Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(7): 15081–15091.
DOI: 10.3934/math.2023770
Received: 28 January 2023
Revised: 24 March 2023
Accepted: 06 April 2023
Published: 23 April 2023

Research article

Special type convex functions on Riemannian manifolds with application

Ehtesham Akhter[1], **Musavvir Ali**[1,*] **and Mohd Bilal**[2]

[1] Department of Mathematics, Aligarh Muslim University, Aligarh-202002, India
[2] Department of Mathematical Sciences, Faculty of Applied Sciences, Umm Al Qura University, Makkah, 21955, KSA

* **Correspondence:** Email: musavvir.alig@gmail.com.

Abstract: In this manuscript, we define a special type convex function on Euclidean space and explore it on the Riemannian manifold. We also detail the fundamental properties of special type convex functions and some examples that illustrate the idea. Moreover, to demonstrate the application to the problems of optimization, these special type convex functions are used.

Keywords: special type convexity; special subgradient; inequality; optimization; manifolds
Mathematics Subject Classification: 52A20, 52A41, 53C20, 53C22

1. Introduction

In nonlinear programming with continuous variables, convexity for sets and functions is crucial. It also has several applications in the fields of operations research, Riemannian manifolds, mathematical economics, engineering, and more. As a result, it's crucial to take into account a larger class of generalized convex functions and search for applicable convexity or generalized convexity criteria.

A few findings in nonlinear analysis and optimization theory have been improved on Riemannian manifolds from Euclidean space. Rapcsák Tamás [1] and Udriste [2] are credited with introducing the geodesic convexity. Later, Pini [3] introduced and Mititelu [4] examined the generalizations of the idea of invexity on a Riemannian manifold. Geodesic invex set, geodesic preinvex, and geodesic invex functions on a Riemannian manifold have all been discussed in [5] by Barani and Pouryayevali.

In the last few years, many researchers developed a large number of generalizations of convex function on Euclidean space and Riemannian manifold. Izhar Ahmad et al. [6] gave the concept of geodesic η-preinvex functions. L. W. Zhou et al. [7] developed the concept of roughly geodesic B-invex and optimization problem on Hadamard manifold. S. L. Chen et al. [8] gave the concept of geodesic B-preinvex functions and multiobjective optimization problems on Riemannian manifolds. R. P. Agrawal et al. [9, 10] developed generalized invex sets and preinvex functions on Riemannian manifolds and geodesic semi-strictly geodesic η-preinvex functions on a manifold. Meraj Ali Khan et al. [11] developed the concept of geodesic r-preinvex functions on the Riemannian manifold. Further, Adem Kılıçman and Wedad Saleh [12] gave the concept of geodesic strongly E-convex sets, and geodesic E-convex function and Izhar Ahmad et al. [13] gave the concept of generalized geodesic convexity on Riemannian manifold.

Motivated by the recent work, we give the new definition known as special type convex function on Euclidean space and explore it on Riemannian manifold. Also, we develop the example of a special type convex function and give some fundamental results in both cases as differentiable and non-differentiable functions on the Riemannian manifold. In the Last, we develop an application of special type convex function on the Riemannian manifold.

2. Preliminaries

Assume that $(\aleph, g)$ is a complete n-dimensional Riemannian manifold with Riemannian connection ∇. Consider $\hbar_1, \hbar_2$ are members in $\aleph$ and also $\delta : [0, 1] \to \aleph$ is a geodesic connecting the members $\hbar_1$, $\hbar_2$, i.e.,

$$\delta_{\hbar_1\hbar_2}(0) = \hbar_1,\ \delta_{\hbar_1\hbar_2}(1) = \hbar_2.$$

Definition 2.1. [2] Let $\mho \subseteq \aleph$ and $\mho$ is known as totally convex if $\mho$ contains each geodesic $\delta_{\hbar_1\hbar_2}$ of manifold $\aleph$ whose end members $\hbar_1$ and $\hbar_2$ are in $\mho$.

For more details, interested reader can see in [2].

Definition 2.2. [14] A subset $\mho$ of $\mathbf{R}^n$. A positive valued function $K : \mho \to \mathbf{R}$ is said to be exponential type convex if

$$K(\varsigma\hbar_1 + (1-\varsigma)\hbar_2) \leq (e^{\varsigma} - 1)K(\hbar_1) + (e^{1-\varsigma} - 1)K(\hbar_2) \tag{2.1}$$

holds $\forall\ \hbar_1, \hbar_2 \in \mho$ and $\forall\ \varsigma \in [0, 1]$.

Now, adding $-K(\hbar_2)$ on both sides of inequality (2.1), we get

$$K(\varsigma\hbar_1 + (1-\varsigma)\hbar_2) - K(\hbar_2) \leq (e^{\varsigma} - 1)K(\hbar_1) + (e^{1-\varsigma} - 2)K(\hbar_2), \tag{2.2}$$

and dividing by ς, we yield

$$\frac{K(\varsigma\hbar_1 + (1-\varsigma)\hbar_2) - K(\hbar_2)}{\varsigma} \leq \frac{(e^{\varsigma} - 1)K(\hbar_1)}{\varsigma} + \frac{(e^{1-\varsigma} - 2)K(\hbar_2)}{\varsigma}, \tag{2.3}$$

for the $\lim\limits_{\varsigma\to 0}$, we have

$$K'(\hbar_2)(\hbar_1 - \hbar_2) \leq K(\hbar_1) + \infty,$$

here, the left hand limit exists and the right hand limit does not exist finitely. But we want that both the limits must exist finitely. For this, we modify the inequality (2.1) as

$$K(\varsigma\hbar_1 + (1-\varsigma)\hbar_2) \leq (e^{\varsigma} - 1)K(\hbar_1) + (1 + e - e^{1-\varsigma})K(\hbar_2), \tag{2.4}$$

then rearrangement in inequality (2.4), division by ς, and taking $\lim\limits_{\varsigma\to 0}$ on both sides, we obtain

$$\lim_{\varsigma\to 0}\frac{K(\varsigma\hbar_1 + (1-\varsigma)\hbar_2) - K(\hbar_2)}{\varsigma} \leq \lim_{\varsigma\to 0}\frac{(e^{\varsigma} - 1)K(\hbar_1)}{\varsigma} + \lim_{\varsigma\to 0}\frac{(e - e^{1-\varsigma})K(\hbar_2)}{\varsigma} \tag{2.5}$$

$$K'(\hbar_2)(\hbar_1 - \hbar_2) \leq K(\hbar_1) + eK(\hbar_2).$$

Hence, limit exists on the both side finitely. Therefore, we will give a new definition as special type convex function.

3. Special type convex functions on Riemannian manifold

Motivated by the Definition (2.2), we introduce special type convex function as in following definition.

Definition 3.1. Let B be a convex set in $\mathbf{R}^n$ and let $K : B \to \mathbf{R}$ be a real-valued function. K is said to be a **special type convex function**, if

$$K(\varsigma\hbar_1 + (1-\varsigma)\hbar_2) \leq (e^{\varsigma} - 1)K(\hbar_1) + (1 + e - e^{1-\varsigma})K(\hbar_2),$$

holds $\forall\, \hbar_1, \hbar_2 \in B,\ \forall\, \varsigma \in [0,1]$.

Consider the following examples :

Example 3.2. The function $K : [0,\infty) \to \mathbf{R}$, defined as

$$K(\hbar) = (\hbar + 1)^2.$$

Example 3.3. The function $K : \mathbf{R} \to \mathbf{R}$ is defined as

$$K(\hbar) = |\hbar| + 1.$$

It can be easily verified that above examples support the definition of special convexity of functions. Special type convex functions on $\mathbf{R}^n$ hold the following properties:

- Let B be a convex set in $\mathbf{R}^n$. If $K_1, K_2 : B \to \mathbf{R}$ be special type convex functions, then their sum $K_1 + K_2$ is also special type convex function on B.
- Let B be a convex set in $\mathbf{R}^n$. If $K : B \to \mathbf{R}$ be a special type convex function and scalar $c \geq 0$, then cK is also special type convex function on B.
- Let B be a convex set in $\mathbf{R}^n$. If $K : B \to \mathbf{R}$ be a special type convex function and $H : \mathbf{R} \to \mathbf{R}$ be linear and increasing function, then their composition $H \circ K$ is also a special type convex function.
- Let B be a convex set in $\mathbf{R}^n$. $K_i : B \to \mathbf{R}$ be an arbitrary family of special type convex functions, restricted to B. Then, $K(\hbar) = \sup_{i \in I} K_i(\hbar)$ is also a special type convex function on B.

Now, we explore the Definition (3.1) of special type convex function on a Riemannian manifold.

Let $\aleph$ be a Riemannian manifold, and also let $\mho$ be a non-empty totally convex subset of $\aleph$ throughout this paper.

Let $(\aleph, g)$ be a complete n-dimensional Riemannian manifold and ∇ be the Riemannian connection on $\aleph$. Suppose $\mho$ is a totally convex subset of $\aleph$. For the elements $\hbar_1, \hbar_2 \in \mho, \delta_{\hbar_1\hbar_2}$ denotes the geodesic connecting the elements $\hbar_1$ to $\hbar_2$.

Definition 3.4. Let $\mho$ be a totally convex set in $\aleph$. Also let $K : \mho \to \mathbf{R}$ be a real-valued function. K is said to be a **special type convex function**, if

$$K(\delta_{\hbar_1\hbar_2}(\varsigma)) \leq (e^{\varsigma} - 1)K(\hbar_2) + (1 + e - e^{1-\varsigma})K(\hbar_1),$$

holds $\forall\, \hbar_1, \hbar_2 \in \mho,\ \forall\, \delta_{\hbar_1\hbar_2} \in \Sigma,\ \forall\, \varsigma \in [0,1]$.

Example 3.5. Let $\aleph = \{e^{i\alpha} : \frac{\pi}{4} < \alpha < \frac{\pi}{3}\}$ be a manifold. Let $\hbar_1 = e^{i\alpha}$ and $\hbar_2 = e^{i\beta}$. So, define a geodesic on $\aleph$ as $\delta_{\hbar_1\hbar_2} : [0,1] \to \aleph$ s.t.

$$\delta_{\hbar_1\hbar_2}(\varsigma) = (\cos((1-\varsigma)\alpha + \varsigma\beta),\ \sin((1-\varsigma)\alpha + \varsigma\beta)),\ \forall\, \varsigma \in [0,1].$$

Now, we define $K : \aleph \to \mathbf{R}$ as

$$K(e^{i\alpha}) = c\ \sin\alpha,\ c > 1.$$

Therefore, we can easily seen that K is special type convex function.

Theorem 3.6. *A function $K : \mho \to \mathbf{R}$ is a special type convex iff $\forall\, \hbar_1, \hbar_2 \in \mho, \forall\, \delta_{\hbar_1\hbar_2} \in \Sigma$ the function $\phi_{\hbar_1\hbar_2} = K \circ \delta_{\hbar_1\hbar_2}$ is special type convex on $[0,1]$.*

Proof. First, we assume that function $\phi_{\hbar_1\hbar_2} : [0,1] \to \mathbf{R}$ is special type convex, then we have

$$\phi_{\hbar_1\hbar_2}((1-\alpha)u_1 + \alpha u_2) \leq (e^{\alpha} - 1)\phi_{\hbar_1\hbar_2}(u_2) + (1 + e - e^{1-\alpha})\phi_{\hbar_1\hbar_2}(u_1),\ \forall\, u_1, u_2 \in [0,1].$$

Particularly for $u_1 = 0, u_2 = 1$, we have

$$\phi_{\hbar_1\hbar_2}(\alpha) \leq (e^{\alpha} - 1)\phi_{\hbar_1\hbar_2}(1) + (1 + e - e^{1-\alpha})\phi_{\hbar_1\hbar_2}(0),$$

i.e.,

$$K(\delta_{\hbar_1\hbar_2}(\alpha)) \leq (e^{\alpha} - 1)K(\hbar_2) + (1 + e - e^{1-\alpha})K(\hbar_1),\ \forall\, \hbar_1, \hbar_2 \in \mho, \forall\, \alpha \in [0,1], \forall\, \delta_{\hbar_1\hbar_2} \in \Sigma.$$

Next, we assume that K is a special type convex function. As $\delta_{\hbar_1\hbar_2} : [0,1] \to \mho$ is a geodesic connecting the members $\hbar_1$ and $\hbar_2$, then the restriction of $\delta_{\hbar_1\hbar_2}$ to $[u_1, u_2]$ meets the members $\delta_{\hbar_1\hbar_2}(u_1)$ and $\delta_{\hbar_1\hbar_2}(u_2)$. We reparametrize this restriction,

$$d(\alpha) = \delta_{\hbar_1\hbar_2}((1-\alpha)u_1 + \alpha u_2)),\ \alpha \in [0,1].$$

Since

$$K(d(\alpha)) \leq (e^{\alpha} - 1)K(d(1)) + (1 + e - e^{1-\alpha})K(d(0)),$$

i.e.,

$$K(\delta_{\hbar_1\hbar_2}((1-\alpha)u_1 + \alpha u_2)) \leq (e^{\alpha} - 1)K(\delta_{\hbar_1\hbar_2}(u_2)) + (1 + e - e^{1-\alpha})K(\delta_{\hbar_1\hbar_2}(u_1)),$$

or

$$\phi_{\hbar_1\hbar_2}((1-\alpha)u_1 + \alpha u_2)) \leq (e^{\alpha} - 1)\phi_{\hbar_1\hbar_2}(u_2) + (1 + e - e^{1-\alpha})\phi_{\hbar_1\hbar_2}(u_1),$$

the function $\phi_{\hbar_1\hbar_2}$ is special convex type on $[0,1]$. □

Theorem 3.7. *Suppose that $\mho$ is subset of $\aleph$ and $\mho$ is totally convex set. A function $K : \mho \to \mathbf{R}$ is special type convex iff its epigraph*

$$Epi(K) = \{(\hbar, \alpha) : K(\hbar) \leq \alpha\} \subset \mho \times \mathbf{R}$$

is a totally convex set.

Proof. Let K be a special type convex function, we have

$$K(\delta_{\hbar_1\hbar_2}(\varsigma)) \quad \leq \quad (e^{\varsigma} - 1)K(\hbar_2) + (1 + e - e^{1-\varsigma})K(\hbar_1).$$

Take $(\hbar_1, \alpha)$ and $(\hbar_2, \beta)$ in $Epi(K)$, so that $K(\hbar_1) \leq \alpha, \;\; K(\hbar_2) \leq \beta$, above inequality can be written as

$$K(\delta_{\hbar_1\hbar_2}(\varsigma)) \quad \leq \quad (e^{\varsigma} - 1)\beta + (1 + e - e^{1-\varsigma})\alpha,$$

$\Longrightarrow$

$$(\delta_{\hbar_1\hbar_2}(\varsigma), (e^{\varsigma} - 1)\beta + (1 + e - e^{1-\varsigma})\alpha) \in Epi(K),$$

so, $Epi(K)$ is a totally convex set.

Conversely, assume that $Epi(K)$ is a totally convex set and $\hbar_1, \hbar_2 \in \mho$. We can have $(\hbar_1, K(\hbar_1))$ and $(\hbar_2, K(\hbar_2))$ in $Epi(K)$. Based on total convexity of $Epi(K)$, it follows

$$(\delta_{\hbar_1\hbar_2}(\varsigma), (e^{\varsigma} - 1)K(\hbar_2) + (1 + e - e^{1-\varsigma})K(\hbar_1)) \in Epi(K),$$

i.e.,

$$K(\delta_{\hbar_1\hbar_2}(\varsigma)) \leq (e^{\varsigma} - 1)K(\hbar_2) + (1 + e - e^{1-\varsigma})K(\hbar_1)$$

and hence K is special type convex function. □

We will now provide a method for constructing special type convex functions on $\aleph$ beginning from totally convex sets in $\aleph \times N$, where $N = (a, \infty)$. Aforementioned, suppose (N, h) is Euclidean space of 1-dimension and $(\aleph \times N, g + h)$ is the product manifold of $(\aleph, g)$ and (N, h). A geodesic connecting the members $(\hbar_1, \alpha)$ and $(\hbar_2, \beta)$ of $\aleph \times N$ is written as $(\delta_{\hbar_1\hbar_2}(\varsigma), (e^{\varsigma} - 1)\beta + (1 + e - e^{1-\varsigma})\alpha)$, where $\delta_{\hbar_1\hbar_2}(\varsigma), \varsigma \in [0, 1]$, is a geodesic in $\aleph$ from $\hbar_1$to $\hbar_2$.

Theorem 3.8. *Let us consider B, a totally convex subset of $\aleph \times N$. The function*

$$K(\hbar) = \inf\{\alpha : (\hbar, \alpha) \in B\}$$

is a special type convex on the projection of B onto $\aleph$.

Proof. $\aleph$ is a convex subset when B is projected onto it. Suppose $(\hbar_1, \alpha)$ and $(\hbar_2, \beta)$ are two members in B. Using the definition of K, we get

$$K(\hbar_1) \leq \alpha, \;\; K(\hbar_2) \leq \beta. \tag{3.1}$$

But B is totally convex, therefore

$$\forall\ \varsigma \in [0, 1], \;\; (\delta_{\hbar_1\hbar_2}(\varsigma), (e^{\varsigma} - 1)\beta + (1 + e - e^{1-\varsigma})\alpha) \in B.$$

Thus,

$$K(\delta_{\hbar_1\hbar_2}(\varsigma)) \leq (e^{\varsigma} - 1)\beta + (1 + e - e^{1-\varsigma})\alpha. \tag{3.2}$$

We find that (3.2) is true whenever (3.1) is true, it means the epigraph $Epi(K)$ to be totally convex set implies K is special type convex function. □

Consider a tangent vector $X \in T_{\hbar_1}\aleph$, s.t. $\|X\| = 1$ and $\delta : (-a, a) \to \mho$ is a geodesic such that $\delta(0) = \hbar_1,\ \dot{\delta}(0) = X$. Consider $K : \mho \to \mathbf{R}$ is a continuous function. We define

$$DK(\hbar_1, X) = \lim_{\varsigma\to 0}\inf \frac{1}{\varsigma^2}[(e^{\frac{1}{2}} - 1)K(\delta(\varsigma)) + (1 + e - e^{\frac{1}{2}})K(\delta(-\varsigma)) - K(\delta(0))],$$

$$\text{and } DK(\hbar_1) = \inf_{\|X\|=1} DK(\hbar_1, X).$$

Theorem 3.9. *Let $\mho$ be a non-empty, totally convex subset of a Riemannian manifold $\aleph$. If K is special type convex function, then $DK(\hbar_1) \geq 0$.*

Proof. Assume that function $K : \mho \to \mathbb{R}$ is special type convex, then for geodesic δ on $\aleph$, from Theorem 3.6, $K \circ \delta : (-a, a) \to \mathbb{R}$ is special convex type, and we have

$$(K \circ \delta)((1 - \sigma)\hbar_1 + \sigma\hbar_2) \leq (e^{\sigma} - 1)(K \circ \delta)(\hbar_2) + (1 + e - e^{1-\sigma})(K \circ \delta)(\hbar_1),$$

for all $\hbar_1, \hbar_2 \in (-a, a)$ and $\sigma \in [0, 1]$. Fixing $\sigma = \frac{1}{2}$ and $\hbar_1 + \hbar_2 = 0$ we find

$$(K(\delta(0)) \leq (e^{\frac{1}{2}} - 1)(K(\delta)(\hbar_2)) + (1 + e - e^{\frac{1}{2}})(K(\delta(-\hbar_2)), \forall\ \hbar_2 \in (-a, a).$$

Consequently $DK(\hbar_1, X) \geq 0$ and $DK(\hbar_1) \geq 0$, where $\hbar_1 = \delta(0)$. □

Let $(\aleph, g)$ be the complete Riemannian manifold in finite dimensions, and let ∇ be the Riemannian connection. Assume that $L : \aleph \to \aleph$ is a diffeomorphism and that $L_*\nabla$ is the connection follows from ∇'s transformation. If δ is the geodesic of $(\aleph, \nabla)$, then $L \circ \delta$ is a geodesic of $(\aleph, L_*\nabla)$.

Theorem 3.10. *For $\mho$ to be a proper totally convex subset of $\aleph$ and $K : \mho \to \mathbb{R}$, a special type convex function. If $L : \aleph \to \aleph$ is a diffeomorphism then $K \circ L^{-1}$ is special type convex on the set $L(\mho)$.*

Proof. Suppose that $\hbar_1, \hbar_2 \in \mho$ and $\delta_{\hbar_1\hbar_2}$ is a geodesic of $\aleph$ which meets the members $\hbar_1$ and $\hbar_2$. Also, let the set $L(\mho)$ is totally convex and the geodesic $L \circ \delta_{\hbar_1\hbar_2}$ meets the members $L(\hbar_1)$ and $L(\hbar_2)$, we get

$$\begin{aligned}(K \circ L^{-1})(L(\delta_{\hbar_1\hbar_2}(\varsigma))) &= K(\delta_{\hbar_1\hbar_2}(\varsigma))\\ &\leq (e^{\varsigma} - 1)K(\hbar_2) + (1 + e - e^{1-\varsigma})K(\hbar_1)\\ &= (e^{\varsigma} - 1)(K \circ L^{-1})(L(\hbar_2)) + (1 + e - e^{1-\varsigma})(K \circ L^{-1})(L(\hbar_1)),\end{aligned}$$

i.e., $K \circ L^{-1}$ is special type convex function on the set $L(\mho)$. □

Theorem 3.11. *Let $K : \mho \to \mathbf{R}$ be a special type convex function defined on the totally convex set $\mho$. Let $G : \mathbf{R} \to \mathbf{R}$ be a linear and increasing function. Then the composite $G \circ K$ is also special type convex function.*

Proof. The proof of this Theorem is quite trivial. □

Theorem 3.12. *If $K_i,\ i = 1, 2, ..., n$ are special convex type functions on $\mho \subset \aleph$ and $c_i \geq 0$, then $\sum_{i=1}^{n} c_i K_i$ is special type convex on $\mho$.*

Proof. By hypothesis we have

$$K_i(\delta_{\hbar_1\hbar_2}(\varsigma)) \leq (e^{\varsigma}-1)K_i(\hbar_2)+(1+e-e^{1-\varsigma})K_i(\hbar_1).$$

It follows that

$$c_iK_i(\delta_{\hbar_1\hbar_2}(\varsigma)) \leq (e^{\varsigma}-1)c_iK_i(\hbar_2)+(1+e-e^{1-\varsigma})c_iK_i(\hbar_1).$$

And

$$\left(\sum_{i=1}^{n} c_iK_i\right)(\delta_{\hbar_1\hbar_2}(\varsigma)) \leq (e^{\varsigma}-1)\left(\sum_{i=1}^{n} c_iK_i\right)(\hbar_2)+(1+e-e^{1-\varsigma})\left(\sum_{i=1}^{n} c_iK_i\right)(\hbar_1).$$

Thus, we achieve the proof. □

Theorem 3.13. *If $K_i : \mho \to \mathbf{R}$ are special type convex functions on the totally convex subset $\mho$ of $\aleph$. The set defined as*

$$W = \{\hbar \in \mho : K_i(\hbar) \leq 0,\ i = 1, 2, ..., n\}$$

is a also totally convex.

Proof. As $K_i(\hbar),\ i = 1, 2, ..., n$, are special type convex functions, then we have

$$K_i(\delta_{\hbar_1\hbar_2}(\varsigma)) \leq (e^{\varsigma}-1)K_i(\hbar_2)+(1+e-e^{1-\varsigma})K_i(\hbar_1),$$

holds $\forall\ \hbar_1, \hbar_2 \in \mho,\ \forall\ \delta_{\hbar_1\hbar_2} \in \Sigma,\ \forall\ \varsigma \in [0, 1]$. This implies that

$$K_i(\delta_{\hbar_1\hbar_2}(\varsigma)) \leq 0,$$

and so, $\delta_{\hbar_1\hbar_2}(\varsigma) \in \mho$. Therefore, $\mho$ is a totally convex set. □

The set $T^*_{\hbar_1}\mho$ of cotangent vectors to $\mho$ at a member $\hbar_1$ is a convex cone in $T^*_{\hbar_1}\aleph$. Let $\hbar_2$ be a generic point in $\mho$ and $\delta_{\hbar_1\hbar_2}(\varsigma),\ \varsigma \in [0, 1]$, be a geodesic such that

$$\delta_{\hbar_1\hbar_2}(0) = \hbar_1,\ \delta_{\hbar_1\hbar_2}(1) = \hbar_2,\ \dot{\delta}_{\hbar_1\hbar_2}(0) = X_{\hbar_1} \in T^*_{\hbar_1}\mho.$$

We denote the collection of all geodesics from $\hbar_1$ and $\hbar_2$ by the symbol Σ.

Definition 3.14. Let $K : \mho \to \mathbf{R}$ be a special convex type function. A 1-form $w_{\hbar_1} \in T^*_{\hbar_1}\mho$ is known as the **special sub-gradient** of K at $\hbar_1$ if

$$K(\hbar_2) + eK(\hbar_1) \geq w_{\hbar_1}(\dot{\delta}_{\hbar_1\hbar_2}(0)),\ \forall\ \hbar_2 \in \mho,\ \forall\ \delta_{\hbar_1\hbar_2} \in \Sigma.$$

Theorem 3.15. *Let $\mho \subset \aleph$ be an open totally convex set and $K : \mho \to \mathbf{R}$ be a differentiable function.*
(a) If the function K is special type convex, then

$$\dot{\delta}_{\hbar_1\hbar_2}(K)(\hbar_1) \leq K(\hbar_2) + eK(\hbar_1),\ \forall\ \hbar_1, \hbar_2 \in \mho,\ \forall\ \delta_{\hbar_1\hbar_2} \in \Sigma.$$

(b) If

$$\dot{\delta}_{\hbar_1\hbar_2}(K)(\hbar_1) \leq K(\hbar_2) + eK(\hbar_1),\ \forall\ \hbar_1, \hbar_2 \in \mho,\ \forall\ \delta_{\hbar_1\hbar_2} \in \Sigma$$

holds, then

$$K(\delta_{\hbar_1\hbar_2}(\varsigma)) \geq -\frac{\varsigma}{e}K(\hbar_2) - \frac{(1-\varsigma)}{e}K(\hbar_1).$$

Proof. First we assume that K is special type convex function, we have

$$K(\delta_{\hbar_1\hbar_2}(\varsigma)) \leq (e^{\varsigma}-1)K(\hbar_2)+(1+e-e^{1-\varsigma})K(\hbar_1),\ 0<\varsigma\leq 1,$$

dividing by ς and taking limit on both sides, we find

$$\lim_{\varsigma\to 0}\frac{K(\delta_{\hbar_1\hbar_2}(\varsigma))-K(\hbar_1)}{\varsigma} \leq \lim_{\varsigma\to 0}\frac{(e^{\varsigma}-1)K(\hbar_2)+(e-e^{1-\varsigma})K(\hbar_1)}{\varsigma}.$$

Taking into account that

$$\lim_{\varsigma\to 0}\frac{K(\delta_{\hbar_1\hbar_2}(\varsigma))-K(\hbar_1)}{\varsigma}=\frac{d}{d\varsigma}K(\delta_{\hbar_1\hbar_2}(\varsigma))|_{\varsigma=0}=\dot{\delta}_{\hbar_1\hbar_2}(0)(K)=\dot{\delta}_{\hbar_1\hbar_2}(K)(\hbar_1)$$

we obtain the relation

$$\dot{\delta}_{\hbar_1\hbar_2}(K)(\hbar_1)\leq K(\hbar_2)+eK(\hbar_1). \tag{3.3}$$

Next, we suppose that $\forall\ \hbar_1,\hbar_2\in\mho,\ \forall\ \delta_{\hbar_1\hbar_2}\in\Sigma$ and the relation (3.3) is holds. Interchanging $\hbar_2$ with $\hbar_1$, we obtain

$$\dot{\delta^t}_{\hbar_2\hbar_1}(K)(\hbar_2)\leq K(\hbar_1)+eK(\hbar_2) \tag{3.4}$$

where

$$\delta^t{}_{\hbar_2\hbar_1}(\varsigma)=\delta_{\hbar_1\hbar_2}(1-\varsigma),\ \varsigma\in[0,1]$$

is a geodesic which meets $\hbar_2$ and $\hbar_1$.

In order to get the point $\delta_{\hbar_1\hbar_2}(\varsigma)$, we fix ς . The geodesic arc which meets $\delta_{\hbar_1\hbar_2}(\varsigma)$ and $\hbar_2$ is the restriction $\delta_{\hbar_1\hbar_2}(u),\ u\in[\varsigma,1]$. Setting $u=\varsigma+s(1-\varsigma),\ s\in[0,1]$, we find the reparametrization

$$\beta(s)=\delta_{\hbar_1\hbar_2}(u(s))=\delta_{\hbar_1\hbar_2}(\varsigma+s(1-\varsigma)),\ s\in[0,1]$$

whence

$$\beta(0)=\delta_{\hbar_1\hbar_2}(\varsigma),\ \frac{d\beta}{ds}(0)=(1-\varsigma)\frac{d\delta_{\hbar_1\hbar_2}}{d\varsigma}(\varsigma).$$

The restriction $\delta^t{}_{\hbar_2\hbar_1}(\varsigma)(u)=\delta_{\hbar_1\hbar_2}(1-u),\ u\in[1-\varsigma,1]$ is the geodesic arc which joins $\delta_{\hbar_1\hbar_2}(\varsigma)$ and $\hbar_1$. Setting $u=1-\varsigma+s\varsigma,\ s\in[0,1]$, we find the reparametrization

$$\gamma(s)=\delta^t{}_{\hbar_2\hbar_1}(1-\varsigma+s\varsigma)=\delta_{\hbar_1\hbar_2}(\varsigma-s\varsigma),\ s\in[0,1]$$

whence

$$\gamma(0)=\delta_{\hbar_1\hbar_2}(\varsigma),\ \frac{d\gamma}{ds}(0)=-\varsigma\frac{d\delta_{\hbar_1\hbar_2}}{d\varsigma}(\varsigma).$$

Replacing, in (3.3), $\hbar_1$ by $\delta_{\hbar_1\hbar_2}(\varsigma)$ and $\dot{\delta}_{\hbar_2\hbar_1}(0)$ by $\frac{d\beta}{ds}(0)$, we get

$$(1-\varsigma)\frac{d\delta_{\hbar_1\hbar_2}}{d\varsigma}(K)(\delta_{\hbar_1\hbar_2}(\varsigma))\leq K(\hbar_2)+eK(\delta_{\hbar_1\hbar_2}(\varsigma)). \tag{3.5}$$

Analogously, substituting, in (3.4) $\delta_{\hbar_1\hbar_2}(\varsigma)$ for $\hbar_2$ and $\frac{d\gamma}{ds}(0)$ for $\dot{\delta^t}_{\hbar_2\hbar_1}(0)$ we obtain

$$-\varsigma\frac{d\delta_{\hbar_1\hbar_2}}{d\varsigma}(K)(\delta_{\hbar_1\hbar_2}(\varsigma))\leq K(\hbar_1)+eK(\delta_{\hbar_1\hbar_2}(\varsigma)). \tag{3.6}$$

Multiplying in (3.5) by ς and in (3.6) by $(1-\varsigma)$, then adding we deduce

$$0 \leq \varsigma K(\hbar_2) + (1-\varsigma)K(\hbar_1) + eK(\delta_{\hbar_1\hbar_2}(\varsigma)).$$

This implies that

$$K(\delta_{\hbar_1\hbar_2}(\varsigma)) \geq -\frac{\varsigma}{e}K(\hbar_2) - \frac{(1-\varsigma)}{e}K(\hbar_1).$$

□

Proposition 3.16. *Let $\mho$ be totally convex subset of $\aleph$ and $\aleph$ be a complete manifold. Let $K : \mho \to \mathbf{R}$ be a differentiable function and there exists a sequence $\{\varsigma_n\}$ of non-negative real number s.t. $\varsigma_n \to 0$ as $n \to 0$ and*

$$K(\delta_{\hbar_1\hbar_2}(\varsigma_n)) \leq (e^{\varsigma_n} - 1)K(\hbar_2) + (1 + e - e^{1-\varsigma_n})K(\hbar_1), \tag{3.7}$$

holds $\forall\, \hbar_1, \hbar_2 \in \mho$, $\forall\, \delta_{\hbar_1\hbar_2} \in \Sigma$, *then*

$$\dot{\delta}_{\hbar_1\hbar_2}(K)(\hbar_1) \leq K(\hbar_2) + eK(\hbar_1).$$

Proof. From (3.7), we have

$$\frac{K(\delta_{\hbar_1\hbar_2}(\varsigma_n)) - K(\hbar_1)}{\varsigma_n} \leq \frac{(e^{\varsigma_n} - 1)}{\varsigma_n}K(\hbar_2) + \frac{(e - e^{1-\varsigma_n})}{\varsigma_n}K(\hbar_1).$$

On using the differentiability of K on $\aleph$, taking the limit $\varsigma_n \to 0$ as $n \to 0$ on the both sides, we obtain

$$\lim_{\varsigma_n \to 0} \frac{K(\delta_{\hbar_1\hbar_2}(\varsigma_n)) - K(\hbar_1)}{\varsigma_n} \leq \lim_{\varsigma_n \to 0} \frac{(e^{\varsigma_n} - 1)}{\varsigma_n}K(\hbar_2) + \lim_{\varsigma \to 0} \frac{(e - e^{1-\varsigma_n})}{\varsigma_n}K(\hbar_1).$$

Taking into account that

$$\lim_{\varsigma_n \to 0} \frac{K(\delta_{\hbar_1\hbar_2}(\varsigma_n)) - K(\hbar_1)}{\varsigma_n} = \frac{d}{d\varsigma_n}K(\delta_{\hbar_1\hbar_2}(\varsigma_n))|_{\varsigma_n=0} = \dot{\delta}_{\hbar_1\hbar_2}(0)(K) = \dot{\delta}_{\hbar_1\hbar_2}(K)(\hbar_1)$$

we obtain the following relation

$$\dot{\delta}_{\hbar_1\hbar_2}(K)(\hbar_1) \leq K(\hbar_2) + eK(\hbar_1).$$

□

Theorem 3.17. *Assume that $\aleph$ is a Riemannian manifold and $\mho$ is a totally convex open subset of $\aleph$, and $K : \mho \to \mathbf{R}$ is a non-negative special type convex function. If $\bar{h} \in \mho$ is a local minimum of the problem*

$$\begin{aligned} &\textit{minimize } K(h) \\ &\textit{subject to } h \in \mho, \end{aligned} \tag{3.8}$$

then $\bar{h}$ is a global minimum of the problem (3.8).

Proof. Let $\bar{\hbar_1} \in \mho$ be a local minimum, then $\exists$ a neighborhood $S_\epsilon(\bar{\hbar_1})$ s.t.

$$K(\bar{\hbar_1}) \leq K(\hbar_1), \;\; \forall\, \hbar_1 \in \mho \cap S_\epsilon(\bar{\hbar_1}). \tag{3.9}$$

Contrarily, assume that $\bar{\hbar_1}$ is not global minimum of K, then $\exists$ a different point $\hbar_2 \in \mho$ s.t.

$$K(\hbar_2) < K(\bar{\hbar_1}). \tag{3.10}$$

Since $\mho$ is a totally convex, so we have $\delta_{\hbar_2\bar{\hbar_1}}(0) = \hbar_2, \; \delta_{\hbar_2\bar{\hbar_1}}(1) = \bar{\hbar_1}, \; \delta_{\hbar_2\bar{\hbar_1}}(\varsigma) \in \mho, \forall \varsigma \in [0,1]$. Now, we choose $\epsilon > 0$ as $d(\delta_{\hbar_2\bar{\hbar_1}}(\varsigma), \bar{\hbar_1}) < \epsilon$, then $\delta_{\hbar_2\bar{\hbar_1}}(\varsigma) \in S_\epsilon(\bar{\hbar_1})$. By the special type convex function of K, using (3.10) and the fact K is non-negative, we get

$$\begin{aligned}
K(\delta_{\hbar_2\bar{\hbar_1}}(\varsigma)) &\leq (e^\varsigma - 1)K(\bar{\hbar_1}) + (1 + e - e^{1-\varsigma})K(\hbar_2) \\
&< (e^\varsigma - 1)K(\bar{\hbar_1}) + (1 + e - e^{1-\varsigma})K(\bar{\hbar_1}) \\
&= (e + e^\varsigma - e^{1-\varsigma})K(\bar{\hbar_1}) \\
&< (e + e^\varsigma)K(\bar{\hbar_1}), \;\; \forall\, \varsigma \in [0,1].
\end{aligned}$$

So, $\forall\, \delta_{\hbar_2\bar{\hbar_1}}(\varsigma) \in \mho \cap S_\epsilon(\bar{\hbar_1}), \; \dfrac{1}{(e + e^\varsigma)} K(\delta_{\hbar_2\bar{\hbar_1}})(\varsigma) < K(\bar{\hbar_1})$, we find a contradiction to (3.10). Hence, our results is proved. □

4. Conclusions

In this paper, we have introduced the idea of special type convex functions on $\mathbf{R}^n$, which we have obtained after the modification in already established definition of exponential type convex functions. We investigate these functions on Riemannian manifolds. We have explored that how such functions can be applied on Riemannian manifolds, constructed examples, and come up with some interesting results. The application of these special type convex functions in solving an optimization problem is also given in the paper.

Acknowledgments

The authors would like to thank the Deanship of Scientific Research at Umm Al-Qura University for supporting this work by Grant Code: 23UQU4330007DSR001.

Conflict of interest

The authors declare no conflict of interest. The funders had no role in the design of the study; in the collection, analyses or interpretation of data; in the writing of the manuscript, or in the decision to publish the results.

References

1. T. Rapcsák, *Smooth nonlinear optimization in R^n*, Dordrecht: Springer Science & Business Media, 2013.
2. C. Udriste, *Convex functions and optimization methods on Riemannian manifolds*, Dordrecht: Springer Science & Business Media, 2013.
3. R. Pini, Convexity along curves and invexity, *Optimization*, **29** (1994), 301–309. https://doi.org/10.1080/02331939408843959
4. S. Mititelu, Generalized invexity and vector optimization on differential manifolds, *Differ. Geom. Dyn. Syst.*, **3** (2001), 21–31.
5. A. Barani, M. R. Pouryayevali, Invex sets and preinvex functions on Riemannian manifolds, *J. Math. Anal. Appl.*, **328** (2007), 767–779. https://doi.org/10.1016/j.jmaa.2006.05.081
6. I. Ahmad, A. Iqbal, S. Ali, On Properties of Geodesic η-Preinvex Functions, *Adv. Oper. Res.*, **2009** (2009), 381831. https://doi.org/10.1155/2009/381831
7. L. W. Zhou, N. J. Huang, Roughly geodesic B-invex and optimization problem on Hadamard manifold, *Taiwanese J. Math.*, **17** (2013), 833–855. https://doi.org/10.11650/tjm.17.2013.1937
8. S. L. Chen, N. J. Huang, D. O'Regan, Geodesic B-preinvex functions and multiobjective optimization problems on Riemannian manifolds, *J. Appl. Math.*, **17** (2014). https://doi.org/10.1155/2014/524698
9. R. P. Agarwal, I. Ahmad, A. Iqbal, S. Ali, Geodesic G-invex sets and semistrictly geodesic η-preinvex functions, *Optimization*, **61** (2012), 1169–1174. https://doi.org/10.1080/02331934.2010.544314
10. R. P. Agarwal, I. Ahmad, A. Iqbal, S. Ali, Generalized invex sets and preinvex functions on Riemannian manifolds, *Taiwanese J. Math.*, **16** (2012), 1719–1732. https://doi.org/10.11650/twjm/1500406792
11. M. A. Khan, I. Ahmad, F. R. Al-Solamy, Geodesic r-preinvex functions on Riemannian manifolds, *J. Inequal. Appl.*, **2014** (2014), 144. https://doi.org/10.1186/1029-242X-2014-144
12. A. Kılıçman, W. Saleh, On geodesic strongly E-convex sets and geodesic strongly E-convex functions, *J. Inequal. Appl.*, **2015** (2015), 297. https://doi.org/10.1186/s13660-015-0824-z
13. I. Ahmad, M. A. Khan, A. A. Ishan, Generalized geodesic convexity on Riemannian manifolds, *Mathematics*, **7** (2019), 547. https://doi.org/10.3390/math7060547
14. M. Kadakal, İ. İşcan, Exponential type convexity and some related inequalities, *J. Inequal. Appl.*, **2020** (2020), 82. https://doi.org/10.1186/s13660-020-02349-1