Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(7): 15035–15057.
DOI: 10.3934/math.2023768
Received: 12 January 2023
Revised: 01 April 2023
Accepted: 12 April 2023
Published: 23 April 2023

Research article

Complicate dynamical analysis of a discrete predator-prey model with a prey refuge

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Abstract: In this paper, some complicated dynamic characteristics are formulated for a discrete predator-prey model with a prey refuge. After studying the local dynamical properties about fixed points, our main purpose is to investigate condition(s) for the occurrence of flip and hopf bifurcations, respectively. Further, by the bifurcation theory, we have studied flip bifurcation at boundary fixed point, and flip and hopf bifurcations at interior fixed point of the discrete model. We have also studied chaos by state feedback control strategy. Furthermore, theoretical results are numerically verified. Finally, we have also discussed the influence of prey refuge in the discrete model.

Keywords: chaos; discrete model; bifurcations; refuge; numerical simulation
Mathematics Subject Classification: 70K50,92D25, 40A05

1. Introduction

The dynamic relationship among predators and prey has for some time been and will keep on being one of the predominant subjects in both mathematical ecology and ecology because of its widespread presence and significance [1]. There are a lot of research articles about the dynamic behavior of predator-prey system without and with various types of functional responses. It is important to note that outcomes of hiding behavior of prey on the dynamics of prey-predator interactions can be predictably significant. Although the effects of prey refuges on population dynamics are actually very complex, they can be divided into two categories for modelling purposes [2]: the first effect, which affects prey growth positively and predator growth negatively, is the reduction of prey mortality as a result of decreased predation success. The second may be the exchange and spin-off of hiding behavior of prey, which may or may not be beneficial for all interacting populations. It is anticipated that, in recent years, most of the work has been done on dynamical characteristics of continuous-time predator-prey system with a prey refuge designated by differential equation without time delay [3, 4]. For instance, Leslie [5, 6] has investigated the following predator-prey model where carrying capacity of the predator's environment is proportional to the number of prey:

$$\begin{cases} \frac{dH}{dt} = (r_1 - a_1 P - b_1 H)\, H, \\ \frac{dP}{dt} = \left(r_2 - a_2 \frac{P}{H}\right) P, \end{cases} \tag{1.1}$$

where H is prey density, P represents predator density, r_1, a_1, b_1, r_2, a_2 are positive constants. Biologically r_2 and r_1 denote intrinsic growth rate of predator and prey, respectively; carrying capacity of prey is $\frac{r_1}{b_1}$ and carrying capacity of predator is $\frac{r_2 H}{a_2}$. Further, Chen et al. [7] have extended the model of Leslie [5, 6] by incorporating a refuge defending mH of the prey, and a resulting continuous-time system designated by differential equations takes the form:

$$\begin{cases} \frac{dH}{dt} = (r_1 - b_1 H)H - a_1(1-m)HP, \\ \frac{dP}{dt} = \left(r_2 - a_2 \frac{P}{(1-m)H}\right) P, \end{cases} \tag{1.2}$$

while $m \in [0, 1)$. This leaves $(1 - m)H$ of the prey available to the predator. In contrast to continuous-time models, discrete-time models designated by maps or different equations are more applicable than differential models, if populations have non-overlapping generations, and these results also provide more efficient computational results for numerical simulations [8, 9]. So, for non-overlapping generations, many mathematicians have investigated the dynamical behavior of discrete-time models as compared to the continuous-time model (see [10–22]). For instance, Zhuang and Wen [23] have studied the local dynamics of the following model which is a discrete form of (1.2) by forward Euler's formula:

$$\begin{cases} H_{t+1} = H_t + hH_t\left(r_1 - b_1 H_t - a_1(1-m)P_t\right), \\ P_{t+1} = P_t + h\left(r_2 - \frac{a_2 P_t}{(1-m)H_t}\right) P_t, \end{cases} \tag{1.3}$$

where h is the step size. More precisely, Zhuang and Wen [23] have proved that boundary and interior fixed points of the discrete model (1.3) is a sink, saddle, source and non-hyperbolic under certain parametric condition(s). Motivated by the mentioned studies, the purpose of this paper is to explore further complicate dynamical analysis of the discrete model (1.3), and so our key contributions in this regard include:

(i) Topological classifications at fixed points of the discrete model (1.3).
(ii) Existence of bifurcations sets at fixed points.
(iii) Detail bifurcation analysis at fixed points of the discrete model (1.3).
(iv) Study of chaos by state feedback control strategy.
(v) Verification of theoretical results numerically.
(vi) Influence of prey refuge.

The organization of the rest of the paper is as fellows: local stability with topological classifications at fixed points of a discrete model (1.3) is explored in Section 2. In Section 3, we have studied detailed bifurcation analysis at fixed points, whereas Section 4 is about the study of chaos control of the discrete model (1.3). The numerical simulation that verifies the correctness of theoretical results are presented in Section 5. In Section 6, we will discuss the influence of prey refuge, whereas the conclusion is given in Section 7.

2. Local stability of fixed points

In this section, we will analyze the local stability of fixed points of discrete model (1.3). Here, due to biological meaning of discrete model (1.3), first we will pick up nonnegative fixed points. It is clear that if $E_{HP}\,(H, P)$ is a fixed point of discrete model (1.3), then

$$\begin{cases} H = H + hH\,(r_1 - b_1 H - a_1(1-m)P)\,, \\ P = P + h\left(r_2 - \frac{a_2 P}{(1-m)H}\right)P. \end{cases} \tag{2.1}$$

From (2.1), the straightforward calculation yields the following results.

Theorem 2.1. (i) $\forall\ r_1, r_2, a_1, a_2, b_1, m, h > 0$, $E_{H0}(\frac{r_1}{b_1}, 0)$ is a boundary fixed point of discrete model (1.3);

(ii) $E^+_{HP}\left(\frac{r_1 a_2}{a_2 b_1 + a_1 r_2 (1-m)^2}, \frac{r_1 r_2 (1-m)}{a_2 b_1 + a_1 r_2 (1-m)^2}\right)$ is the interior fixed point of discrete model (1.3) if $m < 1$.

Now using linearization, at $E_{HP}\,(H, P)$ following variational matrix $V|_{E_{HP}(H,P)}$ is constructed:

$$V|_{E_{HP}(H,P)} = \begin{pmatrix} 1 - 2hb_1 H + hr_1 - a_1 h(1-m)P & -ha_1(1-m)H \\ \frac{a_2 h P^2}{(1-m)H^2} & 1 + hr_2 - \frac{2a_2 hP}{(1-m)H} \end{pmatrix}. \tag{2.2}$$

Now, we will study local stability at boundary and interior fixed points of model (1.3) by the stability theory [24–27]. It should be noted that at $E_{H0}(\frac{r_1}{b_1}, 0)$, (2.2) gives

$$V|_{E_{H0}\left(\frac{r_1}{b_1},0\right)} = \begin{pmatrix} 1 - hr_1 & \frac{ha_1 r_1 (1-m)}{b_1} \\ 0 & 1 + hr_2 \end{pmatrix}. \tag{2.3}$$

The characteristic roots of $V|_{E_{H0}\left(\frac{r_1}{b_1},0\right)}$ are

$$\lambda_1 = 1 - hr_1,\ \lambda_2 = 1 + hr_2. \tag{2.4}$$

Based on (2.4), one can easily obtain the following results.

Theorem 2.2. (A) $\forall\ r_1, r_2, a_1, a_2, b_1, m, h > 0$, $E_{H0}(\frac{r_1}{b_1}, 0)$ of discrete model (1.3) is never sink;

(B) $E_{H0}(\frac{r_1}{b_1}, 0)$ of discrete model (1.3) is a source if

$$r_1 > \frac{2}{h}; \tag{2.5}$$

(C) $E_{H0}\left(\frac{r_1}{b_1}, 0\right)$ of discrete model (1.3) is a saddle if

$$0 < r_1 < \frac{2}{h}; \tag{2.6}$$

(D) $E_{H0}\left(\frac{r_1}{b_1}, 0\right)$ of discrete model (1.3) is non-hyperbolic if

$$r_1 = \frac{2}{h}. \tag{2.7}$$

Now at $E_{HP}^{+}\left(\frac{r_1a_2}{a_2b_1+a_1r_2(1-m)^2}, \frac{r_1r_2(1-m)}{a_2b_1+a_1r_2(1-m)^2}\right)$, (2.2) gives

$$V|_{E_{HP}^{+}\left(\frac{r_1a_2}{a_2b_1+a_1r_2(1-m)^2}, \frac{r_1r_2(1-m)}{a_2b_1+a_1r_2(1-m)^2}\right)} = \begin{pmatrix} 1-\frac{a_2b_1hr_1}{a_2b_1+a_1r_2(1-m)^2} & \frac{-r_1a_1a_2h(1-m)}{a_2b_1+a_1r_2(1-m)^2} \\ \frac{h(1-m)r_2{}^2}{a_2} & 1-hr_2 \end{pmatrix}. \tag{2.8}$$

The characteristic equation of $V|_{E_{HP}^{+}\left(\frac{r_1a_2}{a_2b_1+a_1r_2(1-m)^2}, \frac{r_1r_2(1-m)}{a_2b_1+a_1r_2(1-m)^2}\right)}$ is

$$\lambda^2 - p\lambda + q = 0, \tag{2.9}$$

with

$$\begin{cases} p = 2 - hr_2 - \frac{a_2b_1hr_1}{a_2b_1+a_1r_2(1-m)^2}, \\ q = 1 + hr_2(hr_1 - 1) - \frac{a_2b_1hr_1}{a_2b_1+a_1r_2(1-m)^2}. \end{cases} \tag{2.10}$$

Finally, roots of (2.9) are

$$\lambda_{1,2} = \frac{p \pm \sqrt{\Delta}}{2}, \tag{2.11}$$

where

$$\begin{aligned} \Delta =& p^2 - 4q, \\ =& \left(2 - hr_2 - \frac{a_2b_1hr_1}{a_2b_1 + a_1r_2(1-m)^2}\right)^2 - 4\left(1 + hr_2(hr_1 - 1) - \frac{a_2b_1hr_1}{a_2b_1 + a_1r_2(1-m)^2}\right). \end{aligned} \tag{2.12}$$

Now, following two theorems are established based on sign of Δ, i.e. $\Delta < 0$ and $\Delta \geq 0$, respectively.

Theorem 2.3. If $\Delta = \left(2 - hr_2 - \frac{a_2b_1hr_1}{a_2b_1+a_1r_2(1-m)^2}\right)^2 - 4\left(1 + hr_2(hr_1 - 1) - \frac{a_2b_1hr_1}{a_2b_1+a_1r_2(1-m)^2}\right) < 0$ then at $E_{HP}^{+}\left(\frac{r_1a_2}{a_2b_1+a_1r_2(1-m)^2}, \frac{r_1r_2(1-m)}{a_2b_1+a_1r_2(1-m)^2}\right)$ following results hold:

(A) $E_{HP}^{+}\left(\frac{r_1a_2}{a_2b_1+a_1r_2(1-m)^2}, \frac{r_1r_2(1-m)}{a_2b_1+a_1r_2(1-m)^2}\right)$ of discrete model (1.3) is a stable focus if

$$0 < r_1 < \frac{r_2a_2b_1 + r_2{}^2a_1(1-m)^2}{ha_2b_1r_2 - a_2b_1 + ha_1r_2{}^2(1-m)^2}, \tag{2.13}$$

with

$$h > \frac{a_2b_1}{r_2\left(a_2b_1 + a_1r_2(1-m)^2\right)}; \tag{2.14}$$

(B) $E_{HP}^{+}\left(\frac{r_1a_2}{a_2b_1+a_1r_2(1-m)^2}, \frac{r_1r_2(1-m)}{a_2b_1+a_1r_2(1-m)^2}\right)$ of discrete model (1.3) is **an** unstable focus if

$$r_1 > \frac{r_2a_2b_1 + r_2{}^2a_1(1-m)^2}{ha_2b_1r_2 - a_2b_1 + ha_1r_2{}^2(1-m)^2}; \tag{2.15}$$

(C) $E_{HP}^{+}\left(\frac{r_1a_2}{a_2b_1+a_1r_2(1-m)^2}, \frac{r_1r_2(1-m)}{a_2b_1+a_1r_2(1-m)^2}\right)$ of discrete model (1.3) is non-hyperbolic if

$$r_1 = \frac{r_2a_2b_1 + r_2{}^2a_1(1-m)^2}{ha_2b_1r_2 - a_2b_1 + ha_1r_2{}^2(1-m)^2}. \tag{2.16}$$

Theorem 2.4. If $\Delta = \left(2 - hr_2 - \frac{a_2b_1hr_1}{a_2b_1+a_1r_2(1-m)^2}\right)^2 - 4\left(1 + hr_2(hr_1 - 1) - \frac{a_2b_1hr_1}{a_2b_1+a_1r_2(1-m)^2}\right) \geq 0$ then at $E^+_{HP}\left(\frac{r_1a_2}{a_2b_1+a_1r_2(1-m)^2}, \frac{r_1r_2(1-m)}{a_2b_1+a_1r_2(1-m)^2}\right)$ following results hold:

(A) $E^+_{HP}\left(\frac{r_1a_2}{a_2b_1+a_1r_2(1-m)^2}, \frac{r_1r_2(1-m)}{a_2b_1+a_1r_2(1-m)^2}\right)$ of discrete model (1.3) is a stable node if

$$r_1 < \frac{2(hr_2 - 2)\left(a_2b_1 + a_1r_2(1 - m)^2\right)}{h^2r_2\left(a_2b_1 + a_1r_2(1 - m)^2\right) - 2a_2b_1h}, \tag{2.17}$$

with

$$h > \max\left\{\frac{2}{r_2}, \frac{2a_2b_1}{r_2\left(a_2b_1 + a_1r_2(1 - m)^2\right)}\right\}; \tag{2.18}$$

(B) $E^+_{HP}\left(\frac{r_1a_2}{a_2b_1+a_1r_2(1-m)^2}, \frac{r_1r_2(1-m)}{a_2b_1+a_1r_2(1-m)^2}\right)$ of discrete model (1.3) is **an** unstable node if

$$r_1 > \frac{2(hr_2 - 2)\left(a_2b_1 + a_1r_2(1 - m)^2\right)}{h^2r_2\left(a_2b_1 + a_1r_2(1 - m)^2\right) - 2a_2b_1h}; \tag{2.19}$$

(C) $E^+_{HP}\left(\frac{r_1a_2}{a_2b_1+a_1r_2(1-m)^2}, \frac{r_1r_2(1-m)}{a_2b_1+a_1r_2(1-m)^2}\right)$ of discrete model (1.3) is non-hyperbolic if

$$r_1 = \frac{2(hr_2 - 2)\left(a_2b_1 + a_1r_2(1 - m)^2\right)}{h^2r_2\left(a_2b_1 + a_1r_2(1 - m)^2\right) - 2a_2b_1h}. \tag{2.20}$$

3. Analysis of bifurcation

In this section, we study the flip bifurcation at boundary fixed point $E_{H0}(\frac{r_1}{b_1}, 0)$, and flip and hopf bifurcations at $E^+_{HP}\left(\frac{r_1a_2}{a_2b_1+a_1r_2(1-m)^2}, \frac{r_1r_2(1-m)}{a_2b_1+a_1r_2(1-m)^2}\right)$ of discrete model (1.3) by center manifold theorem and bifurcation theory [28–32]. If condition (2.7) of Theorem 2.2 holds, then from (2.4) one gets $\lambda_1|_{(2.7)} = -1$ but $\lambda_2|_{(2.7)} = 1 + hr_2 \neq 1$ or -1, which implies that at $E_{H0}(\frac{r_1}{b_1}, 0)$ discrete model (1.3) may undergoes flip bifurcation if $(h, r_1, r_2, b_1, a_1, a_2, m)$ located in the set:

$$\mathcal{F}|_{E_{H0}(\frac{r_1}{b_1},0)} = \left\{(h, r_1, r_2, b_1, a_1, a_2, m),\ r_1 = \frac{2}{h}\right\}. \tag{3.1}$$

But following this theorem, it follows that if $(h, r_1, r_2, b_1, a_1, a_2, m) \in \mathcal{F}|_{E_{H0}\left(\frac{r_1}{b_1},0\right)}$ then discrete model (1.3) must undergo flip bifurcation.

Theorem 3.1. Discrete model (1.3) undergoes flip bifurcation if $(h, r_1, r_2, b_1, a_1, a_2, m) \in \mathcal{F}|_{E_{H0}\left(\frac{r_1}{b_1},0\right)}$.

Proof. Since, with respect to $P = 0$, discrete model (1.3) is invariant. So, if it is restricted on $P = 0$ then

$$H_{t+1} = H_t + hH_t(r_1 - b_1H_t). \tag{3.2}$$

Now from (3.2), we denote

$$f(r_1, H) := H + hH(r_1 - b_1H). \tag{3.3}$$

Now if $r_1 = r_1{}^* = \frac{2}{h}$ and $H = H^* = \frac{r_1}{b_1}$ then from (3.3) one gets

$$\left.\frac{\partial f}{\partial H}\right|_{r_1=r_1{}^*=\frac{2}{h},\ H=H^*=\frac{r_1}{b_1}} := -1, \tag{3.4}$$

$$\left.\frac{\partial^2 f}{\partial H^2}\right|_{r_1=r_1{}^*=\frac{2}{h},\ H=H^*=\frac{r_1}{b_1}} := -2hb_1 \neq 0, \tag{3.5}$$

and

$$\left.\frac{\partial f}{\partial r_1}\right|_{r_1=r_1{}^*=\frac{2}{h},\ H=H^*=\frac{r_1}{b_1}} := \frac{2}{b_1} \neq 0. \tag{3.6}$$

From (3.4)–(3.6), and Table 4.4.1 of [30] one can concluded that if $(h, r_1, r_2, b_1, a_1, a_2, m) \in \mathcal{F}|_{E_{H0}\left(\frac{r_1}{b_1},0\right)}$ then flip bifurcation must exist at $E_{H0}(\frac{r_1}{b_1}, 0)$. Additionally $\Omega_1 = \left.\left(\frac{\partial^2 f}{\partial H^2}\right)\left(\frac{\partial f}{\partial r_1}\right)+2\left(\frac{\partial^2 f}{\partial H \partial r_1}\right)\right|_{r_1=r_1{}^*=\frac{2}{h},\ H=H^*=\frac{r_1}{b_1}} = -2h$, $\Omega_2 = \left(\frac{\partial^3 f}{\partial H^3}\right)+3\left(\left(\frac{\partial^2 f}{\partial H^2}\right)\right)^2 = 12h^2b_1^2$ and finally $\Omega_1\Omega_2 = -24b_1^2h^3 < 0$ which implies that at $E_{H0}(\frac{r_1}{b_1}, 0)$ discrete model (1.3) undergoes supercritical flip bifurcation. □

Now if $\Delta = \left(2 - hr_2 - \frac{a_2b_1hr_1}{a_2b_1+a_1r_2(1-m)^2}\right)^2 - 4\left(1 + hr_2(hr_1 - 1) - \frac{a_2b_1hr_1}{a_2b_1+a_1r_2(1-m)^2}\right) < 0$ then roots of (2.11) are pair of complex conjugate satisfying $\left|\lambda_{1,2}\right|_{(2.16)} = 1$ which implies that hopf bifurcation may exists if $(h, r_1, r_2, b_1, a_1, a_2, m)$ are locate in the set:

$$\mathcal{N}|_{E^+_{HP}\left(\frac{r_1a_2}{a_2b_1+a_1r_2(1-m)^2},\frac{r_1r_2(1-m)}{a_2b_1+a_1r_2(1-m)^2}\right)} = \left\{(h, r_1, r_2, b_1, a_1, a_2, m) : \Delta < 0 \text{ and } r_1 = \frac{r_2a_2b_1 + r_2{}^2a_1(1-m)^2}{ha_2b_1r_2 - a_2b_1 + ha_1r_2{}^2(1-m)^2}\right\}. \tag{3.7}$$

But following result follows that if $(h, r_1, r_2, b_1, a_1, a_2, m) \in \mathcal{N}|_{E^+_{HP}\left(\frac{r_1a_2}{a_2b_1+a_1r_2(1-m)^2},\frac{r_1r_2(1-m)}{a_2b_1+a_1r_2(1-m)^2}\right)}$ then at $E^+_{HP}\left(\frac{r_1a_2}{a_2b_1+a_1r_2(1-m)^2}, \frac{r_1r_2(1-m)}{a_2b_1+a_1r_2(1-m)^2}\right)$ discrete model (1.3) undergoes hopf bifurcation.

Theorem 3.2. If $(h, r_1, r_2, b_1, a_1, a_2, m) \in \mathcal{N}|_{E^+_{HP}\left(\frac{r_1a_2}{a_2b_1+a_1r_2(1-m)^2},\frac{r_1r_2(1-m)}{a_2b_1+a_1r_2(1-m)^2}\right)}$ then at interior equilibrium $E^+_{HP}\left(\frac{r_1a_2}{a_2b_1+a_1r_2(1-m)^2}, \frac{r_1r_2(1-m)}{a_2b_1+a_1r_2(1-m)^2}\right)$, discrete model (1.3) undergo hopf bifurcation.

Proof. Since $(h, r_1, r_2, b_1, a_1, a_2, m) \in \mathcal{N}|_{E^+_{HP}\left(\frac{r_1a_2}{a_2b_1+a_1r_2(1-m)^2},\frac{r_1r_2(1-m)}{a_2b_1+a_1r_2(1-m)^2}\right)}$ and clearly r_1 is the bifurcation parameter. So, if r_1 varies in a small neighborhood of r_1^*, *i.e*, $r_1 = r_1^* + \epsilon$ where $\epsilon << 1$ then model (1.3) becomes

$$\begin{cases} H_{t+1} = H_t + hH_t\left((r_1{}^* + \epsilon) - b_1H_t - a_1(1-m)P_t\right), \\ P_{t+1} = P_t + h\left(r_2 - \frac{a_2P_t}{(1-m)H_t}\right)P_t, \end{cases} \tag{3.8}$$

with $E^+_{HP}\left(\frac{(r_1^*+\epsilon)a_2}{a_2b_1+a_1r_2(1-m)^2}, \frac{(r_1^*+\epsilon)r_2(1-m)}{a_2b_1+a_1r_2(1-m)^2}\right)$ is the interior fixed point. Now the roots of characteristic equation of $V|_{E^+_{HP}\left(\frac{(r_1^*+\epsilon)a_2}{a_2b_1+a_1r_2(1-m)^2},\frac{(r_1^*+\epsilon)r_2(1-m)}{a_2b_1+a_1r_2(1-m)^2}\right)}$ at $E^+_{HP}\left(\frac{(r_1^*+\epsilon)a_2}{a_2b_1+a_1r_2(1-m)^2}, \frac{(r_1^*+\epsilon)r_2(1-m)}{a_2b_1+a_1r_2(1-m)^2}\right)$ of model (3.8) is

$$\lambda_{1,2} = \frac{p(\epsilon) \pm \iota\sqrt{4q(\epsilon) - p^2(\epsilon)}}{2}, \tag{3.9}$$

where

$$\begin{cases} p(\epsilon) = 2 - hr_2 - \frac{ha_2b_1(r_1^*+\epsilon)}{a_2b_1+a_1r_2(1-m)^2}, \\ q(\epsilon) = 1 + hr_2\left(h(r_1^*+\epsilon)-1\right) - \frac{a_2b_1h(r_1^*+\epsilon)}{a_2b_1+a_1r_2(1-m)^2}. \end{cases} \tag{3.10}$$

From (3.9) and (3.10) we have

$$|\lambda_{1,2}| = \sqrt{1 + hr_2\left(h(r_1^*+\epsilon)-1\right) - \frac{a_2b_1h(r_1^*+\epsilon)}{a_2b_1+a_1r_2(1-m)^2}}, \tag{3.11}$$

with $\frac{d|\lambda_{1,2}|}{d\epsilon}|_{\epsilon=0} = \frac{1}{2}\left(h^2r_2 - \frac{a_2b_1h}{a_2b_1+a_1r_2(1-m)^2}\right) \neq 0$. Moreover, for occurrence of hopf bifurcation at $E^+_{HP}\left(\frac{r_1a_2}{a_2b_1+a_1r_2(1-m)^2}, \frac{r_1r_2(1-m)}{a_2b_1+a_1r_2(1-m)^2}\right)$ it is also require that $\lambda^v_{1,2} \neq 1,\ v = 1,\cdots,4$ if $\epsilon = 0$ that corresponds to $p(0) \neq -2, 0, 1, 2$. But $q(0) = 1$ if (2.16) holds, and so $p(0) \neq -2, 2$. Hence $p(0) \neq 0, 1$ which gives

$$h \neq \frac{2}{r_2}, \frac{2a_2b_1 + 2a_1r_2(1-m)^2}{r_2a_2b_1 + a_1r_2{}^2(1-m)^2 + a_2b_1}, \frac{a_2b_1 + a_1r_2(1-m)^2}{r_2a_2b_1 + a_1r_2{}^2(1-m)^2 + a_2b_1r_1}. \tag{3.12}$$

Now using the following transformations in order to transform $E^+_{HP}\left(\frac{r_1a_2}{a_2b_1+a_1r_2(1-m)^2}, \frac{r_1r_2(1-m)}{a_2b_1+a_1r_2(1-m)^2}\right)$ of (3.8) to origin:

$$\begin{cases} U_t = H_t - H^*, \\ V_t = P_t - P^*, \end{cases} \tag{3.13}$$

with $H^* = \frac{r_1a_2}{a_2b_1+a_1r_2(1-m)^2}$ *and* $P^* = \frac{r_1r_2(1-m)}{a_2b_1+a_1r_2(1-m)^2}$. From (3.13) and (3.8) one writes

$$\begin{cases} U_{t+1} = U_t + h(U_t+H^*)\left(r_1 - b_1(U_t+H^*) - a_1(1-m)(V_t+P^*)\right), \\ V_{t+1} = V_t + h(V_t+P^*)\left(r_2 - \frac{V_t+P^*}{(1-m)(U_t+H^*)}\right). \end{cases} \tag{3.14}$$

Now if $\epsilon = 0$ then we will explore normal form of (3.14). On expanding (3.14) up to order-2^{nd} at $F_{00}(0,0)$ we write

$$\begin{cases} U_{t+1} = \alpha_{11}U_t + \alpha_{12}V_t + \alpha_{13}U_t^2 + \alpha_{14}U_tV_t, \\ V_{t+1} = \alpha_{21}U_t + \alpha_{22}V_t + \alpha_{23}U_t^2 + \alpha_{24}U_tV_t + \alpha_{25}V_t^2, \end{cases} \tag{3.15}$$

with

$$\begin{cases} \alpha_{11} = 1 - 2hb_1H^* + hr_1 - a_1h(1-m)P^*,\ \alpha_{12} = -a_1(1-m)hH^*, \\ \alpha_{13} = -hb_1,\ \alpha_{14} = -a_1h(1-m),\ \alpha_{21} = \frac{ha_2P^{*2}}{(1-m)H^{*2}}, \\ \alpha_{22} = 1 + hr_2 - \frac{2ha_2P^*}{(1-m)H^*},\ \alpha_{23} = -\frac{ha_2P^{*2}}{(1-m)H^{*3}},\ \alpha_{24} = \frac{2ha_2P^*}{(1-m)H^{*3}}, \\ \alpha_{25} = -\frac{ha_2}{(1-m)H^*}. \end{cases} \tag{3.16}$$

Now using the following transformation, we transform linear part of (3.15) to canonical form

$$\begin{pmatrix} U_t \\ V_t \end{pmatrix} := \begin{pmatrix} \alpha_{12} & 0 \\ \eta - \alpha_{11} & -\zeta \end{pmatrix} \begin{pmatrix} H_t \\ P_t \end{pmatrix}, \tag{3.17}$$

with

$$\begin{cases} \eta = \frac{1}{2}\left(2 - hr_2 - \frac{a_2 b_1 h r_1}{a_2 b_1 + a_1 r_2 (1-m)^2}\right), \\ \zeta = \frac{h}{2}\sqrt{-{r_2}^2 - \frac{{a_2}^2 {b_1}^2 {r_1}^2}{\left[a_2 b_1 + a_1 r_2 (1-m)^2\right]^2} + 2 r_1 r_2 \left(2 - \frac{a_2 b_1}{a_2 b_1 + a_1 r_2 (1-m)^2}\right)}. \end{cases} \tag{3.18}$$

From (3.17) and (3.15) we write

$$\begin{cases} H_{t+1} = \eta H_t - \zeta P_t + \bar{F}(H_t, P_t), \\ P_{t+1} = \zeta H_t + \eta P_t + \bar{G}(H_t, P_t), \end{cases} \tag{3.19}$$

where

$$\begin{cases} \bar{F}(H_t, P_t) = r_{11} H_t^2 + r_{12} H_t P_t, \\ \bar{G}(H_t, P_t) = r_{21} H_t^2 + r_{22} H_t P_t + r_{23} P_t^2, \end{cases} \tag{3.20}$$

and

$$\begin{cases} r_{11} = \alpha_{12}\alpha_{13} + \alpha_{14}(\eta - \alpha_{11}),\ r_{12} = -\alpha_{14}\zeta, \\ r_{21} = \frac{1}{\zeta}\left[(\zeta - \alpha_{11})(\alpha_{12}\alpha_{13} - \alpha_{12}\alpha_{24}) - {\alpha_{12}}^2\alpha_{23} + (\alpha_{14} - \alpha_{25})(\eta - \alpha_{11})^2\right], \\ r_{22} = \alpha_{12}\alpha_{24} - (\eta - \alpha_{11})(\alpha_{14} - 2\alpha_{25}),\ r_{23} = -\alpha_{25}\zeta. \end{cases} \tag{3.21}$$

From (3.20) we have

$$\begin{cases} \frac{\partial^2 \bar{F}}{\partial H_t^2}\Big|_{F_{00}(0,0)} = 2r_{11},\ \frac{\partial^2 \bar{F}}{\partial H_t \partial P_t}\Big|_{F_{00}(0,0)} = r_{12},\ \frac{\partial^2 \bar{F}}{\partial P_t^2}\Big|_{F_{00}(0,0)} = 0, \\ \frac{\partial^3 \bar{F}}{\partial H_t^3}\Big|_{F_{00}(0,0)} = 0,\ \frac{\partial^3 \bar{F}}{\partial H_t^2 \partial P_t}\Big|_{F_{00}(0,0)} = 0,\ \frac{\partial^3 \bar{F}}{\partial H_t \partial P_t^2}\Big|_{F_{00}(0,0)} = 0, \\ \frac{\partial^3 \bar{F}}{\partial P_t^3}\Big|_{F_{00}(0,0)} = 0,\ \frac{\partial^2 \bar{G}}{\partial H_t^2}\Big|_{F_{00}(0,0)} = 2r_{21},\ \frac{\partial^2 \bar{G}}{\partial H_t \partial P_t}\Big|_{F_{00}(0,0)} = r_{22}, \\ \frac{\partial^2 \bar{G}}{\partial P_t^2}\Big|_{F_{00}(0,0)} = 2r_{23},\ \frac{\partial^3 \bar{G}}{\partial H_t^3}\Big|_{F_{00}(0,0)} = 0,\ \frac{\partial^3 \bar{G}}{\partial H_t^2 \partial P_t}\Big|_{F_{00}(0,0)} = 0, \\ \frac{\partial^3 \bar{G}}{\partial H_t \partial P_t^2}\Big|_{F_{00}(0,0)} = 0,\ \frac{\partial^3 \bar{G}}{\partial P_t^3}\Big|_{F_{00}(0,0)} = 0. \end{cases} \tag{3.22}$$

Finally, following condition required to be non-zero for (3.19) undergo hopf bifurcation

$$\ell = -\Re\left(\frac{(1-2\lambda)\bar{\lambda}^2}{1-\lambda}\tau_{11}\tau_{20}\right) - \frac{1}{2}|\tau_{11}|^2 - |\tau_{02}|^2 + \Re\left(\bar{\lambda}\tau_{21}\right), \tag{3.23}$$

where

$$\begin{cases}\tau_{02} = \frac{1}{8}\left(\frac{\partial^2 \bar{F}}{\partial H_t^2} - \frac{\partial^2 \bar{F}}{\partial P_t^2} - 2\frac{\partial^2 \bar{G}}{\partial H_t \partial P_t} + \iota\left(\frac{\partial^2 \bar{G}}{\partial H_t^2} - \frac{\partial^2 \bar{G}}{\partial P_t^2} + 2\frac{\partial^2 \bar{F}}{\partial H_t \partial P_t}\right)\right)\Big|_{F_{00}(0,0)}, \\ \tau_{11} = \frac{1}{4}\left(\frac{\partial^2 \bar{F}}{\partial H_t^2} + \frac{\partial^2 \bar{F}}{\partial P_t^2} + \iota\left(\frac{\partial^2 \bar{G}}{\partial H_t^2} + \frac{\partial^2 \bar{G}}{\partial P_t^2}\right)\right)\Big|_{F_{00}(0,0)}, \\ \tau_{20} = \frac{1}{8}\left(\frac{\partial^2 \bar{F}}{\partial H_t^2} - \frac{\partial^2 \bar{F}}{\partial P_t^2} + 2\frac{\partial^2 \bar{G}}{\partial H_t \partial P_t} + \iota\left(\frac{\partial^2 \bar{G}}{\partial H_t^2} - \frac{\partial^2 \bar{G}}{\partial P_t^2} - 2\frac{\partial^2 \bar{F}}{\partial H_t \partial P_t}\right)\right)\Big|_{F_{00}(0,0)}, \\ \tau_{21} = \frac{1}{16}\left(\frac{\partial^3 \bar{F}}{\partial H_t^3} + \frac{\partial^3 \bar{F}}{\partial P_t^3} + \frac{\partial^3 \bar{G}}{\partial H_t^2 \partial P_t} + \frac{\partial^3 \bar{G}}{\partial P_t^3} \right. \\ \qquad \left. + \iota\left(\frac{\partial^3 \bar{G}}{\partial H_t^3} + \frac{\partial^3 \bar{G}}{\partial H_t \partial P_t^2} - \frac{\partial^3 \bar{F}}{\partial H_t^2 \partial P_t} - \frac{\partial^3 \bar{F}}{\partial P_t^3}\right)\right)\Big|_{F_{00}(0,0)}. \end{cases} \tag{3.24}$$

The calculation yields

$$\begin{cases}\tau_{02} = \frac{1}{4}\left[r_{11} - r_{22} + \iota(r_{21} - r_{23} + r_{12})\right], \\ \tau_{11} = \frac{1}{2}\left[r_{11} + \iota(r_{21} + r_{23})\right], \\ \tau_{20} = \frac{1}{4}\left[r_{11} + r_{22} + \iota(r_{21} - r_{23} - r_{12})\right],\ \tau_{21} = 0. \end{cases} \tag{3.25}$$

From (3.25) and (3.23) if we get $\ell \neq 0$ as $(h, r_1, r_2, b_1, a_1, a_2, m) \in \mathcal{N}|_{E^+_{HP}\left(\frac{r_1 a_2}{a_2 b_1 + a_1 r_2(1-m)^2}, \frac{r_1 r_2(1-m)}{a_2 b_1 + a_1 r_2(1-m)^2}\right)}$ then at $E^+_{HP}\left(\frac{r_1 a_2}{a_2 b_1 + a_1 r_2(1-m)^2}, \frac{r_1 r_2(1-m)}{a_2 b_1 + a_1 r_2(1-m)^2}\right)$ discrete model (1.3) undergoes hopf bifurcation. Further supercritical (respectively subcritical) hopf bifurcation take place if $\ell < 0$ (respectively $\ell > 0$). □

Now if $\Delta = \left(2 - hr_2 - \frac{a_2 b_1 h r_1}{a_2 b_1 + a_1 r_2(1-m)^2}\right)^2 - 4\left(1 + hr_2(hr_1 - 1) - \frac{a_2 b_1 h r_1}{a_2 b_1 + a_1 r_2(1-m)^2}\right) > 0$ and condition (2.20) of Theorem 2.4 holds then $\lambda_1|_{(2.20)} = -1$ but $\lambda_2|_{(2.20)} = 3 - hr_2 - \frac{2a_2 b_1 (hr_2 - 2)}{a_1 h(1-m)^2 r_2{}^2 + a_2 b_1 (hr_2 - 2)} \neq 1$ or -1 that concludes the fact that at $E^+_{HP}\left(\frac{r_1 a_2}{a_2 b_1 + a_1 r_2(1-m)^2}, \frac{r_1 r_2(1-m)}{a_2 b_1 + a_1 r_2(1-m)^2}\right)$ discrete model (1.3) may undergoes flip bifurcation if $(h, r_1, r_2, b_1, a_1, a_2, m)$ are located in the set:

$$\mathcal{F}|_{E^+_{HP}\left(\frac{r_1 a_2}{a_2 b_1 + a_1 r_2(1-m)^2}, \frac{r_1 r_2(1-m)}{a_2 b_1 + a_1 r_2(1-m)^2}\right)} = \Bigg\{(h, r_1, r_2, b_1, a_1, a_2, m) : \Delta > 0 \text{ and } r_1 = \frac{2(hr_2 - 2)\left(a_2 b_1 + a_1 r_2(1-m)^2\right)}{h^2 r_2\left(a_2 b_1 + a_1 r_2(1-m)^2\right) - 2a_2 b_1 h}\Bigg\}. \tag{3.26}$$

But following result follows that if $(h, r_1, r_2, b_1, a_1, a_2, m) \in \mathcal{F}|_{E^+_{HP}\left(\frac{r_1 a_2}{a_2 b_1 + a_1 r_2(1-m)^2}, \frac{r_1 r_2(1-m)}{a_2 b_1 + a_1 r_2(1-m)^2}\right)}$ then at $E^+_{HP}\left(\frac{r_1 a_2}{a_2 b_1 + a_1 r_2(1-m)^2}, \frac{r_1 r_2(1-m)}{a_2 b_1 + a_1 r_2(1-m)^2}\right)$ discrete model (1.3) undergoes flip bifurcation.

Theorem 3.3. If $(h, r_1, r_2, b_1, a_1, a_2, m) \in \mathcal{F}|_{E^+_{HP}\left(\frac{r_1 a_2}{a_2 b_1 + a_1 r_2(1-m)^2}, \frac{r_1 r_2(1-m)}{a_2 b_1 + a_1 r_2(1-m)^2}\right)}$ then discrete model (1.3) undergoes flip bifurcation.

Proof. If r_1 varies in a small neighborhood of r_1^* then model (1.3) takes the form (3.8) which further transform into the following form

$$\begin{cases} U_{t+1} = \alpha_{11} U_t + \alpha_{12} V_t + \alpha_{13} U_t^2 + \alpha_{14} U_t V_t + K_{01} U_t \epsilon, \\ V_{t+1} = \alpha_{21} U_t + \alpha_{22} V_t + \alpha_{23} U_t^2 + \alpha_{24} U_t V_t + \alpha_{25} V_t^2, \end{cases} \tag{3.27}$$

where

$$
\begin{cases}
\alpha_{11}=1-2hb_1H^*+hr_1^*-a_1h(1-m)P^*,\\
\alpha_{12}=-a_1h(1-m)H^*,\\
\alpha_{13}=-hb_1,\ \alpha_{14}=-a_1h(1-m),\ K_{11}=h,\\
\alpha_{21}=\frac{ha_2P^{*2}}{(1-m)H^{*2}},\\
\alpha_{22}=1+hr_2-\frac{2ha_2P^*}{(1-m)H^*},\\
\alpha_{23}=-\frac{ha_2P^{*2}}{(1-m)H^{*3}},\\
\alpha_{24}=\frac{2ha_2P^*}{(1-m)H^{*3}},\ \alpha_{25}=-\frac{ha_2}{(1-m)H^*},
\end{cases}
\tag{3.28}
$$

by (3.13). By

$$
\begin{pmatrix} U_t \\ V_t \end{pmatrix} := \begin{pmatrix} \alpha_{12} & \alpha_{12} \\ -1-\alpha_{11} & \lambda_2-\alpha_{11} \end{pmatrix}\begin{pmatrix} H_t \\ P_t \end{pmatrix},
\tag{3.29}
$$

(3.27) becomes

$$
\begin{pmatrix} H_{t+1} \\ P_{t+1} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} H_t \\ P_t \end{pmatrix} + \begin{pmatrix} \widehat{F}(U_t,V_t,\epsilon) \\ \widehat{G}(U_t,V_t,\epsilon) \end{pmatrix},
\tag{3.30}
$$

where

$$
\begin{cases}
\widehat{F}=\frac{\alpha_{13}(\lambda_2-\alpha_{11})-\alpha_{12}\alpha_{23}}{\alpha_{12}(1+\lambda_2)}U_t^2+\frac{\alpha_{14}(\lambda_2-\alpha_{11})-\alpha_{12}\alpha_{24}}{\alpha_{12}(1+\lambda_2)}U_tV_t+\frac{K_{11}(\lambda_2-\alpha_{11})}{\alpha_{12}(1+\lambda_2)}\epsilon U_t-\frac{\alpha_{25}}{1+\lambda_2}V_t{}^2,\\
\widehat{G}=\frac{\alpha_{13}(1+\alpha_{11})+\alpha_{12}\alpha_{23}}{\alpha_{12}(1+\lambda_2)}U_t^2+\frac{\alpha_{14}(1+\alpha_{11})+\alpha_{12}\alpha_{24}}{\alpha_{12}(1+\lambda_2)}U_tV_t+\frac{K_{11}(1+\alpha_{11})}{\alpha_{12}(1+\lambda_2)}\epsilon U_t+\frac{\alpha_{25}}{1+\lambda_2}V_t{}^2,\\
U_t=\alpha_{12}H_t+\alpha_{12}P_t,\ V_t=-(1+\alpha_{11})H_t+(\lambda_2-\alpha_{11})P_t,\\
U_t^2=\alpha_{12}^2\left(H_t^2+2H_tP_t+P_t^2\right),\\
V_t^2=(1+\alpha_{11})^2H_t^2+(\lambda_2-\alpha_{11})^2P_t^2-2(1+\alpha_{11})(\lambda_2-\alpha_{11})H_tP_t,\\
U_tV_t=-\alpha_{12}(1+\alpha_{11})H_t^2+(\alpha_{12}(\lambda_2-\alpha_{11})-\alpha_{12}(1+\alpha_{11}))H_tP_t+\alpha_{12}(\lambda_2-\alpha_{11})P_t^2,\\
U_t\epsilon=\alpha_{12}H_t\epsilon+\alpha_{12}P_t\epsilon.
\end{cases}
\tag{3.31}
$$

Now center manifold $M^cF_{00}(0,0)$ of (3.30) at $F_{00}(0,0)$ is determined in a neighborhood of ϵ, and therefore mathematical expression for $M^cF_{00}(0,0)$ is

$$
M^cF_{00}(0,0)=\left\{(H_t,P_t):P_t=C_0\epsilon+C_1H_t^2+C_2H_t\epsilon+C_3\epsilon^3+O\left((|H_t|+|\epsilon|)^3\right)\right\},
\tag{3.32}
$$

where the computation yields

$$
\begin{cases}
C_0=0,\\
C_1=\frac{1+\alpha_{11}}{1-\lambda_2^2}\Big(\alpha_{12}\alpha_{13}+(\alpha_{25}-\alpha_{14})(1+\alpha_{11})-\alpha_{12}\alpha_{24}\Big)+\frac{1}{1-\lambda_2^2}\alpha_{12}^2\alpha_{23},\\
C_2=\frac{K_{11}(1+\alpha_{11})}{1-\lambda_2^2},\\
C_3=0.
\end{cases}
\tag{3.33}
$$

Finally, (3.30) restricted to $M^cF_{00}(0,0)$ is

$$
f(H_t)=-H_t+h_1H_t^2+h_2H_t\epsilon+h_3H_t^2\epsilon+h_4H_t\epsilon^2+h_5H_t^3+O\left((|H_t|+|\epsilon|)^4\right),
\tag{3.34}
$$

where

$$\begin{cases} h_1 = \frac{1}{1+\lambda_2}\left[\alpha_{12}\alpha_{13}\left(\lambda_2-\alpha_{11}\right)-\left(1+\alpha_{11}\right)\left(\alpha_{14}\left(\lambda_2-\alpha_{11}\right)-\alpha_{12}\alpha_{24}+\alpha_{25}\left(1+\alpha_{11}\right)\right)\right.\\ \qquad \left.-\alpha_{12}^2\alpha_{23}\right], \\ h_2 = \frac{K_{11}(\lambda_2-\alpha_{11})}{1+\lambda_2}, \\ h_3 = \frac{1}{1+\lambda_2}\left[2C_2\alpha_{12}\alpha_{13}(\lambda_2-\alpha_{11})+C_2\alpha_{14}(\lambda_2-\alpha_{11})(\lambda_2-2\alpha_{11}-1)\right.\\ \qquad +C_1K_{11}(\lambda_2-\alpha_{11})\ -2C_2\alpha_{23}\alpha_{12}^2-C_2\alpha_{12}\alpha_{24}(\lambda_2-2\alpha_{11}-1)\\ \qquad \left.+2C_2\alpha_{25}(1+\alpha_{11})(\lambda_2-\alpha_{11})\right], \\ h_4 = \frac{C_2K_{11}(\lambda_2-\alpha_{11})}{1+\lambda_2}, \\ h_5 = \frac{1}{1+\lambda_2}\left[2C_1\alpha_{12}\alpha_{13}\left(\lambda_2-\alpha_{11}\right)-2C_1\alpha_{23}\alpha_{12}^2\right.\\ \qquad +C_1\alpha_{14}(\lambda_2-\alpha_{11})(\lambda_2-2\alpha_{11}-1)-\ C_1\alpha_{12}\alpha_{24}(\lambda_2-2\alpha_{11}-1)\\ \qquad \left.+2C_1\alpha_{25}(1+\alpha_{11})(\lambda_2-\alpha_{11})\right]. \end{cases} \tag{3.35}$$

So, following discriminatory quantities are non-zero for existence of the flip bifurcation:

$$\begin{cases} J_1 = \left(\frac{\partial^2 f}{\partial H_t \partial\epsilon}+\frac{1}{2}\frac{\partial f}{\partial\epsilon}\frac{\partial^2 f}{\partial H_t^2}\right)\Big|_{F_{00}(0,0)}, \\ J_2 = \left(\frac{1}{6}\frac{\partial^3 f}{\partial H_t^3}+\left(\frac{1}{2}\frac{\partial^2 f}{\partial H_t^2}\right)^2\right)\Big|_{F_{00}(0,0)}. \end{cases} \tag{3.36}$$

The simplification yields

$$J_1 = \frac{h(2-hr_2)\left(a_1hr_2^2(1-m)^2+a_2b_1(hr_2-2)\right)}{a_1hr_2^2(1-m)^2(hr_2-4)+a_2b_1(hr_2-2)^2} \neq 0, \tag{3.37}$$

and

$$\begin{aligned} J_2 = &\frac{Aa_1h(1-m)}{Ba_2(hr_2-2)\left(a_1hr_2^2(1-m)^2(hr_2-4)+a_2b_1(hr_2-2)^2\right)}\\ &\times\left[a_1^2h^5r_2^6(1-m)^4+a_2^2b_1(hr_2-2)^2\left(-4+h^2r_2\left(-r_2+b_1(hr_2-2)\right)\right)\right.\\ &\left.+a_1a_2hr_2^2(1-m)^2(hr_2-2)\left(8+hr_2\left(-6-hr_2+2b_1h(hr_2-1)\right)\right)\right]\\ &+\left[\frac{\begin{array}{c}2a_1h(1-m)\left(a_1^2h^4r_2^5(1-m)^4+a_2^2b_1(hr_2-2)^3\right.\\ \left.+a_1a_2h^3r_2^3(1-m)^3(-r_2+b_1(hr_2-2))\right)\end{array}}{a_2(hr_2-2)\left(a_1hr_2^2(1-m)^2(hr_2-4)+a_2b_1(hr_2-2)^2\right)}\right]^2, \end{aligned} \tag{3.38}$$

where the involved quantities $A = a_1h^3r_2^3(1-m)^2 + a_2\left(-8+hr_2\left(6-2hr_2+b_1h(hr_2-2)\right)\right)$ and $B = a_2(hr_2-2)^2\left(a_2b_1+a_1r_2(1-m)^2\right)\left(a_1hr_2^2(1-m)^2(hr_2-4)+a_2b_1(hr_2-2)^2\right)$. From (3.38) if one gets $J_2 \neq 0$ as $(h, r_1, r_2, b_1, a_1, a_2, m) \in \mathcal{F}|_{E^+_{HP}\left(\frac{r_1a_2}{a_2b_1+a_1r_2(1-m)^2}, \frac{r_1r_2(1-m)}{a_2b_1+a_1r_2(1-m)^2}\right)}$ then at equilibrium $E^+_{HP}\left(\frac{r_1a_2}{a_2b_1+a_1r_2(1-m)^2}, \frac{r_1r_2(1-m)}{a_2b_1+a_1r_2(1-m)^2}\right)$ discrete model (1.3) undergoes flip bifurcation. Additionally, period-2 points from $E^+_{HP}\left(\frac{r_1a_2}{a_2b_1+a_1r_2(1-m)^2}, \frac{r_1r_2(1-m)}{a_2b_1+a_1r_2(1-m)^2}\right)$ are stable (respectively unstable) if $J_2 > 0$ (respectively $J_2 < 0$). □

4. Chaos control

In the literature, there are many techniques to control chaos for the discrete-time models like state feedback control, pole placement and hybrid control methods [33–37]. In our understudied discrete-time model (1.3), we will use the state feedback control strategy that stabilizes the chaotic orbits at an unstable equilibrium point $E^+_{HP}\left(\frac{r_1a_2}{a_2b_1+a_1r_2(1-m)^2}, \frac{r_1r_2(1-m)}{a_2b_1+a_1r_2(1-m)^2}\right)$. It is important here to mention that in this method, the chaotic discrete-time model (1.3) is transformed into a piecewise linear system to attain an optimal controller that minimizes the upper bound, and then solving the optimization problem under certain constraints. So, on adding control force U_t to model (1.3), it becomes

$$\begin{cases} H_{t+1} = H_t + hH_t\left(r_1 - b_1H_t - a_1(1-m)P_t\right) + U_t, \\ P_{t+1} = P_t + h\left(r_2 - \frac{a_2P_t}{(1-m)H_t}\right)P_t, \end{cases} \tag{4.1}$$

where $U_t = -k_1\left(H_t - H^*\right) - k_2\left(P_t - P^*\right)$ and $k_{1,2}$ are control parameters. Now for (4.1), the variational matrix $V^C|_{E^+_{HP}\left(\frac{r_1a_2}{a_2b_1+a_1r_2(1-m)^2}, \frac{r_1r_2(1-m)}{a_2b_1+a_1r_2(1-m)^2}\right)}$ is

$$J^C|_{E^+_{HP}\left(\frac{r_1a_2}{a_2b_1+a_1r_2(1-m)^2}, \frac{r_1r_2(1-m)}{a_2b_1+a_1r_2(1-m)^2}\right)} = \begin{pmatrix} \ell_{11} - k_1 & \ell_{12} - k_2 \\ \ell_{21} & \ell_{22} \end{pmatrix}, \tag{4.2}$$

where

$$\begin{cases} \ell_{11} = \frac{a_2b_1 - a_2b_1hr_1 + a_1r_2(1-m)^2}{a_2b_1+a_1r_2(1-m)^2}, \\ \ell_{12} = \frac{-ha_1r_1a_2(1-m)}{a_2b_1+a_1r_2(1-m)^2}, \\ \ell_{21} = \frac{hr_2^2(1-m)}{a_2}, \\ \ell_{22} = 1 - hr_2. \end{cases} \tag{4.3}$$

Now if $\lambda_{1,2}$ are roots of characteristic equation of $V^C|_{E_{HP}(H,P)}$ then

$$\lambda_1 + \lambda_2 = \ell_{11} + \ell_{22} - k_1, \tag{4.4}$$

and

$$\lambda_1\lambda_2 = \ell_{22}(\ell_{11} - k_1) - \ell_{21}(\ell_{12} - k_2). \tag{4.5}$$

Since marginal stability determined from the conditions $\lambda_1 = \pm 1,\ \lambda_1\lambda_2 = 1$ which further implies the fact that $|\lambda_{1,2}| < 1$. If $\lambda_1\lambda_2 = 1$ then from (4.5) we get

$$L_1 : a_2(hr_2 - 1)k_1 + hr_2^2(1-m)k_2 - \frac{a_2\left(a_2b_1h(r_1+r_2) + ha_1r_2^2(1-m)^2(1-hr_1) - h^2a_2b_1r_1r_2\right)}{a_2b_1 + a_1r_2(1-m)^2} = 0. \tag{4.6}$$

If $\lambda_1 = 1$ then from (4.4) and (4.5) we get

$$L_2 : a_2k_1 + r_2(1-m)k_2 + \frac{hr_1a_2^2b_1}{a_2b_1 + a_1r_2(1-m)^2} = 0. \tag{4.7}$$

Finally, if $\lambda_1 = -1$ then from (4.4) and (4.5) we get

$$\begin{aligned} L_3 :& a_2(hr_2 - 2)k_1 + hr_2^2(1-m)k_2 \\ &- \frac{a_2(hr_2-2)\left(2a_2b_1 + 2a_1r_2(1-m)^2 - a_2b_1hr_1\right) - h^2a_1r_1r_2^2(1-m)^2}{a_2b_1 + a_1r_2(1-m)^2} = 0. \end{aligned} \tag{4.8}$$

Therefore, from (4.6)–(4.8) lines $L_1,\ L_2$ and L_3 in (k_1, k_2)-plane gives the triangular region that further gives $|\lambda_{1,2}| < 1$.

5. Numerical simulations

In this section, we will give some numerical simulation to verify theoretical results.

5.1. Flip bifurcation at $E_{H0}(\frac{r_1}{b_1}, 0)$

If $h = 1.4,\ b_1 = 0.18,\ a_1 = 0.3,\ m = 0.8,\ r_2 = 0.27,\ a_2 = 0.14$ then from (2.7) one gets: $r_1 = 1.4285714285714286$. From theoretical point of view, $E_{H0}\left(\frac{r_1}{b_1}, 0\right)$ of discrete model (1.3) undergoes a flip bifurcation if $r_1 = 1.4285714285714286$. So, if $r_1 = {r_1}^* = \frac{2}{h} = 1.4285714285714286$ and $H = H^* = \frac{r_1}{b_1} = 7.936507936507937$ then from (3.4)–(3.6) the computation yields:

$$\left.\frac{\partial f}{\partial H}\right|_{{r_1}^*=1.4285714285714286,\ H^*=7.936507936507937} = -1, \tag{5.1}$$

$$\left.\frac{\partial^2 f}{\partial H^2}\right|_{{r_1}^*=1.4285714285714286,\ H^*=7.936507936507937} = -0.504 \neq 0, \tag{5.2}$$

and

$$\left.\frac{\partial f}{\partial r_1}\right|_{{r_1}^*=1.4285714285714286,\ H^*=7.936507936507937} = 11.11111111111111 \neq 0. \tag{5.3}$$

Equations (5.1)–(5.3) indicate that non-degenerate conditions hold, and so at $E_{H0}\left(\frac{r_1}{b_1}, 0\right) = E_{H0}(7.936507936507937, 0)$ discrete model (1.3) undergoes flip bifurcation. In addition, the simple calculation also yields $\Omega_1 = \left.\left(\frac{\partial^2 f}{\partial H^2}\frac{\partial f}{\partial r_1} + 2\frac{\partial^2 f}{\partial H \partial r_1}\right)\right|_{{r_1}^*=1.4285714285714286, H^*=7.936507936507937} = -2.8$ and $\Omega_2 = \left.\left(\frac{\partial^3 f}{\partial H^3} + 3\left(\frac{\partial^2 f}{\partial H^2}\right)^2\right)\right|_{{r_1}^*=1.4285714285714286,\ H^*=7.936507936507937} = 0.7620480000000001$. Finally, $\Omega_1\Omega_2 = -2.1337344 < 0$ which shows that model (1.3) undergoes supercritical flip bifurcation. Hence Maximum Lyapunov exponents (M. L. E.) and flip bifurcation diagram at $E_{H0}(\frac{r_1}{b_1}, 0)$ are drawn in Figure 1.

5.2. Hopf bifurcation at $E^+_{HP}\left(\frac{r_1 a_2}{a_2 b_1 + a_1 r_2 (1-m)^2}, \frac{r_1 r_2 (1-m)}{a_2 b_1 + a_1 r_2 (1-m)^2}\right)$

If $h = 1.39,\ b_1 = 0.56,\ a_1 = 0.61,\ m = 0.48,\ r_2 = 1.05,\ a_2 = 1.65$ then from (2.16) we get $r_1 = 1.7008190945069108$, and so $E^+_{HP}\left(\frac{r_1 a_2}{a_2 b_1 + a_1 r_2 (1-m)^2}, \frac{r_1 r_2 (1-m)}{a_2 b_1 + a_1 r_2 (1-m)^2}\right)$ of discrete model (1.3) is a stable (respectively, an unstable) focus if $0 < r_1 < 1.7008190945069108$ (respectively $r_1 > 1.7008190945069108$). For this if $r_1 = 1.686 < 1.7008190945069108$ then Figure 2a indicates that $E^+_{HP}\left(\frac{r_1 a_2}{a_2 b_1 + a_1 r_2 (1-m)^2}, \frac{r_1 r_2 (1-m)}{a_2 b_1 + a_1 r_2 (1-m)^2}\right) = E^+_{HP}(2.5354742181672614, 0.8390114685571666)$ of discrete model (1.3) is a stable focus, that means that all orbits goes to the interior fixed point $E^+_{HP}(2.5354742181672614, 0.8390114685571666)$ and additionally Figure 2b–2h also indicates same nature of solution if r_1 respectively are $r_1 = 1.62,\ 1.58,\ 1.66,\ 1.695,\ 1.698,\ 1.0,\ 1.6 < 1.7008190945069108$. Furthermore, if $r_1 = 1.72 > 1.7008190945069108$ then Figure 3a indicates that $E^+_{HP}(2.586604777726981, 0.8559310355387465)$ of discrete model (1.3) changes the nature of solution and as a result stable curves take place. Hereafter it is shown numerically that under consideration model undergoes supercritical hopf bifurcation if $r_1 = 1.72 > 1.7008190945069108$, i.e., $\ell < 0$. Therefore, if $r_1 = 1.72$ then $\frac{d|\lambda_{1,2}|}{d\epsilon}|_{\epsilon=0} = 0.4255699494665087 > 0$, and additionally from (3.9) and (3.25) we get

$$\lambda_{1,2} = -0.736456579491341 \pm 0.6885427710325857\iota, \tag{5.4}$$

and

$$\begin{aligned} \tau_{02} &= 0.5213959228438938 - 0.10672543334655603\iota, \\ \tau_{11} &= 0.38280879877518403 + 1.112417243978212\iota, \\ \tau_{20} &= -0.13858712406870982 - 0.10672543334655603\iota, \\ \tau_{21} &= 0. \end{aligned} \tag{5.5}$$

On substituting (5.4) and (5.5) in (3.23) we get $\ell = -1.1066864173641395 < 0$ which confirms the correctness of theoretical results and so supercritical hopf bifurcation takes place. Similarly Figure 3b–3h also shown same nature of solution if r_1 respectively are $r_1 = 1.735, 1.75, 1.776, 1.797, 1.83 > 1.7008190945069108$ and so for $r_1 = 1.735, 1.75, 1.776, 1.797, 1.83, 1.8, 2.1 > 1.7008190945069108$ model (1.3) undergoes supercritical hopf bifurcation with $\ell < 0$ (see Table 1). The M. L. E. and bifurcation diagrams are plotted in Figure 4.

Table 1. Numerical values of ℓ for $r_1 > 1.7008190945069108$.

Bifurcation values if $r_1 > 1.7008190945069108$	Corresponding value of ℓ
1.72	$-1.1066864173641395 < 0$
1.735	$-1.1405324390187181 < 0$
1.75	$-1.175118297518472 < 0$
1.776	$-1.23678224560618 < 0$
1.797	$-1.2881322661012926 < 0$
1.83	$-1.3714964471022881 < 0$
1.8	$-1.2955778257273525 < 0$
2.1	$-2.1388681583975675 < 0$

5.3. Flip bifurcation at $E^+_{HP}\left(\frac{r_1 a_2}{a_2 b_1 + a_1 r_2 (1-m)^2}, \frac{r_1 r_2 (1-m)}{a_2 b_1 + a_1 r_2 (1-m)^2}\right)$

If $h = 1.4, b_1 = 0.18, a_1 = 0.3, m = 0.8, r_2 = 0.27, a_2 = 0.14$ then from non-hyperbolic condition (2.20) we get $r_1 = 1.6620447594316734$. Theoretically, if $r_1 = 1.6620447594316734$ then at interior fixed point $E^+_{HP}\left(\frac{r_1 a_2}{a_2 b_1 + a_1 r_2 (1-m)^2}, \frac{r_1 r_2 (1-m)}{a_2 b_1 + a_1 r_2 (1-m)^2}\right)$, discrete model (1.3) undergoes a flip bifurcation, i.e., if $r_1 = 1.6620447594316734$ then from (3.37) one gets $\jmath_1 = 1.455433013797801 \neq 0$. Further from (3.38) we get $\jmath_2 = 0.025512710056662495 > 0$ which represent that stable period-2 points bifurcate from the interior fixed point $E^+_{HP}\left(\frac{r_1 a_2}{a_2 b_1 + a_1 r_2 (1-m)^2}, \frac{r_1 r_2 (1-m)}{a_2 b_1 + a_1 r_2 (1-m)^2}\right)$, and hence M. L. E. and flip bifurcation diagram are drawn in Figure 5.

5.4. Simulation for correctness of theoretical results in Section 4

If $h = 0.4, b_1 = 0.58, a_1 = 0.03, m = 0.15, r_2 = 1.1, a_2 = 0.14, r_1 = 0.04$ then from (4.6)–(4.8) we get

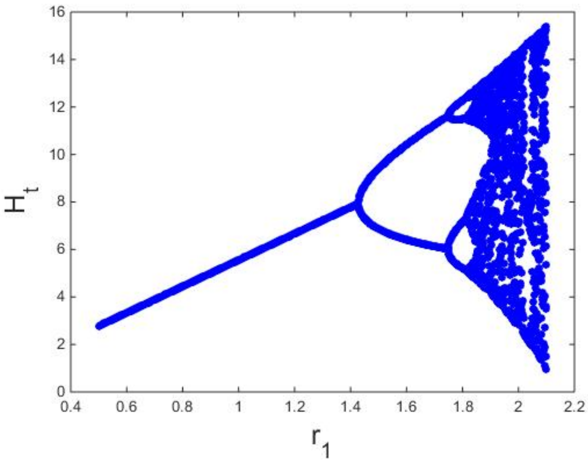
$$L_1 : -0.02035549527096175 - 0.0784k_1 + 0.4114000000000001k_2 = 0, \tag{5.6}$$

$$L_2 : 0.0017315657947973438 + 0.14k_1 + 0.935k_2 = 0, \tag{5.7}$$

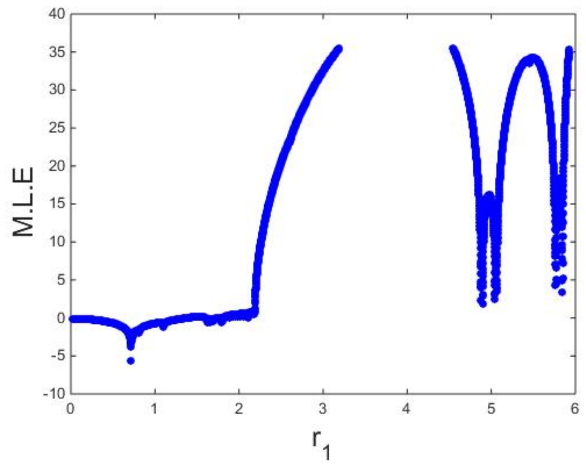
and

$$L_3 : 0.4356966934336102 - 0.21840000000000004k_1 + 0.4114000000000001k_2 = 0. \tag{5.8}$$

The lines (5.6)–(5.8) define a triangular region that gives $|\lambda_{1,2}| < 1$ (See Figure 6). Finally, t vs H_t and P_t have drawn for (4.1) with respectively $k_{1,2} = -0.15083845871908988, 0.02073349564264731$, which predict that unstable trajectories are stabilized (See Figure 7).

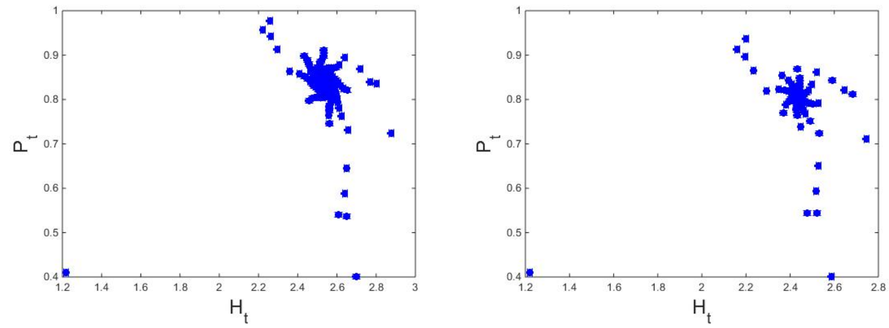


(a)

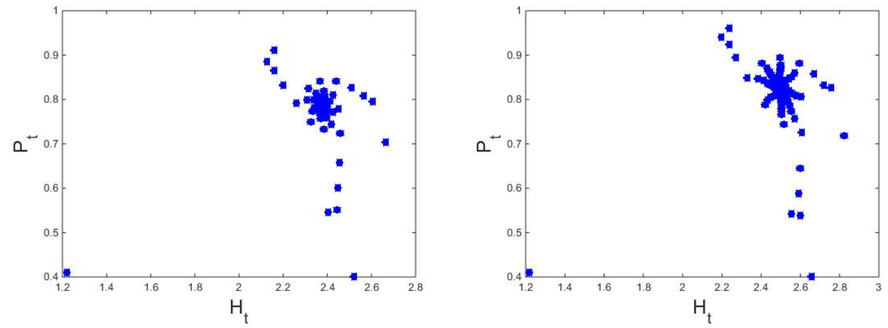


(b)

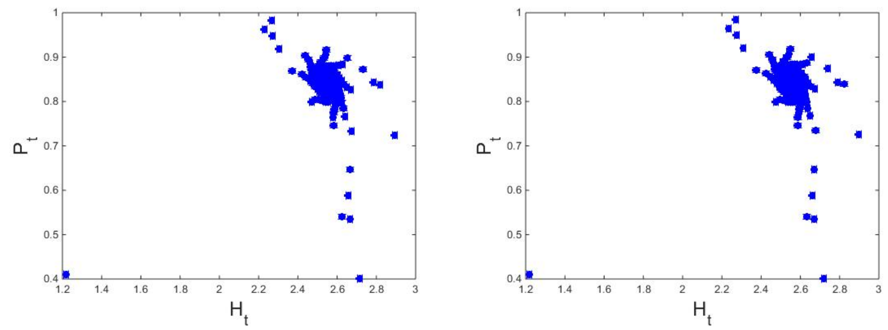
Figure 1. 1a Flip bifurcation diagrams at $E_{H0}(\frac{r_1}{b_1}, 0)$ of discrete model (1.3) with $r_1 \in [0.5, 2.1]$ 1b M. L. E. corresponding to 1a with $(0.5, 0)$.



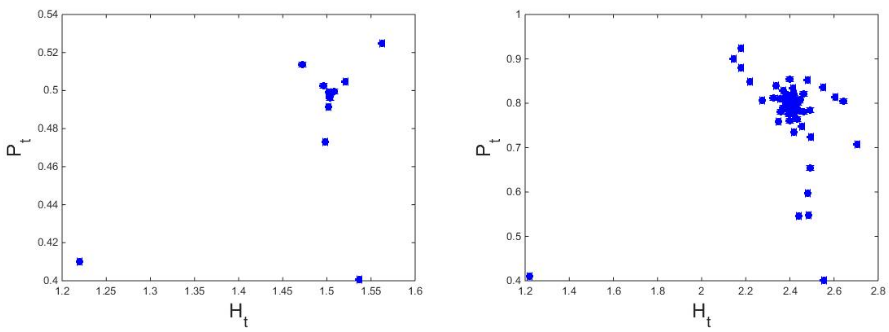
(a) $r_1 = 1.686$ **(b)** $r_1 = 1.62$



(c) $r_1 = 1.58$ **(d)** $r_1 = 1.66$

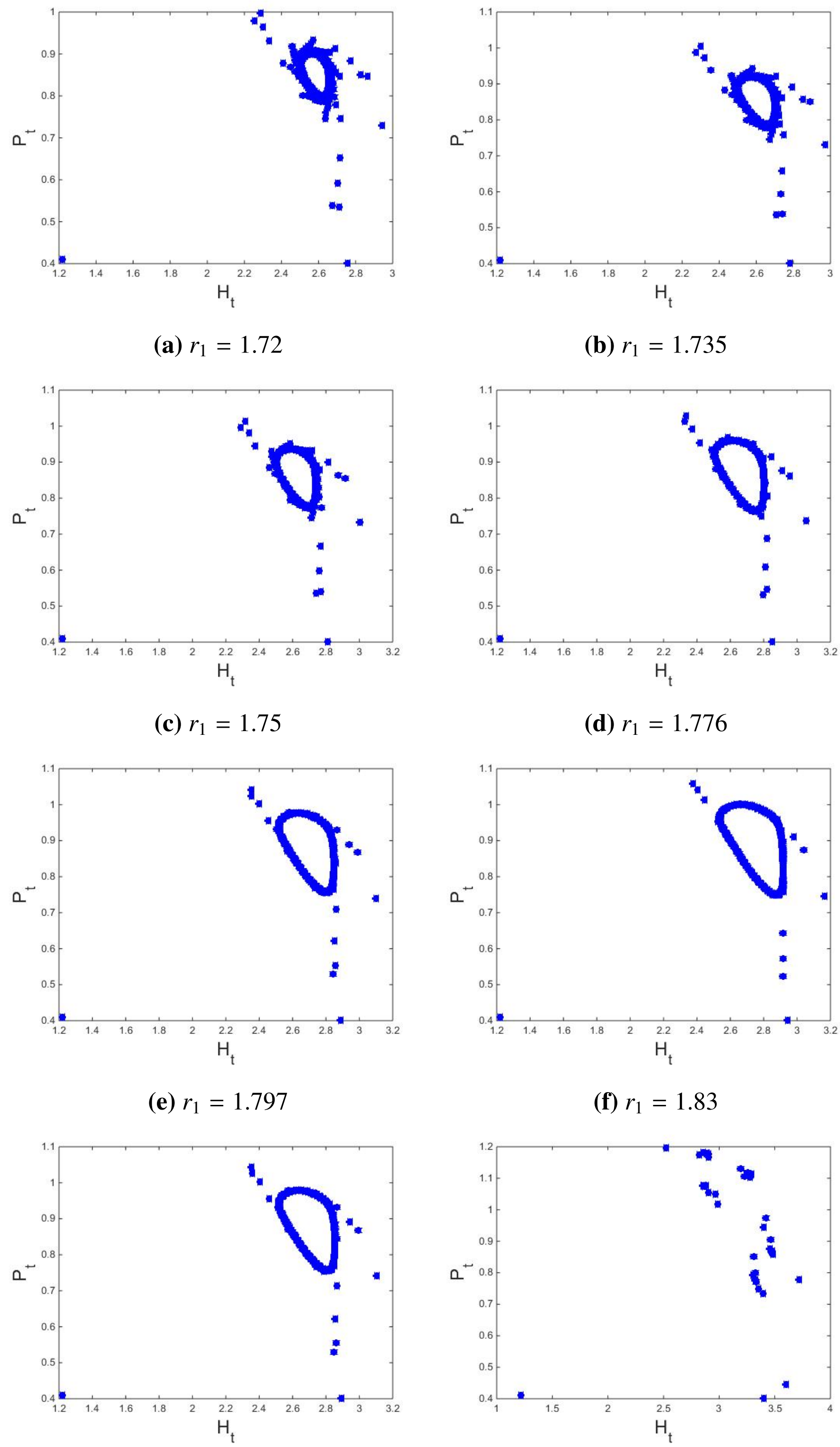


(e) $r_1 = 1.695$ **(f)** $r_1 = 1.698$



(g) $r_1 = 1.0$ **(h)** $r_1 = 1.6$

Figure 2. Stable focus of discrete model (1.3) with $(1.22, 0.41)$.



(a) $r_1 = 1.72$ **(b)** $r_1 = 1.735$

(c) $r_1 = 1.75$ **(d)** $r_1 = 1.776$

(e) $r_1 = 1.797$ **(f)** $r_1 = 1.83$

(g) $r_1 = 1.8$ **(h)** $r_1 = 2.1$

Figure 3. Unstable focus of discrete model (1.3) with $(1.22, 0.41)$.

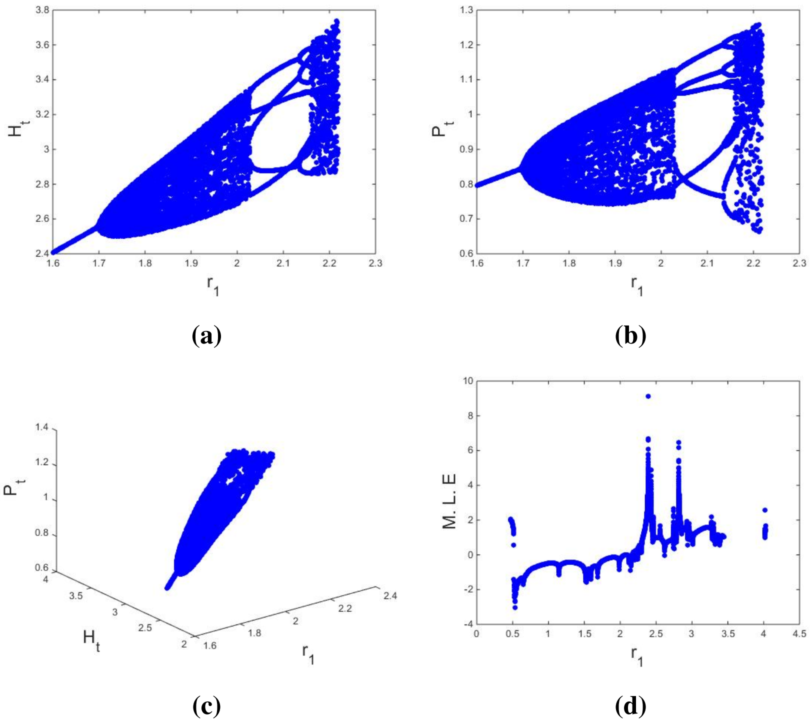


Figure 4. 4a–4c Hopf bifurcation diagrams of discrete model (1.3) with $r_1 \in [1.0, 2.05]$ 4d M. L. E. corresponding to 4a–4c.

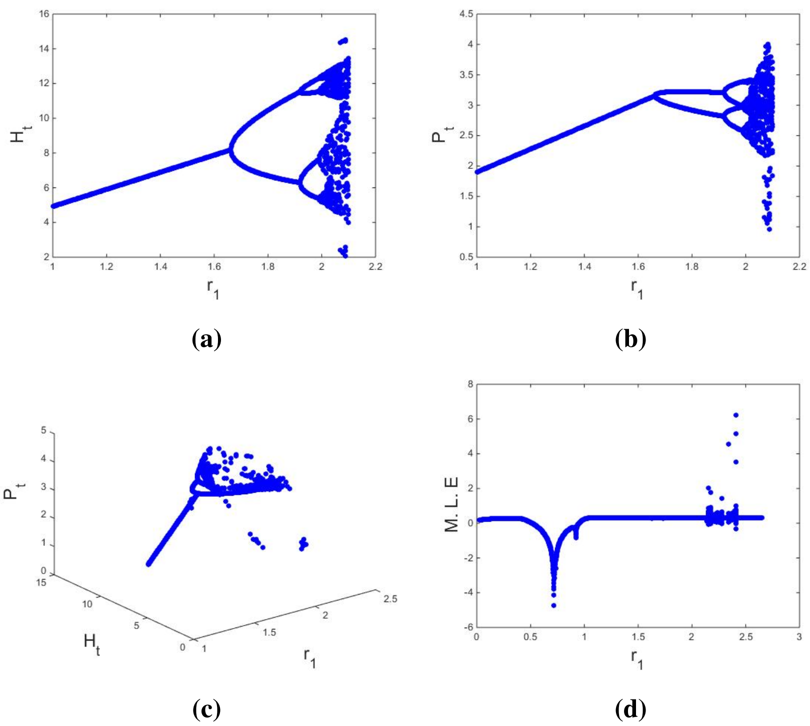
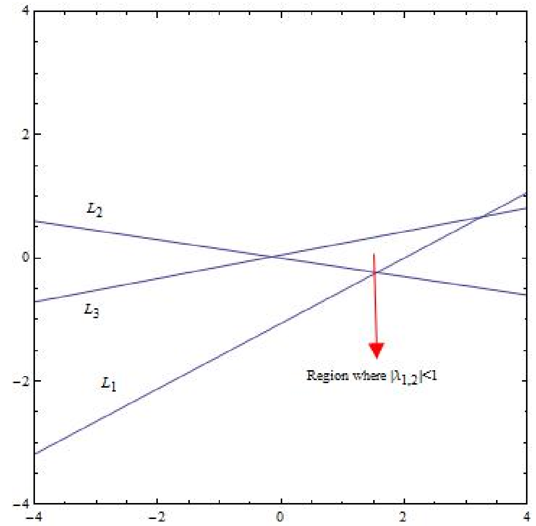
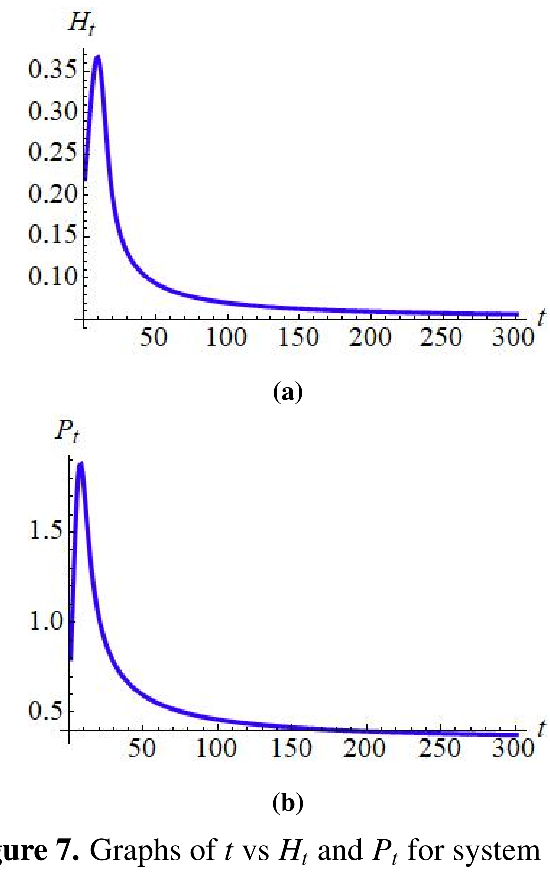


Figure 5. 5a–5c Flip bifurcation diagrams of discrete model (1.3) with $r_1 \in [1.0, 2.1]$ 5d M. L. E. corresponding to 5a–5c with $(1.54, 0.005)$.



(a)

Figure 6. Region of stability where $|\lambda_{1,2}| < 1$.



(a)

(b)

Figure 7. Graphs of t vs H_t and P_t for system (4.1).

6. Influence of prey refuge

In this section, the following two cases are to be considered:

Case I: Prey densities increase due to the influence of prey refuge

For this, from P-component of interior fixed point $E^+_{HP}\left(\frac{r_1a_2}{a_2b_1+a_1r_2(1-m)^2}, \frac{r_1r_2(1-m)}{a_2b_1+a_1r_2(1-m)^2}\right)$, one can observe that if $m \in (0,1)$, then following inequality holds obviously:

$$a_2b_1 + a_1r_2(1-m)^2 < a_2b_1 + a_1r_2. \tag{6.1}$$

From (6.1), the H-component of interior fixed point $E^+_{HP}\left(\frac{r_1a_2}{a_2b_1+a_1r_2(1-m)^2}, \frac{r_1r_2(1-m)}{a_2b_1+a_1r_2(1-m)^2}\right)$ give

$$\frac{r_1a_2}{a_2b_1 + a_1r_2(1-m)^2} > \frac{r_1a_2}{a_2b_1 + a_1r_2}, \tag{6.2}$$

which implies that for fixed refuge, the prey refuge can increase the prey densities. Furthermore, $\frac{d}{dm}\left(\frac{r_1a_2}{a_2b_1+a_1r_2(1-m)^2}\right) = \frac{2r_1r_2a_1a_2}{\left(a_2b_1+a_1r_2(1-m)^2\right)^2} > 0 \ \forall\ m \in (0,1)$. This shows the fact that $\frac{r_1a_2}{a_2b_1+a_1r_2(1-m)^2}$ is strictly increasing function of m, that is, increasing the amount of refuge results the increase of prey densities.

Case II: Predator densities decreases due to the influence of prey refuge

Again from the P-component of interior fixed point $E^+_{HP}\left(\frac{r_1a_2}{a_2b_1+a_1r_2(1-m)^2}, \frac{r_1r_2(1-m)}{a_2b_1+a_1r_2(1-m)^2}\right)$, one has $\frac{d}{dm}\left(\frac{r_1r_2(1-m)}{a_2b_1+a_1r_2(1-m)^2}\right) = \frac{r_1r_2\left(a_1r_2(1-m)^2-a_2b_1\right)^2}{\left(a_2b_1+a_1r_2(1-m)^2\right)^2}$. Additionally if $a_1r_2 - a_2b_1 \leq 0$, that is, $a_1r_2 \leq a_2b_1$ then $\frac{d}{dm}\left(\frac{r_1r_2(1-m)}{a_2b_1+a_1r_2(1-m)^2}\right) = \frac{r_1r_2\left(a_1r_2(1-m)^2-a_2b_1\right)^2}{\left(a_2b_1+a_1r_2(1-m)^2\right)^2} < 0 \ \forall\ m \in (0,1)$. This implies that $\frac{r_1r_2(1-m)}{a_2b_1+a_1r_2(1-m)^2}$ is strictly non-increasing function of m, that is, predator densities decreases due to the influence of prey refuge. Moreover $\frac{r_1r_2(1-m)}{a_2b_1+a_1r_2(1-m)^2}$ has maximum value $\frac{r_1r_2}{a_2b_1+a_1r_2}$ at $m = 0$.

7. Conclusions

In this paper, we have investigated local behavior at fixed points, chaos and bifurcation of a discrete time model (1.3). More precisely, it is shown that $\forall\ r_1,\ r_2,\ a_1,\ a_2,\ b_1,\ m,\ h > 0$, model (1.3) has boundary fixed point $E_{H0}\left(\frac{r_1}{b_1}, 0\right)$ and if $m < 1$ then it has interior fixed point $E^+_{HP}\left(\frac{r_1a_2}{a_2b_1+a_1r_2(1-m)^2}, \frac{r_1r_2(1-m)}{a_2b_1+a_1r_2(1-m)^2}\right)$. Further at $E_{H0}\left(\frac{r_1}{b_1}, 0\right)$ and $E^+_{HP}\left(\frac{r_1a_2}{a_2b_1+a_1r_2(1-m)^2}, \frac{r_1r_2(1-m)}{a_2b_1+a_1r_2(1-m)^2}\right)$ the local dynamical characteristics have been studied, and proved that $E_{H0}\left(\frac{r_1}{b_1}, 0\right)$ of discrete model (1.3) is never sink; source if $r_1 > \frac{2}{h}$; saddle if $0 < r_1 < \frac{2}{h}$ and non-hyperbolic if $r_1 = \frac{2}{h}$. Moreover interior fixed point $E^+_{HP}\left(\frac{r_1a_2}{a_2b_1+a_1r_2(1-m)^2}, \frac{r_1r_2(1-m)}{a_2b_1+a_1r_2(1-m)^2}\right)$ of discrete mathematical model (1.3) is a stable focus if $0 < r_1 < \frac{r_2a_2b_1+r_2{}^2a_1(1-m)^2}{ha_2b_1r_2-a_2b_1+ha_1r_2{}^2(1-m)^2}$ with (2.14) holds; an unstable focus if $r_1 > \frac{r_2a_2b_1+r_2{}^2a_1(1-m)^2}{ha_2b_1r_2-a_2b_1+ha_1r_2{}^2(1-m)^2}$; non-hyperbolic if $r_1 = \frac{r_2a_2b_1+r_2{}^2a_1(1-m)^2}{ha_2b_1r_2-a_2b_1+ha_1r_2{}^2(1-m)^2}$; stable node if $0 < \frac{2(hr_2-2)\left(a_2b_1+a_1r_2(1-m)^2\right)}{h^2r_2\left(a_2b_1+a_1r_2(1-m)^2\right)-2a_2b_1h}$ with (2.18) holds; an unstable node if $r_1 > \frac{2(hr_2-2)\left(a_2b_1+a_1r_2(1-m)^2\right)}{h^2r_2\left(a_2b_1+a_1r_2(1-m)^2\right)-2a_2b_1h}$ and non-hyperbolic if $r_1 = \frac{2(hr_2-2)\left(a_2b_1+a_1r_2(1-m)^2\right)}{h^2r_2\left(a_2b_1+a_1r_2(1-m)^2\right)-2a_2b_1h}$. We have also explored existence of bifurcation scenarios at fixed points, and proved that flip bifurcation exists at $E_{H0}\left(\frac{r_1}{b_1}, 0\right)$ if

$(h, r_1, r_2, b_1, a_1, a_2, m) \in \mathcal{F}|_{E_{H0}\left(\frac{r_1}{b_1},0\right)} = \left\{(h, r_1, r_2, b_1, a_1, a_2, m),\ r_1 = \frac{2}{h}\right\}$. It is proved that at $E^+_{HP}\left(\frac{r_1 a_2}{a_2 b_1 + a_1 r_2(1-m)^2}, \frac{r_1 r_2(1-m)}{a_2 b_1 + a_1 r_2(1-m)^2}\right)$ model (1.3) undergoes hopf bifurcation if $(h, r_1, r_2, b_1, a_1, a_2, m) \in \mathcal{N}|_{E^+_{HP}\left(\frac{r_1 a_2}{a_2 b_1 + a_1 r_2(1-m)^2}, \frac{r_1 r_2(1-m)}{a_2 b_1 + a_1 r_2(1-m)^2}\right)} = \left\{(h, r_1, r_2, b_1, a_1, a_2, m) : \Delta < 0 \text{ and } r_1 = \frac{r_2 a_2 b_1 + r_2{}^2 a_1(1-m)^2}{h a_2 b_1 r_2 - a_2 b_1 + h a_1 r_2{}^2(1-m)^2}\right\}$ and flip bifurcation if $(h, r_1, r_2, b_1, a_1, a_2, m) \in \mathcal{F}|_{E^+_{HP}\left(\frac{r_1 a_2}{a_2 b_1 + a_1 r_2(1-m)^2}, \frac{r_1 r_2(1-m)}{a_2 b_1 + a_1 r_2(1-m)^2}\right)} = \left\{(h, r_1, r_2, b_1, a_1, a_2, m) : \Delta > 0 \text{ and } r_1 = \frac{2(hr_2-2)\left(a_2 b_1 + a_1 r_2(1-m)^2\right)}{h^2 r_2\left(a_2 b_1 + a_1 r_2(1-m)^2\right) - 2 a_2 b_1 h}\right\}$. By state feedback control strategy, chaos at $E^+_{HP}\left(\frac{r_1 a_2}{a_2 b_1 + a_1 r_2(1-m)^2}, \frac{r_1 r_2(1-m)}{a_2 b_1 + a_1 r_2(1-m)^2}\right)$ in discrete model (1.3) is also investigated. Next numerically verified theoretical results. Our numerical simulation reveals that if parameter crosses $(h, r_1, r_2, b_1, a_1, a_2, m) \in \mathcal{N}|_{E^+_{HP}\left(\frac{r_1 a_2}{a_2 b_1 + a_1 r_2(1-m)^2}, \frac{r_1 r_2(1-m)}{a_2 b_1 + a_1 r_2(1-m)^2}\right)}$ then model (1.3) undergoes the supercritical Neimark-Sacker bifurcation at $E^+_{HP}\left(\frac{r_1 a_2}{a_2 b_1 + a_1 r_2(1-m)^2}, \frac{r_1 r_2(1-m)}{a_2 b_1 + a_1 r_2(1-m)^2}\right)$, and so biologically this implies that there exists a periodic or quasi-periodic oscillation between prey and predator populations. Furthermore if $(h, r_1, r_2, b_1, a_1, a_2, m) \in \mathcal{F}|_{E^+_{HP}\left(\frac{r_1 a_2}{a_2 b_1 + a_1 r_2(1-m)^2}, \frac{r_1 r_2(1-m)}{a_2 b_1 + a_1 r_2(1-m)^2}\right)}$ then model undergoes the flip bifurcation which indicates that the prey population will not remain steady, resulting in a biological imbalance in the ecosystem. Finally, we have also discussed the influence of prey refuge in the understudied discrete model.

Conflict of interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

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