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Research article

Solvability and representations of the general solutions to some nonlinear difference equations of second order

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Abstract: We give detailed theoretical explanations for getting the closed-form formulas and representations for the general solutions to four special cases of a class of nonlinear difference equations of second order considered in the literature, present an extension of the class of difference equations which is solvable in closed form, analyze some results on the long-term behavior of the solutions to the class of equations, and give some results on convergence.

Keywords: difference equation; solvable equation; closed-form formula for solutions; bilinear difference equation
Mathematics Subject Classification: 39A20

1. Introduction

1.1. Notation

As usual, throughout the paper, the set of all positive natural numbers is denoted by $\mathbb{N}$, the set of all whole numbers is denoted by $\mathbb{Z}$, whereas the set of real numbers is denoted by $\mathbb{R}$. If $k \in \mathbb{Z}$ is fixed, then by $\mathbb{N}_k$ we denote the set

$$\{j \in \mathbb{Z} : j \geq k\}.$$

If $k, l \in \mathbb{Z}$ where $k \leq l$, then the notation $j = \overline{k, l}$ is used instead of using the following phrase/notation: $k \leq j \leq l$ for $j \in \mathbb{Z}$. If $l \in \mathbb{Z}$, then we regard that

$$\prod_{j=l}^{l-1} a_j = 1,$$

where $a_j \in \mathbb{R}$ is a member of a finite or infinite sequence of numbers and the index $j \in I \subseteq \mathbb{Z}$.

1.2. Little on history and some classical closed-form formulas

Difference equations and systems of difference equations appeared in some classical problems in combinatorics, probability and economics. To solve some of the practical problems in these scientific areas, it has been of a great importance to know some closed-form formulas for the solutions of the difference equations which serve as models for the problems. The following papers and books [7, 10, 12, 21–24] contain some of the oldest results on solvability of difference equations and their applications (see also the references therein). Since that time have appeared many books containing chapters devoted to the solvability and their applications such as [8, 15, 25, 26, 28, 50].

De Moivre solved the equation

$$x_{n+2} - px_{n+1} - qx_n = 0, \quad n \in \mathbb{N}_0, \tag{1.1}$$

as well as the corresponding linear difference equations with constant coefficients of the order three and four (see [10, 12]), whereas Bernoulli in [7] presented a method for solving the linear difference equations with constant coefficients of any order.

The formula

$$x_n = \frac{(x_1 - t_2x_0)t_1^n - (x_1 - t_1x_0)t_2^n}{t_1 - t_2}, \quad n \in \mathbb{N}_0, \tag{1.2}$$

where t_j, $j = 1, 2$, are the zeros of the polynomial

$$P_{p,q}(t) = t^2 - pt - q, \tag{1.3}$$

is a closed-form formula for the general solution to Eq (1.1) under the assumptions:

$$p \in \mathbb{R}, \quad q \in \mathbb{R} \setminus \{0\} \quad \text{and} \quad p^2 + 4q \neq 0.$$

If

$$p \in \mathbb{R}, \quad q \in \mathbb{R} \setminus \{0\} \quad \text{and} \quad p^2 + 4q = 0,$$

then we have

$$x_n = ((x_1 - t_1x_0)\,n + t_1x_0)\,t_1^{n-1}, \quad n \in \mathbb{N}_0. \tag{1.4}$$

In this case the zeros of (1.3) are

$$t_1 = t_2 = \frac{p}{2}.$$

Classical formulas (1.2) and (1.4) are frequently used in the literature. This will be the case also in the present paper.

One of the first nonlinear difference equations for which was found the general solution in a closed form is the bilinear one

$$y_{n+1} = \frac{\alpha y_n + \beta}{\gamma y_n + \delta}, \quad n \in \mathbb{N}_0. \tag{1.5}$$

See, for example, [1, 8, 9, 20–22, 25, 27, 28, 43, 44, 49], where some applications of the closed-form formulas can be found.

For some recent results on solvability and related topics see, for instance, [14, 29, 30, 32–35, 40–49] and the references therein.

1.3. Motivation

The following class of nonlinear difference equations of second order

$$x_{n+1} = ax_n + \frac{bx_nx_{n-1}}{cx_n + dx_{n-1}}, \quad n \in \mathbb{N}_0, \tag{1.6}$$

where $a, b, c, d,\ x_{-j} \in \mathbb{R},\ j = 0, 1$, was considered in [11], where several claims were formulated and were also given some closed-form formulas for solutions of several special cases of Eq (1.6), but without providing any theory or explanations related to the formulas. It has been noticed that many of the papers of this type have various type of problems (see, for instance, [43, 44, 49]).

1.4. Aim of the paper

We provide some detailed theoretical explanations for getting the closed-form formulas and representations for the general solutions to the four special cases of Eq (1.6) considered in [11], and give some natural proofs of the results which where not proved therein, that is, without using only the method of mathematical induction, and show that all the difference equations are special cases of a general class of difference equations which is solvable in closed form. We also show that the main results on the long-term behavior, that is, the ones on local and global stability, of the solutions to Eq (1.6) formulated therein are not correct. Finally, we give some results on convergence of solutions to Eq (1.6), under some assumptions related to the ones posed in [11].

2. On some formulas for solutions to special cases of Eq (1.6)

Closed-form formulas for solutions to four special cases of Eq (1.6) were given in [11]. The formulas for two of these equations were proved by the method of mathematical induction, whereas the formulas for the other two ones were even not proved. It was only said therein that the cases can be treated similarly. Beside this, nothing was said about the methods which were used for getting the formulas.

2.1. On four special cases of Eq (1.6) and the closed-form formulas

The following four special cases of Eq (1.6) were considered in [11]:

$$x_{n+1} = x_n + \frac{x_nx_{n-1}}{x_n + x_{n-1}}, \quad n \in \mathbb{N}_0, \tag{2.1}$$

$$x_{n+1} = x_n + \frac{x_nx_{n-1}}{x_n - x_{n-1}}, \quad n \in \mathbb{N}_0, \tag{2.2}$$

$$x_{n+1} = x_n - \frac{x_nx_{n-1}}{x_n + x_{n-1}}, \quad n \in \mathbb{N}_0, \tag{2.3}$$

$$x_{n+1} = x_n - \frac{x_nx_{n-1}}{x_n - x_{n-1}}, \quad n \in \mathbb{N}_0. \tag{2.4}$$

It is claimed therein that solutions to Eq (2.1) are given by the formula

$$x_n = x_0 \prod_{j=1}^{n} \frac{A_jx_0 + 2B_jx_{-1}}{B_jx_0 + A_jx_{-1}}, \quad n \in \mathbb{N}_0, \tag{2.5}$$

where A_j and B_j are the solutions to the equation

$$y_{n+1} = 2y_n + y_{n-1}, \quad n \in \mathbb{N}_0, \tag{2.6}$$

with the initial values

$$y_{-1} = -1, \quad y_0 = 1, \tag{2.7}$$

and

$$y_{-1} = 1, \quad y_0 = 0,$$

respectively, that the solutions to Eq (2.2) are given by the formulas

$$x_{2n-1} = \frac{x_0^{2n}}{x_{-1}^{n-1}(x_0 - x_{-1})^n}, \quad n \in \mathbb{N}_0, \tag{2.8}$$

$$x_{2n} = \frac{x_0^{2n+1}}{(x_{-1}(x_0 - x_{-1}))^n}, \quad n \in \mathbb{N}_0, \tag{2.9}$$

that the solutions to Eq (2.3) are given by the formula

$$x_n = \frac{x_0^{n+1}}{\prod_{j=1}^{n}(x_0 j + x_{-1})}, \quad n \in \mathbb{N}_0, \tag{2.10}$$

and that the solutions to Eq (2.4) are given by the formulas

$$x_{2n-1} = \frac{x_0^n}{x_{-1}^{n-1}}\left(\frac{x_0 - 2x_{-1}}{x_0 - x_{-1}}\right)^n, \quad n \in \mathbb{N}_0, \tag{2.11}$$

$$x_{2n} = \frac{x_0^{n+1}}{x_{-1}^n}\left(\frac{x_0 - 2x_{-1}}{x_0 - x_{-1}}\right)^n, \quad n \in \mathbb{N}_0. \tag{2.12}$$

2.2. *Explanations for above formulas for solutions to Eqs* (2.1)–(2.4)

Here we present some very detailed explanations how the closed-form formulas and representations given in (2.5), (2.8)–(2.12), for the general solutions to the corresponding difference equations given in (2.1)–(2.4), can be obtained in some natural ways, where an inductive argument is not the only used method in obtaining and verifying closed-form formulas, which occurs in the investigation. In fact, one of our aims is to eliminate any inductive argument as much as is possible. In the present investigation, we employ some methods, ideas and tricks related to the ones, for example, in [14, 42–47, 49].

On Eq (2.1). First note that

$$B_1 = 2B_0 + B_{-1} = 1.$$

Hence, we have

$$B_0 = 0 \quad \text{and} \quad B_1 = 1. \tag{2.13}$$

The solution to Eq (1.1) with these initial values is a sort of a fundamental solution to the difference equation. Some explanations for the claim follow.

Let

$$(s_n)_{n\in\mathbb{N}_0} = (s_n(p,q))_{n\in\mathbb{N}_0}$$

be the solution to Eq (1.1) satisfying the initial conditions

$$x_0 = 0 \quad \text{and} \quad x_1 = 1. \tag{2.14}$$

If $p^2 + 4q \neq 0$, then we have

$$s_n = \frac{t_1^n - t_2^n}{t_1 - t_2}, \quad n \in \mathbb{N}_0, \tag{2.15}$$

where t_1 and t_2 are the zeros of polynomial (1.3).

From (1.2) and (2.15) we see that the solution to Eq (1.1) with the initial values x_0 and x_1, can be written in the form

$$x_n = x_1 s_n + q x_0 s_{n-1}, \quad n \in \mathbb{N}_0. \tag{2.16}$$

Here we naturally regard that

$$s_{-1} = \frac{s_1 - p s_0}{q} = \frac{1}{q},$$

so that formula (2.16) holds also for $n = 0$. Let us mention that the formula holds also in the case $p^2 + 4q = 0$. Namely, in this case we have

$$s_n = n t_1^{n-1}, \quad n \in \mathbb{N}_0$$

and (1.4) holds.

Consider Eq (1.5) under the assumptions:

$$\alpha, \beta, \gamma, \delta, y_0 \in \mathbb{R}, \quad \gamma \neq 0 \quad \text{and} \quad \alpha\delta \neq \beta\gamma.$$

Employing the change of variables

$$\frac{z_n}{z_{n+1}} = \frac{1}{\gamma y_n + \delta}, \quad n \in \mathbb{N}_0, \tag{2.17}$$

the equation is transformed to

$$z_{n+1} - (\alpha + \delta) z_n + (\alpha\delta - \beta\gamma) z_{n-1} = 0, \quad n \in \mathbb{N}. \tag{2.18}$$

Thus from (2.16) we have

$$z_n = z_1 s_n + z_0 (\beta\gamma - \alpha\delta) s_{n-1}, \quad n \in \mathbb{N}_0, \tag{2.19}$$

where

$$s_n = s_n(\alpha + \delta, \beta\gamma - \alpha\delta).$$

Relations (2.17)–(2.19) together with some calculations imply

$$y_n = \frac{(\alpha y_0 + \beta) s_n + y_0 (\beta\gamma - \alpha\delta) s_{n-1}}{(\gamma y_0 - \alpha) s_n + s_{n+1}}, \quad n \in \mathbb{N}_0. \tag{2.20}$$

Now, we apply the analysis in the case of Eq (2.1). If in the equation we use the change of variables

$$y_n = \frac{x_n}{x_{n-1}}, \quad n \in \mathbb{N}_0, \tag{2.21}$$

we get the following special case of Eq (1.5)

$$y_{n+1} = \frac{y_n + 2}{y_n + 1}, \quad n \in \mathbb{N}_0.$$

The corresponding associate Eq (2.18) is the following

$$z_{n+1} - 2z_n - z_{n-1} = 0, \quad n \in \mathbb{N}, \tag{2.22}$$

from which together with (2.13) we have

$$B_n = s_n(2, 1), \quad n \in \mathbb{N}_{-1}. \tag{2.23}$$

From (2.20) and since $\alpha = \gamma = \delta = 1$ and $\beta = 2$, we have

$$y_n = \frac{(s_n + s_{n-1})y_0 + 2s_n}{s_n y_0 + s_{n+1} - s_n}, \quad n \in \mathbb{N}_0,$$

from which together with (2.21) it follows that

$$x_n = \frac{(s_n + s_{n-1})x_0 + 2s_n x_{-1}}{s_n x_0 + (s_{n+1} - s_n)x_{-1}} x_{n-1}, \quad n \in \mathbb{N}_0. \tag{2.24}$$

From (2.23), (2.24), since

$$A_n = A_1 s_n + A_0 s_{n-1} = s_n + s_{n-1}, \quad n \in \mathbb{N}_0,$$

(here we have also used the fact that $A_1 = 2A_0 + A_{-1} = 1$; see (2.7)), and the fact that s_n is a solution to Eq (2.22) it easily follows that

$$x_n = \frac{A_n x_0 + 2B_n x_{-1}}{B_n x_0 + A_n x_{-1}} x_{n-1}, \quad n \in \mathbb{N}_0, \tag{2.25}$$

from which formula (2.5) follows.

Remark 2.1. Note that from (2.25) it follows the formula

$$x_n = x_{-1} \prod_{j=0}^{n} \frac{A_j x_0 + 2B_j x_{-1}}{B_j x_0 + A_j x_{-1}}, \quad n \in \mathbb{N}_{-1},$$

which is a bit better closed-form formula for solutions to Eq (2.1), than the one given in (2.5).

On Eq (2.2). First note that Eq (2.2) can be written in the following form

$$x_{n+1} = \frac{x_n^2}{x_n - x_{n-1}}, \quad n \in \mathbb{N}_0,$$

from which for all the solutions such that $x_n \neq 0$, $n \in \mathbb{N}_0$, we have

$$\frac{x_n}{x_{n+1}} = 1 - \frac{x_{n-1}}{x_n}, \quad n \in \mathbb{N}_0. \tag{2.26}$$

Hence, the sequence

$$y_n = \frac{x_{n-1}}{x_n}, \quad n \in \mathbb{N}_0,$$

satisfies the relation

$$y_{n+1} = 1 - y_n, \quad n \in \mathbb{N}_0,$$

from which it follows that

$$y_{n+1} = y_{n-1}, \quad n \in \mathbb{N},$$

that is, the sequence $(y_n)_{n\in\mathbb{N}_0}$ is two-periodic.

Hence, we have

$$\frac{x_{2m-j-1}}{x_{2m-j}} = \frac{x_{-j-1}}{x_{-j}}, \quad m \in \mathbb{N}_0,\ j = -1, 0,$$

from which it follows that

$$x_{2m} = \frac{x_0}{x_{-1}} x_{2m-1}, \quad m \in \mathbb{N}_0,$$

and

$$x_{2m-1} = \frac{x_1}{x_0} x_{2m-2} = \frac{x_0}{x_0 - x_{-1}} x_{2m-2}, \quad m \in \mathbb{N},$$

and consequently

$$x_{2m} = \frac{x_0^2}{x_{-1}(x_0 - x_{-1})} x_{2m-2}, \quad m \in \mathbb{N}, \tag{2.27}$$

and

$$x_{2m-1} = \frac{x_0^2}{x_{-1}(x_0 - x_{-1})} x_{2m-3}, \quad m \in \mathbb{N}. \tag{2.28}$$

From (2.27) and (2.28) we obtain

$$x_{2m} = x_0 \left(\frac{x_0^2}{x_{-1}(x_0 - x_{-1})} \right)^m, \quad m \in \mathbb{N}_0,$$

and

$$x_{2m-1} = x_{-1} \left(\frac{x_0^2}{x_{-1}(x_0 - x_{-1})} \right)^m, \quad m \in \mathbb{N}_0,$$

from which the formulas in (2.8) and (2.9) immediately follow.

On Eq (2.3). First note that Eq (2.3) can be written in the following form

$$x_{n+1} = \frac{x_n^2}{x_n + x_{n-1}}, \quad n \in \mathbb{N}_0,$$

from which for all the solutions such that $x_n \neq 0$, $n \in \mathbb{N}_0$, we have

$$\frac{x_n}{x_{n+1}} = \frac{x_{n-1}}{x_n} + 1, \quad n \in \mathbb{N}_0. \tag{2.29}$$

Hence, the sequence

$$y_n = \frac{x_{n-1}}{x_n}, \quad n \in \mathbb{N}_0,$$

satisfies the relation

$$y_{n+1} = y_n + 1, \quad n \in \mathbb{N}_0,$$

from which it follows that

$$y_n = n + y_0, \quad n \in \mathbb{N}_0,$$

that is,

$$\frac{x_{n-1}}{x_n} = n + \frac{x_{-1}}{x_0}, \quad n \in \mathbb{N}_0.$$

Hence, we have

$$x_n = \frac{x_0}{x_0 n + x_{-1}} x_{n-1}, \quad n \in \mathbb{N}_0, \tag{2.30}$$

and consequently

$$x_n = x_0 \prod_{j=1}^{n} \frac{x_0}{x_0 j + x_{-1}}, \quad n \in \mathbb{N}_0,$$

from which formula (2.10) immediately follows.

Remark 2.2. Note that from (2.30) it follows the formula

$$x_n = x_{-1} \frac{x_0^{n+1}}{\prod_{j=0}^{n}(x_0 j + x_{-1})}, \quad n \in \mathbb{N}_{-1},$$

which is a bit better closed-form formula for solutions to Eq (2.3), than the one given in (2.10).

On Eq (2.4). First note that Eq (2.4) can be written in the following form

$$x_{n+1} = x_n \frac{x_n - 2x_{n-1}}{x_n - x_{n-1}}, \quad n \in \mathbb{N}_0.$$

Let

$$y_n = \frac{x_n}{x_{n-1}}, \quad n \in \mathbb{N}_0.$$

Then, the sequence $(y_n)_{n \in \mathbb{N}_0}$ satisfies the bilinear difference equation

$$y_{n+1} = \frac{y_n - 2}{y_n - 1}, \quad n \in \mathbb{N}_0,$$

from which along with the formula where index n is replaced with $n-1$, it follows that

$$y_{n+1} = y_{n-1}, \quad n \in \mathbb{N}_0,$$

that is, the sequence y_n is two-periodic.

Hence, we have

$$x_{2m} = \frac{x_0}{x_{-1}} x_{2m-1}, \quad m \in \mathbb{N}_0,$$

and

$$x_{2m-1} = \frac{x_1}{x_0} x_{2m-2} = \frac{x_0 - 2x_{-1}}{x_0 - x_{-1}} x_{2m-2}, \quad m \in \mathbb{N},$$

from which it follows that

$$\begin{aligned} x_{2m-1} &= \left(\frac{x_0(x_0 - 2x_{-1})}{x_{-1}(x_0 - x_{-1})}\right) x_{2m-3}, \quad m \in \mathbb{N},\\ x_{2m} &= \left(\frac{x_0(x_0 - 2x_{-1})}{x_{-1}(x_0 - x_{-1})}\right) x_{2m-2}, \quad m \in \mathbb{N}, \end{aligned}$$

and consequently

$$\begin{aligned} x_{2m-1} &= x_{-1}\left(\frac{x_0(x_0 - 2x_{-1})}{x_{-1}(x_0 - x_{-1})}\right)^m, \quad m \in \mathbb{N}_0,\\ x_{2m} &= x_0\left(\frac{x_0(x_0 - 2x_{-1})}{x_{-1}(x_0 - x_{-1})}\right)^m, \quad m \in \mathbb{N}_0, \end{aligned}$$

from which the closed-form formulas for the general solution of Eq (2.4) given in (2.11) and (2.12) immediately follow.

3. Solvability of an extension of Eq (1.6)

Solvability of Eq (1.6) can be treated in some general ways. Namely, the following equation

$$x_{n+1} = f^{-1}\left(f(x_n)\frac{\alpha f(x_n) + \beta f(x_{n-1})}{\gamma f(x_n) + \delta f(x_{n-1})}\right), \quad n \in \mathbb{N}_0, \tag{3.1}$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{R}$, $\gamma^2 + \delta^2 \neq 0$, $f : \mathbb{R} \to \mathbb{R}$ is a function, is a natural extension of Eq (1.6). Indeed, note that Eq (1.6) can be written in the form

$$x_{n+1} = x_n \frac{acx_n + (ad + b)x_{n-1}}{cx_n + dx_{n-1}}, \quad n \in \mathbb{N}_0,$$

from which it follows that the difference equation is obtained from the Eq (3.1) with

$$f(x) \equiv x, \quad \alpha = ac, \quad \beta = ad + b, \quad \gamma = c \quad \text{and} \quad \delta = d.$$

The following result has been recently proved in [47].

Theorem 3.1. *Let $\alpha,\beta,\gamma,\delta \in \mathbb{R}$, $\alpha^2+\beta^2 \neq 0 \neq \gamma^2+\delta^2$, f be a homeomorphism of $\mathbb{R}$ such that $f(0)=0$. Then Eq* (3.1) *is solvable in closed-form. Moreover, the following statements hold.*

(a) If $\alpha\delta=\beta\gamma$, $\alpha=0$ or $\gamma=0$, then the general solution to Eq (3.1) *is given by the formula*

$$x_n = f^{-1}\left(\left(\frac{\beta}{\delta}\right)^n f(x_0)\right), \quad n \in \mathbb{N}_0. \tag{3.2}$$

(b) If $\alpha\delta=\beta\gamma$, $\beta=0$ or $\delta=0$, then the general solution to Eq (3.1) *is given by the formula*

$$x_n = f^{-1}\left(\left(\frac{\alpha}{\gamma}\right)^n f(x_0)\right), \quad n \in \mathbb{N}_0. \tag{3.3}$$

(c) If $\alpha\delta=\beta\gamma$, $\alpha\beta\gamma\delta \neq 0$, then the general solution to Eq (3.1) *is given by formula* (3.2), *which in this case matches with formula* (3.3).

(d) If $\alpha\delta \neq \beta\gamma$, $\gamma=0$, $\alpha=\delta$, then the general solution to Eq (3.1) *is given by the formula*

$$x_n = f^{-1}\left(f(x_{-1})\prod_{j=0}^{n}\left(\frac{\beta}{\delta}j+\frac{f(x_0)}{f(x_{-1})}\right)\right), \tag{3.4}$$

for $n \in \mathbb{N}_{-1}$.

(e) If $\alpha\delta \neq \beta\gamma$, $\gamma=0$, $\alpha \neq \delta$, then the general solution to Eq (3.1) *is given by the formula*

$$x_n = f^{-1}\left(f(x_{-1})\prod_{j=0}^{n}\left(\beta\frac{(\alpha/\delta)^j-1}{\alpha-\delta}+\left(\frac{\alpha}{\delta}\right)^j\frac{f(x_0)}{f(x_{-1})}\right)\right), \tag{3.5}$$

for $n \in \mathbb{N}_{-1}$.

(f) If $\alpha\delta \neq \beta\gamma$, $\gamma \neq 0$, $\Delta := (\alpha+\delta)^2-4(\alpha\delta-\beta\gamma) \neq 0$, then the general solution to Eq (3.1) *is given by the formula*

$$x_n = f^{-1}\left(f(x_{-1})\prod_{j=0}^{n}\left(\frac{(\frac{f(x_0)}{f(x_{-1})}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{j+1}-(\frac{f(x_0)}{f(x_{-1})}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{j+1}}{(\frac{f(x_0)}{f(x_{-1})}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{j}-(\frac{f(x_0)}{f(x_{-1})}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{j}}-\frac{\delta}{\gamma}\right)\right), \tag{3.6}$$

for $n \in \mathbb{N}_{-1}$, where

$$\lambda_1 = \frac{\alpha+\delta+\sqrt{\Delta}}{2\gamma} \quad \textit{and} \quad \lambda_2 = \frac{\alpha+\delta-\sqrt{\Delta}}{2\gamma}.$$

(g) If $\alpha\delta \neq \beta\gamma$, $\gamma \neq 0$, $\Delta := (\alpha+\delta)^2-4(\alpha\delta-\beta\gamma) = 0$, then the general solution to Eq (3.1) *is given by the formula*

$$x_n = f^{-1}\left(f(x_{-1})\prod_{j=0}^{n}\left(\frac{((f(x_0)+(\frac{\delta}{\gamma}-\lambda_1)f(x_{-1}))(j+1)+\lambda_1 f(x_{-1}))\lambda_1}{(f(x_0)+(\frac{\delta}{\gamma}-\lambda_1)f(x_{-1}))j+\lambda_1 f(x_{-1})}-\frac{\delta}{\gamma}\right)\right), \tag{3.7}$$

for $n \in \mathbb{N}_{-1}$, where $\lambda_1 = \frac{\alpha+\delta}{2\gamma}$.

Remark 3.1. From Theorem 3.1 it follows that Eq (1.6) is solvable in closed form. By using the corresponding formulas in (3.2)–(3.7), after some calculations can be obtained some closed-form formulas for solutions to Eqs (2.1)–(2.4). The closed-form formulas in (2.8)–(2.11) can be obtained relatively easy. Regarding formula (2.5), since it is a representation of the general solution of Eq (2.1), it needs some further works which we have conducted in the previous section.

Remark 3.2. The above analyses and results refers to well-defined solutions. It is obvious that not for all initial values solutions to the equations are defined. In the case of Eq (3.1) for a well-defined solution it must be

$$\gamma f(x_n) + \delta f(x_{n-1}) \neq 0$$

for every $n \in \mathbb{N}_0$.

4. On some results on local and global stability in [11]

Here we discuss the results on local and global stability solutions of Eq (1.6) formulated in [11]. Results on long term behaviour of solutions to difference equations and systems, including the ones on local and especially on global stability, are of a great importance. Some of them can be found, for instance, in [1, 2, 5, 6, 9, 13, 16–20, 25, 27, 31, 33, 36, 38–40] (see also the related references therein).

4.1. On equilibria of Eq (1.6)

In [11] were first studied the equilibria of Eq (1.6). Let $\bar{x}$ be an equilibrium of the equation. Then it must be

$$\bar{x} = a\bar{x} + \frac{b\bar{x}^2}{(c+d)\bar{x}}. \tag{4.1}$$

The relation in (4.1) shows that $\bar{x}$ cannot be equal to zero. This was not noticed in [11]. Not noticing this fact the author of [11] multiplied both sides in (4.1) by $\bar{x}$ and obtained a relation from which he concluded that it must be $\bar{x} = 0$, if

$$(c+d)(1-a) \neq b, \tag{4.2}$$

which leads to a contradiction. In this case, (1.6) simply does not have an equilibrium.

Thus, Theorem 1 in [11] tries to show that a wrong equilibrium point of the equation is locally asymptotically stable under the condition

$$b < (1-a)(c+d),$$

a statement which makes no sense.

Relation (4.1) is also not defined if $c + d = 0$, so if we assume that

$$c + d \neq 0, \tag{4.3}$$

from (4.1) we have

$$\bar{x}((c+d)(1-a) - b) = 0.$$

Thus, if

$$(c + d)(1 - a) - b = 0, \tag{4.4}$$

any $\bar{x} \neq 0$ is an equilibrium of (1.6).

This is a typical situation for the difference equations whose right-hand side is a homogeneous function of order one on the diagonal.

4.2. On a claim on global stability

The main result in [11] on the long-term behavior of positive solutions to Eq (1.6) should have been Theorem 2 therein. The theorem is on global convergence of the solutions to the difference equation. Here is the claim.

Claim 1. *Let* $\min\{a, b, c, d\} > 0$*, then the equilibrium point* $\bar{x} = 0$ *of Eq* (1.6) *is global attractor.*

As we have shown $\bar{x} = 0$ is not an equilibrium point of Eq (1.6), so the claim has a problem. Moreover, the claim is even wrong since all well-defined solutions to the equation need not be convergent. Indeed, if

$$(ac + d)^2 \neq -4bc,$$

then by Theorem 3.1 (f) the general solution to Eq (1.6) is given by the formula

$$x_n = x_{-1} \prod_{j=0}^{n} \left(\frac{(x_0 + (\frac{d}{c} - \lambda_2)x_{-1})\lambda_1^{j+1} - (x_0 + (\frac{d}{c} - \lambda_1)x_{-1})\lambda_2^{j+1}}{(x_0 + (\frac{d}{c} - \lambda_2)x_{-1})\lambda_1^{j} - (x_0 + (\frac{d}{c} - \lambda_1)x_{-1})\lambda_2^{j}} - \frac{d}{c} \right), \tag{4.5}$$

for $n \in \mathbb{N}_0$, where

$$\lambda_1 = \frac{ac + d + \sqrt{(ac + d)^2 + 4bc}}{2c}$$

and

$$\lambda_2 = \frac{ac + d - \sqrt{(ac + d)^2 + 4bc}}{2c}.$$

Let

$$y_n = \frac{(x_0 + (\frac{d}{c} - \lambda_2)x_{-1})\lambda_1^{n+1} - (x_0 + (\frac{d}{c} - \lambda_1)x_{-1})\lambda_2^{n+1}}{(x_0 + (\frac{d}{c} - \lambda_2)x_{-1})\lambda_1^{n} - (x_0 + (\frac{d}{c} - \lambda_1)x_{-1})\lambda_2^{n}} - \frac{d}{c}, \quad n \in \mathbb{N}_0. \tag{4.6}$$

If

$$x_0 + \left(\frac{d}{c} - \lambda_2\right)x_{-1} \neq 0, \tag{4.7}$$

then by letting $n \to +\infty$ in relation (4.6), it is not difficult to see that the following relation holds

$$\lim_{n\to+\infty} y_n = \lambda_1 - \frac{d}{c} = \frac{ac - d + \sqrt{(ac + d)^2 + 4bc}}{2c}. \tag{4.8}$$

Assume that a, b, c, d satisfy the condition

$$\frac{ac - d + \sqrt{(ac+d)^2 + 4bc}}{2c} > 1,$$

and that x_{-1}, x_0 are positive numbers satisfying condition (4.7), then from (4.8) and since

$$x_n = x_{-1}\prod_{j=0}^{n} y_j, \quad n \in \mathbb{N}_{-1},$$

we have

$$\lim_{n\to+\infty} x_n = +\infty. \tag{4.9}$$

Relation (4.9) shows that many of the solutions to such chosen special cases of equation (1.6) are not only divergent but are even unbounded, showing that Claim 1 is not true.

For example, if

$$a = 2, \quad b = 1, \quad c = 1 \quad \text{and} \quad d = 2,$$

then we have

$$x_n = x_{-1}\prod_{j=0}^{n}\left(\frac{(x_0 + (2-\lambda_2)x_{-1})\lambda_1^{j+1} - (x_0 + (2-\lambda_1)x_{-1})\lambda_2^{j+1}}{(x_0 + (2-\lambda_2)x_{-1})\lambda_1^{j} - (x_0 + (2-\lambda_1)x_{-1})\lambda_2^{j}} - 2\right), \tag{4.10}$$

for $n \in \mathbb{N}_0$, where

$$\lambda_1 = 2 + \sqrt{5} \quad \text{and} \quad \lambda_2 = 2 - \sqrt{5},$$

from which when

$$\frac{x_0}{x_{-1}} \neq \lambda_2 - 2 = -\sqrt{5},$$

and if x_n is a well-defined solution, it follows that

$$\lim_{n\to+\infty}\frac{(x_0 + (2-\lambda_2)x_{-1})\lambda_1^{n+1} - (x_0 + (2-\lambda_1)x_{-1})\lambda_2^{n+1}}{(x_0 + (2-\lambda_2)x_{-1})\lambda_1^{j} - (x_0 + (2-\lambda_1)x_{-1})\lambda_2^{j}} - 2 = \sqrt{5} > 1. \tag{4.11}$$

From (4.10) and (4.11) we have that for such chosen solutions relation (4.9) holds. Hence, the solutions are not convergent.

4.3. On a result on boundedness

Beside above mentioned results, in [11] was proved the following simple result on the boundedness of positive solutions to Eq (1.6).

Theorem 4.1. *Every (positive) solution of Eq* (1.6) *is bounded if*

$$a + \frac{b}{d} < 1. \tag{4.12}$$

This result is an immediate consequence of the most simple comparison result in the theory of difference equations. Namely, if a positive sequence $(x_n)_{n\in\mathbb{N}_0}$ satisfies the inequality

$$x_{n+1} \le x_n, \quad n \in \mathbb{N}_0,$$

then it is bounded.

For some other extensions of the result and various methods for proving boundedness of solutions to nonlinear difference equations, see, for instance, [3–5, 13, 36–41] and the related references therein.

Bearing in mind that from (1.6) for every positive solution to the equation we obviously have

$$x_{n+1} \le ax_n + \frac{bx_nx_{n-1}}{dx_{n-1}} = \left(a + \frac{b}{d}\right)x_n \le x_n, \quad n \in \mathbb{N}_0, \tag{4.13}$$

the result immediately follows.

Remark 4.1. Note that the argument in (4.13) holds if

$$0 \le a + \frac{b}{d} \le 1, \tag{4.14}$$

which was not noticed in [11]. This means that Theorem 4.1 also holds if condition (4.12) is replaced by (4.14). A natural generalization of the boundedness result under condition (4.12) frequently appears in the literature (see, e.g., [37, Theorem 1]).

Remark 4.2. Note that if condition (4.12) holds, then for every positive solution to Eq (1.6) we have

$$x_{n+1} \le \left(a + \frac{b}{d}\right)x_n, \quad n \in \mathbb{N}_0,$$

from which it follows that

$$x_n \le \left(a + \frac{b}{d}\right)^n x_0, \quad n \in \mathbb{N}_0. \tag{4.15}$$

From inequality (4.15), condition (4.12), and the positivity of the sequence x_n, it follows that

$$\lim_{n\to+\infty} x_n = 0.$$

Hence, the following simple result on convergence holds, which was also not noticed in [11].

Theorem 4.2. *Assume that*

$$\min\{a, b, c, d\} > 0 \tag{4.16}$$

and that inequality (4.12) *holds. Then every positive solution to Eq* (1.6) *converges to zero.*

Remark 4.3. Note that from (1.6) for every positive solution $(x_n)_{n\in\mathbb{N}_{-1}}$ to the equation we have

$$x_{n+1} = x_n \frac{acx_n + (ad+b)x_{n-1}}{cx_n + dx_{n-1}} \le x_n \frac{\max\{ac, ad+b\}}{\min\{c,d\}}, \quad n \in \mathbb{N}_0. \tag{4.17}$$

From (4.17) we have

$$x_n \le \left(\frac{\max\{ac, ad+b\}}{\min\{c,d\}}\right)^n x_0, \quad n \in \mathbb{N}_0. \tag{4.18}$$

Employing estimate (4.18) and the arguments in Remarks 4.1 and 4.2, we see that the following result holds.

Theorem 4.3. *Assume that condition* (4.16) *holds. Then the following statements hold.*

(a) If

$$\frac{\max\{ac, ad+b\}}{\min\{c,d\}} \le 1,$$

then every positive solution to Eq (1.6) *is bounded.*

(b) If

$$\frac{\max\{ac, ad+b\}}{\min\{c,d\}} < 1,$$

then every positive solution to Eq (1.6) *converges to zero.*

5. Conclusions

We provide some detailed theoretical explanations for getting the closed-form formulas and representations for the general solutions to four special cases of a difference equation in the literature, without using only the method of mathematical induction, and conducted some analyses which show that investigations of difference equations should be conducted more carefully than it is frequently done in the literature. The methods and ideas given in the paper can be used in many similar situations and should be useful to a wide audience.

Conflict of interest

The author declares no conflict of interest.

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