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Research article

Numerical approximation of Atangana-Baleanu Caputo derivative for space-time fractional diffusion equations

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Abstract: In this study, we attempt to obtain the approximate solution for the time-space fractional linear and nonlinear diffusion equations. A finite difference approach is given for the solution of both linear and nonlinear fractional order diffusion problems. The Riesz fractional derivative in space is specifically approximated using the centered difference scheme. A system of Atangana-Baleanu Caputo equations that have been converted through spatial discretization is solved using a newly developed modified Simpson's 1/3 formula. A study of the proposed scheme is done to ascertain its stability and convergence. It has been shown that for mesh size h and time steps δt the recommended method converges at a rate of $O(\delta t^2 + h^2)$. Based on graphic results and numerical examples, the application of the model is also examined.

Keywords: fractional diffusion equation; numerical approximation; Atangana-Baleanu Caputo derivative; non-singular kernel; stability-convergence
Mathematics Subject Classification: 34K28, 35-XX

1. Introduction

Fractional calculus has recently piqued the interest of an increasing number of scientists since it is an important tool for modeling many systems in science and engineering. The increased degree of freedom that comes with using fractional derivatives allows for more realistic models of various systems, including viscoelastic materials, anomalous diffusion processes, control systems with long-term memory and more. Furthermore, the use of fractional derivatives can lead to more accurate predictions and a better understanding of complex physical, biological and engineering systems. In conclusion, fractional calculus has become an indispensable tool in modern science and engineering, providing researchers with a powerful framework for modeling complex systems and describing the behavior of phenomena that cannot be captured by traditional integer-order models. Engineering, geophysics, biology, control theory, special functions, viscoelasticity, electricity, fluid dynamics and mechanics all use fractional differential equations [1–7].

Diffusion is a widespread and important physical phenomenon that is relevant in many areas of science and engineering. It refers to the spreading of matter, energy or information over space and time, and it plays a crucial role in a wide range of processes, such as heat transfer, mass transfer, fluid dynamics and more. For example, [8] contains a comprehensive derivation of the nonlinear diffusion equation. The space-time fractional diffusion equation derived from the classic diffusion equation has received a lot of attention for replacing the fractional equivalents of the integer order space and time derivatives [9–11]. In various domains, it has frequently been employed to investigate the unusual diffusive processes connected to fractional sub-diffusion and super-diffusion [12, 13].

Fractional-order partial differential equations (FPDEs) are widely used to model complex physical, biological and engineering systems that exhibit memory and non-local behavior. Unlike traditional partial differential equations, which are usually of integer order, FPDEs are based on derivatives of fractional order. This means that they can capture the long-term effects of past events on the current state of a system, making them very useful for modeling phenomena, such as diffusion, turbulence, viscoelasticity and more. The use of FPDEs has led to new mathematical techniques and theories, such as fractional calculus, which is a branch of mathematics that deals with the study of fractional derivatives and integrals. Additionally, numerical methods have been developed to solve FPDEs, including finite difference, finite element and spectral methods. Despite the progress that has been made, there are still many open questions and challenges when it comes to FPDEs. Nevertheless, their impact and importance continue to grow, making them a popular and active area of research in both mathematics and the applied sciences. Caputo and Fabrizio proposed a novel fractional order differential operator based on the exponential kernel in 2015 [14]. Liu et al. [15] proposed a novel second-order algorithm to approximate the fractional-order Caputo-Fabrizio derivative. To address a number of restrictions in the field of fractional differentiation and fractional models, researchers have proposed non-singular and non-local kernels for the formulation of fractional integration/differentiation operators. Recent literature contains research on fractional operators with non-singular kernels. However, there are surprisingly few works that discuss the numerical solutions of such fractional-order differential equations with nonsingular kernel. For differential equations with the Atangana-Baleanu Caputo fractional-order derivative, only a few numbers of precise numerical approaches have been applied to date. An improved formulation for fractional derivatives based on the non-singular and non-local kernel was proposed by Atangana and Baleanu [16]. These newly proposed non-singular operators have a great probability of application for modeling in a number of applied sciences fields. The space-time fractional diffusion equation with the ABC derivative is the subject of this paper. This fractional diffusion equation classification has been linked to a number of investigations. Using the Banach fixed-point theorem, existence and uniqueness findings for linear and nonlinear differential equations with the ABC fractional derivative were established in [17]. The Chebyshev collocation method was then used to construct a numerical methodology. Different diffusion techniques specified by the Atangana-Baleanu-Caputo derivative are explored in order to analyze the forms of diffusion processes using the Laplace transform method [18]. The authors of [19] devised a difference scheme for the nonlinear reaction-diffusion equation and nonlinear integro reaction-diffusion equation involving the ABC derivative, using Taylor series to cope with time direction in fractional differential terms and spatial discretization with quasi wavelet. The authors of [20] proposed ABC derivative approximation methods established on the finite difference and Taylor expansion methods, which are then used to find the solution of advection-diffusion problems with fractional derivative. The Legendre polynomial was used to build a numerical technique to find the solution of a non-linear Burgers-Huxley equation and a non-linear reaction-diffusion equation with the ABC derivative in [21]. The authors of [22] applied the q-homotopy analysis transform approach to provide an approximate/analytical solution to the nonlinear ABC fractional diffusion problem. The authors of [23] analyzed a nonlinear fractional order differential equation with the Atangana-Baleanu operator, employing the renowned Adams-Bashforth numerical approach to solve the equation system. The authors of [24] investigated the issue of locating an unidentified source term for the ABC fractional derivative, using the generalized Tikhonov approach to regularize the unstable solution to the issue. Numerical solutions of fractional differential equations with nonsingular kernels, particularly those involving the Atangana-Baleanu-Caputo derivative, have been the subject of limited research to date. This is partly due to the mathematical complexity of such equations and the difficulty of developing numerical methods that can handle them effectively.

However, the lack of precise numerical approaches and the need for more accurate and efficient methods have motivated researchers to investigate this area further. The development of a new numerical framework that can handle fractional differential equations involving the Atangana-Baleanu Caputo derivative with higher accuracy and higher order of convergence would be a significant contribution to the field. It is important to note that the development of such a numerical framework would have a wide range of potential applications in various fields, such as physics, engineering, finance and more, where fractional differential equations are used to model complex systems. Furthermore, it would help advance our understanding of these systems and enable us to make more accurate predictions and design better solutions. Thus, we aim to develop a new numerical framework for fractional differential equations involving the Atangana-Baleanu-Caputo derivative with better accuracy and higher order of convergence.

Our aim of this research is to study and then try to find the numerical solution of the following time-space fractional diffusion equation involving the Atangana-Baleanu derivative in the Caputo sense

$$^{ABC}D_t^{\varrho} z(v,t) = U_\eta \frac{\partial^\eta}{\partial|v|^\eta} z(v,t) + k(v,t), \quad c < v < d, \quad 0 < t \le T, \tag{1.1}$$

$$^{ABC}D_t^{\varrho} z(v,t) = U_\eta \frac{\partial^\eta}{\partial|v|^\eta} z(v,t) + k(z(v,t),v,t), \quad c < v < d, \quad 0 < t \le T, \tag{1.2}$$

along with the given initial and boundary conditions:

$$\begin{aligned} z(v,0) &= \psi(v), \quad c \le v \le d, \\ z(c,t) &= z(d,t) = 0, \quad 0 \le t \le T, \end{aligned} \tag{1.3}$$

where $1 < \eta \le 2$, $0 < \varrho \le 1$, $^{ABC}D^{\varrho}$ is the Atangana-Balanu derivative of fractional order ϱ and $\frac{\partial^\eta}{\partial|y|^\eta}$ is the Riesz fractional order derivative.

The following is how the rest of the article is organized: Section 2 contains the basic definitions and lemmas that are required to obtain the main conclusions. Section 3 presents a numerical approximation of both the linear and non-linear Atangana-Baleanu-Caputo fractional order diffusion equations. A complete study on stability and convergence analysis of the proposed numerical scheme is presented in Section 4. In Section 5, some numerical examples are presented. Section 6 has the conclusion.

2. Preliminaries

Some definitions and lemmas are given in this section which are important and will be used accordingly in this research.

Definition 2.1. The definition of Atangana-Baleanu-Caputo derivative and integral of $z(v,t)$ of order ϱ are defined [16] as under:

For $z \in \mathbb{H}^1(0,T)$ and $0 < \varrho < 1$,

$$ {}^{ABC}D^{\varrho} z(v,t) := \frac{P(\varrho)}{1-\varrho}\int_0^t E_{\varrho}[-\frac{\varrho}{1-\varrho}(t-\xi)^{\varrho}]\dot{z}(v,\xi)d\xi, \tag{2.1} $$

$$ {}^{ABC}I^{\varrho} z(v,t) := \frac{1-\varrho}{P(\varrho)}z(v,t) + \frac{\varrho}{P(\varrho)\Gamma(\varrho)}\int_0^t (t-\xi)^{\varrho-1}z(v,\xi)d\xi, \tag{2.2} $$

where E_{ϱ} is the Mittag-Leffler function and $P(\varrho)$ with $P(\varrho)|_{\varrho=0,1} = 1$ represents the normalization function.

Definition 2.2. The η^{th} order Riesz fractional derivative is defined as [25]:

$$ \frac{\partial^{\eta}}{\partial |v|^{\eta}}k = -\frac{1}{2cos\frac{\pi\eta}{2}}[{}_{-\infty}D_v^{\eta}k +{}_v D_{+\infty}^{\eta}k]. \tag{2.3} $$

The symmetric Riesz derivative with a fractional order η can be approximated using the following fractional centred difference formula can be used [26]:

$$ \Delta_h^{\eta}k(v) = \sum_{r=-\infty}^{\infty}\frac{(-1)^r\Gamma(\eta+1)}{\Gamma(\frac{\eta}{2}-r+1)\Gamma(\frac{\eta}{2}+r+1)}k(v-rh), \quad \eta > -1. \tag{2.4} $$

Then, the Riesz fractional derivative of order η with $1 < \eta \leq 2$ and $c = \infty, d = -\infty$ is given as

$$ \lim_{h\to 0}\frac{\Delta_h^{\eta}}{h^{\eta}}k(v) = \lim_{h\to 0}\frac{1}{h^{\eta}}\sum_{r=-\infty}^{\infty}\frac{(-1)^r\Gamma(\eta+1)}{\Gamma(\frac{\eta}{2}-r+1)\Gamma(\frac{\eta}{2}+r+1)}k(v-rh), \quad \eta > -1. \tag{2.5} $$

Lemma 2.1. *[27] Let $k \in C^5(R)$, all of its derivatives up to the 5^{th} order belong to $L^1(R)$ and the fractional central difference is given as*

$$ \Delta_h^{\eta}k(v) = \sum_{r=-\infty}^{\infty}\frac{(-1)^r\Gamma(\eta+1)}{\Gamma(\frac{\eta}{2}-r+1)\Gamma(\frac{\eta}{2}+r+1)}k(v-rh). \tag{2.6} $$

Then,

$$ -\frac{\Delta_h^{\eta}k(v)}{h^{\eta}} = \frac{\partial^{\eta}k}{\partial|v|^{\eta}} + O(h^2), \quad when \quad h \to 0, \tag{2.7} $$

and for $1 < \eta \leq 2$ Riesz fractional derivative is denoted by $\frac{\partial^{\eta}}{\partial|v|^{\eta}}$ with $h = \frac{(d-c)}{M}$.

As $k(v) = 0$ for $v \notin [c,d]$, we obtain

$$ \frac{\partial^{\eta}}{\partial|v|^{\eta}}k(v) = -\frac{1}{h^{\eta}}\sum_{r=\frac{-(b-v)}{h}}^{r=\frac{(r-a)}{h}}\frac{(-1)^r\Gamma(\eta+1)}{\Gamma(\frac{\eta}{2}-r+1)\Gamma(\frac{\eta}{2}+r+1)}k(v-rh) + O(h^2). \tag{2.8} $$

Lemma 2.2. *[27] Assume that the centred finite difference approximation's coefficients are*

$$\omega_r^\eta = \frac{(-1)^r\Gamma(\eta+1)}{\Gamma(\frac{\eta}{2}-r+1)\Gamma(\frac{\eta}{2}+r+1)}, \quad r = 0, \pm1, \pm2, ...,$$

for order $\varrho > -1$. Then,

(1) $\omega_0^\eta = \frac{\Gamma(\eta+1)}{\Gamma(\frac{\eta}{2}+1)^2} \geq 0$,
(2) $\omega_{k+1}^\eta = \frac{k-\frac{\eta}{2}}{(\frac{\eta}{2}+k+1)}\omega_k^\eta \leq 0$, *for* $k = \pm1, \pm2, ...$,
(3) $\omega_k^\eta = \omega_{-k}^\eta \leq 0, \quad |k| \geq 1$,
(4) $\sum_{-\infty}^{\infty} \omega_k^\eta = 0$,
(5) $\sum_{k=-r+n}^{n} \omega_k^\eta > 0$, *with $n < r$ for all $n, r \in Z^+$.*

3. Approximation by finite difference

In order to numerically solve the diffusion equations (1.1) and (1.2) along with its conditions (1.3), let the space be divided in equidistant points $v_r = rh$, where $r = 0, 1, 2, ..., M$ and $h = \frac{(d-c)}{M}$. Let the numerical solutions to $z(v_r, t)$ be denoted by $z_r(t)$. Discretizing the fractional derivative $\frac{\partial^\eta}{\partial|v|^\eta}$ in truncated bounded domain,

$$\delta_v^\eta z_r(t) = -\frac{1}{h^\eta}\sum_{r=-M+r}^{r} \omega_k^\eta z_{r-k}(t) + O(h^2). \tag{3.1}$$

We use approximation to achieve the following result for Riesz fractional derivative, as defined in Eq (3.1),

$${}^{ABC}D^\varrho z_r(t) = -\frac{U_\eta}{h^\eta}\sum_{k=-M+r}^{r} \omega_k^\eta z_{r-k}(t) + k(v_r, t), \quad 0 < t < T, \quad 1 \leqslant r \leqslant M-1, \tag{3.2}$$

$$z_r(0) = \psi(v_r), \quad 0 \leq r \leq M,$$
$$z_0(t) = z_M(t) = 0, \quad 0 \leq t \leq T.$$

As follows, write the given system as an equivalent ABC integral equation,

$$z_r(t) = z_r(0) - \frac{U_\eta(1-\varrho)}{h^\eta P(\varrho)}\sum_{k=-M+r}^{r} \omega_k^\eta z_{r-k}(t) - \frac{U_\eta\varrho}{h^\eta P(\varrho)\Gamma(\varrho)}\sum_{k=-M+r}^{r} \omega_k^\eta \int_0^t (t-s)^{\varrho-1} z_{r-k}(s)ds + \tilde{k}_r(t), \tag{3.3}$$

where $1 \leq r \leq M-1$, $0 < t < T$,

$$\tilde{k}_r(t) = \frac{1-\varrho}{P(\varrho)}k_r(t) + \frac{\varrho}{P(\varrho)\Gamma(\varrho)}\int_0^t (t-s)^{\varrho-1} k_r(s)ds.$$

3.1. Modified Simpson's fractional formula: Linear case

This section will go over the modified Simpson's fractional formula for solving Eq (1.1) in the linear case. To begin, define the grid $t_n = n\delta t$, with equal intervals $n = \overline{0, N}$ and $\delta t = T/N$. Let z_r^ζ, the numerical approximations of $z(v_r, t_\zeta)(\zeta = \overline{0, n-1})$, and $\tilde{k}_r(t_n) = \tilde{k}_r^n, n = \overline{0, N}$. To discretize the integrand in (3.3), quadratic interpolation will be utilized for finding the numerical solution Z^n as

$$\begin{aligned} z_r(t_n) = z_r(0) - \frac{U_\eta(1-\varrho)}{h^\eta P(\varrho)} \sum_{k=-M+r}^{r} \omega_k^\eta z_{r-k}(t_n) \\ - \frac{U_\eta \varrho}{h^\eta P(\varrho)\Gamma(\varrho)} \sum_{k=-M+r}^{r} \omega_k^\eta \sum_{\zeta=0}^{n-1} \int_{t_\zeta}^{t_{\zeta+1}} (t_{\zeta+1} - s)^{\varrho-1} z_{r-k}(s) ds + \tilde{k}_r(t_n). \end{aligned}$$

Approximating $z_{r-k}(s)$ on each subinterval $[t_\zeta, t_{\zeta+1}]$, using the following piecewise quadratic interpolation polynomial.

$$\begin{aligned} z_{r-k}(s) \approx & \frac{(t_{\zeta+1} - s)(t_{\zeta+\frac{1}{2}} - s)}{(t_\zeta - t_{\zeta+1})(t_\zeta - t_{\zeta+\frac{1}{2}})} z_{r-k}(t_\zeta) + \frac{(t_{\zeta+1} - s)(t_\zeta - s)}{(t_{\zeta+\frac{1}{2}} - t_{\zeta+1})(t_{\zeta+\frac{1}{2}} - t_\zeta)} z_{r-k}(t_{\zeta+\frac{1}{2}}) \\ & + \frac{(t_{\zeta+\frac{1}{2}} - s)(t_\zeta - s)}{(t_{\zeta+1} - t_\zeta)(t_{\zeta+1} - t_{\zeta+\frac{1}{2}})} z_{r-k}(t_{\zeta+1}), \end{aligned}$$

where $t_{\zeta+\frac{1}{2}} = \frac{t_\zeta + t_{\zeta+1}}{2}$ is mid point of the interval $[t_\zeta, t_{\zeta+1}]$, we get

$$\begin{aligned} z_r(t_n) \approx & z_r(0) - \frac{U_\eta(1-\varrho)}{h^\eta P(\varrho)} \sum_{k=-M+r}^{r} \omega_k^\eta z_{r-k}(t_n) - \frac{U_\eta \varrho}{h^\eta P(\varrho)\Gamma(\varrho)} \sum_{k=-M+r}^{r} \omega_k^\eta \sum_{\zeta=0}^{n-1} \int_{t_\zeta}^{t_{\zeta+1}} (t_{\zeta+1} - s)^{\varrho-1} \\ & \times \left(\frac{(t_{\zeta+1} - s)(t_{\zeta+\frac{1}{2}} - s)}{(t_\zeta - t_{\zeta+1})(t_\zeta - t_{\zeta+\frac{1}{2}})} z_{r-k}(t_\zeta) + \frac{(t_{\zeta+1} - s)(t_\zeta - s)}{(t_{\zeta+\frac{1}{2}} - t_{\zeta+1})(t_{\zeta+\frac{1}{2}} - t_\zeta)} z_{r-k}(t_{\zeta+\frac{1}{2}}) \right. \\ & \left. + \frac{(t_{\zeta+\frac{1}{2}} - s)(t_\zeta - s)}{(t_{\zeta+1} - t_\zeta)(t_{\zeta+1} - t_{\zeta+\frac{1}{2}})} z_{r-k}(t_{\zeta+1}) \right) ds + \tilde{k}_r^n. \end{aligned}$$

Now approximating the value of $z_{r-k}(t_{\zeta+\frac{1}{2}})$ using interpolation as

$$z_{r-k}(t_{\zeta+\frac{1}{2}}) = \frac{1}{2} z_{r-k}(t_\zeta) + \frac{1}{2} z_{r-k}(t_{\zeta+1}), \tag{3.4}$$

and putting in the above equation, we obtain

$$
\begin{aligned}
z_r(t_n) \approx& z_r(0) - \frac{U_\eta(1-\varrho)}{h^\eta P(\varrho)} \sum_{k=-M+r}^{r} \omega_k^\eta z_{r-k}(t_n) - \frac{U_\eta \varrho}{h^\eta P(\varrho)\Gamma(\varrho)} \sum_{k=-M+r}^{r} \omega_k^\eta \sum_{\zeta=0}^{n-1} \int_{t_\zeta}^{t_{\zeta+1}} (t_{\zeta+1}-s)^{\varrho-1} \\
&\times \left(\frac{(t_{\zeta+1}-s)(t_{\zeta+\frac{1}{2}}-s)}{(t_\zeta - t_{\zeta+1})(t_\zeta - t_{\zeta+\frac{1}{2}})} z_{r-k}(t_\zeta) + \frac{(t_{\zeta+1}-s)(t_\zeta - s)}{(t_{\zeta+\frac{1}{2}} - t_{\zeta+1})(t_{\zeta+\frac{1}{2}} - t_\zeta)} \left(\frac{1}{2} z_{r-k}(t_\zeta) + \frac{1}{2} z_{r-k}(t_{\zeta+1}) \right) \right. \\
&\left. + \frac{(t_{\zeta+\frac{1}{2}}-s)(t_\zeta - s)}{(t_{\zeta+1} - t_\zeta)(t_{\zeta+1} - t_{\zeta+\frac{1}{2}})} z_{r-k}(t_{\zeta+1}) \right) ds + \tilde{k}_r^n \\
=& z_r(0) - \frac{U_\eta(1-\varrho)}{h^\eta P(\varrho)} \sum_{k=-M+r}^{r} \omega_k^\eta z_{r-k}(t_n) - \frac{U_\eta \varrho}{h^\eta P(\varrho)\Gamma(\varrho)} \sum_{k=-M+r}^{r} \omega_k^\eta \sum_{\zeta=0}^{n-1} \int_{t_\zeta}^{t_{\zeta+1}} (t_{\zeta+1}-s)^{\varrho-1} \\
&\times \left(\left(\frac{(t_{\zeta+1}-s)(t_{\zeta+\frac{1}{2}}-s)}{(t_\zeta - t_{\zeta+1})(t_\zeta - t_{\zeta+\frac{1}{2}})} + \frac{(t_{\zeta+1}-s)(t_\zeta - s)}{2(t_{\zeta+\frac{1}{2}} - t_{\zeta+1})(t_{\zeta+\frac{1}{2}} - t_\zeta)} \right) z_{r-k}(t_\zeta) \right. \\
&\left. + \left(\frac{(t_{\zeta+\frac{1}{2}}-s)(t_\zeta - s)}{(t_{\zeta+1} - t_\zeta)(t_{\zeta+1} - t_{\zeta+\frac{1}{2}})} + \frac{(t_{\zeta+1}-s)(t_\zeta - s)}{2(t_{\zeta+\frac{1}{2}} - t_{\zeta+1})(t_{\zeta+\frac{1}{2}} - t_\zeta)} \right) z_{r-k}(t_{\zeta+1}) \right) ds + \tilde{k}_r^n \\
=& z_r(0) - \frac{U_\eta(1-\varrho)}{h^\eta P(\varrho)} \sum_{k=-M+r}^{r} \omega_k^\eta z_{r-k}(t_n) - \frac{(\delta t)^\varrho U_\eta \varrho}{h^\eta P(\varrho)\Gamma(\varrho+3)} \sum_{k=-M+r}^{r} \omega_k^\eta \sum_{\zeta=0}^{n-1} b_\zeta z_{r-k}(t_\zeta) \\
&- \frac{(\delta t)^\varrho U_\eta \varrho}{h^\eta P(\varrho)\Gamma(\varrho+3)} \sum_{k=-M+r}^{r} \omega_k^\eta b_n z_{r-k}(t_n) + \tilde{k}_r^n,
\end{aligned}
$$

where $b_\zeta (\zeta = 0, 1, 2, ..., n)$ represents the weights of the modified Simpson's fractional method given as

$$
\begin{aligned}
b_0 =& (\varrho+1)(\varrho+2)(n+1)^\varrho + (\varrho+2)(n)^{\varrho+1} - (\varrho+2)(n+1)^{\varrho+1}, \\
b_\zeta =& -2(\varrho+2)(n+1-\zeta)^{\varrho+1} + (\varrho+2)(n+2-\zeta)^{\varrho+1} + (\varrho+2)(n-\zeta)^{\varrho+1}, \quad 1 \le \zeta \le n-1 \\
b_n =& 2+\varrho.
\end{aligned}
\tag{3.5}
$$

The following numerical technique using the above approximation, for fractional order linear diffusion equation constructed on the modified Simpson's fractional formula is obtained.

$$
z_r^n + \sigma \sum_{k=-M+r}^{r} \omega_k^\eta z_{r-k}^n = z_r(0) - \mu \sum_{k=-M+r}^{r} \omega_k^\eta \sum_{\zeta=0}^{n-1} b_\zeta z_{r-k}^\zeta + \tilde{k}_r^n, \tag{3.6}
$$

where $\beta = \frac{U_\eta}{h^\eta P(\varrho)}$, $\sigma = \beta - \beta\varrho + \mu b_n$ and $\mu = \frac{\varrho\beta(\delta t)^\varrho}{\Gamma(\varrho+3)}$.

The following matrix form can be used to write the system (3.6),

$$
(\mathrm{I} + \sigma D) Z^n = \overline{\psi} - \mu D \sum_{\zeta=0}^{n-1} b_\zeta Z^\zeta + \tilde{K}^n, \tag{3.7}
$$

where

$$
\begin{aligned}
&\tilde{K}^n = (\tilde{k}_1^n, \tilde{k}_2^n, ..., \tilde{k}_{(M-1)}^n)^T, \quad Z^n = (z_1^n, z_2^n, ..., z_{(M-1)}^n)^T, \\
&\overline{\psi} = (\psi_1, \psi_2, ..., \psi_{M-1}), \quad \psi_r = \psi(v_r), \quad 1 \le r \le M-1,
\end{aligned}
$$

with identity matrix I of order $(M-1)\times(M-1)$ and D is the $(M-1)\times(M-1)$ matrix, which satisfy

$$D=\begin{bmatrix} \omega_0^\eta & \omega_{-1}^\eta & \omega_{-2}^\eta & \omega_{-3}^\eta & \dots & \omega_{-M+2}^\eta \\ \omega_1^\eta & \omega_0^\eta & \omega_{-1}^\eta & \omega_{-2}^\eta & \dots & \omega_{-M+3}^\eta \\ \omega_2^\eta & \omega_1^\eta & \omega_0^\eta & \omega_{-1}^\eta & \dots & \omega_{-M+4}^\eta \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \omega_{M-3}^\eta & \dots & \omega_2^\eta & \omega_1^\eta & \omega_0^\eta & \omega_1^\eta \\ \omega_{M-2}^\eta & \omega_{M-3}^\eta & \dots & \omega_2^\eta & \omega_1^\eta & \omega_0^\eta \end{bmatrix}.$$

3.2. Modified Simpson's fractional method: Non-linear case

Now considering the non-linear diffusion equation (1.2) having fractional order in both space and time with the given initial and boundary conditions (1.3) for $0<\varrho\le 1$ and $0<\eta\le 2$,

$$^{ABC}D_t^\varrho z(v,t)=U_\eta\frac{\partial^\eta}{\partial|v|^\eta}z(v,t)+k(z(v,t),v,t),\quad c<v<d,\quad 0<t<T. \tag{3.8}$$

The same spatial and temporal discretization results in

$$^{ABC}D_t^\varrho z_r(t)=-\frac{U_\eta}{h^\eta}\sum_{k=-M+r}^{r}\omega_k^\eta z_{r-k}(t)+k(z(v,t),v,t),\quad 1\le r\le M,\quad 0<t<T.$$

After integrating both sides of the preceding equation, we get

$$\begin{aligned} z_r(t)=&z_r(0)-\frac{U_\eta(1-\varrho)}{h^\eta P(\varrho)}\sum_{k=-M+r}^{r}\omega_k^\eta z_{r-k}(t)-\frac{U_\eta\varrho}{h^\eta P(\varrho)\Gamma(\varrho)}\sum_{k=-M+r}^{r}\omega_k^\eta\int_0^t(t-s)^{\varrho-1}z_{r-k}(s)ds \\ &+\frac{1-\varrho}{P(\varrho)}k(z_r(t),v_r,t)+\frac{\varrho}{P(\varrho)\Gamma(\varrho)}\int_0^t(t-s)^{\varrho-1}k(z_r(s),v_r,s)ds. \end{aligned}\tag{3.9}$$

By discretizing the integrals given in (3.9) using quadratic interpolation, we obtain

$$\begin{aligned} z_r(t_n)=&z_r(0)-\frac{U_\eta(1-\varrho)}{h^\eta P(\varrho)}\sum_{k=-M+r}^{r}\omega_k^\eta z_{r-k}(t_n)-\frac{U_\eta\varrho}{h^\eta P(\varrho)\Gamma(\varrho)}\sum_{k=-M+r}^{r}\omega_k^\eta\sum_{\zeta=0}^{n-1}\int_{t_\zeta}^{t_{\zeta+1}}(t_{\zeta+1}-s)^{\varrho-1}z_{r-k}(s)ds \\ &+\frac{1-\varrho}{P(\varrho)}k(z_r(t_n),v_r,t_n)+\frac{\varrho}{P(\varrho)\Gamma(\varrho)}\sum_{\zeta=0}^{n-1}\int_{t_\zeta}^{t_{\zeta+1}}(t_{\zeta+1}-s)^{\varrho-1}k(z_r(s),v_r,s)ds \\ \approx&z_r(0)-\frac{U_\eta(1-\varrho)}{h^\eta P(\varrho)}\sum_{k=-M+r}^{r}\omega_k^\eta z_{r-k}(t_n)-\frac{U_\eta\varrho}{h^\eta P(\varrho)\Gamma(\varrho)}\sum_{k=-M+r}^{r}\omega_k^\eta\sum_{\zeta=0}^{n-1}\int_{t_\zeta}^{t_{\zeta+1}}(t_{\zeta+1}-s)^{\varrho-1} \\ &\times\Bigg(\frac{(t_{\zeta+1}-s)(t_{\zeta+\frac{1}{2}}-s)}{(t_\zeta-t_{\zeta+1})(t_\zeta-t_{\zeta+\frac{1}{2}})}z_{r-k}(t_\zeta)+\frac{(t_{\zeta+1}-s)(t_\zeta-s)}{(t_{\zeta+\frac{1}{2}}-t_{\zeta+1})(t_{\zeta+\frac{1}{2}}-t_\zeta)}z_{r-k}(t_{\zeta+\frac{1}{2}}) \\ &+\frac{(t_{\zeta+\frac{1}{2}}-s)(t_\zeta-s)}{(t_{\zeta+1}-t_\zeta)(t_{\zeta+1}-t_{\zeta+\frac{1}{2}})}z_{r-k}(t_{\zeta+1})\Bigg)ds+\frac{1-\varrho}{P(\varrho)}k(z_r(t_n),v_r,t_n) \\ &+\frac{\varrho}{P(\varrho)\Gamma(\varrho)}\sum_{\zeta=0}^{n-1}\int_{t_\zeta}^{t_{\zeta+1}}(t_{\zeta+1}-s)^{\varrho-1}\Bigg(\frac{(t_{\zeta+1}-s)(t_{\zeta+\frac{1}{2}}-s)}{(t_\zeta-t_{\zeta+1})(t_\zeta-t_{\zeta+\frac{1}{2}})}k(z_r(t_\zeta),v_r,t_\zeta) \\ &+\frac{(t_{\zeta+1}-s)(t_\zeta-s)}{(t_{\zeta+\frac{1}{2}}-t_{\zeta+1})(t_{\zeta+\frac{1}{2}}-t_\zeta)}k(z_r(t_{\zeta+\frac{1}{2}}),v_r,t_{\zeta+\frac{1}{2}})+\frac{(t_{\zeta+\frac{1}{2}}-s)(t_\zeta-s)}{(t_{\zeta+1}-t_\zeta)(t_{\zeta+1}-t_{\zeta+\frac{1}{2}})}k(z_r(t_{\zeta+1}),v_r,t_{\zeta+1})\Bigg)ds. \end{aligned}$$

Now approximating the value of $z_{r-k}(t_{\zeta+\frac{1}{2}})$ using interpolation as

$$z_{r-k}(t_{\zeta+\frac{1}{2}}) = \frac{1}{2}z_{r-k}(t_\zeta) + \frac{1}{2}z_{r-k}(t_{\zeta+1}), \tag{3.10}$$

and integrating, we obtain

$$\begin{aligned} z_r(t_n) =& z_r(0) - \frac{U_\eta(1-\varrho)}{h^\eta P(\varrho)} \sum_{k=-M+r}^{r} \omega_k^\eta z_{r-k}(t_n) - \frac{(\delta t)^\varrho U_\eta \varrho}{h^\eta P(\varrho)\Gamma(\varrho+3)} \sum_{k=-M+r}^{r} \omega_k^\eta \sum_{\zeta=0}^{n-1} b_\zeta z_{r-k}(t_\zeta) \\ &- \frac{(\delta t)^\varrho U_\eta \varrho}{h^\eta P(\varrho)\Gamma(\varrho+3)} \sum_{k=-M+r}^{r} \omega_k^\eta b_n z_{r-k}(t_n) + \frac{1-\varrho}{P(\varrho)} k(z_r(t_n), v_r, t_n) \\ &+ \frac{(\delta t)^\varrho \varrho}{P(\varrho)\Gamma(\varrho+3)} \sum_{\zeta=0}^{n-1} b_\zeta k(z_r(t_n), v_r, t_n) + \frac{(\delta t)^\varrho \varrho b_n}{P(\varrho)\Gamma(\varrho+3)} k(z_r(t_n), v_r, t_n). \end{aligned}$$

The following numerical technique is obtained from the above approximation for non-linear diffusion equation based on the modified Simpson's fractional formula:

$$z_r^n = z_r(0) - \mu \sum_{k=-M+r}^{r} \omega_k^\eta \sum_{\zeta=0}^{n-1} b_\zeta z_{r-k}^\zeta - \sigma \sum_{k=-M+r}^{r} \omega_k^\eta z_{r-k}^n + \varpi \sum_{\zeta=0}^{n-1} b_\zeta k(z_r^\zeta, v_r, t_\zeta) + \varphi k(z_r^n, v_r, t_n), \tag{3.11}$$

where $\beta = \frac{U_\eta}{h^\eta P(\varrho)}$, $\sigma = \beta - \beta\varrho + \mu b_n$ and $\mu = \frac{\varrho\beta(\delta t)^\varrho}{\Gamma(\varrho+3)}$, $\varphi = \frac{1-\varrho}{P(\varrho)} + \frac{\varrho(\delta t)^\varrho b_n}{P(\varrho)\Gamma(\varrho+3)}$, $\varpi = \frac{\varrho(\delta t)^\varrho}{P(\varrho)\Gamma(\varrho+3)}$.

The following matrix form can be used to write the scheme (3.11),

$$Z^n + \sigma b_n D Z^n - \varphi K(Z^n) = \overline{\psi} - \mu D \sum_{\zeta=0}^{n-1} b_\zeta Z^\zeta + \varpi \sum_{\zeta=0}^{n-1} b_\zeta K(Z^\zeta), \tag{3.12}$$

where

$$K(Z^n) = (k(z_1^n), k(z_2^n), ..., k(z_{M-1}^n))^T.$$

4. Stability and convergence

We'll discuss the stability as well as convergence of the fractional Simpson's scheme in this section. Let the approximate solution be denoted by $\hat{z}_r^n$ for numerical scheme (3.6), and assume

$$\epsilon_r^n = \hat{z}_r^n - z_r^n, \quad n = \overline{0, N}, \quad r = \overline{1, M-1}.$$

Putting $\mathbb{R}^n = (\epsilon_1^n, \epsilon_2^n, ..., \epsilon_{M-1}^n), \quad n = \overline{0, N}$, and assume that

$$\mathbb{R}_\infty^n = \max_{1\le r\le M-1} |\epsilon_r^n| = |\epsilon_{\hat{r}}^n|.$$

Theorem 4.1. *The proposed numerical scheme(3.6) is unconditionally stable with*

$$\mathbb{R}_\infty^n \le \mathbb{R}_\infty^0, \quad n = \overline{0, N}.$$

Proof. From (3.6),

$$\epsilon_r^n + \mu \sum_{k=-M+r}^{r} \omega_k^\eta \sum_{\zeta=0}^{n-1} b_\zeta \epsilon_{r-k}^\zeta + \sigma \sum_{k=-M+r}^{r} \omega_k^\eta b_n \epsilon_{r-k}^n = \epsilon_r^0. \tag{4.1}$$

Taking $|z_1| - |z_2| \leq |z_1 - z_2|$ with (4.1), gives

$$\begin{aligned}
\mathbb{R}_\infty^n &= |\epsilon_{\hat{r}}^n| \\
&\leq |\epsilon_{\hat{r}}^n| + \mu \sum_{k=-M+r}^{r} \omega_k^\eta \sum_{\zeta=0}^{n-1} b_\zeta |\epsilon_{\hat{r}}^\zeta| + \sigma \sum_{k=-M+r}^{r} \omega_k^\eta b_n |\epsilon_{\hat{r}}^n| \\
&= \left(|\epsilon_{\hat{r}}^n| + \sigma b_n \omega_0^\eta |\epsilon_{\hat{r}}^n| + \mu \omega_0^\eta \sum_{\zeta=0}^{n-1} b_\zeta |\epsilon_{\hat{r}}^\zeta| \right) + \mu \sum_{k=-M+r, k\neq 0}^{r} \omega_k^\eta \sum_{\zeta=0}^{n-1} b_\zeta |\epsilon_{\hat{r}}^\zeta| + \sigma b_n \sum_{k=-M+r, k\neq 0}^{r} \omega_k^\eta |\epsilon_{\hat{r}}^n| \\
&\leq \left(|\epsilon_{\hat{r}}^n| + \sigma b_n \omega_0^\eta |\epsilon_{\hat{r}}^n| + \mu \omega_0^\eta \sum_{\zeta=0}^{n-1} b_\zeta |\epsilon_{\hat{r}}^\zeta| \right) + \mu \sum_{k=-M+r, k\neq 0}^{r} \omega_k^\eta \sum_{\zeta=0}^{n-1} b_\zeta |\epsilon_{\hat{r}-k}^\zeta| + \sigma b_n \sum_{k=-M+r, k\neq 0}^{r} \omega_k^\eta |\epsilon_{\hat{r}-k}^n| \\
&\leq \left| \left(\epsilon_{\hat{r}}^n + \sigma b_n \omega_0^\eta \epsilon_{\hat{r}}^n + \mu \omega_0^\eta \sum_{\zeta=0}^{n-1} b_\zeta \epsilon_{\hat{r}}^\zeta \right) + \mu \sum_{k=-M+r, k\neq 0}^{r} \omega_k^\eta \sum_{\zeta=0}^{n-1} b_\zeta \epsilon_{\hat{r}-k}^\zeta + \sigma b_n \sum_{k=-M+r, k\neq 0}^{r} \omega_k^\eta \epsilon_{\hat{r}-k}^n \right| \\
&= \left| \epsilon_{\hat{r}}^n + \mu \sum_{k=-M+r}^{r} \omega_k^\eta \sum_{\zeta=0}^{n-1} b_\zeta \epsilon_{\hat{r}-k}^\zeta + \sigma \sum_{k=-M+r}^{r} \omega_k^\eta b_n \epsilon_{\hat{r}-k}^n \right| \\
&= |\epsilon_{\hat{r}}^0|.
\end{aligned}$$

Hence we get, $\mathbb{R}_\infty^n \leq \mathbb{R}_\infty^0, \quad n = \overline{0, N}$. □

To show that fractional Simpson's formula (3.7) is convergent, we first offer these lemmas.

Lemma 4.1. *[28] Let $g(t) \in C^2([0, T])$, then there exists a constant C such that*

$$\left| \frac{1}{\Gamma(\varrho)} \int_{t_o}^{t_{\zeta+1}} (t_{\zeta+1} - s)^{\varrho-1} g(s) ds - (\delta t)^\varrho \left(\sum_{j=0}^{\zeta+1} b_j g(t_j) \right) \right| \leq \frac{T^\varrho C (\delta t)^2}{2! \Gamma(\varrho + 1)}.$$

Lemma 4.2. *For $0 < \varrho < 2$, the order of magnitude of the weights b_ζ, $\zeta = 0, 1, 2, ..., n-2$ given in (3.6) is*

$$b_\zeta = O((n - \zeta)^{\varrho-1}).$$

Proof. It is deduced from the definition of $b_\zeta (\zeta = 1, 2, ..., n-2)$ that

$$b_\zeta = \frac{(n-\zeta)^{\varrho-1}}{\Gamma(\varrho+3)} \left[-2(\varrho+2)(n-\zeta)^2 \left(1 + \frac{1}{n-\zeta}\right)^{\varrho+1} + (\varrho+2)(n-\zeta)^2 \left(1 + \frac{2}{n-\zeta}\right)^{\varrho+1} + (\varrho+2)(n-\zeta)^2 \right],$$

$$
\begin{aligned}
b_\zeta =& \frac{(\varrho+2)(n-\zeta)^{\varrho-1}}{\Gamma(\varrho+3)}\Bigg[-2\Bigg((n-\zeta)^2+(\varrho+1)(n-\zeta)+\frac{(\varrho+1)\varrho}{2!}+\frac{(\varrho+1)\varrho(\varrho-1)}{3!}\left(\frac{1}{n-\zeta}\right)\\
&+\frac{(\varrho+1)\varrho(\varrho-1)(\varrho-2)}{4!}\left(\frac{1}{n-\zeta}\right)^2+\ldots\Bigg)+(n-\zeta)^2+2(\varrho+1)(n-\zeta)^2+\varrho(\varrho+1)\\
&+\frac{(\varrho+1)\varrho(\varrho-1)}{3!}\left(\frac{2}{n-\zeta}\right)+\frac{(\varrho+1)\varrho(\varrho-1)(\varrho-2)}{4!}\left(\frac{2}{n-\zeta}\right)^2+\ldots+(n-\zeta)^2\Bigg],
\end{aligned}
$$

implies

$$
\begin{aligned}
b_\zeta =& \frac{(\varrho+2)(n-\zeta)^{\varrho-1}}{\Gamma(\varrho+3)}\Bigg[\frac{2(\varrho+1)\varrho(\varrho-1)(\varrho-2)}{4!}\left(\frac{1}{n-\zeta}\right)^2+\frac{6(\varrho+1)\varrho(\varrho-1)(\varrho-2)(\varrho-3)}{5!}\left(\frac{1}{n-\zeta}\right)^3\\
&+\frac{14(\varrho+1)\varrho(\varrho-1)(\varrho-2)(\varrho-3)(\varrho-4)}{6!}\left(\frac{1}{n-\zeta}\right)^4+\ldots\Bigg].
\end{aligned}
$$

We can see that, as $0<\varrho<1$, so the coefficients related to the power of $\frac{1}{n-\zeta}$ are smaller than 1. Thus the above given series is said to be convergent and the proof is completed for $b_\zeta(\zeta=1,2,...,n-2)$. We can neglect the proof for b_n since they can be proved using same method. □

To establish the main result, following inequality named as Gronwall inequality [28] is needed.

Lemma 4.3. *Assume that $C_1>0$ not depending on δt, $C_2\geq 0$ and $\{u_n\}$ satisfy*

$$
|u_n| \leqslant (\delta t)^\varrho C_1\sum_{\zeta=0}^{n-1}(n-\zeta)^{\varrho-1}|u_\zeta|+C_2,\quad \zeta=0,1,...,n-1,\quad n\delta t\leq T,
$$

with $0<\varrho\leq 1$. Then,

$$
|u_n| \leqslant C_2E_\varrho(C_1\Gamma(\varrho)T^\varrho),\quad n\delta t\leq T, \tag{4.2}
$$

where the parameter E_ϱ is the Mittage-Leffler function. For the particular case $\varrho=1$, inequality (4.2) becomes

$$
|u_n| \leqslant C_2e^{C_1T},\quad n\delta t\leq T.
$$

By setting $e_r^n=z(v_r,t_n)-z_r^n$ and $E^n=(e_1^n,e_2^n,e_3^n,...,e_{M-1}^n)^T$, then $e_r^0=0$ and using Eq (3.11) and Lemmas 2.1 and 4.1, the error e_r^n gives

$$
\begin{aligned}
&e_r^n+\mu\sum_{k=-M+r}^{r}\omega_k^\eta\sum_{\zeta=0}^{n-1}b_\zeta e_{r-k}^\zeta+\sigma\sum_{k=-M+r}^{r}\omega_k^\eta b_n e_{r-k}^n\\
&=\varpi\sum_{\zeta=0}^{n-1}b_\zeta[k(z(v_r,t_\zeta),v_r,t_\zeta)-k(z_r^\zeta,v_r,t_\zeta)]\\
&\quad+\varphi[k(z_r^n,v_r,t_n)-k(z(v_r,t_n),v_r,t_n)]+\mathrm{C}((\delta t)^2+h^2),\quad r=\overline{1,M-1}.
\end{aligned} \tag{4.3}
$$

Let us suppose that

$$
\mathbb{E}_\infty^n=\max_{1\leq r\leq M-1}|e_r^n|.
$$

For convenience, the preceding can be rephrased as follows:

$$
\mathbb{E}_\infty^n=|e_{\hat{r}}^n|.
$$

Theorem 4.2. *Let the fractional diffusion equation's solution be smooth and that the function $k(z, v, t)$ meets the Lipschitz condition with $\mathcal{L}$ being a constant,*

$$|k(z_1, v, t) - k(z_2, v, t)| \leq \mathcal{L}|z_1 - z_2|, \quad \forall z_1, z_2. \tag{4.4}$$

Then, the fractional numerical scheme given in (3.7) for sufficiently small t is convergent. Which means a constant $\mathrm{C}^* > 0$ *exists such that*

$$\mathrm{C}^*((\delta t)^2 + h^2) \geq \mathbb{E}_\infty^n, \quad n = \overline{1, N}.$$

Proof. Using Eqs (4.3) and (4.4), we get

$$\left| e_r^n + \mu \sum_{k=-M+r}^{r} \omega_k^\eta \sum_{\zeta=0}^{n-1} b_\zeta e_{r-k}^\zeta + \sigma \sum_{k=-M+r}^{r} \omega_k^\eta b_n e_{r-k}^n \right| \leq \varpi \mathcal{L} \sum_{\zeta=0}^{n-1} b_\zeta |e_r^\zeta| + \varphi \mathcal{L} |e_r^n| + \mathrm{C}((\delta t)^2 + h^2). \tag{4.5}$$

Using above inequality (4.5) and taking $|z_1| - |z_2| \leq |z_1 - z_2|$ along with the Lemmas 4.1 and 4.2, we get

$$\begin{aligned}
\mathbb{E}_\infty^n &= |e_{\hat{r}}^n| \\
&\leq |e_{\hat{r}}^n| + \mu \sum_{k=-M+r}^{r} \omega_k^\eta \sum_{\zeta=0}^{n-1} b_\zeta |e_{\hat{r}}^\zeta| + \sigma \sum_{k=-M+r}^{r} \omega_k^\eta b_n |e_{\hat{r}}^\zeta| \\
&= \left(|e_{\hat{r}}^n| + \sigma b_n \omega_0^\eta |e_{\hat{r}}^n| + \mu \omega_0^\eta \sum_{\zeta=0}^{n-1} b_\zeta |e_{\hat{r}}^\zeta| \right) + \mu \sum_{k=-M+r, k\neq 0}^{r} \omega_k^\eta \sum_{\zeta=0}^{n-1} b_\zeta |e_{\hat{r}}^\zeta| + \sigma b_n \sum_{k=-M+r, k\neq 0}^{r} \omega_k^\eta |e_{\hat{r}}^\zeta| \\
&\leq \left(|e_{\hat{r}}^n| + \sigma b_n \omega_0^\eta |e_{\hat{r}}^n| + \mu \omega_0^\eta \sum_{\zeta=0}^{n-1} b_\zeta |e_{\hat{r}}^\zeta| \right) + \mu \sum_{k=-M+r, k\neq 0}^{r} \omega_k^\eta \sum_{\zeta=0}^{n-1} b_\zeta |e_{\hat{r}-k}^\zeta| + \sigma b_n \sum_{k=-M+r, k\neq 0}^{r} \omega_k^\eta |e_{\hat{r}-k}^\zeta| \\
&\leq \left| \left(e_{\hat{r}}^n + \sigma b_n \omega_0^\eta e_{\hat{r}}^n + \mu \omega_0^\eta \sum_{\zeta=0}^{n-1} b_\zeta e_{\hat{r}}^\zeta \right) + \mu \sum_{k=-M+r, k\neq 0}^{r} \omega_k^\eta \sum_{\zeta=0}^{n-1} b_\zeta e_{\hat{r}-k}^\zeta + \sigma b_n \sum_{k=-M+r, k\neq 0}^{r} \omega_k^\eta e_{\hat{r}-k}^\zeta \right| \\
&= \left| e_{\hat{r}}^n + \mu \sum_{k=-M+r}^{r} \omega_k^\eta \sum_{\zeta=0}^{n-1} b_\zeta e_{\hat{r}-k}^\zeta + \sigma \sum_{k=-M+r}^{r} \omega_k^\eta b_n e_{\hat{r}-k}^\zeta \right| \\
&\leq \varpi \mathcal{L} \sum_{\zeta=0}^{n-1} b_\zeta |e_{\hat{r}}^\zeta| + \varphi \mathcal{L} |e_{\hat{r}}^n| + \mathrm{C}((\delta t)^2 + h^2) \\
&\leq \varpi \mathcal{L} \sum_{\zeta=0}^{n-1} (n - \zeta)^{\varrho - 1} |e_{\hat{r}}^\zeta| + \varphi \mathcal{L} |e_{\hat{r}}^n| + \mathrm{C}((\delta t)^2 + h^2),
\end{aligned}$$

this yeilds

$$|e_{\hat{r}}^n| \left(1 - \frac{((1-\varrho)\Gamma(\varrho+3) + \varrho(\delta t)^\varrho)\mathcal{L}}{P(\varrho)\Gamma(\varrho+3)} \right) \leq \frac{\varrho(\delta t)^\varrho \mathcal{L}}{P(\varrho)\Gamma(\varrho+3)} \sum_{\zeta=0}^{n-1} (n-\zeta)^{\varrho-1} |e_{\hat{r}}^\zeta| + \mathrm{C}((\delta t)^2 + h^2).$$

Now a constant C_1 exists for sufficiently small t such that

$$\left(1 - \frac{((1-\varrho)\Gamma(\varrho+3) + \varrho(\delta t)^\varrho)\mathcal{L}}{P(\varrho)\Gamma(\varrho+3)} \right)^{-1} \leq \mathrm{C}_1,$$

thus

$$|e_{\hat{r}}^n| \leq C_1 \frac{\varrho(\delta t)^\varrho \mathcal{L}}{P(\varrho)\Gamma(\varrho+3)} \sum_{\zeta=0}^{n-1} (n-\zeta)^{\varrho-1} |e_{\hat{r}}^\zeta| + C^*((\delta t)^2 + h^2).$$

Now from Gronwall inequality presented in Lemma 4.3, we get

$$C^*((\delta t)^2 + h^2) \geq E_\infty^n, \quad n = \overline{0, N}.$$

Hence the scheme (3.7) converges. □

5. Examples

Example 5.1. Consider the following ABC fractional order linear diffusion equation:

$$^{ABC}D_t^\varrho z(v,t) = U_\eta \frac{\partial^\eta}{\partial |v|^\eta} z(v,t) + k(v,t), \tag{5.1}$$

with $k(v,t) = t\sin(\pi v)$, along with the given initial and boundary conditions,

$$\begin{aligned} z(v,0) &= 0, \quad 0 \leq v \leq 1, \\ z(0,t) = z(1,t) &= 0, \quad 0 \leq t \leq T. \end{aligned} \tag{5.2}$$

Example 5.2. Consider the following time-space ABC fractional order non-linear diffusion equation,

$$^{ABC}D_t^\varrho z(v,t) = U_\eta \frac{\partial^\eta}{\partial |v|^\eta} z(v,t) + k(z(v,t),v,t), \tag{5.3}$$

with

$$\begin{aligned} k(z(v,t),v,t) =& \frac{200t^{2-\varrho}}{\Gamma(3-\varrho)} v^2(1-v)^2 - 9(t^2+1)^2 v^4 (1-v)^4 + z^2 \\ &+ \frac{100(t^2+1)U_\eta}{\cos(\pi\eta/2)} \left(\frac{v^{2-\eta} + (1-v)^{2-\eta}}{\Gamma(3-\eta)} - \frac{6v^{3-\eta} + 6(1-v)^{3-\eta}}{\Gamma(4-\eta)} + \frac{12v^{4-\eta} + 12(1-v)^{4-\eta}}{\Gamma(5-\eta)} \right), \end{aligned}$$

along with the given initial and boundary conditions

$$\begin{aligned} z(v,0) &= 3v^2(1-v)^2, \quad 0 \leq v \leq 1, \\ z(0,t) = 0, \quad z(1,t) &= 0, \quad 0 \leq t \leq 1. \end{aligned} \tag{5.4}$$

The solution profiles of the diffusion equations with fractional order given in Examples 5.1 and 5.2 are shown in Figures 1–6. Figures 1 and 4 depict the result of change in fractional order for space with fixed ϱ=0.9 and varying η. Figures 2 and 5 show the effects of varying ϱ, the fractional order in time with fixed η=1.5. It can be seen that the fractional orders ϱ and η influences how the solutions take on their shape. Figures 3 and 6 show the effect of coefficient U_η on the solution profile of all the given examples for $\varrho = 0.9$ and $\eta = 1.5$. This means that the diffusion coefficient also affects the dynamics of the fractional diffusion equation. In the absence of exact solution, we calculated the temporal error using $error = |z_{(h,\delta t/i)} - z_{(h,\delta t/2i)}|, \quad i = 1,2,3,4$ and for spatial error we calculated as, *error* = $|z_{(h/i,\delta t)} - z_{(h/2i,\delta t)}|$, $\quad i = 1,2,3,4$. The estimated order of convergence(EOC) is obtained by calculating the sequence of ratios for time as

$$EOC = \frac{|z_{(h,\delta t/i)} - z_{(h,\delta t/2i)}|}{|z_{(h,\delta t/2i)} - z_{(h,\delta t/4i)}|}, \quad i = 1,2,4,$$

and for space as

$$EOC = \frac{|z_{(h/i,\delta t)} - z_{(h/2i,\delta t)}|}{|z_{(h/2i,\delta t)} - z_{(h/4i,\delta t)}|}, \quad i = 1,2,4.$$

The estimated order of convergence with respect to time and space for Examples (5.1) and (5.2) are presented in Tables 1 and 2 respectively.

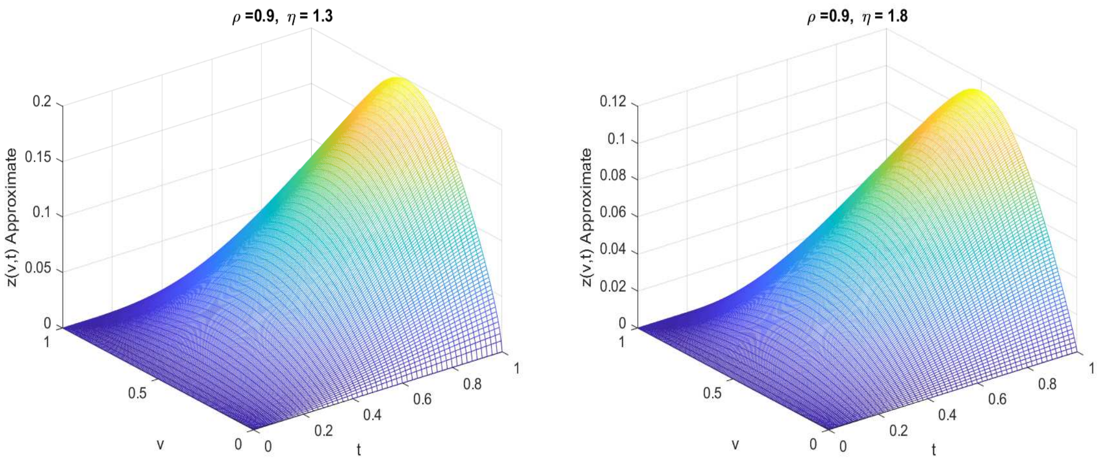

Figure 1. Numerical solutions for t = 0.02, h = 0.005, ϱ = 0.9 and η = 1.3, 1.8.

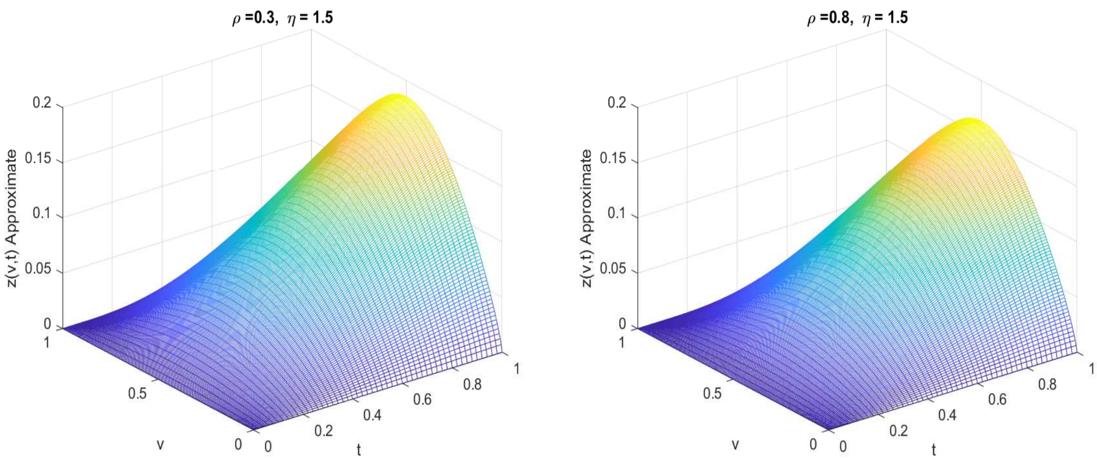

Figure 2. Numerical solutions for t= 0.02, h = 0.005, η = 1.5 and ϱ = 0.3, 0.8.

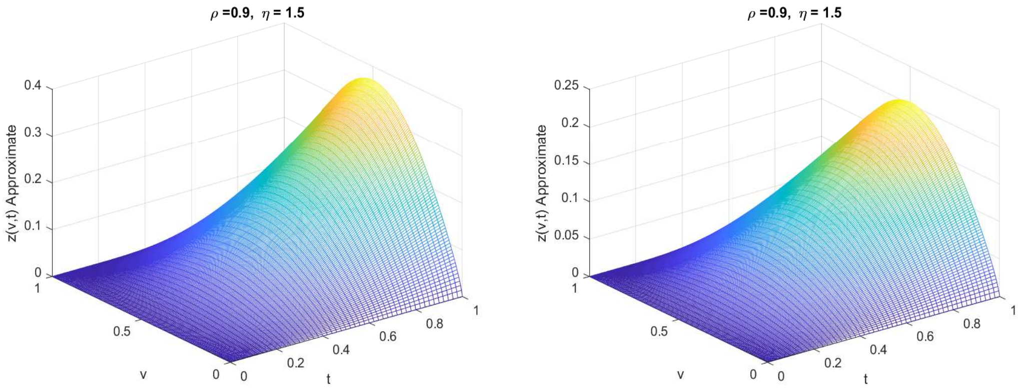

Figure 3. Effect of different values of diffusion coefficient U_η for t=0.02, h=0.005, ϱ=0.9 and η=1.5.

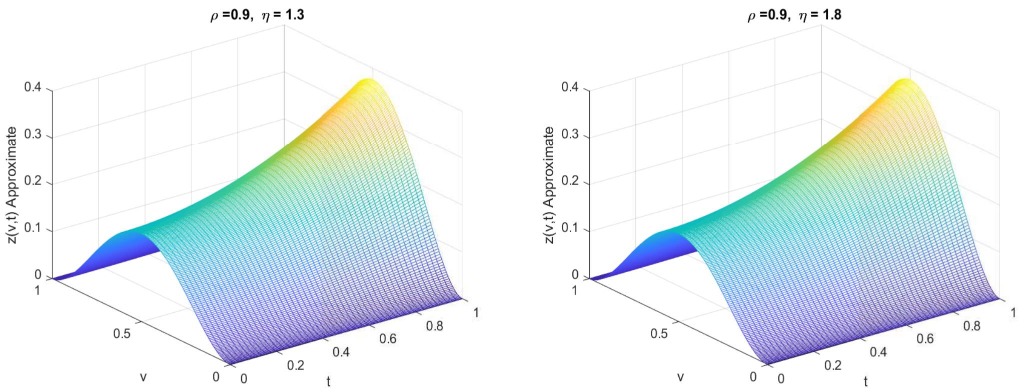

Figure 4. Numerical solutions for U_η=1, t=0.02, h=0.005, ϱ=0.9 and η=1.3, 1.8.

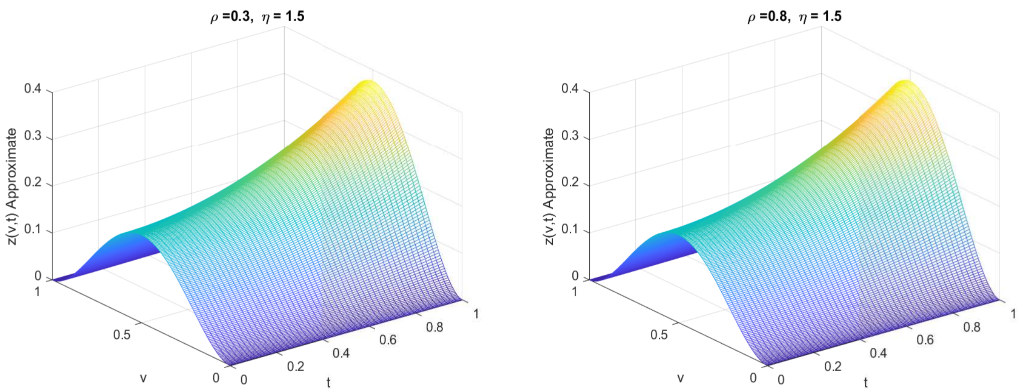

Figure 5. Numerical solutions for U_η=1, t=0.02, h=0.005, η=1.5 and ϱ=0.3, 0.8.

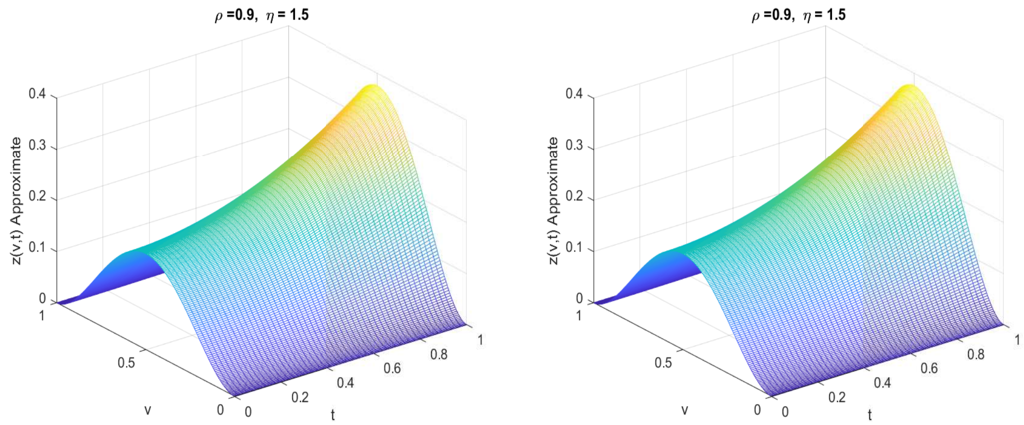

Figure 6. Effect of varying diffusion coefficient U_η for t=0.02, h=0.005, ϱ=0.9 and η=1.5.

Table 1. The errors and estimated order of convergence order when $\varrho = 0.9$, $\eta = 1.9$ and $T = 1$.

EOC w.r.t time with $h = 0.025$			EOC w.r.t space with $\delta t = 0.02$		
δ t	*Error*	order	h	*Error*	order
1/10	1.33×10^{-2}		1/10	6.38×10^{-2}	
1/20	6.40×10^{-3}	2.0801	1/20	3.32×10^{-2}	1.9243
1/40	3.10×10^{-3}	2.0825	1/40	1.71×10^{-2}	1.9354
1/80	1.50×10^{-3}	2.0652	1/80	8.80×10^{-3}	1.9355

Table 2. The errors and estimated order of convergence order when $\varrho = 0.3$, $\eta = 1.5$ and $T = 1$.

EOC w.r.t time with $h = 0.025$			EOC w.r.t space with $\delta t = 0.002$		
δt	Error	order	h	Error	order
1/10	2.9478×10^{-5}		1/10	4.772×10^{-3}	
1/20	8.2291×10^{-6}	1.8408	1/20	1.0745×10^{-3}	2.1508
1/40	2.2648×10^{-6}	1.8613	1/40	2.4564×10^{-4}	2.1290
1/80	9.2017×10^{-7}	1.8706	1/80	5.7170×10^{-5}	2.1033

6. Conclusions

With the spectral fractional Reisz derivative in space and the Atangana-Baleanu Caputo fractional derivative in time, we proposed a quick approach for effective and precise solutions for the linear and non-linear time-space fractional order diffusion equations. In order to establish a numerical framework for the time-space linear fractional diffusion problem, we merged the modified version of Simpson's 1/3 formula with Riesz fractional derivative in space approximated by centered difference. Following that, we expanded the plan to attempt to solve the fractional diffusion equation in a non-linear case. To determine the stability, a study of the suggested scheme was conducted. It has been demonstrated that the suggested technique has a convergence rate of $O(\delta t^2 + h^2)$ with time step δt and mesh size h. Our quick technique works well to simulate the fractional diffusion equation and is simple to use. The impact of fractional order in both time and space, as well as the effects of changing the diffusion coefficient U_η on the fractional diffusion equation, are illustrated using numerical simulations. Numerical schemes for both linear and non-linear time-space fractional diffusion are developed in this work. In future, these methods can be applied to a class of linear and non-linear fractional differential equations, such as Schrodinger, Korteweg-De Vries (KdV) and Burger's equations.

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Conflict of interest

The authors declare no conflict of interest.

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