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Research article

System decomposition method-based global stability criteria for T-S fuzzy Clifford-valued delayed neural networks with impulses and leakage term

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Abstract: This paper investigates the global asymptotic stability problem for a class of Takagi-Sugeno fuzzy Clifford-valued delayed neural networks with impulsive effects and leakage delays using the system decomposition method. By applying Takagi-Sugeno fuzzy theory, we first consider a general form of Takagi-Sugeno fuzzy Clifford-valued delayed neural networks. Then, we decompose the considered n-dimensional Clifford-valued systems into $2^m n$-dimensional real-valued systems in order to avoid the inconvenience caused by the non-commutativity of the multiplication of Clifford numbers. By using Lyapunov-Krasovskii functionals and integral inequalities, we derive new sufficient criteria to guarantee the global asymptotic stability for the considered neural networks. Further, the results of this paper are presented in terms of real-valued linear matrix inequalities, which can be directly solved using the MATLAB LMI toolbox. Finally, a numerical example is provided with their simulations to demonstrate the validity of the theoretical analysis.

Keywords: global asymptotic stability; Clifford-valued neural networks; Lyapunov-Krasovskii functionals; Takagi-Sugeno fuzzy; impulses
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1. Introduction

The study of neural networks (NNs) has attracted considerable attention among researchers over the past two decades because it plays an important role in a wide range of applications such as associative memory, automatic control, pattern recognition, image processing, secure communication and optimization problems, see for examples [1–7]. Recently, the extension of real-valued NNs such as complex-valued NNs and quaternion-valued NNs have attracted significant attention due to their ability to solve a variety of engineering problems [8–13, 22, 23]. It is important to point out that all of these applications depend on the stability of the equilibrium of NNs. Thus, the stability analysis is a necessary step for the design and applications of NNs. As a result, a number of theoretical results concerning the stability analysis of NNs using Lyapunov-Krasovskii functionals (LKFs) and linear matrix inequalities (LMIs) recently have been published [24–27].

On the other hand, Clifford algebra (geometric algebra) is an effective and powerful framework that can be used to represent and solve geometrical problems, and as a result, it is successfully applied to neural computing, computer and robot vision, and other engineering problems [28, 29]. As such, Clifford-valued NNs have become one of the most active research fields, as they are generalizations of real-valued, complex-valued, and quaternion-valued NNs [30, 31]. Moreover, Clifford-valued NNs have been proven to be superior to real-valued, complex-valued, and quaternion-valued NNs in dealing with multidimensional data as well as spatial geometric transformations [29,31–33]. However, Clifford-valued NNs have more complicated dynamical properties than traditional networks. Hence, only a few studies have been conducted on the dynamics of Clifford-valued NNs [34–38]. For example, by using the system decomposition method, some sufficient conditions are derived in terms of real-valued LMI to ensure the global stability of Clifford-valued recurrent NNs in [31]. By utilizing the homeomorphism principle and the Cauchy-Schwarz algorithm, sufficient conditions for the global stability of Clifford-valued neutral-type NNs are obtained in [35]. Based on the LKF and LMI approach, the existence and global asymptotic stability of Clifford-valued NNs with impulsive effects have been established in [37]. Some other related results can be found in [32–34, 36].

The Takagi-Sugeno (T-S) fuzzy model is introduced in [39], which has performed as an effective tool for modeling and analyzing complex nonlinear systems. It is worth mentioning that the T-S fuzzy model has the advantage of being able to approximate a nonlinear system with linear models. Unlike typical NN structures, T-S fuzzy NNs have fuzzy operations and they are able to preserve the direct correlation between the cells [40,41]. Recently, T-S fuzzy NNs have become one of the most important research topics, and many studies have proposed T-S fuzzy logic into NNs in order to enhance the performance of NNs [42–44]. For example, by employing the LKFs and matrix inequality technique, the authors of [43] have determined the exponential convergence for T-S fuzzy complex-valued NNs including impulsive effects and time delays. By decomposing the original Clifford-valued NNs into $2^m n$-dimensional real-valued NNs the authors of [44] have derived the global asymptotic stability of T-S fuzzy Clifford-valued NNs with time-varying delays and impulses.

On the other hand, time delays inherently occur in NN implementations and they can cause undesirable system behaviours. Therefore, it is essential to study how delays affect the dynamics of the system. Recently, a lot of research results have been published regarding the dynamical analysis of NNs by considering various time delays [31–33, 45–47, 53, 54]. In addition, time delay in the leakage term also has a great impact on the dynamics of NNs. Hence, it is essential to study how time delays and leakage terms affect the system's dynamics [11, 24, 34, 35, 49, 50]. Similar to time delays, impulsive perturbations also affect the dynamics of NNs. Therefore, it is important to consider the impulsive effects when analysing the dynamics of NNs [48–52].

By the above discussions, we aim to investigate the global asymptotic stability of T-S fuzzy Clifford-valued delayed NNs with impulses by applying the system decomposition method. To the best of our knowledge, few studies have investigated the stability analysis of Clifford-valued NNs with time delays by the decomposition method. However, T-S fuzzy Clifford-valued NNs with leakage delays and impulses have not been fully explored and are not receiving much attention, which motivates us to investigate this paper. This paper has the following main merits: 1) To represent more realistic dynamics of Clifford-valued NNs, we present a general form of T-S fuzzy Clifford-valued NNs with time delays and impulsive effects. 2) The system decomposition method is employed to examine the global asymptotically stability of T-S fuzzy Clifford-valued NNs. 3) By considering suitable LKFs that contain double integral terms and by employing integral inequalities, enhanced stability conditions for the concerned NN model are derived in terms of real-valued LMIs, which could be verified directly by the MATLAB LMI toolbox.

The paper is structured as follows: Section 2 provides the problem model. Section 3 gives the main results of this paper. Section 4 discusses a numerical example that demonstrate the feasibility of the derived results. Section 5 shows the conclusion of this paper.

2. Preliminaries and model description

2.1. Notations

The Clifford algebra over $\mathbb{R}$ is defined as $\mathcal{A}$ with m generators. Let $\mathbb{R}^n$, $\mathcal{A}^n$ denote the n-dimensional real and real Clifford vector space, respectively. $\mathbb{R}^{n\times m}$, $\mathcal{A}^{n\times m}$ denote the set of all $n \times m$ real and real Clifford matrices, respectively. The superscript T and $*$ denote, respectively, the matrix transposition and involution transposition. The matrix $\mathbf{P} > 0$ ($\mathbf{P} < 0$) means that $\mathbf{P}$ is the positive (negative) definite matrix. $\star$ denotes the elements below the main diagonal of a symmetric matrix. I is the identity matrix with appropriate dimensions.

2.2. Clifford algebra

The real Clifford algebra over $\mathbb{R}^m$ is defined as

$$\mathcal{A} = \Big\{ \sum_{A\subseteq\{1,2,\dots,m\}} a^A e_A,\ a^A \in \mathbb{R}\Big\},$$

where $e_A = e_{r_1}e_{r_2}\dots e_{r_\nu}$ with $A = \{r_1, r_2, \dots, r_\nu\}$, $1 \le r_1 < r_2 < \dots < r_\nu \le m$. The Clifford generators $e_\emptyset = e_0 = 1$ and $e_r = e_{\{r\}},\ r = 1, 2, \dots, m$ are assumed to satisfy $e_ie_j + e_je_i = 0,\ i \neq j,\ i, j = 1, 2, \dots, m$, $e_i^2 = -1,\ i = 1, 2,\ \dots, m$. Moreover, the product of Clifford generators of e_4, e_5, e_6, e_7 can be defined as $e_4e_5e_6e_7 = e_{4567}$. Let $\Gamma = \{\emptyset, 1, 2, \dots, A, \dots, 12\dots m\}$, we have $\mathcal{A} = \Big\{\sum\limits_A a^Ae_A,\ a^A \in \mathbb{R}\Big\}$, where $\sum\limits_A$ denotes $\sum\limits_{A\in\Gamma}$ and $\mathcal{A}$ is isomorphic to $\mathbb{R}^{2^m}$. The involution of Clifford number $u = \sum\limits_A u^Ae_A$ is defined by $\bar{u} = \sum\limits_A u^A\bar{e}_A$, where $\bar{e}_A = (-1)^{\frac{\varrho[A](\varrho[A]+1)}{2}} e_A$ and $\varrho[A] = 0$ if $A = \emptyset$. $\varrho[A] = \nu$ if $A = r_1r_2\dots r_\nu$ and

$e_A\bar{e}_A = \bar{e}_A e_A = 1$. The Clifford-valued function defined as $u = \sum_A u^A e_A : \mathbb{R} \to \mathcal{A}$, where $u^A : \mathbb{R} \to \mathbb{R}$, and its derivative is given by $\frac{du(t)}{dt} = \sum_A \frac{du^A(t)}{dt} e_A$, where $A \in \Gamma$ and $\sum_A$ is the short form of $\sum_{A\in\Gamma}$. We refer the reader to [29–31] for more information about Clifford algebra.

2.3. Problem definition

Consider the following Clifford-valued NNs with time-varying delays and leakage terms

$$\begin{cases} \dot{u}(t) = -\mathbf{D}u(t-\sigma) + \mathbf{A}f(u(t-\tau(t))) + \mathbf{L},\ t \geq 0, \\ u(t) = \psi(t),\ t \in [-\rho, 0], \end{cases} \tag{2.1}$$

where $u(t) = (u_1(t), ..., u_n(t))^T \in \mathcal{A}^n$ denotes the neuron state vector; $\mathbf{D} = diag(d_1, ..., d_n) \in \mathbf{R}^{n\times n}$ is the self feedback connection weight matrix with $d_i > 0$ $(i = 1, 2, ..., n)$; $\mathbf{A} = (a_{ij})_{n\times n} \in \mathcal{A}^{n\times n}$ is the delayed connection weight matrix; $f(u(t-\tau(t))) = \big(f_1(u_1(t-\tau(t))), ..., f_n(u_n(t-\tau(t)))\big)^T : \mathcal{A}^n \to \mathcal{A}^n$ is the Clifford-valued activation function; $\mathbf{L} = (l_1, ..., l_n)^T \in \mathcal{A}^n$ is the Clifford-valued external input vector; $\sigma > 0$ denotes the leakage delay; $\tau(t)$ denotes the time-varying delays satisfies $0 \leq \tau(t) \leq \tau$, $\dot{\tau}(t) \leq \mu < 1$ where τ and μ are real constants; $\psi(t)$ is the initial condition which is continuously differential on $t \in [-\rho, 0]$ and $\rho = max\{\sigma, \tau\}$.

Assumption 1: For all $j = 1, ..., n$, the neuron activation functions $f_j(\cdot)$ is continuous and bounded, there exist positive diagonal matrix $\mathbf{K} = diag\{k_1, ..., k_n\}$ such that

$$|f_j(x) - f_j(y)|_{\mathcal{A}} \leq k_j|x-y|_{\mathcal{A}},\ \forall x, y \in \mathcal{A}.$$

It is obvious from Assumption 1 that,

$$(f(x) - f(y))^*(f(x) - f(y)) \leq (x-y)^*\mathbf{K}^T\mathbf{K}(x-y). \tag{2.2}$$

Theorem 2.1. *(Existence of equilibrium point) Under Assumption 1, there exists an equilibrium point $u^* \in \mathcal{A}^n$ of NNs* (2.1) *if*

$$-\mathbf{D}u^* + \mathbf{A}f(u^*) + \mathbf{L} = 0. \tag{2.3}$$

Proof: Since the activation function of NNs (2.1) is bounded, there exist constants $\mathbf{K}_j$ such that, $|f_j(u_j)|_{\mathcal{A}} \leq \mathbf{K}_j$ for all $u_j \in \mathcal{A}$, $j = 1, 2, ..., n$. Let $\mathbf{K} = (\sum_{j=1}^{n} \mathbf{K}_j^2)^{\frac{1}{2}}$, then $||f(u)||_{\mathcal{A}} \leq \mathbf{K}$ for $u = (u_1, u_2, ..., u_n) \in \mathcal{A}^n$. According to the self-feedback connection weight matrix $\mathbf{D} > 0$ that $\mathbf{D}$ is invertible. We denote $\Omega = \{u \in \mathcal{A}^n : ||u|| \leq ||\mathbf{D}^{-1}||(||\mathbf{A}||\mathbf{K} + ||\mathbf{J}||)\}$ and define the map $\mathcal{A}^n \longrightarrow \mathcal{A}^n$ by

$$\mathcal{H}(u) = \mathbf{D}^{-1}(\mathbf{A}f(u) + \mathbf{J}).$$

Here, $\mathcal{H}$ is a continuous map and by applying $||f(u)||_{\mathcal{A}} \leq \mathbf{K}$, we obtain that,

$$||\mathcal{H}(u)|| \leq ||\mathbf{D}^{-1}||(||\mathbf{A}||\mathbf{K} + ||\mathbf{J}||).$$

Thus, $\mathcal{H}$ maps Ω into itself. By Brouwers fixed point theorem, it can be derived that there exist a fixed point u^* of $\mathcal{H}$, satisfying

$$\mathbf{D}^{-1}(\mathbf{A}f(u^*) + \mathbf{J}) = u^*.$$

Pre multiplying by $\mathbf{D}$ on two sides, gives

$$-\mathbf{D}u^* + \mathbf{A}f(u^*) + \mathbf{J} = 0$$

which is equivalent to $-\mathbf{D}u^* + \mathbf{A}f(u^*) + \mathbf{J} = 0$. This completes the proof.

Remark 2.2. *According to the above Assumption 1, this paper assumes that Clifford-valued activation functions satisfy Lipschitz conditions. Similar to previous results [12,13], we consider the boundedness of activation functions in order to derive the existence of the equilibrium point. Obviously, this assumption of boundedness can lead to limitations in choosing activation functions. Therefore, the boundedness of solutions is one of the most important aspects of the systems that needs to be taken into account, see for examples [14–21].*

Conveniently, we transform $v(t) = u(t) - u^*$ to shift the equilibrium point. Then, NN (2.1) can be re-written as

$$\dot{v}(t) = -\mathbf{D}v(t-\sigma) + \mathbf{A}g(v(t-\tau(t))),\ t \geq 0, \tag{2.4}$$

where $v(t)$ is the state vector, $\varphi(t) = \psi(t) - u^*$ is the initial condition and the transformed activation function $g(v(\cdot)) = f((u(\cdot)) + u^* + \mathbf{L}) - f(u^* + \mathbf{L})$ satisfies

$$|g_j(x) - g_j(y)|_{\mathcal{A}} \leq k_j|x-y|_{\mathcal{A}},\ \forall x, y \in \mathcal{A},\ j = 1, ..., n. \tag{2.5}$$

Based on [39–43], the T-S fuzzy Clifford-valued NNs can be shown as follows

Plant Rule p:
If $\chi_1(t)$ is ϖ_{p1} and $\chi_2(t)$ is ϖ_{p2} and ... and $\chi_g(t)$ is ϖ_{pg}, **Then**

$$\begin{cases} \dot{v}(t) = -\mathbf{D}_p v(t-\sigma) + \mathbf{A}_p g(v(t-\tau(t))),\ t \geq 0, \\ v(t) = \varphi(t),\ t \in [-\rho, 0], \end{cases} \tag{2.6}$$

where $\chi_r(t),\ r = 1, ..., g$ is the premise variable; $\varpi_{pr},\ p = 1, ..., m; r = 1, ..., g$ is the fuzzy set and m is the total number of If-Then rules. Using the fuzzy model, the final outcome of the T-S fuzzy Clifford-valued NN can be determined as follows

$$\begin{cases} \dot{v}(t) = \dfrac{\sum\limits_{p=1}^{m} h_p(\chi(t))\Big\{ -\mathbf{D}_p v(t-\sigma) + \mathbf{A}_p g(v(t-\tau(t)))\Big\}}{\sum\limits_{p=1}^{m} h_p(\chi(t))},\ t \geq 0, \\ v(t) = \varphi(t),\ t \in [-\rho, 0], \end{cases} \tag{2.7}$$

or equivalently

$$\begin{cases} \dot{v}(t) = \sum\limits_{p=1}^{m} \lambda_p(\chi(t))\Big\{ -\mathbf{D}_p v(t-\sigma) + \mathbf{A}_p g(v(t-\tau(t)))\Big\},\ t \geq 0, \\ v(t) = \varphi(t),\ t \in [-\rho, 0], \end{cases} \tag{2.8}$$

where $\chi(t) = (\chi_1(t), ..., \chi_g(t))^T$, $\lambda_p(\chi(t)) = \frac{h_p(\chi(t))}{\sum\limits_{p=1}^{m} h_p(\chi(t))}$ and $h_p(\chi(t)) = \prod\limits_{r=1}^{g} \varpi_{pr}(\chi(t))$. The term $\varpi_{pr}(\chi(t))$ is the grade membership of $\chi_r(t)$ in ϖ_{pr}. From the fuzzy set theory, we have $h_p(\chi(t)) \geq 0,\ p = 1, ..., m$ and $\sum\limits_{p=1}^{m} \lambda_p(\chi(t)) = 1$ for all $t \geq 0$.

When the Clifford-valued NNs (2.8) is incorporated with impulse effects, we have

$$\begin{cases} \dot{v}(t) = \ \sum\limits_{p=1}^{m} \lambda_p(\chi(t)) \Big\{ -\mathbf{D}_p v(t-\sigma) + \mathbf{A}_p g(v(t-\tau(t))) \Big\},\ t \geq 0,\ t \neq t_k, \\ \triangle v(t_k) = v(t_k^+) - v(t_k^-) = \mathbf{I}_k(v(t_k^-)),\ t = t_k,\ k \in \mathbb{Z}_+, \\ v(t) = \varphi(t),\ t \in [-\rho, 0], \end{cases} \tag{2.9}$$

where $\triangle v(t_k) = v(t_k^+) - v(t_k^-)$ is the impulse at moments t_k and $v(t_k^+)$ and $v(t_k^-)$ denotes the right and left hand limits of $v(t_k)$, respectively. In addition, $\mathbf{I}_k = diag\{\mathbf{I}_1, ..., \mathbf{I}_n\} \in \mathbb{R}^{n\times n}$ denotes the impulsive matrix and the impulse time t_k satisfies $0 = t_1 < t_2 < ...\ t_k < ... \to \infty$ and $\inf\limits_{k\in\mathbb{Z}_+} \{t_k - t_{k-1}\} > 0$.

3. Main results

First, we use $e_A\bar{e}_A = \bar{e}_A e_A = 1$ to rewrite the original Clifford-valued NNs. Similar to the papers [30,31,35,37], it is simple to obtain a unique $\mathbf{G}^C$ satisfying $\mathbf{G}^C e_C g^A e_A = (-1)^{\varrho(B.\bar{A})}\mathbf{G}^C g^A e_B = \mathbf{G}^{B.\bar{A}} g^A e_B$, which implies the following transformation NNs (3.1).

The second term in NN (2.9) can be defined as

$$\begin{aligned} \mathbf{A}_p g(v(t-\tau(t))) =& \sum_C \mathbf{A}_p^C e_C \sum_B g^B(v(t-\tau(t)))e_B \\ =& \sum_A \sum_B (-1)^{\varrho[A.\bar{B}]} \mathbf{A}_p^{A.\bar{B}} (-1)^{\varrho[A.\bar{B}]} e_A \bar{e}_B g^B(v(t-\tau(t)))e_B \\ =& (-1)^{2\varrho[A.\bar{B}]} \sum_A \sum_B \mathbf{A}_p^{A.\bar{B}} g^B(v(t-\tau(t)))e_A\bar{e}_B e_B \\ =& \sum_A \sum_B \mathbf{A}_p^{A.\bar{B}} g^B(v(t-\tau(t)))e_A. \end{aligned}$$

Then, we can decompose NN (2.9) into the following real-valued one:

$$\begin{cases} \dot{v}^A(t) = \sum\limits_{p=1}^{m} \lambda_p(\chi(t)) \Big\{ -\mathbf{D}_p v^A(t-\sigma) + \sum\limits_A \mathbf{A}_p^{A\cdot\bar{B}} g^A(v(t-\tau(t))) \Big\},\ t \geq 0,\ t \neq t_k, \\ \triangle v^A(t_k) = v^A(t_k^+) - v^A(t_k^-) = \mathbf{I}_k(v^A(t_k^-)),\ t = t_k,\ k \in \mathbb{Z}_+,\ A \in \Gamma, \\ v^A(t) = \varphi^A(t),\ t \in [-\rho, 0], \end{cases} \tag{3.1}$$

where

$$\begin{aligned} v^A(t-\sigma) =& (v_1^A(t-\sigma), ..., v_n^A(t-\sigma))^T,\ v(t-\sigma) = \sum_A v^A(t-\sigma)e_A, \\ g(v(t-\tau(t))) =& \sum_B g^B(v^{C_1}(t-\tau(t)), ..., v^{C_{2m}}(t-\tau(t)))e_B = \sum_B g^B(v(t-\tau(t)))e_B, \end{aligned}$$

$$\mathbf{A}_p = \sum_C \mathbf{A}_p^C e_C,\ \mathbf{A}_p^{A.\bar{B}} = (-1)^{\varrho[A.\bar{B}]}\mathbf{A}_p^C,\ e_A\bar{e}_B = (-1)^{\varrho[A.\bar{B}]}e_C.$$

According to Clifford algebra, NN (3.1) can be expressed as a new real-valued NNs. Let

$$\hat{v}(t) = ((v^0(t))^T, ..., (v^A(t))^T, ..., (v^{12...m}(t))^T)^T \in \mathbb{R}^{2^m n},$$

$$\hat{g}(\hat{v}(t-\tau(t))) = ((g^0(v(t-\tau(t))))^T, ..., (g^A(v(t-\tau(t))))^T, ..., (g^{12...m}(v(t-\tau(t))))^T)^T \in \mathbb{R}^{2^m n},$$

$$\hat{\mathbf{D}}_p = \begin{pmatrix} \mathbf{D}_p & 0 & \dots & 0 \\ 0 & \mathbf{D}_p & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathbf{D}_p \end{pmatrix}_{2^m n \times 2^m n},$$

$$\hat{\mathbf{I}}_k = \begin{pmatrix} \mathbf{I}_k & 0 & \dots & 0 \\ 0 & \mathbf{I}_k & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathbf{I}_k \end{pmatrix}_{2^m n \times 2^m n},$$

$$\hat{\mathbf{A}}_p = \begin{pmatrix} \mathbf{A}_p^0 & \dots & \mathbf{A}_p^{\overline{A}} & \dots & \mathbf{A}_p^{\overline{12...m}} \\ \mathbf{A}_p^1 & \dots & \mathbf{A}_p^{1.\overline{A}} & \dots & \mathbf{A}_p^{1.\overline{12...m}} \\ \vdots & \dots & \vdots & \ddots & \vdots \\ \mathbf{A}_p^{12...m} & \dots & \mathbf{A}_p^{12...m.\overline{A}} & \dots & \mathbf{A}_p^{12...m.\overline{12...m}} \end{pmatrix}_{2^m n \times 2^m n},$$

$$\hat{\varphi}(t) = [(\varphi^0(t))^T, ..., (\varphi^A(t))^T, ...(\varphi^{12...m}(t))^T]^T \in \mathbb{R}^{2^m n}.$$

Then, NN (3.1) can be written as

$$\begin{cases} \dot{\hat{v}}(t) = \sum_{p=1}^{m} \lambda_p(\chi(t)) \Big\{ -\hat{\mathbf{D}}_p \hat{v}(t-\sigma) + \hat{\mathbf{A}}_p \hat{g}(\hat{v}(t-\tau(t))) \Big\},\ t \geq 0,\ t \neq t_k, \\ \triangle \hat{v}(t_k) = \hat{v}(t_k^+) - \hat{v}(t_k^-) = \hat{\mathbf{I}}_k(\hat{v}(t_k^-)),\ t = t_k,\ k \in \mathbb{Z}_+, \\ \hat{v}(t) = \hat{\varphi}(t),\ t \in [-\rho, 0]. \end{cases} \tag{3.2}$$

Furthermore, (2.2) can be written in the following form:

$$(\hat{g}(x) - \hat{g}(y))^T(\hat{g}(x) - \hat{g}(y)) \leq (x-y)^T \hat{\mathbf{K}}(x-y), \tag{3.3}$$

where $\hat{\mathbf{K}} = \begin{pmatrix} \mathbf{K}^T\mathbf{K} & 0 & \dots & 0 \\ 0 & \mathbf{K}^T\mathbf{K} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathbf{K}^T\mathbf{K} \end{pmatrix}_{2^m n \times 2^m n}.$

Assumption 2: The impulsive effects $\hat{\mathbf{I}}_k(\hat{v}(t_k^-))$ are assumed to satisfy the following conditions

$$\triangle \hat{v}(t_k) = \hat{\mathbf{I}}_k(\hat{v}(t_k^-)) = -\hat{\mathbf{J}}_k \Big\{ \hat{v}(t_k^-) - \hat{\mathbf{D}}_p \int_{t_k-\sigma}^{t_k} \hat{v}(s)ds \Big\},\ k \in \mathbb{Z}_+,$$

where $\hat{\mathbf{J}}_k = \begin{pmatrix} \mathbf{J}_k & 0 & \dots & 0 \\ 0 & \mathbf{J}_k & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathbf{J}_k \end{pmatrix}_{2^m n \times 2^m n}$ and $\mathbf{J}_k = diag\{\mathbf{J}_1, ..., \mathbf{J}_n\} \in \mathbb{R}^{n \times n}$.

Lemma 3.1. *[53] For any constant positive definite matrix* $\mathbf{M} = \mathbf{M}^T \in \mathbb{R}^{2^m n \times 2^m n}$*, the following inequality is true for all continuously differentiable function* $\hat{v}(\alpha)$ *in* $[\eta_1, \eta_2] \in \mathbb{R}^{2^m n}$

$$-(\eta_2 - \eta_1) \int_{t-\eta_2}^{t-\eta_1} \hat{v}^T(\alpha)\mathbf{M}\hat{v}(\alpha)d\alpha \leq -\Big(\int_{t-\eta_2}^{t-\eta_1} \hat{v}(\alpha)d\alpha\Big)^T \mathbf{M}\Big(\int_{t-\eta_2}^{t-\eta_1} \hat{v}(\alpha)d\alpha\Big).$$

Lemma 3.2. *[54] For any constant positive definite matrix* $\mathbf{M} = \mathbf{M}^T \in \mathbb{R}^{2^m n \times 2^m n}$*, any constant matrix* $\mathbf{X} \in \mathbb{R}^{2^n \times 2^m n}$*, any vector* $\theta_1, \theta_2 \in \mathbb{R}^{2^m n}$*, and* $\vartheta \in (0, 1)$*, such that* $\begin{pmatrix} \mathbf{M} & \mathbf{X} \\ \mathbf{X}^T & \mathbf{M} \end{pmatrix} > 0$*, the following condition holds*

$$\frac{1}{\vartheta}\theta_1^T\mathbf{M}\theta_1 + \frac{1}{1-\vartheta}\theta_2^T\mathbf{M}\theta_2 \geq \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}^T \begin{pmatrix} \mathbf{M} & \mathbf{X} \\ \mathbf{X}^T & \mathbf{M} \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}.$$

Global asymptotic stability analysis

In this subsection, we will derive the sufficient criteria to assure the global asymptotic stability of the considered NNs (3.2) using the LKFs and LMI method.

Theorem 3.3. *Suppose Assumptions 1 and 2 holds. The NN model* (3.2) *is globally asymptotically stable if there exist positive definite symmetric matrices* $\mathbf{P}$, $\mathbf{Q}_1$, $\mathbf{Q}_2$, $\mathbf{Q}_3$, $\mathbf{R}_1$, $\mathbf{R}_2$, $\mathbf{U}$, $\begin{pmatrix} \mathbf{S}_{11} & \mathbf{S}_{12} \\ \star & \mathbf{S}_{22} \end{pmatrix} > 0$ *and* $\begin{pmatrix} \mathbf{T}_{11} & \mathbf{T}_{12} \\ \star & \mathbf{T}_{22} \end{pmatrix} > 0$*, any matrix* $\mathbf{X}$ *and scalars* $\epsilon_1 > 0$ *such that the following LMIs hold for all* $p = 1, 2, ..., m$*:*

$$\begin{pmatrix} \mathbf{P} & (I - \hat{\mathbf{J}}_k)^T\mathbf{P} \\ \star & \mathbf{P} \end{pmatrix} \geq 0, \ k \in \mathbb{Z}_+, \tag{3.4}$$

$$\Xi_p = \begin{pmatrix} (\Theta_{i,j,p})_{6\times 6} & (\tau\mathbf{R}_1 + \sigma\mathbf{R}_2)\Pi^T & (\sqrt{\tau}\mathbf{S}_{22} + \sqrt{\sigma}\mathbf{T}_{22})\Pi^T \\ \star & -\mathbf{R}_1 - \mathbf{R}_2 & 0 \\ \star & \star & -\mathbf{S}_{22} - \mathbf{T}_{22} \end{pmatrix} < 0, \tag{3.5}$$

where $\Theta_{1,1,p} = -\mathbf{P}\hat{\mathbf{D}}_p - \hat{\mathbf{D}}_p\mathbf{P} + \mathbf{Q}_1 + \mathbf{Q}_2 + \mathbf{Q}_3 - \mathbf{R}_1 - \mathbf{R}_2 + \sigma^2\mathbf{U}$, $\Theta_{1,2,p} = \mathbf{R}_1^T - \mathbf{X} + \mathbf{S}_{12}^T$, $\Theta_{1,3,p} = \mathbf{X}$, $\Theta_{1,4,p} = \mathbf{R}_2 + \mathbf{T}_{12}^T$, $\Theta_{1,5,p} = \mathbf{P}\hat{\mathbf{A}}_p$, $\Theta_{1,6,p} = \hat{\mathbf{D}}_p^T\mathbf{P}\hat{\mathbf{D}}_p$, $\Theta_{2,2,p} = -(1-\mu)\mathbf{Q}_1 - \mathbf{R}_1 - \mathbf{R}_1 + \mathbf{X} + \mathbf{X}^T + \tau\mathbf{S}_{11} - 2\mathbf{S}_{12}^T + \epsilon_1\hat{\mathbf{K}}$, $\Theta_{2,3,p} = \mathbf{R}_1 - \mathbf{X}$, $\Theta_{3,3,p} = -\mathbf{Q}_2 - \mathbf{R}_1$, $\Theta_{4,4,p} = -\mathbf{Q}_3 - \mathbf{R}_2 + \sigma\mathbf{T}_{11} - 2\mathbf{T}_{12}^T$, $\Theta_{5,5,p} = -\epsilon_1 I$, $\Theta_{5,6,p} = -\hat{\mathbf{A}}_p^T\mathbf{P}\hat{\mathbf{D}}_p$, $\Theta_{6,6,p} = -\mathbf{U}$, $\Pi = [0 \ \ 0 \ \ 0 \ \ -\hat{\mathbf{D}}_p^T \ \ \hat{\mathbf{A}}_p^T \ \ 0]^T$.

Proof: Construct the following LKF for NN model (3.2):

$$\mathbf{V}(t, v(t), p) = \sum_{i=1}^{6} \mathbf{V}_i(t, v(t), p) \tag{3.6}$$

where

$$\mathbf{V}_1(t, v(t), p) = \Big(\hat{v}(t) - \hat{\mathbf{D}}_p \int_{t-\sigma}^{t} \hat{v}(s)ds\Big)^T \mathbf{P}\Big(\hat{v}(t) - \hat{\mathbf{D}}_p \int_{t-\sigma}^{t} \hat{v}(s)ds\Big),$$

$$\mathbf{V}_2(t, v(t), p) = \int_{t-\tau(t)}^{t} \hat{v}^T(s)\mathbf{Q}_1\hat{v}(s)ds + \int_{t-\tau}^{t} \hat{v}^T(s)\mathbf{Q}_2\hat{v}(s)ds$$

$$
\begin{aligned}
&\quad + \int_{t-\sigma}^{t} \hat{v}^T(s)\mathbf{Q}_3\hat{v}(s)ds,\\
\mathbf{V}_3(t, v(t), p) &= \tau \int_{t-\tau}^{t} (s-(t-\tau))\dot{\hat{v}}^T(s)\mathbf{R}_1\dot{\hat{v}}(s)ds\\
&\quad + \sigma \int_{t-\sigma}^{t} (s-(t-\sigma))\dot{\hat{v}}^T(s)\mathbf{R}_2\dot{\hat{v}}(s)ds,\\
\mathbf{V}_4(t, v(t), p) &= \int_0^t \int_{u-\tau(u)}^{u} \begin{pmatrix} \hat{v}(u-\tau(u)) \\ \dot{\hat{v}}(s) \end{pmatrix}^T \begin{pmatrix} \mathbf{S}_{11} & \mathbf{S}_{12} \\ \star & \mathbf{S}_{22} \end{pmatrix} \begin{pmatrix} \hat{v}(u-\tau(u)) \\ \dot{\hat{v}}(s) \end{pmatrix} dsdu,\\
&\quad + \int_0^t \int_{u-\sigma}^{u} \begin{pmatrix} \hat{v}(u-\sigma) \\ \dot{\hat{v}}(s) \end{pmatrix}^T \begin{pmatrix} \mathbf{T}_{11} & \mathbf{T}_{12} \\ \star & \mathbf{T}_{22} \end{pmatrix} \begin{pmatrix} \hat{v}(u-\sigma) \\ \dot{\hat{v}}(s) \end{pmatrix} dsdu,\\
\mathbf{V}_5(t, v(t), p) &= \int_{t-\tau}^{t} (s-(t-\tau))\dot{\hat{v}}^T(s)\mathbf{S}_{22}\dot{\hat{v}}(s)ds\\
&\quad + \int_{t-\sigma}^{t} (s-(t-\sigma))\dot{\hat{v}}^T(s)\mathbf{T}_{22}\dot{\hat{v}}(s)ds,\\
\mathbf{V}_6(t, v(t), p) &= \sigma \int_{t-\sigma}^{t} (s-(t-\sigma))\hat{v}^T(s)\mathbf{U}\hat{v}(s)ds.
\end{aligned}
$$

When $t = t_k,\ k \in \mathbb{Z}_+$, we can compute

$$
\begin{aligned}
\hat{v}(t_k) - \hat{\mathbf{D}}_p \int_{t_k-\sigma}^{t_k} \hat{v}(s)ds &= \hat{v}(t_k^-) - \hat{\mathbf{J}}_k\Big(\hat{v}(t_k^-) - \hat{\mathbf{D}}_p \int_{t_k-\sigma}^{t_k} \hat{v}(s)ds\Big) - \hat{\mathbf{D}}_p \int_{t_k-\sigma}^{t_k} \hat{v}(s)ds\\
&= \hat{v}(t_k^-) - \hat{\mathbf{J}}_k\hat{v}(t_k^-) + \hat{\mathbf{J}}_k\hat{\mathbf{D}}_p \int_{t_k-\sigma}^{t_k} \hat{v}(s)ds - \hat{\mathbf{D}}_p \int_{t_k-\sigma}^{t_k} \hat{v}(s)ds\\
&= (I - \hat{\mathbf{J}}_k)\hat{v}(t_k^-) - (I - \hat{\mathbf{J}}_k)\hat{\mathbf{D}}_p \int_{t_k-\sigma}^{t_k} \hat{v}(s)ds\\
&= (I - \hat{\mathbf{J}}_k)[\hat{v}(t_k^-) - \hat{\mathbf{D}}_p \int_{t_k-\sigma}^{t_k} \hat{v}(s)ds]. \qquad (3.7)
\end{aligned}
$$

Moreover, it follows from (3.4) that

$$
\begin{aligned}
&\begin{pmatrix} \mathbf{P} & (I - \hat{\mathbf{J}}_k)^T\mathbf{P} \\ \star & \mathbf{P} \end{pmatrix} \geq 0\\
\Leftrightarrow &\begin{pmatrix} I & -(I - \hat{\mathbf{J}}_k)^T \\ 0 & I \end{pmatrix} \begin{pmatrix} \mathbf{P} & (I - \hat{\mathbf{J}}_k)^T\mathbf{P} \\ \star & \mathbf{P} \end{pmatrix} \begin{pmatrix} I & 0 \\ -(I - \hat{\mathbf{J}}_k) & I \end{pmatrix} \geq 0\\
\Leftrightarrow &\begin{pmatrix} \mathbf{P} - (I - \hat{\mathbf{J}}_k)^T\mathbf{P}(I - \hat{\mathbf{J}}_k) & 0 \\ \star & \mathbf{P} \end{pmatrix} \geq 0\\
\Leftrightarrow &\ \mathbf{P} - (I - \hat{\mathbf{J}}_k)^T\mathbf{P}(I - \hat{\mathbf{J}}_k) \geq 0. \qquad (3.8)
\end{aligned}
$$

Combining (3.7) and (3.8), we have

$$
\mathbf{V}_1(t_k, v(t_k^-), p) = \Big(\hat{v}(t_k) - \hat{\mathbf{D}}_p \int_{t_k-\sigma}^{t_k} \hat{v}(s)ds\Big)^T \mathbf{P}\Big(\hat{v}(t_k) - \hat{\mathbf{D}}_p \int_{t_k-\sigma}^{t_k} \hat{v}(s)ds\Big)
$$

$$
\begin{aligned}
&= \Big(\hat{v}(t_k^-) - \hat{\mathbf{D}}_p \int_{t_k-\sigma}^{t_k} \hat{v}(s)ds\Big)^T (I - \hat{\mathbf{J}}_k)^T \mathbf{P} (I - \mathbf{J}_k) \Big(\hat{v}(t_k^-) - \hat{\mathbf{D}}_p \int_{t_k-\sigma}^{t_k} \hat{v}(s)ds\Big) \\
&\leq \Big(\hat{v}(t_k^-) - \hat{\mathbf{D}}_p \int_{t_k-\sigma}^{t_k} \hat{v}(s)ds\Big)^T \mathbf{P} \Big(\hat{v}(t_k^-) - \hat{\mathbf{D}}_p \int_{t_k-\sigma}^{t_k} \hat{v}(s)ds\Big) \\
\mathbf{V}_1(t_k, v(t_k^-), p) &\leq \mathbf{V}_1(t_k^-, v(t_k^-), p).
\end{aligned} \tag{3.9}
$$

It is easy to verify that $\mathbf{V}_2(t_k, v(t), p) \leq \mathbf{V}_2(t_k^-, v(t_k^-), p)$, $\mathbf{V}_3(t_k, v(t), p) \leq \mathbf{V}_3(t_k^-, v(t_k^-), p)$, $\mathbf{V}_4(t_k, v(t), p) \leq \mathbf{V}_4(t_k^-, v(t_k^-), p)$, $\mathbf{V}_5(t_k, v(t), p) \leq \mathbf{V}_5(t_k^-, v(t_k^-), p)$ and $\mathbf{V}_6(t_k, v(t), p) \leq \mathbf{V}_6(t_k^-, v(t_k^-), p)$ which implies that

$$
\mathbf{V}(t_k, v(t), p) \leq \mathbf{V}(t_k^-, v(t_k^-), p), \ k \in \mathbb{Z}_+. \tag{3.10}
$$

When $t \neq t_k, \ k \in \mathbb{Z}_+$, we can compute the upper right derivative of (3.6) along the trajectories of (3.2), we have

$$
D^+\mathbf{V}(t, v(t), p) = \sum_{i=1}^{6} D^+\mathbf{V}_i(t, v(t), p), \tag{3.11}
$$

where

$$
\begin{aligned}
D^+\mathbf{V}_1(t, v(t), p) &= \Big(\hat{v}(t) - \hat{\mathbf{D}}_p \int_{t-\sigma}^{t} \hat{v}(s)ds\Big)^T \mathbf{P}\Big(\dot{\hat{v}}(t) - \mathbf{D}_p \hat{v}(t) + \hat{\mathbf{D}}_p \hat{v}(t-\sigma)\Big) \\
&\quad + \Big(\dot{\hat{v}}(t) - \mathbf{D}_p \hat{v}(t) + \hat{\mathbf{D}}_p \hat{v}(t-\sigma)\Big)^T \mathbf{P}\Big(\hat{v}(t) - \hat{\mathbf{D}}_p \int_{t-\sigma}^{t} \hat{v}(s)ds\Big) \\
&= \Big(\hat{v}(t) - \hat{\mathbf{D}}_p \int_{t-\sigma}^{t} \hat{v}(s)ds\Big)^T \mathbf{P}\Big(\sum_{p=1}^{m} \lambda_p(\chi(t)) \Big\{ - \hat{\mathbf{D}}_p \hat{v}(t) + \hat{\mathbf{A}}_p \hat{g}(\hat{v}(t - \tau(t))) \Big\}\Big) \\
&\quad + \Big(\sum_{p=1}^{m} \lambda_p(\chi(t)) \Big\{ - \hat{\mathbf{D}}_p \hat{v}(t) + \hat{\mathbf{A}}_p \hat{g}(\hat{v}(t - \tau(t))) \Big\}\Big)^T \mathbf{P}\Big(\hat{v}(t) - \hat{\mathbf{D}}_p \int_{t-\sigma}^{t} \hat{v}(s)ds\Big),
\end{aligned} \tag{3.12}
$$

$$
\begin{aligned}
D^+\mathbf{V}_2(t, v(t), p) &= \hat{v}^T(t)(\mathbf{Q}_1 + \mathbf{Q}_2 + \mathbf{Q}_3)\hat{v}(t) - (1 - \dot{\tau}(t))\hat{v}^T(t - \tau(t))\mathbf{Q}_1 \hat{v}(t - \tau(t)) \\
&\quad - \hat{v}^T(t - \tau)\mathbf{Q}_2 \hat{v}(t - \tau) - \hat{v}^T(t - \sigma)\mathbf{Q}_3 \hat{v}(t - \sigma),
\end{aligned} \tag{3.13}
$$

$$
D^+\mathbf{V}_3(t, v(t), p) = \dot{\hat{v}}^T(t)(\tau^2 \mathbf{R}_1 + \sigma^2 \mathbf{R}_2)\dot{\hat{v}}(t) - \tau \int_{t-\tau}^{t} \dot{\hat{v}}^T(s)\mathbf{R}_1 \dot{\hat{v}}(s)ds - \sigma \int_{t-\sigma}^{t} \dot{\hat{v}}^T(s)\mathbf{R}_2 \dot{\hat{v}}(s)ds. \tag{3.14}
$$

The first integral term in (3.14) can be defined as

$$
-\tau \int_{t-\tau}^{t} \dot{\hat{v}}^T(s)\mathbf{R}_1 \dot{\hat{v}}(s)ds = -\int_{t-\tau}^{t-\tau(t)} \dot{\hat{v}}^T(s)\mathbf{R}_1 \dot{\hat{v}}(s)ds - \int_{t-\tau(t)}^{t} \dot{\hat{v}}^T(s)\mathbf{R}_1 \dot{\hat{v}}(s)ds. \tag{3.15}
$$

By applying Lemma (3.1) in the following forms

$$
\begin{aligned}
-\tau \int_{t-\tau}^{t} \dot{\hat{v}}^T(s)\mathbf{R}_1 \dot{\hat{v}}(s)ds &\leq -\frac{\tau}{\tau - \tau(t)} \Big(\int_{t-\tau}^{t-\tau(t)} \dot{\hat{v}}(s)ds\Big)^T \mathbf{R}_1 \Big(\int_{t-\tau}^{t-\tau(t)} \dot{\hat{v}}(s)ds\Big) \\
&\quad - \frac{\tau}{\tau(t)} \Big(\int_{t-\tau(t)}^{t} \dot{\hat{v}}(s)ds\Big)^T \mathbf{R}_1 \Big(\int_{t-\tau(t)}^{t} \dot{\hat{v}}(s)ds\Big)
\end{aligned}
$$

$$
\begin{aligned}
&= -\Big(\int_{t-\tau}^{t-\tau(t)} \dot{\hat{v}}(s)ds\Big)^T \mathbf{R}_1\Big(\int_{t-\tau}^{t-\tau(t)} \dot{\hat{v}}(s)ds\Big) \\
&\quad - \frac{\tau(t)}{\tau-\tau(t)}\Big(\int_{t-\tau}^{t-\tau(t)} \dot{\hat{v}}(s)ds\Big)^T \mathbf{R}_1\Big(\int_{t-\tau}^{t-\tau(t)} \dot{\hat{v}}(s)ds\Big) \\
&\quad - \Big(\int_{t-\tau(t)}^{t} \dot{\hat{v}}(s)ds\Big)^T \mathbf{R}_1\Big(\int_{t-\tau(t)}^{t} \dot{\hat{v}}(s)ds\Big) \\
&\quad - \frac{\tau-\tau(t)}{\tau(t)}\Big(\int_{t-\tau(t)}^{t} \dot{\hat{v}}(s)ds\Big)^T \mathbf{R}_1\Big(\int_{t-\tau(t)}^{t} \dot{\hat{v}}(s)ds\Big).
\end{aligned}
\tag{3.16}
$$

If $\begin{pmatrix} \mathbf{R}_1 & \mathbf{X} \\ \mathbf{X}^T & \mathbf{R}_1 \end{pmatrix} \geq 0$, by Lemma (3.2), the following inequality true:

$$
\begin{pmatrix} \sqrt{\frac{\tau(t)}{\tau-\tau(t)}}\Big(\int_{t-\tau}^{t-\tau(t)} \dot{\hat{v}}(s)ds\Big) \\ \sqrt{\frac{\tau-\tau(t)}{\tau(t)}}\Big(\int_{t-\tau(t)}^{t} \dot{\hat{v}}(s)ds\Big) \end{pmatrix}^T \begin{pmatrix} \mathbf{R}_1 & \mathbf{X} \\ \mathbf{X}^T & \mathbf{R}_1 \end{pmatrix} \begin{pmatrix} \sqrt{\frac{\tau(t)}{\tau-\tau(t)}}\Big(\int_{t-\tau}^{t-\tau(t)} \dot{\hat{v}}(s)ds\Big) \\ \sqrt{\frac{\tau-\tau(t)}{\tau(t)}}\Big(\int_{t-\tau(t)}^{t} \dot{\hat{v}}(s)ds\Big) \end{pmatrix} \geq 0,
\tag{3.17}
$$

which implies

$$
\begin{aligned}
&-\frac{\tau(t)}{\tau-\tau(t)}\Big(\int_{t-\tau}^{t-\tau(t)} \dot{\hat{v}}(s)ds\Big)^T \mathbf{R}_1\Big(\int_{t-\tau}^{t-\tau(t)} \dot{\hat{v}}(s)ds\Big) - \frac{\tau-\tau(t)}{\tau(t)}\Big(\int_{t-\tau(t)}^{t} \dot{\hat{v}}(s)ds\Big)^T \mathbf{R}_1\Big(\int_{t-\tau(t)}^{t} \dot{\hat{v}}(s)ds\Big) \\
&\leq -\Big(\int_{t-\tau}^{t-\tau(t)} \dot{\hat{v}}(s)ds\Big)^T \mathbf{X}\Big(\int_{t-\tau(t)}^{t} \dot{\hat{v}}(s)ds\Big) - \Big(\int_{t-\tau(t)}^{t} \dot{\hat{v}}(s)ds\Big)^T \mathbf{X}^T\Big(\int_{t-\tau}^{t-\tau(t)} \dot{\hat{v}}(s)ds\Big).
\end{aligned}
\tag{3.18}
$$

Combining (3.16) and (3.18), we have

$$
\begin{aligned}
-\tau\int_{t-\tau}^{t} \dot{\hat{v}}^T(s)\mathbf{R}_1\dot{\hat{v}}(s)ds \leq &-\Big(\int_{t-\tau}^{t-\tau(t)} \dot{\hat{v}}(s)ds\Big)^T \mathbf{R}_1\Big(\int_{t-\tau}^{t-\tau(t)} \dot{\hat{v}}(s)ds\Big) - \Big(\int_{t-\tau(t)}^{t} \dot{\hat{v}}(s)ds\Big)^T \mathbf{R}_1\Big(\int_{t-\tau(t)}^{t} \dot{\hat{v}}(s)ds\Big) \\
&- \Big(\int_{t-\tau}^{t-\tau(t)} \dot{\hat{v}}(s)ds\Big)^T \mathbf{X}\Big(\int_{t-\tau(t)}^{t} \dot{\hat{v}}(s)ds\Big) - \Big(\int_{t-\tau(t)}^{t} \dot{\hat{v}}(s)ds\Big)^T \mathbf{X}^T\Big(\int_{t-\tau}^{t-\tau(t)} \dot{\hat{v}}(s)ds\Big).
\end{aligned}
\tag{3.19}
$$

By applying Lemma (3.1), the second integral term in (3.14) can be defined as

$$
-\sigma\int_{t-\sigma}^{t} \dot{\hat{v}}^T(s)\mathbf{R}_2\dot{\hat{v}}(s)ds \leq -\Big(\int_{t-\sigma}^{t} \dot{\hat{v}}(s)ds\Big)^T \mathbf{R}_2\Big(\int_{t-\sigma}^{t} \dot{\hat{v}}(s)ds\Big).
\tag{3.20}
$$

$$
\begin{aligned}
D^+\mathbf{V}_4(t, v(t), p) = &\int_{t-\tau(t)}^{t} \begin{pmatrix} \hat{v}(t-\tau(t)) \\ \dot{\hat{v}}(s) \end{pmatrix}^T \begin{pmatrix} \mathbf{S}_{11} & \mathbf{S}_{12} \\ \star & \mathbf{S}_{22} \end{pmatrix} \begin{pmatrix} \hat{v}(t-\tau(t)) \\ \dot{\hat{v}}(s) \end{pmatrix} ds \\
&+ \int_{t-\sigma}^{t} \begin{pmatrix} \hat{v}(t-\sigma) \\ \dot{\hat{v}}(s) \end{pmatrix}^T \begin{pmatrix} \mathbf{T}_{11} & \mathbf{T}_{12} \\ \star & \mathbf{T}_{22} \end{pmatrix} \begin{pmatrix} \hat{v}(t-\sigma) \\ \dot{\hat{v}}(s) \end{pmatrix} ds \\
= &\ \tau(t)\hat{v}^T(t-\tau(t))\mathbf{S}_{11}\hat{v}(t-\tau(t)) + 2\hat{v}^T(t)\mathbf{S}_{12}^T\hat{v}(t-\tau(t))
\end{aligned}
$$

$$
\begin{aligned}
&- 2\hat{v}^T(t-\tau(t))\mathbf{S}_{12}^T\hat{v}(t-\tau(t)) + \int_{t-\tau(t)}^{t} \dot{\hat{v}}^T(s)\mathbf{S}_{22}\dot{\hat{v}}(s)ds \\
&+ \sigma\hat{v}^T(t-\sigma)\mathbf{T}_{11}\hat{v}(t-\sigma) + 2\hat{v}^T(t)\mathbf{T}_{12}^T\hat{v}(t-\sigma) \\
&- 2\hat{v}^T(t-\sigma)\mathbf{T}_{12}^T\hat{v}(t-\sigma) + \int_{t-\sigma}^{t} \dot{\hat{v}}^T(s)\mathbf{T}_{22}\dot{\hat{v}}(s)ds,
\end{aligned} \tag{3.21}
$$

$$
\begin{aligned}
D^+\mathbf{V}_5(t, v(t), p) = {}& \dot{\hat{v}}^T(t)(\tau\mathbf{S}_{22} + \sigma\mathbf{T}_{22})\dot{\hat{v}}(t) - \int_{t-\tau(t)}^{t} \dot{\hat{v}}^T(s)\mathbf{S}_{22}\dot{\hat{v}}(s)ds \\
&- \int_{t-\sigma}^{t} \dot{\hat{v}}^T(s)\mathbf{T}_{22}\dot{\hat{v}}(s)ds,
\end{aligned} \tag{3.22}
$$

$$
D^+\mathbf{V}_6(t, v(t), p) = \sigma^2\hat{v}^T(t)\mathbf{U}\hat{v}(t) - \sigma\int_{t-\sigma}^{t} \hat{v}^T(s)\mathbf{U}\hat{v}(s)ds. \tag{3.23}
$$

By applying Lemma (3.1), we get

$$
D^+\mathbf{V}_6(t, v(t), p) \leq \sigma^2\hat{v}^T(t)\mathbf{U}\hat{v}(t) - \Big(\int_{t-\sigma}^{t} \hat{v}(s)ds\Big)^T\mathbf{U}\Big(\int_{t-\sigma}^{t} \hat{v}(s)ds\Big). \tag{3.24}
$$

There exist positive scalar $\epsilon_1 > 0$. By Assumption 1, we have

$$
0 \leq \epsilon_1[\hat{v}^T(t-\tau(t))\hat{\mathbf{K}}\hat{v}(t-\tau(t)) - \hat{g}^T(\hat{v}(t-\tau(t)))\hat{g}(\hat{v}(t-\tau(t)))]. \tag{3.25}
$$

Combining (3.11)–(3.25), we have

$$
\begin{aligned}
D^+\mathbf{V}(t, v(t), p) \leq {}& \sum_{p=1}^{m} \lambda_p(\chi(t))\Big\{\zeta^T(t)[(\Theta_{i,j,p})_{6\times6} \\
&+ \Pi^T(\tau^2\mathbf{R}_1 + \sigma^2\mathbf{R}_2 + \tau\mathbf{S}_{22} + \sigma\mathbf{T}_{22})\Pi]\zeta(t)\Big\},
\end{aligned} \tag{3.26}
$$

where $\zeta(t) = [\hat{v}^T(t), \hat{v}^T(t-\tau(t)), \hat{v}^T(t-\tau), \hat{v}^T(t-\sigma), \hat{g}^T(\hat{v}(t-\tau(t))), \int_{t-\sigma}^{t} \hat{v}^T(s)ds]^T$.
Using the Schur complement it can be derived from (3.26) that

$$
D^+\mathbf{V}(t, v(t), p) \leq \sum_{p=1}^{m} \lambda_p(\chi(t))\Big\{\zeta^T(t)\Xi_p\zeta(t)\Big\}. \tag{3.27}
$$

From condition (3.5), we have

$$
D^+\mathbf{V}(t, v(t), p) \leq -\varsigma\zeta^T(t)\zeta(t) \leq -\varsigma||\hat{v}(t)||^2 < 0, \tag{3.28}
$$

for any $\hat{v}(t) \neq 0$, where $\varsigma = \varsigma_{min}(-\Xi_p) > 0$. This implies that the equilibrium point of NN (3.2) is globally asymptotically stable. This completes the proof of Theorem (3.3).

Remark 3.4. *When the impulsive effect is absent, NN* (3.2) *reduces as follows:*

$$\begin{cases}\dot{\hat{v}}(t) = \displaystyle\sum_{p=1}^{m}\lambda_p(\chi(t))\Big\{-\hat{\mathbf{D}}_p\hat{v}(t-\sigma)+\hat{\mathbf{A}}_p\hat{g}(\hat{v}(t-\tau(t)))\Big\},\ t\geq 0,\\ \hat{v}(t)=\hat{\varphi}(t),\ t\in[-\rho,0],\end{cases} \tag{3.29}$$

Corollary 3.5. *Suppose Assumptions 1 holds. The NN model* (3.29) *is globally asymptotically stable if there exist positive definite symmetric matrices* $\mathbf{P}$, $\mathbf{Q}_1$, $\mathbf{Q}_2$, $\mathbf{Q}_3$, $\mathbf{R}_1$, $\mathbf{R}_2$, $\mathbf{U}$, $\begin{pmatrix}\mathbf{S}_{11} & \mathbf{S}_{12}\\ \star & \mathbf{S}_{22}\end{pmatrix} > 0$ *and* $\begin{pmatrix}\mathbf{T}_{11} & \mathbf{T}_{12}\\ \star & \mathbf{T}_{22}\end{pmatrix} > 0$, *any matrix* $\mathbf{X}$ *and scalars* $\epsilon_1 > 0$ *such that the following LMIs hold for all* $p = 1,2,...,m$:

$$\Xi_p = \begin{pmatrix}(\Theta_{i,j,p})_{6\times 6} & (\tau\mathbf{R}_1+\sigma\mathbf{R}_2)\Pi^T & (\sqrt{\tau}\mathbf{S}_{22}+\sqrt{\sigma}\mathbf{T}_{22})\Pi^T\\ \star & -\mathbf{R}_1-\mathbf{R}_2 & 0\\ \star & \star & -\mathbf{S}_{22}-\mathbf{T}_{22}\end{pmatrix}<0, \tag{3.30}$$

where $\Theta_{1,1,p} = -\mathbf{P}\hat{\mathbf{D}}_p-\hat{\mathbf{D}}_p\mathbf{P}+\mathbf{Q}_1+\mathbf{Q}_2+\mathbf{Q}_3-\mathbf{R}_1-\mathbf{R}_2+\sigma^2\mathbf{U}$, $\Theta_{1,2,p} = \mathbf{R}_1^T-\mathbf{X}+\mathbf{S}_{12}^T$, $\Theta_{1,3,p} = \mathbf{X}$, $\Theta_{1,4,p} = \mathbf{R}_2+\mathbf{T}_{12}^T$, $\Theta_{1,5,p} = \mathbf{P}\hat{\mathbf{A}}_p$, $\Theta_{1,6,p} = \hat{\mathbf{D}}_p^T\mathbf{P}\hat{\mathbf{D}}_p$, $\Theta_{2,2,p} = -(1-\mu)\mathbf{Q}_1-\mathbf{R}_1-\mathbf{R}_1+\mathbf{X}+\mathbf{X}^T+\tau\mathbf{S}_{11}-2\mathbf{S}_{12}^T+\epsilon_1\hat{\mathbf{K}}$, $\Theta_{2,3,p} = \mathbf{R}_1-\mathbf{X}$, $\Theta_{3,3,p} = -\mathbf{Q}_2-\mathbf{R}_1$, $\Theta_{4,4,p} = -\mathbf{Q}_3-\mathbf{R}_2+\sigma\mathbf{T}_{11}-2\mathbf{T}_{12}^T$, $\Theta_{5,5,p} = -\epsilon_1 I$, $\Theta_{5,6,p} = -\hat{\mathbf{A}}_p^T\mathbf{P}\hat{\mathbf{D}}_p$, $\Theta_{6,6,p} = -\mathbf{U}$, $\Pi = [0\ \ 0\ \ 0\ \ -\hat{\mathbf{D}}_p^T\ \ \hat{\mathbf{A}}_p^T\ \ 0]^T$.

Proof: Take $\mathbf{V}_1(t,v(t),p)$, $\mathbf{V}_2(t,v(t),p)$, $\mathbf{V}_3(t,v(t),p)$, $\mathbf{V}_4(t,v(t),p)$, $\mathbf{V}_5(t,v(t),p)$, $\mathbf{V}_6(t,v(t),p)$ same as in LKF (3.6). The remaining proof is similar to that in Theorem (3.3), and so it is omitted.

Remark 3.6. *When the leakage term is absent, NN* (3.29) *decreases as follows:*

$$\begin{cases}\dot{\hat{v}}(t) = \displaystyle\sum_{p=1}^{m}\lambda_p(\chi(t))\Big\{-\hat{\mathbf{D}}_p\hat{v}(t)+\hat{\mathbf{A}}_p\hat{g}(\hat{v}(t-\tau(t)))\Big\},\ t\geq 0,\\ \hat{v}(t)=\hat{\varphi}(t),\ t\in[-\tau,0]\end{cases} \tag{3.31}$$

Corollary 3.7. *Suppose Assumptions 1 holds. The NN model* (3.31) *is globally asymptotically stable if there exist positive definite symmetric matrices* $\mathbf{P}$, $\mathbf{Q}_1$, $\mathbf{Q}_2$, $\mathbf{R}_1$ *and* $\begin{pmatrix}\mathbf{S}_{11} & \mathbf{S}_{12}\\ \star & \mathbf{S}_{22}\end{pmatrix} > 0$, *any matrix* $\mathbf{X}$ *and scalars* $\epsilon_1 > 0$ *such that the following LMIs hold for all* $p = 1,2,...,m$:

$$\bar{\Xi}_p = \begin{pmatrix}(\bar{\Theta}_{i,j,p})_{4\times 4} & \tau\mathbf{R}_1\bar{\Pi}^T & \sqrt{\tau}\mathbf{S}_{22}\bar{\Pi}^T\\ \star & -\mathbf{R}_1 & 0\\ \star & \star & -\mathbf{S}_{22}\end{pmatrix}<0, \tag{3.32}$$

where $\bar{\Theta}_{1,1,p} = -\mathbf{P}\hat{\mathbf{D}}_p-\hat{\mathbf{D}}_p\mathbf{P}+\mathbf{Q}_1+\mathbf{Q}_2-\mathbf{R}_1$, $\bar{\Theta}_{1,2,p} = \mathbf{R}_1^T-\mathbf{X}+\mathbf{S}_{12}^T$, $\bar{\Theta}_{1,3,p} = \mathbf{X}$, $\bar{\Theta}_{1,4,p} = \mathbf{P}\hat{\mathbf{A}}_p$, $\bar{\Theta}_{2,2,p} = -(1-\mu)\mathbf{Q}_1-\mathbf{R}_1-\mathbf{R}_1+\mathbf{X}+\mathbf{X}^T+\tau\mathbf{S}_{11}-2\mathbf{S}_{12}^T+\epsilon_1\hat{\mathbf{K}}$, $\bar{\Theta}_{2,3,p} = \mathbf{R}_1-\mathbf{X}$, $\bar{\Theta}_{3,3,p} = -\mathbf{R}_1$, $\bar{\Theta}_{4,4,p} = -\epsilon_1 I$, $\bar{\Pi} = [-\hat{\mathbf{D}}_p^T\ \ 0\ \ 0\ \ \hat{\mathbf{A}}_p^T]^T$.

Proof: Construct the following LKF for NN model (3.31):

$$\mathbf{V}(t,v(t),p) = \sum_{i=1}^{5}\mathbf{V}_i(t,v(t),p) \tag{3.33}$$

where

$$
\begin{aligned}
\mathbf{V}_1(t, v(t), p) &= \hat{v}(t)^T \mathbf{P} \hat{v}(t),\\
\mathbf{V}_2(t, v(t), p) &= \int_{t-\tau(t)}^{t} \hat{v}^T(s)\mathbf{Q}_1 \hat{v}(s)ds + \int_{t-\tau}^{t} \hat{v}^T(s)\mathbf{Q}_2 \hat{v}(s)ds,\\
\mathbf{V}_3(t, v(t), p) &= \tau \int_{t-\tau}^{t} (s-(t-\tau))\dot{\hat{v}}^T(s)\mathbf{R}_1 \dot{\hat{v}}(s)ds,\\
\mathbf{V}_4(t, v(t), p) &= \int_0^t \int_{u-\tau(u)}^{u} \begin{pmatrix} \hat{v}(u-\tau(u)) \\ \dot{\hat{v}}(s) \end{pmatrix}^T \begin{pmatrix} \mathbf{S}_{11} & \mathbf{S}_{12} \\ \star & \mathbf{S}_{22} \end{pmatrix} \begin{pmatrix} \hat{v}(u-\tau(u)) \\ \dot{\hat{v}}(s) \end{pmatrix} dsdu,\\
\mathbf{V}_5(t, v(t), p) &= \int_{t-\tau}^{t} (s-(t-\tau))\dot{\hat{v}}^T(s)\mathbf{S}_{22} \dot{\hat{v}}(s)ds.
\end{aligned}
$$

The remaining proof is similar to that in Theorem (3.3), and so it is omitted.

Remark 3.8. *According to our knowledge, there are no studies that have compared the global asymptotic stability criteria for time-varying delays, impulse effects as well as leakage terms among the obtained global asymptotic stability criteria for T-S fuzzy Clifford-valued NNs, which shows the novelty of this paper.*

4. Numerical example

This section provides a numerical example to demonstrate the validity of the obtained results.
Example 1: Let $p = 1, 2$. Consider the following plant rules for T-S fuzzy Clifford-valued NNs.

$$
\begin{cases}
\dot{v}(t) = \sum_{p=1}^{2} \lambda_p(\chi(t)) \Big\{ -\mathbf{D}_p v(t-\sigma) + \mathbf{A}_p g(v(t-\tau(t))) \Big\},\ t \geq 0,\ t \neq t_k,\\
\triangle v(t_k) = v(t_k^+) - v(t_k^-) = \mathbf{I}_k v(t_k^-),\ t = t_k,\ k \in \mathbb{Z}_+,\\
v(t) = \varphi(t),\ t \in [-\rho, 0],
\end{cases}
\tag{4.1}
$$

Plant Rule 1: **If** $\chi_1(t)$ is ϖ_{11}, **Then**

$$
\begin{cases}
\dot{v}(t) = -\mathbf{D}_1 v(t-\sigma) + \mathbf{A}_1 g(v(t-\tau(t))),\ t \geq 0,\ t \neq t_k,\\
\triangle v(t_k) = v(t_k^+) - v(t_k^-) = \mathbf{I}_k v(t_k^-),\ t = t_k,\ k \in \mathbb{Z}_+,\\
v(t) = \varphi(t),\ t \in [-\rho, 0],
\end{cases}
$$

Plant Rule 2: **If** $\chi_1(t)$ is ϖ_{21}, **Then**

$$
\begin{cases}
\dot{v}(t) = -\mathbf{D}_2 v(t-\sigma) + \mathbf{A}_2 g(v(t-\tau(t))),\ t \geq 0,\ t \neq t_k,\\
\triangle v(t_k) = v(t_k^+) - v(t_k^-) = \mathbf{I}_k v(t_k^-),\ t = t_k,\ k \in \mathbb{Z}_+,\\
v(t) = \varphi(t),\ t \in [-\rho, 0],
\end{cases}
$$

where ϖ_{11} is $v_1(t) \leq 1$, ϖ_{21} is $v_1(t) > 1$, and in which the following parameters are used

$$
\mathbf{D}_1 = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix},\ \mathbf{D}_2 = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix},
$$

$$\mathbf{A}_1 = \begin{pmatrix} 0.3e_0 + 2e_1 & 0.2e_0 + 0.4e_2 - 0.7e_{12} \\ 0.06e_0 - 0.3e_2 + 0.05e_{12} & 0.2e_0 + 0.2e_1 + 0.06e_{12} \end{pmatrix},$$
$$\mathbf{A}_2 = \begin{pmatrix} 0.2e_0 + e_1 & 0.1e_0 + 0.3e_2 - 0.6e_{12} \\ 0.05e_0 - 0.2e_2 + 0.4e_{12} & 0.1e_0 + 0.1e_1 + 0.05e_{12} \end{pmatrix},$$
$$\mathbf{I}_k = \begin{pmatrix} -0.5 & 0 \\ 0 & -0.5 \end{pmatrix},\ \mathbf{K} = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}.$$

The Clifford generators are $e_1^2 = e_2^2 = e_{12}^2 = e_1e_2e_{12} = -1$, $e_1e_2 = -e_2e_1 = e_{12}$, $e_1e_{12} = -e_{12}e_1 = -e_2$, $e_2e_{12} = -e_{12}e_2 = e_1$, $\dot{v}_1(t) = \dot{v}_1^0(t)e_0 + \dot{v}_1^1(t)e_1 + \dot{v}_1^2(t)e_2 + \dot{v}_1^{12}(t)e_{12}$, $\dot{v}_2(t) = \dot{v}_2^0(t)e_0 + \dot{v}_2^1(t)e_1 + \dot{v}_2^2(t)e_2 + \dot{v}_2^{12}(t)e_{12}$. According to the definitions, we have

$$\mathbf{A}_1^0 = \begin{pmatrix} 0.3 & 0.2 \\ 0.06 & 0.2 \end{pmatrix},\ \mathbf{A}_1^1 = \begin{pmatrix} 2 & 0 \\ 0 & 0.2 \end{pmatrix},\ \mathbf{A}_1^2 = \begin{pmatrix} 0 & 0.4 \\ -0.3 & 0 \end{pmatrix},$$
$$\mathbf{A}_1^{12} = \begin{pmatrix} 0 & -0.7 \\ 0.5 & 0.06 \end{pmatrix},\ \mathbf{A}_2^0 = \begin{pmatrix} 0.2 & 0.1 \\ 0.05 & 0.1 \end{pmatrix},\ \mathbf{A}_2^1 = \begin{pmatrix} 1 & 0 \\ 0 & 0.1 \end{pmatrix},$$
$$\mathbf{A}_2^2 = \begin{pmatrix} 0 & 0.3 \\ -0.2 & 0 \end{pmatrix},\ \mathbf{A}_2^{12} = \begin{pmatrix} 0 & -0.6 \\ 0.4 & 0.05 \end{pmatrix},$$

and

$$\hat{\mathbf{A}}_1 = \begin{pmatrix} \mathbf{A}_1^0 & \mathbf{A}_1^{\bar{1}} & \mathbf{A}_1^{\bar{2}} & \mathbf{A}_1^{\bar{12}} \\ \mathbf{A}_1^1 & \mathbf{A}_1^{1.\bar{1}} & \mathbf{A}_1^{1.\bar{2}} & \mathbf{A}_1^{1.\bar{12}} \\ \mathbf{A}_1^2 & \mathbf{A}_1^{2.\bar{1}} & \mathbf{A}_1^{2.\bar{2}} & \mathbf{A}_1^{2.\bar{12}} \\ \mathbf{A}_1^{12} & \mathbf{A}_1^{12.\bar{1}} & \mathbf{A}_1^{12.\bar{2}} & \mathbf{A}_1^{12.\bar{12}} \end{pmatrix} = \begin{pmatrix} \mathbf{A}_1^0 & -\mathbf{A}_1^1 & -\mathbf{A}_1^2 & -\mathbf{A}_1^{12} \\ \mathbf{A}_1^1 & \mathbf{A}_1^0 & -\mathbf{A}_1^{12} & \mathbf{A}_1^2 \\ \mathbf{A}_1^2 & \mathbf{A}_1^{12} & \mathbf{A}_1^0 & -\mathbf{A}_1^1 \\ \mathbf{A}_1^{12} & -\mathbf{A}_1^2 & \mathbf{A}_1^1 & \mathbf{A}_1^0 \end{pmatrix},$$
$$\hat{\mathbf{A}}_2 = \begin{pmatrix} \mathbf{A}_2^0 & \mathbf{A}_2^{\bar{1}} & \mathbf{A}_2^{\bar{2}} & \mathbf{A}_2^{\bar{12}} \\ \mathbf{A}_2^1 & \mathbf{A}_2^{1.\bar{1}} & \mathbf{A}_2^{1.\bar{2}} & \mathbf{A}_2^{1.\bar{12}} \\ \mathbf{A}_2^2 & \mathbf{A}_2^{2.\bar{1}} & \mathbf{A}_2^{2.\bar{2}} & \mathbf{A}_2^{2.\bar{12}} \\ \mathbf{A}_2^{12} & \mathbf{A}_2^{12.\bar{1}} & \mathbf{A}_2^{12.\bar{2}} & \mathbf{A}_2^{12.\bar{12}} \end{pmatrix} = \begin{pmatrix} \mathbf{A}_2^0 & -\mathbf{A}_2^1 & -\mathbf{A}_2^2 & -\mathbf{A}_2^{12} \\ \mathbf{A}_2^1 & \mathbf{A}_2^0 & -\mathbf{A}_2^{12} & \mathbf{A}_2^2 \\ \mathbf{A}_2^2 & \mathbf{A}_2^{12} & \mathbf{A}_2^0 & -\mathbf{A}_2^1 \\ \mathbf{A}_2^{12} & -\mathbf{A}_2^2 & \mathbf{A}_2^1 & \mathbf{A}_2^0 \end{pmatrix}.$$

Choose the time-varying delay as $\tau(t) = 0.5 + 0.2\ sin(t)$, which implies that the maximum permissible upper bound is $\tau = 0.7$. It is observable that $\dot{\tau}(t) \leq \mu = 0.2\ cos(t) = 0.2$. The premise variable $\chi(t)$ is chosen as a state-dependent term, that is, $\chi(t) = v_1(t)$. Using the same procedure as in [41], the membership functions can be obtained from the property of $\lambda_1(v_1(t)) + \lambda_2(v_1(t)) = 1$, where $\lambda_1(v_1(t)) = \frac{1}{1+e^{-v_1(t)}}$, $\lambda_2(v_1(t)) = 1 - \frac{1}{1+e^{-v_1(t)}}$. The LMI conditions (3.4) and (3.5) in Theorem (3.3) are verified using MATLAB LMI toolbox with $t_{\min} = -4.0770 \times 10^{-004}$.

Under the initial conditions $\varphi_1(t) = -0.9e_0 + e_1 - 0.2e_2 - 1.6e_{12}$, $\varphi_2(t) = -e_0 - e_1 + 1.8e_2 + 2e_{12}$, the time responses of states $v_i^0(t)$, $v_i^1(t)$, $v_i^2(t)$, $v_i^{12}(t)$, $i = 1, 2$ are shown in Figures (1)–(6).

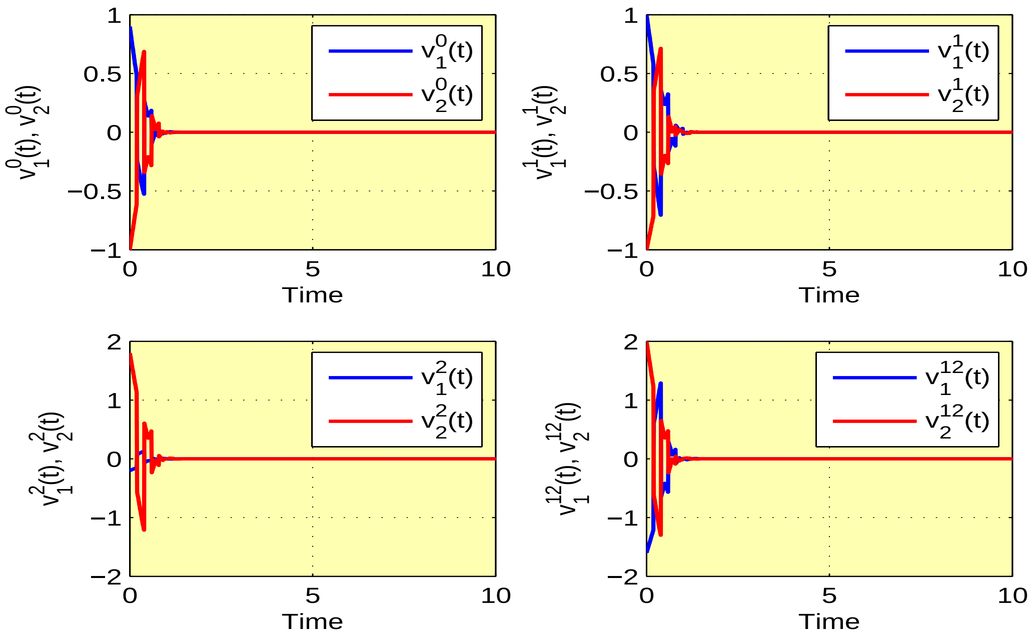

Figure 1. The time responses of states $v_i^0(t)$, $v_i^1(t)$, $v_i^2(t)$, $v_i^{12}(t)$, $i = 1, 2$ of NNs (4.1) with $v_1(t_k) = -0.5v_1(t_k^-)$, $v_2(t_k) = -0.5v_2(t_k^-)$ and $\sigma = 0.10$.

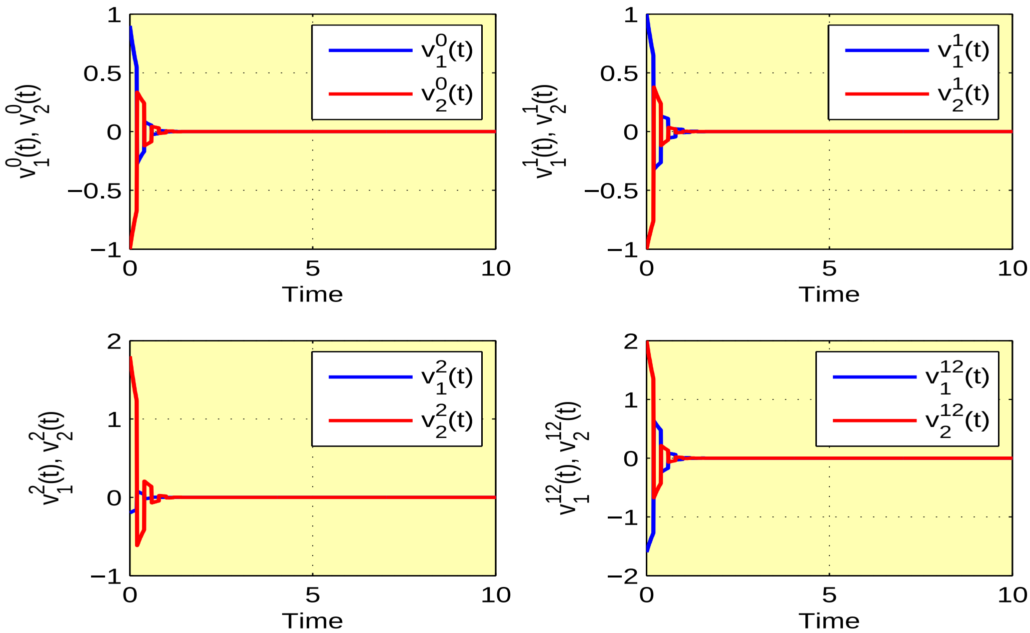

Figure 2. The time responses of states $v_i^0(t)$, $v_i^1(t)$, $v_i^2(t)$, $v_i^{12}(t)$, $i = 1, 2$ of NNs (4.1) with $v_1(t_k) = -0.5v_1(t_k^-)$, $v_2(t_k) = -0.5v_2(t_k^-)$ and $\sigma = 0$.

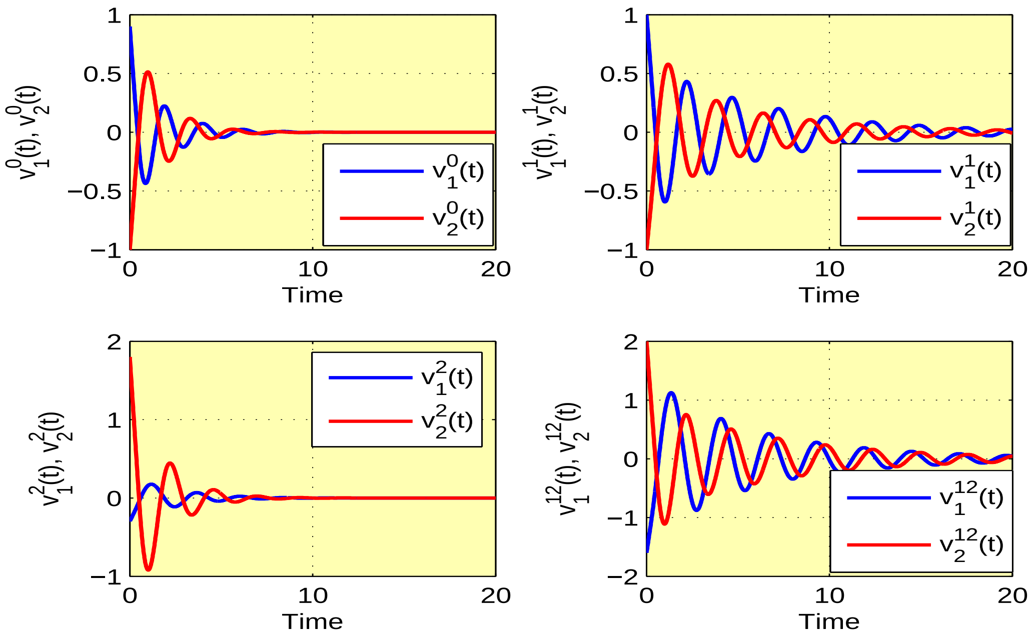

Figure 3. The time responses of states $v_i^0(t)$, $v_i^1(t)$, $v_i^2(t)$, $v_i^{12}(t)$, $i = 1, 2$ of NNs (4.1) with $v_1(t_k) = v_2(t_k) = 0$ and $\sigma = 0.12$.

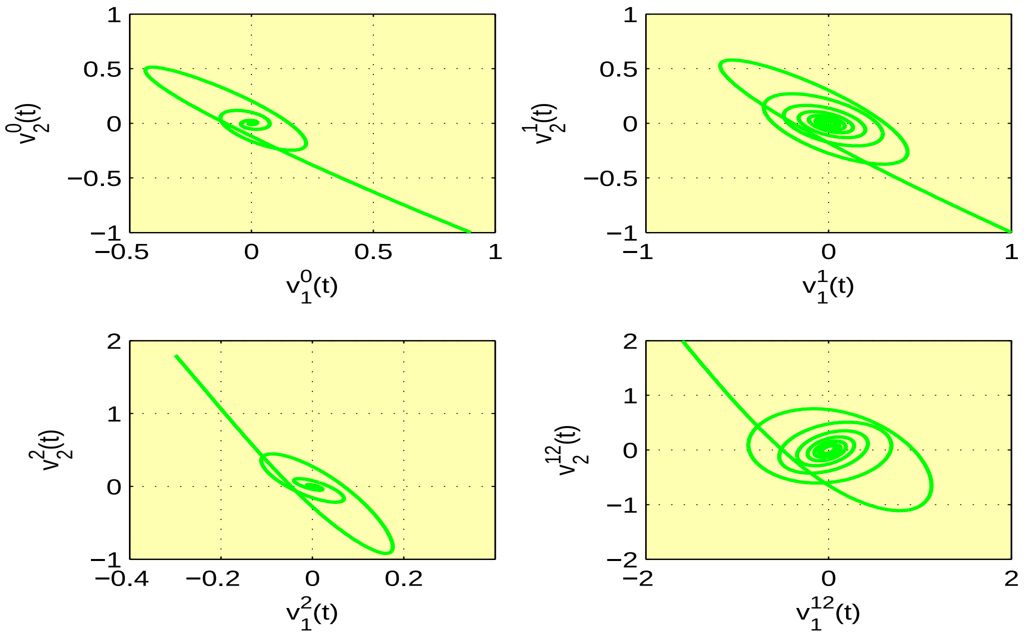

Figure 4. The time responses of states $v_i^0(t)$, $v_i^1(t)$, $v_i^2(t)$, $v_i^{12}(t)$, $i = 1, 2$ of NNs (4.1) with $v_1(t_k) = v_2(t_k) = 0$ and $\sigma = 0.12$.

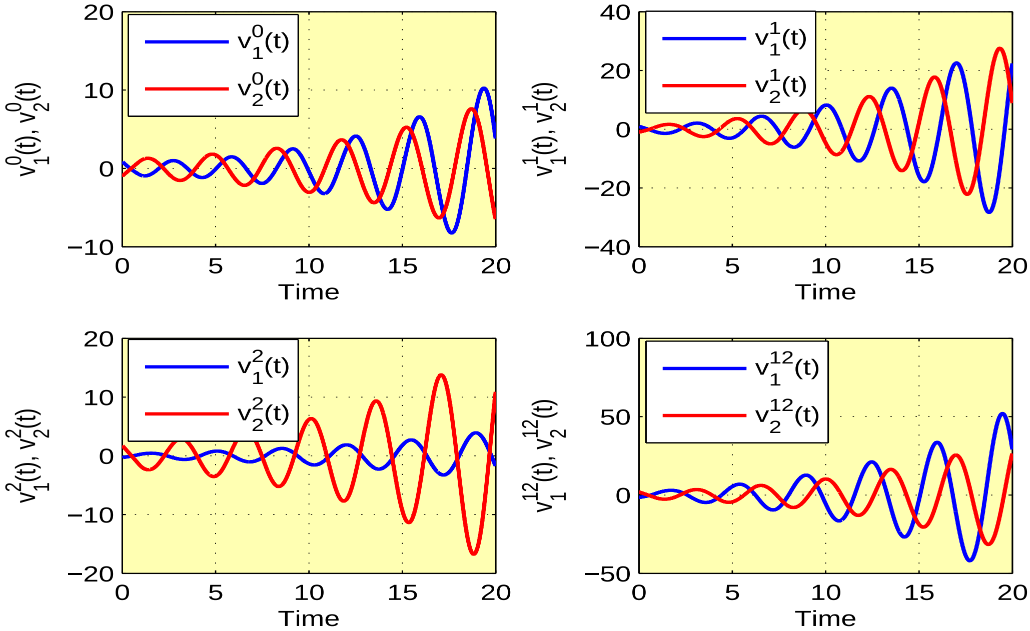

Figure 5. The time responses of states $v_i^0(t)$, $v_i^1(t)$, $v_i^2(t)$, $v_i^{12}(t)$, $i = 1, 2$ of NNs (4.1) with $v_1(t_k) = v_2(t_k) = 0$ and $\sigma = 0.15$ in a 2-dimensional space.

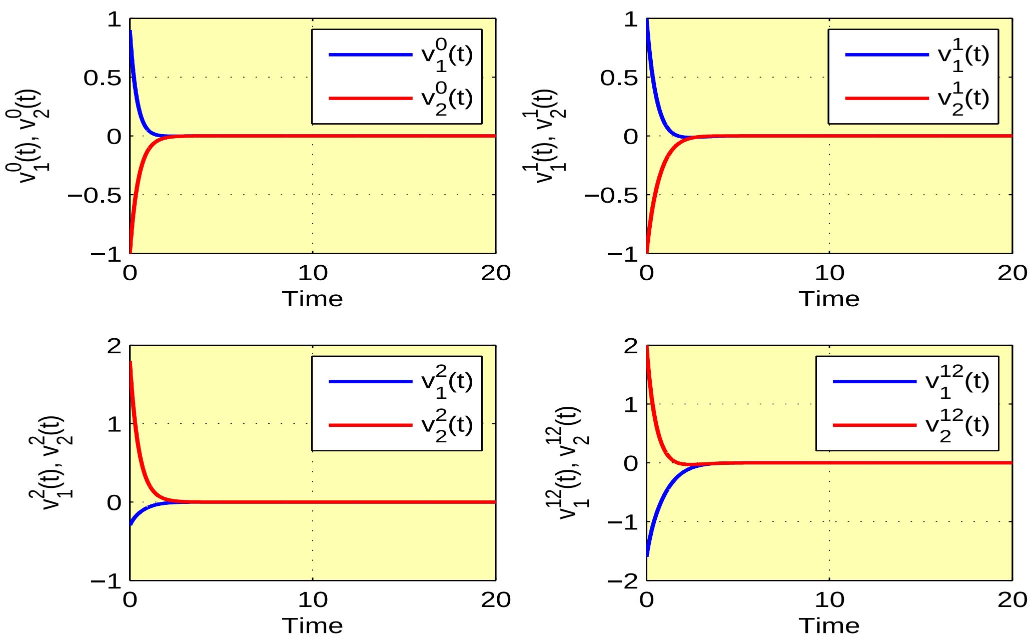

Figure 6. The time responses of states $v_i^0(t)$, $v_i^1(t)$, $v_i^2(t)$, $v_i^{12}(t)$, $i = 1, 2$ of NNs (4.1) with $v_1(t_k) = v_2(t_k) = 0$ and $\sigma = 0$.

From the above analysis, all the conditions associated with Theorem (3.3) are satisfied, then the equilibrium point of NNs (4.1) is globally asymptotically stable.

Remark 4.1. *From example 1, it is clear that the stability behaviour of the considered T-S Clifford-valued NNs has highly dependent on time delays in the leakage term. For instance, when* $\sigma = 0$*, the time response of the states of NNs* (4.1) *approaches the equilibrium point, as shown in Figure* (6)*. When* $\sigma = 0.12$ *is increased, the time responses of the states of NNs* (4.1) *oscillate, as illustrated in Figure* (3) *and Figure* (4)*. When* $\sigma = 0.15$ *is constantly increased, the time response of the states of NNs* (4.1) *becomes unstable, as illustrated in Figure* (5)*.*

5. Conclusions

In this paper, the problem of global asymptotic stability of T-S Clifford-valued fuzzy delayed NNs with impulsive effects and leakage term has been investigated. By applying T-S fuzzy theory, we first considered a general form of T-S fuzzy Clifford-valued NNs with time-varying delays. Then, we decomposed the original Clifford-valued NNs into the $2^m n$-dimensional real-valued NNs in order to solve the non-commutativity issue pertaining Clifford numbers. By considering appropriate LKFs and integral inequalities, new sufficient criteria are obtained to guarantee the global asymptotic stability of the considered networks. Furthermore, the results of this paper are presented in the form of LMIs, which can be solved using the MATLAB LMI toolbox. Finally, a numerical example is presented with their simulations to demonstrate the validity of the theoretical analysis.

By applying the main results of this paper, we can analyze various dynamical behaviors of T-S fuzzy Clifford-valued NNs including finite-time stability, passivity, state estimation, synchronization, and others. There are certain advancements worth investigating further in this proposed area of research. We will soon attempt to examine the finite-time dissipativity of T-S fuzzy Clifford-valued NNs with time delays.

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Conflict of interest

The authors declare no conflict of interest.

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