left|\int_0^T\int_{\mathbb{R}}B_\varepsilon g(x)f'(p_\varepsilon)\,dtdx\right|\\
\le\;& c_0\int_0^T\int_{\mathbb{R}}g(x)\left[(\alpha_1+1)\,|p_\varepsilon|+1\right]dtdx\\
\le\;& c_0\int_0^T\left((\alpha_1+1)\|\,g(x)\|_{L^2(\mathbb{R})}\,\|p_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}+\|g(x)\|_{L^1(\mathbb{R})}\right)dt\\
\le\;& c_0T. \qquad (2.26)
\end{aligned}
$$

Applying (2.18)–(2.26) yields

$$
\frac{1-\alpha_1}{2}\int_0^T\int_{\mathbb{R}}|p_\varepsilon|^2 f(x)(1+|p_\varepsilon|^{\alpha_1})dtdx\le c,
$$

where $c>0$ relies only on $T>0,a,b,\alpha_1$ and $\|V_0\|_{H^1(\mathbb{R})}$. Furthermore, we have

$$
\int_0^T\int_a^b\left|\frac{\partial V_\varepsilon}{\partial x}(t,x)\right|^{2+\alpha_1}dxdt\le\int_0^T\int_{\mathbb{R}}|p_\varepsilon|\,g(x)\,(|p_\varepsilon|+1)^{\alpha_1+1}\,dtdx\le\frac{2c}{(1-\alpha_1)}.
$$

The proof of (2.13) is completed.

Lemma 2.4. *For $(t,x)\in(0,\infty)\times\mathbb{R}$, provided that $V_\varepsilon=V_\varepsilon(t,x)$ satisfy problem (2.3), then*

$$
\frac{\partial V_\varepsilon(t,x)}{\partial x}\le\frac{2}{t}+c, \qquad (2.27)
$$

in which positive constant $c=c(\|V_0\|_{H^1(\mathbb{R})})$.

Proof. Using Lemma 2.2 gives rise to

$$
\frac{\partial p_\varepsilon}{\partial t}+(V_\varepsilon+\frac{\alpha_0}{\alpha})\frac{\partial p_\varepsilon}{\partial x}-\varepsilon\frac{\partial^2 p_\varepsilon}{\partial x^2}+\frac{1}{2}p_\varepsilon^2=B_\varepsilon(t,x)\le c. \qquad (2.28)
$$

Assume that $H=H(t)$ satisfies the problem

$$
\frac{dH}{dt}+\frac{1}{2}H^2=c,\quad t>0,\quad H(0)=\left\|\frac{\partial V_{\varepsilon,0}}{\partial x}\right\|_{L^\infty}.
$$

Due to (2.28), we know that $H = H(t)$ is a supersolution* of parabolic equation (2.4) associated with initial value $\frac{\partial V_{\varepsilon,0}}{\partial x}$. Utilizing the comparison principle for parabolic equations arises

$$p_\varepsilon(t,x) \le H(t).$$

We choose the function $F(t) := \frac{2}{t} + \sqrt{2c}, t > 0$. Since $\frac{dF}{dt}(t) + \frac{1}{2}F^2(t) - c = \frac{2\sqrt{2c}}{t} > 0$, we conclude

$$H(t) \le F(t),$$

which finishes the proof of (2.27).

Lemma 2.5. *There exists a subsequence* $\left\{\varepsilon_j\right\}_{j\in\mathbb{N}}$, $\varepsilon_j \to 0$ *and* $V(t,x) \in L^\infty\left([0,\infty); H^1(\mathbb{R})\right) \cap H^1([0,T]\times\mathbb{R})$, *for every* $T \ge 0$, *such that*

$$\begin{aligned} &V_{\varepsilon_j} \rightharpoonup V \ \text{ in } H^1([0,T]\times\mathbb{R}),\\ &V_{\varepsilon_j} \to V \ \text{ in } L^\infty_{loc}([0,\infty)\times\mathbb{R}). \end{aligned}$$

The proof of Lemma 2.5 can be found in Coclite el al. [4].

Lemma 2.6. *Assume* $V_0 \in H^1(\mathbb{R})$. *Then* $\{B_\varepsilon(t,x)\}_\varepsilon$ *is uniformly bounded in* $W^{1,1}_{loc}([0,\infty)\times\mathbb{R})$. *Moreover, there has a sequence* $\left\{\varepsilon_j\right\}_{j\in N}$, $\varepsilon_j \to 0$ *to guarantee that*

$$B_{\varepsilon_j} \to B \ \text{ strongly in } L^r_{loc}([0,T)\times\mathbb{R}),$$

where function $B \in L^\infty([0,T); W^{1,\infty}(\mathbb{R}))$ *and* $1 < r < \infty$.

The standard proof of Lemma 2.6 can be found in [4]. We omit its proof here.

For conciseness, we use overbars to denote weak limits which are taken in the space $L^r[(0,\infty)\times\mathbb{R})$ with $1 < r < 3$.

Lemma 2.7. *There exist a sequence* $\left\{\varepsilon_j\right\}_{j\in N}$ *tending to zero and two functions* $p \in L^r_{loc}([0,\infty)\times\mathbb{R})$, $\overline{p^2} \in L^{r_1}_{loc}([0,\infty)\times\mathbb{R})$ *such that*

$$p_{\varepsilon_j} \rightharpoonup p \ \text{ in } L^r_{loc}([0,\infty)\times\mathbb{R}), \quad p_{\varepsilon_j} \overset{\star}{\rightharpoonup} p \ \text{ in } L^\infty_{loc}\left([0,\infty); L^2(\mathbb{R})\right), \tag{2.29}$$

$$p^2_{\varepsilon_j} \rightharpoonup \overline{p^2} \ \text{ in } L^{r_1}_{loc}([0,\infty)\times\mathbb{R}) \tag{2.30}$$

for each $1 < r < 3$ *and* $1 < r_1 < \frac{3}{2}$. *In addition, it holds that*

$$p^2(t,x) \le \overline{p^2}(t,x), \tag{2.31}$$

$$\frac{\partial V}{\partial x} = p \quad \text{in the sense of distribution.} \tag{2.32}$$

*The supersolution is defined by $\sup_{x\in\mathbb{R}} p_\varepsilon(t,x)$. If there exists a point (t,x_0) such that $\sup_{x\in\mathbb{R}} p_\varepsilon(t,x)) = p_\varepsilon(t,x_0)$, then $\frac{\partial p_\varepsilon(t,x_0)}{\partial x} = 0$ and $\frac{\partial^2 p_\varepsilon(t,x_0)}{\partial x^2} < 0$.

Proof. Lemmas 2.1 and 2.2 validate (2.29) and (2.30). The weak convergence in (2.30) ensures the reasonableness of (2.31). Using Lemma 2.5 and (2.29) derives that (2.32) holds.

For conciseness in the following discussion, we denote $\left\{p_{\varepsilon_j}\right\}_{j\in N}$, $\left\{V_{\varepsilon_j}\right\}_{j\in N}$ and $\left\{B_{\varepsilon_j}\right\}_{j\in N}$ by $\{p_\varepsilon\}_{\varepsilon>0}$, $\{V_\varepsilon\}_{\varepsilon>0}$ and $\{B_\varepsilon\}_{\varepsilon>0}$. Assume that $F \in C^1(\mathbb{R})$ is an arbitrary convex function with F' being bounded, Lipschitz continuous on $\mathbb{R}$. Using (2.29) derives that

$$F(p_\varepsilon) \rightharpoonup \overline{F(p)} \text{ in } L^r_{loc}([0,\infty)\times\mathbb{R}),$$
$$F(p_\varepsilon) \overset{\star}{\rightharpoonup} \overline{F(p)} \text{ in } L^\infty_{loc}\left([0,\infty); L^2(\mathbb{R})\right).$$

Multiplying (2.4) by $F'(p_\varepsilon)$ yields

$$\begin{aligned} &\frac{\partial}{\partial t}F(p_\varepsilon) + \frac{\partial}{\partial x}\left((V_\varepsilon + \frac{\alpha_0}{\alpha})F(p_\varepsilon)\right) - \varepsilon\frac{\partial^2}{\partial x^2}F(p_\varepsilon) + \varepsilon F''(p_\varepsilon)\left(\frac{\partial p_\varepsilon}{\partial x}\right)^2 \\ = \;& p_\varepsilon F(p_\varepsilon) - \frac{1}{2}F'(p_\varepsilon)p_\varepsilon^2 + B_\varepsilon F'(p_\varepsilon). \end{aligned} \tag{2.33}$$

Lemma 2.8. *Suppose that $F \in C^1(\mathbb{R})$ is a convex function with F' being bounded, Lipschitz continuous on $\mathbb{R}$. In the sense of distribution, then*

$$\frac{\overline{\partial F(p)}}{\partial t} + \frac{\partial}{\partial x}\left((V_\varepsilon + \frac{\alpha_0}{\alpha})\overline{F(p)}\right) \le \overline{pF(p)} - \frac{1}{2}\overline{F'(p)p^2} + B\overline{F'(p)}, \tag{2.34}$$

where $\overline{pF(p)}$ and $\overline{F'(p)p^2}$ represent the weak limits of $p_\varepsilon F(p_\varepsilon)$ and $F'(p_\varepsilon)p_\varepsilon^2$ in $L^{r_1}_{loc}([0,\infty)\times\mathbb{R})$, $1 < r_1 < \frac{3}{2}$, respectively.

Proof. Applying Lemmas 2.5 and 2.7, letting $\varepsilon \to 0$ in (2.33) and noticing the convexity of function F, we finish the proof of (2.34).

Lemma 2.9. *[4] Almost everywhere in $[0,\infty)\times\mathbb{R}$, it has*

$$p = p_+ + p_- = \overline{p_+} + \overline{p_-}, \quad p^2 = (p_+)^2 + (p_-)^2, \quad \overline{p^2} = \overline{(p_+)^2} + \overline{(p_-)^2},$$

where $\eta_+ := \eta_{\chi[0,+\infty)}(\eta)$, $\eta_- := \eta_{\chi(-\infty,0]}(\eta)$, $\eta \in \mathbb{R}$.

Using Lemmas 2.4 and 2.7 leads to

$$p_\varepsilon, \quad p \le \frac{2}{t} + c, \quad 0 < t < T.$$

Lemma 2.10. *For $t \ge 0, x \in \mathbb{R}$, in the sense of distribution, it holds that*

$$\frac{\partial p}{\partial t} + \frac{\partial}{\partial x}\left((V_\varepsilon + \frac{\alpha_0}{\alpha})p\right) = \frac{1}{2}\overline{p^2} + B(t,x). \tag{2.35}$$

Proof. Making use of (2.4), Lemmas 2.5–2.7, we derive that (2.35) holds by letting $\varepsilon \to 0$.

Lemma 2.11. *Provided that $F \in C^1(\mathbb{R})$ is a convex function with $F' \in L^\infty(\mathbb{R})$, for every $T > 0$, in the sense of distribution, then*

$$\frac{\partial F(p)}{\partial t} + \frac{\partial}{\partial x}\left((V_\varepsilon + \frac{\alpha_0}{\alpha})F(p)\right) = pF(p) + (\frac{1}{2}\overline{p^2} - p^2)F'(p) + BF'(p).$$

Proof. Suppose that $\{w_\delta\}_\delta$ is a kind of mollifiers defined in $(-\infty,\infty)$. Let $p_\delta(t,x) := (p(t,\cdot)\star w_\delta)(x)$ in which $\star$ denotes the convolution with respect to variable x. Using (2.35) yields

$$\begin{aligned}\frac{\partial F(p_\delta)}{\partial t} &= F'(p_\delta)\frac{\partial p_\delta}{\partial t} = F'(p_\delta)\left(-\frac{\partial}{\partial x}\left((V_\varepsilon+\frac{\alpha_0}{\alpha})p\right)\star w_\delta + \frac{1}{2}\bar{p}^2\star w_\delta + B\star w_\delta\right)\\ &= F'(p_\delta)\left[(-V_\varepsilon+\frac{\alpha_0}{\alpha})p_x\star w_\delta - Vp^2\star w_\delta\right] + F'(p_\delta)\left(\frac{1}{2}V\overline{q^2}\star w_\delta + B\star w_\delta\right).\end{aligned} \tag{2.36}$$

Utilizing the assumptions on F and F' and letting $\delta\to 0$ in (2.36), we complete the proof.

Following the ideas in [21], we hope that the weak convergence of p_ε should be strong convergence in (2.30). The strong convergence leads to the existence of global weak solution for system (2.1).

Lemma 2.12. *[4] Assume $V_0\in H^1(\mathbb{R})$. Then*

$$\lim_{t\to 0}\int_{\mathbb{R}} p^2(t,x)dx = \lim_{t\to 0}\int_{\mathbb{R}}\overline{p^2}(t,x)dx = \int_{\mathbb{R}}\left(\frac{\partial V_0}{\partial x}\right)^2 dx.$$

Lemma 2.13. *[4] If $V_0\in H^1(\mathbb{R})$, $L>0$, then*

$$\lim_{t\to 0}\int_{\mathbb{R}}\left(\overline{F_L^{\pm}(p)}(t,x) - F_L^{\pm}(p)(t,x)\right)dx = 0,$$

where

$$F_L(\rho) := \begin{cases}\frac{1}{2}\rho^2, & \text{if } |\rho|\le L,\\ L|\rho| - \frac{1}{2}L^2, & \text{if } |\rho|>L,\end{cases} \tag{2.37}$$

$F_L^+(\rho) = F_L(\rho)\chi_{[0,\infty)}(\rho)$ and $F_L^-(\rho) = F_L(\rho)\chi_{(-\infty,0]}(\rho)$, $\rho\in(-\infty,\infty)$.

Lemma 2.14. *[4] Let $L>0$. For $F_L(\rho)$ defined in (2.37), then*

$$\begin{cases}F_L(\rho) = \frac{1}{2}\rho^2 - \frac{1}{2}(L-|\rho|)^2\chi_{(-\infty,-L)\cap(L,\infty)}(\rho),\\ F_L'(\rho) = \rho + (L-|\rho|)\,\mathrm{sign}(\rho)\chi_{(-\infty,-L)\cap(L,\infty)}(\rho),\\ F_L^+(\rho) = \frac{1}{2}(\rho_+)^2 - \frac{1}{2}(L-\rho)^2\chi_{(L,\infty)}(\rho),\\ \left(F_L^+\right)'(\rho) = \rho_+ + (L-\rho)\chi_{(L,\infty)}(\rho),\\ F_L^-(\rho) = \frac{1}{2}(\rho_-)^2 - \frac{1}{2}(L+\rho)^2\chi_{(-\infty,-L)}(\rho),\\ \left(F_L^-\right)'(\rho) = \rho_- - (L+\rho)\chi_{(-\infty,-L)}(\rho).\end{cases}$$

Lemma 2.15. *Assume $V_0\in H^1(\mathbb{R})$. For almost all $t>0$, then*

$$\frac{1}{2}\int_{\mathbb{R}}\left(\overline{(p_+)^2} - p_+^2\right)(t,x)dx \le \int_0^t\int_{\mathbb{R}} B(s,x)\left[\overline{p_+}(s,x) - p_+(s,x)\right]dxds.$$

Lemma 2.16. *Assume $V_0\in H^1(\mathbb{R})$. For almost all $t>0$, then*

$$\begin{aligned}&\int\left(\overline{F_L^-(p)} - F_L^-(p)\right)(t,x)dx\\ \le&\frac{L^2}{2}\int_0^t\int_{\mathbb{R}}\overline{(L+p)\chi_{(-\infty,-L)}(p)}dxds - \frac{L^2}{2}\int_0^t\int_{\mathbb{R}}(L+p)\chi_{(-\infty,-L)}(p)dxds\\ &+L\int_0^t\int_{\mathbb{R}}\left[\overline{F_L^-(p)} - F_L^-(p)\right]dxds + \frac{L}{2}\int_0^t\int_{\mathbb{R}}\left(\overline{p_+^2} - p_+^2\right)dxds\\ &+\int_0^t\int_{\mathbb{R}} B(t,x)\left(\overline{\left(F_L^-\right)'(p)} - \left(F_L^-\right)'(p)\right)dxds.\end{aligned}$$

Using Lemmas 2.8 and 2.11–2.14, the proofs of Lemmas 2.15 and 2.16 are analogous to those of Lemmas 4.4 and 4.5 in Tang et al. [19]. Here we omit their proofs.

Lemma 2.17. *Assume $V_0 \in H^1(\mathbb{R})$. Almost everywhere in $[0,\infty)\times(-\infty,\infty)$, it holds that*

$$\overline{p^2} = p^2. \tag{2.38}$$

Proof. Using Lemmas 2.15 and 2.16 arises

$$\begin{aligned}
&\int_{\mathbb{R}}\left(\frac{1}{2}\left[\overline{(p_+)^2}-(p_+)^2\right]+\left[\overline{F_L^-}-F_L^-\right]\right)(t,x)dx\\
\leq&\frac{L^2}{2}\int_0^t\int_{\mathbb{R}}\overline{(L+p)\chi_{(-\infty,-L)}(p)}dxds-\frac{L^2}{2}\int_0^t\int_{\mathbb{R}}(L+p)\chi_{(-\infty,-L)}(p)dxds\\
&+L\int_0^t\int_{\mathbb{R}}\left[\overline{F_L^-(p)}-F_L^-(p)\right]dxds+\frac{L}{2}\int_0^t\int_{\mathbb{R}}\left(\overline{p_+^2}-p_+^2\right)dxds\\
&+\int_0^t\int_{\mathbb{R}}B(s,x)\left([\overline{p_+}-p_+]+\left[\overline{\left(F_L^-\right)'(p)}-\left(F_L^-\right)'(p)\right]\right)dxds.
\end{aligned} \tag{2.39}$$

Applying Lemma 2.6 drives that there has a constant constant $N>0$ to ensure

$$\|B(t,x)\|_{L^\infty([0,T)\times\mathbb{R})}\leq N. \tag{2.40}$$

Using Lemmas 2.9 and 2.14 yields

$$\begin{cases}
p_+ + \left(F_L^-\right)'(p) = p-(L+p)\chi_{(-\infty,-L)},\\
\overline{p_+}+\overline{\left(F_L^-\right)'(p)} = p-\overline{(L+p)\chi_{(-\infty,-L)}(p)}.
\end{cases} \tag{2.41}$$

Since the map $\rho\to\rho_+ + \left(F_L^-\right)'(\rho)$ is convex, it holds that

$$\begin{aligned}
0 &\leq [\overline{p_+}-p_+]+[\overline{(F_L^-)'(p)}-(F_L^-)'(p)]\\
&= (L+p)\chi_{(-\infty,-L)}-\overline{(L+p)\chi_{(-\infty,-L)}(p)}.
\end{aligned} \tag{2.42}$$

Using (2.40) gives rise to

$$\begin{aligned}
&B(s,x)\left([\overline{p_+}-p_+]+\left[\overline{\left(F_L^-\right)'(p)}-\left(F_L^-\right)'(p)\right]\right)\\
\leq&-N\left(\overline{(L+p)\chi_{(-\infty,-L)}(p)}-(L+p)\chi_{(-\infty,-L)}(p)\right).
\end{aligned} \tag{2.43}$$

Since $\rho\to(L+\rho)\chi_{(-\infty,-L)}(\rho)$ is concave, letting L be sufficiently large, we have

$$\begin{aligned}
&\frac{L^2}{2}\overline{(L+p)\chi_{(-\infty,-L)}(p)}-\frac{L^2}{2}(L+p)\chi_{(-\infty,-L)}(p)+B(s,x)\left([\overline{p_+}-p_+]+\left[\overline{(F_L^-)'(p)}-\left(F_L^-\right)'(p)\right]\right)\\
\leq&\left(\frac{L^2}{2}-N\right)\left(\overline{(L+p)\chi_{(-\infty,-L)}(p)}-(L+p)\chi_{(-\infty,-L)}(p)\right)\leq 0.
\end{aligned} \tag{2.44}$$

Using (2.39)–(2.44) yields

$$\begin{aligned} 0 &\leq \int_{\mathbb{R}} \left(\frac{1}{2} \left[\overline{(p_+)^2} - (p_+)^2 \right] + \left[\overline{F_L^-(p)} - F_L^-(p) \right] \right) (t,x) dx \\ &\leq L \int_0^t \int_{\mathbb{R}} \left(\frac{1}{2} \left[\overline{(p_+)^2} - p_+^2 \right] + \left[\overline{F_L^-(p)} - F_L^-(p) \right] \right) ds dx, \end{aligned}$$

which together with the Gronwall inequality yields

$$0 \leq \int_{\mathbb{R}} \left(\frac{1}{2} \left[\overline{(p_+)^2} - (p_+)^2 \right] + \left[\overline{F_R^-(p)} - F_R^-(p) \right] \right) (t,x) dx \leq 0. \tag{2.45}$$

Using the Fatou lemma, Lemma 2.9 and (2.45), sending $L \to \infty$, it holds that

$$0 \leq \int_{\mathbb{R}} \left(\overline{p^2} - p^2 \right) (t,x) dx \leq 0, \quad t > 0,$$

which finishes the proof of (2.38).

3. Main result and its proof

Theorem 3.1. *Assume that $V_0(x) \in H^1(\mathbb{R})$. Then system (2.1) has at least a global weak solution $V(t,x)$. Furthermore, this weak solution possesses the features:*
(a) For $(t,x) \in [0,\infty) \times \mathbb{R}$, there exists a positive constant $c = c(\|V_0\|_{H^1(\mathbb{R})})$ such that

$$\frac{\partial V(t,x)}{\partial x} \leq \frac{2}{t} + c. \tag{3.1}$$

(b) If $a, b \in \mathbb{R},\ a < b$, for any $0 < \alpha_1 < 1$ and $T > 0$, it holds that

$$\int_0^T \int_a^b \left| \frac{\partial V(t,x)}{\partial x} \right|^{2+\alpha_1} dx dt \leq c_0, \tag{3.2}$$

where positive constant c_0 relies on α_1, k, T, a, b and $\|V_0\|_{H^1(\mathbb{R})}$.

Proof. Utilizing (2.3), (2.5) and Lemma 2.5, we derive (1) and (2) in Definition 2.1. From Lemma 2.17, we have

$$p_\varepsilon^2 \to p^2 \text{ in } L^1_{loc}([0,\infty) \times \mathbb{R}).$$

Employing Lemmas 2.5 and 2.6 results in that V is a global weak solution to system (2.2). Making use of Lemmas 2.3 and 2.4 gives rise to inequalities (3.1) and (3.2). The proof is finished.

4. Conclusions

In this work, we study the rotation-Camassa-Holm (RCH) model (1.1), a nonlinear equation describing the motion of equatorial water waves with the Coriolis effect due to the Earth's rotation. The presence of the term V_{xxx} in the RCH equation leads to difficulties of establishing estimates of solutions for the viscous approximation. To overcome these difficulties, we establish a high order integrable estimate and show that $\frac{\partial V(t,x)}{\partial x}$ possesses an upper bound. Using these two estimates and the viscous approximation technique, we examine the existence of $H^1(\mathbb{R})$ global weak solutions to the RCH equation without the sign condition.

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Conflict of interest

The authors declare no conflicts of interest.

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