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Research article

Orientable vertex transitive embeddings of K_p

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Abstract: In [*J. Combin. Theory Ser. B*, **99** (2009), 447–454)], Li characterized the classification of vertex-transitive embeddings of complete graphs, and proposed how to enumerate such maps. In this paper, we study the counting problem of orientable vertex-transitive embeddings of K_p, where $p \geq 5$ is a prime. Moreover, we obtain the number of non-isomorphic orientable vertex-transitive complete maps with p vertices.

Keywords: vertex-transitive; maps; complete graphs; Frobenius groups
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1. Introduction

If a graph can be embedded in a surface, then naturally there will be a problem: how many non-isomorphic ways can it be done. One of the main aims of topological graph theory is to enumerate all the symmetrical embeddings of a given class of graphs in closed surfaces, see [10, 14, 15]. We will restrict our attention here to the orientable vertex-transitive embeddings of K_p, where $p \geq 5$ is a prime.

An *orientable map* is a 2-cell embedding of a finite graph in an orientable surface. That is, 'drawing' a graph $\Gamma = (V, E)$ into an orientable surface $\mathcal{S}$ such that both any two edges do not intersect except for the end point and E divides the surface $\mathcal{S}$ into discs. So an embedding divides the surface into open discs, called *faces*, the set of faces is denoted by F, and the map is denoted by $\mathcal{M} = (V, E, F)$. The graph of a map is called the *underlying graph*, and the orientable surface is called the *supporting surface* of the map. For convenience, a map $\mathcal{M}$ is called a *complete map* if its underlying graph is a complete graph K_n.

An incident triple (v, e, f) is called a *flag*. An automorphism of a map $\mathcal{M}$ is a permutation of the flags which preserves the incident relation. So it is exactly an automorphism of the underlying graph which preserves the supporting surface. All automorphisms of $\mathcal{M}$ form the automorphism group $\mathsf{Aut}(\mathcal{M})$.

A map $\mathcal{M}$ is said to be *G-vertex-transitive* (or a *vertex-transitive embedding* of its underlying graph) if $G \leq \mathsf{Aut}(\mathcal{M})$ is transitive on the vertex set V; if in addition G also preserves the orientation of the supporting surface, then $\mathcal{M}$ is called *orientable vertex-transitive*. Similarly, *orientable arc-transitive* maps are defined.

Recent development of the theory of maps was closely related to the theory of map colorings, with the topic of highly 'symmetrical' maps always at the center of interest and recent investigation began with Biggs [1, 2]. In the past fifty years, plenty of results about 'symmetrical' maps have been obtained, see [14, 19–21] and references therein. Particularly, see [1, 6, 14, 15] for arc transitive maps, see [17, 20, 23, 24] for vertex transitive maps, see [13, 22] for edge transitive maps. Very recently, some special families of edge-transitive maps underlying complete bipartite graphs are classified in [7–9, 25], and for more information about the embeddings of complete graphs, see [12, 16, 18].

Two maps $\mathcal{M}_1$ and $\mathcal{M}_2$ are *isomorphic*, denoted by $\mathcal{M}_1 \cong \mathcal{M}_2$, if there is a one-to-one correspondence from the vertices of $\mathcal{M}_1$ to the vertices of $\mathcal{M}_2$ that maps flags to flags. It follows that $\mathsf{Aut}\mathcal{M}_1 \cong \mathsf{Aut}\mathcal{M}_2$ if $\mathcal{M}_1 \cong \mathcal{M}_2$. Recall that $\phi(n)$ is the Euler phi-function, i.e. the number of positive integers which is less than and coprime to n, where n is a positive integer.

The purpose of this paper is to enumerate the number of orientable vertex-transitive maps with underlying graphs being complete graphs K_p, where $p \geq 5$ is a prime. The following theorem is the main result.

Theorem 1.1. *Let $\mathcal{M}$ be an orientable vertex transitive map with underlying graph K_p, where $p \geq 5$ is a prime. Let $G = \mathsf{Aut}(\mathcal{M})$. Then $\mathcal{M}$ is a Cayley map of $\mathbb{Z}_p$, $G \cong \mathbb{Z}_p : G_\alpha$ is a Frobenius group, where G_α is a cyclic group for each $\alpha \in V$.*

Further, if $G_\alpha \cong \mathbb{Z}_k$ acting on the neighborhood of α has r orbits with $(k, p) = 1$, $rk = p - 1$ and $r \geq 2$ a prime, then the number of non-isomorphic orientable vertex transitive maps of K_p equals

$$\frac{|\mathcal{A}_r| - |\mathcal{A}_1|}{r}$$

where $|\mathcal{A}_r| = (r-1)!k^{r-1}\phi(k)$ and $|\mathcal{A}_1| = \phi(p-1)$.

With regard to the Theorem 1.1, we can deduce the following similar conclusions when r is a composite integer.

Corollary 1.2. *If $r = p_1p_2$ with p_i different prime and $i = 1, 2$, then the number of non-isomorphic orientable vertex transitive maps of K_p equals $\frac{|\mathcal{A}_{p_1p_2}|-|\mathcal{A}_{p_1}|-|\mathcal{A}_{p_2}|+|\mathcal{A}_1|}{p_1p_2}$.*

Corollary 1.3. *If $r = p_1^2p_2$ with p_i different prime and $i = 1, 2$, then the number of non-isomorphic orientable vertex transitive maps of K_p equals $\frac{|\mathcal{A}_{p_1^2p_2}|-|\mathcal{A}_{p_1p_2}|-|\mathcal{A}_{p_1^2}|+|\mathcal{A}_{p_1}|}{p_1^2p_2}$.*

This paper is organized as follows. After this introductory section, some preliminary results are given in Section 2, then the enumeration of the different and non-isomorphic orientable vertex-transitive complete maps is given in Section 3 and Section 4, respectively. We give the complete proof of Theorem 1.1 in Section 5. Finally, we present conclusions for the paper in Section 6.

2. Preliminaries

In this section, we need some notations for convenience which will be used in the following discussion.

Let $F = \mathbb{F}_p$ be the field of order p with $p \geq 5$ a prime. Let $F^+ = \mathbb{F}_p^+$ and $F^\times = \mathbb{F}_p^\times$ be the additive group and the multiplicative group of F, respectively. It follows that

$$F^+ \cong \mathbb{Z}_p,\ F^\times \cong \mathbb{Z}_{p-1}.$$

Let $\mathbf{0}$ be the identity of F^+. Let $F^\#$ be the set of all non-identity elements of F^+, namely, $F^\# = F^+ \setminus \{\mathbf{0}\}$. Then the complete graph K_p may be represented as a Cayley graph

$$\mathsf{K}_p = \mathsf{Cay}(F^+, F^\#).$$

A *Cayley map* $\mathcal{M}$ is an embedding of a Cayley graph $\Sigma = \mathsf{Cay}(H, S)$ into a surface such that $\mathsf{Aut}(\mathcal{M})$ contains a subgroup N acting regularly on the vertices, and $\mathcal{M}$ is called a Cayley map of N (or a *Cayley embedding* of Σ with respect to N).

For a vertex v, a cyclic permutation of the neighbor set $\Gamma(v)$ of v is called a *rotation* at v and denoted by R_v. A *rotation system* $R(\Gamma)$ of a graph Γ is the set of rotations at all vertices, that is $R(\Gamma) = \{R_v\}_{v \in V}$. Hence each rotation system $R(\Gamma)$ defines an orientable embedding of Γ, refer to [3, pp.104-108]. Noting that the vertex rotations R_v can be regarded as permutations not only of the neighborhood $\Gamma(v)$ but also the generating set S, so Cayley maps have another equivalent definitions [11]. A map with underlying graph being Cayley graph $\Sigma = \mathsf{Cay}(H, S)$ is a *Cayley map* if the induced local cyclic permutations of S are all the same. Moreover, each circular permutation ρ of $F^\#$ gives rise to a unique orientable Cayley embedding of K_p with the underlying graph $\Gamma = \mathsf{Cay}(F^+, F^\#)$.

3. Enumeration of different embeddings

In this section, we determine enumeration of different vertex transitive embeddings of K_p. Now, we begin by citing the well-known conclusion about vertex transitive maps.

Lemma 3.1. ([17, Lemma 2.2]) *Let $\mathcal{M}$ be an orientable vertex transitive map. Let $G = \mathsf{Aut}(\mathcal{M})$. Then the stabilizer $G_\alpha \cong \mathbb{Z}_k$ or D_{2k} for a vertex α, and each orbit of G_α acting on the neighborhood of α has length k.*

Next, by [17, Theorem 1.1] and [3, Lemma 5.4.1], we can obtain the following lemma.

Lemma 3.2. *Let $\mathcal{M}$ be an orientable vertex transitive embedding with underlying graph K_p, where $p \geq 5$ is a prime. Let $G = \mathsf{Aut}(\mathcal{M})$. Then $\mathcal{M}$ is a Cayley map of $\mathbb{Z}_p$, $G \cong \mathbb{Z}_p{:}\mathbb{Z}_k$ is a Frobenius group, and $G_\alpha \cong \mathbb{Z}_k$ such that $(k, p) = 1$.*

Assume that $G_\alpha \cong \mathbb{Z}_k := \langle a \rangle$ with $o(a) \geq 2$. Then by Lemma 3.2, $G \cong F^+{:}\langle a \rangle$ is a Frobenius group. It follows that $\langle a \rangle$ is half-transitive on F^+, and $|\langle a \rangle| = k$ is a divisor of $|F^+| - 1 = p - 1$. Let $r = (p-1)/k$. Thus G_α acting on $\Gamma(\alpha)$ has r orbits with $r \geq 2$, and we get the lemma as follows.

Lemma 3.3. *If $G \cong F^+{:}\mathbb{Z}_k$, then there are exactly $(r-1)!k^{r-1}\phi(k)$ different orientable vertex-transitive embeddings of K_p.*

Proof. Taking $\alpha = \mathbf{0}$ for convenience with $\mathbf{0}$ the identity element of F^+, then $G_{\mathbf{0}}$ partitions $\Gamma(\mathbf{0})$ into r orbits, and by Lemma 3.1, the length of each orbit is k. Since $F^\#$ be the set of all non-identity elements of F^+, we have $|F^\#| = p - 1$ and

$$F^\# = \Delta_1 \dot\cup \Delta_2 \dot\cup \cdots \dot\cup \Delta_r,$$

where Δ_i is an orbit of $G_{\mathbf{0}}$ acting on $\Gamma(\mathbf{0})$, $|\Delta_i| = k$ with $1 \leq i \leq r$ and $r \geq 2$.

Note that the vertex $\mathbf{0}$ and the neighbors can be lied on a disc such that $\mathbf{0}$ is in the centre and the neighbors of $\mathbf{0}$ are around $\mathbf{0}$. Without loss of generality, we may assume that the $p-1$ neighbors of $\mathbf{0}$ (i.e. all the elements of $F^{\#}$) are in clockwise order around $\mathbf{0}$, say $\beta_1, \beta_2, \cdots, \beta_{p-1}$. Viewing

$$\rho := (\beta_1, \beta_2, \cdots, \beta_{p-1})$$

as a circular permutation of $F^{\#}$. Since the number of the circular permutations of $F^{\#}$ equals the number of arrangements of β_i, we can obtain that to determine the number of the orientable vertex transitive complete maps is only need to determine the different choices of β_i with $1 \leq i \leq p-1$.

Setting $\beta_1 = \mathbf{1}$ and $\beta_1 \in \Delta_1$ for convenience, where $\mathbf{1}$ is the identity element of $F^{\times}$. If $\beta_2 \in \Delta_1$, then $\mathbf{1}^{a^j} = \beta_2$ for some a^j, where $0 < j < k$. Correspondingly, $a^j : \beta_2 \mapsto \beta_3 \mapsto \beta_4 \mapsto \cdots \mapsto \mathbf{1}$. Then $G_{\mathbf{0}}$ acting on $\Gamma(\mathbf{0})$ has only one orbit which is a contradiction. Thus $\beta_2 \notin \Delta_1$. Hence $\beta_2 \in \Delta_i$ with $2 \leq i \leq r$. It follows that β_2 has $(r-1)k$ different choices.

Setting $\beta_2 \in \Delta_2$ for convenience. If $\beta_3 \in \Delta_2$, then $\beta_2^{a^s} = \beta_3$ for some a^s, where $0 < s < k$. Correspondingly, there has $a^s : \beta_2 \mapsto \beta_3 \mapsto \beta_4 \mapsto \cdots \mapsto \mathbf{1}$. Then $\beta_1 \in \Delta_2$, and $G_{\mathbf{0}}$ acting on $\Gamma(\mathbf{0})$ has only one orbit which is a contradiction. Thus $\beta_3 \notin \Delta_2$.

If $\beta_3 \in \Delta_1$, then $\mathbf{1}^{a^t} = \beta_3$ for some a^t, where $0 < t < k$. Correspondingly, there are $a^t : \mathbf{1} \mapsto \beta_3 \mapsto \beta_5 \mapsto \cdots \mapsto \mathbf{1}$, and $a^t : \beta_2 \mapsto \beta_4 \mapsto \beta_6 \mapsto \cdots \mapsto \beta_2$. Thus $G_{\mathbf{0}}$ acting on $\Gamma(\mathbf{0})$ has two orbits, namely, $r = 2$. Since the number of generators of $G_{\mathbf{0}}$ is $\phi(k)$, we obtain that a^t has $\phi(k)$ different choices. Notice that $G_{\mathbf{0}}$ is cyclic, then except $\mathbf{1}, \beta_2$, the remaining vertices of Δ_1, Δ_2 can be obtained by $\mathbf{1}, \beta_2$ through the conjugate action of a, $a^2, \cdots, a^{k-1}$, respectively. So

$$\rho|_{r=2} := (\mathbf{1}, \beta_2, \mathbf{1}^a, \beta_2^a, \cdots, \mathbf{1}^{a^{k-1}}, \beta_2^{a^{k-1}})$$

is a circular permutation of $F^{\#}$. It follows that the number of $\rho|_{r=2}$ is determined by the choices of $\mathbf{1}$, β_2, a. Thus the number of $\rho|_{r=2}$ equals $(r-1)k \cdot \phi(k) = k\phi(k)$. Let the corresponding maps generated by $\rho|_{r=2}$ be

$$\mathcal{M}_2(\mathbf{1}, \beta_2, a).$$

Hence the number of vertex transitive maps of K_p equals $k\phi(k)$ if $r = 2$.

Now, suppose that $r \geq 3$. Then $\beta_3 \notin \Delta_1$, $\beta_3 \notin \Delta_2$ and $\beta_3 \in \Delta_i$ with $3 \leq i \leq r$. Taking $\beta_3 \in \Delta_3$ for convenience, and β_3 has $(r-2)k$ different choices. If $\beta_4 \in \Delta_3$, then $\beta_3^{a^l} = \beta_4$ for some a^l, where $0 < l < k$. It follows that there has $a^l : \beta_3 \mapsto \beta_4 \mapsto \beta_5 \mapsto \cdots \mapsto \mathbf{1}$. So $G_{\mathbf{0}}$ acting on $\Gamma(\mathbf{0})$ has one orbit which is a contradiction as $r \geq 3$. Thus $\beta_4 \notin \Delta_3$.

If $\beta_4 \in \Delta_2$, then $\beta_2^{a^{l'}} = \beta_4$ for some $a^{l'}$, where $0 < l' < k$. Correspondingly, there have $a^{l'} : \beta_2 \mapsto \beta_4 \mapsto \beta_6 \mapsto \cdots \mapsto \beta_2$ and $a^{l'} : \mathbf{1} \mapsto \beta_3 \mapsto \beta_5 \mapsto \cdots \mapsto \mathbf{1}$. We obtain that $G_{\mathbf{0}}$ acting on $\Gamma(\mathbf{0})$ has two orbits which is a contradiction as $r \geq 3$. Thus $\beta_4 \notin \Delta_2$.

If $\beta_4 \in \Delta_1$, then $\mathbf{1}^{a^{l''}} = \beta_4$ for some $a^{l''}$, where $0 < l'' < k$. Correspondingly, there have $a^{l''} : \mathbf{1} \mapsto \beta_4 \mapsto \beta_7 \mapsto \cdots \mapsto \mathbf{1}$, $a^{l''} : \beta_2 \mapsto \beta_5 \mapsto \beta_8 \mapsto \cdots \mapsto \beta_2$ and $a^{l''} : \beta_3 \mapsto \beta_6 \mapsto \beta_9 \mapsto \cdots \mapsto \beta_3$. Thus $G_{\mathbf{0}}$ acting on $\Gamma(\mathbf{0})$ has three orbits, that is, $r = 3$. Note that $G_{\mathbf{0}}$ is cyclic, then except $\mathbf{1}, \beta_2$ and β_3, the remaining vertices of Δ_1, Δ_2 and Δ_3 can be obtained by $\mathbf{1}, \beta_2$ and β_3 through the conjugate action of a,

$a^2, \cdots, a^{k-1}$, respectively. So

$$\rho|_{r=3} := (\mathbf{1}, \beta_2, \beta_3, \mathbf{1}^a, \beta_2^a, \beta_3^a, \cdots, \mathbf{1}^{a^{k-1}}, \beta_2^{a^{k-1}}, \beta_3^{a^{k-1}})$$

is a circular permutation of $F^{\#}$. It follows that the number of $\rho|_{r=3}$ is determined by the choices of $\mathbf{1}$, β_2, β_3 and a. Further, the number of $\rho|_{r=3}$ equals

$$(r-1)k \cdot (r-2)k \cdot \phi(k) = 2!k^2\phi(k).$$

Let the corresponding maps generated by $\rho|_{r=3}$ be

$$\mathcal{M}_3(\mathbf{1}, \beta_2, \beta_3, a).$$

Thus the number of orientable vertex transitive maps of K_p equals $2!k^2\phi(k)$ if $r = 3$.

Next, suppose that $r \geq 4$. Then $\beta_4 \notin \Delta_1, \beta_4 \notin \Delta_2, \beta_4 \notin \Delta_3$ and $\beta_4 \in \Delta_i$ with $4 \leq i \leq r$. Taking $\beta_4 \in \Delta_4$ for convenience, and β_4 has $(r-3)k$ different choices. If $\beta_5 \in \Delta_4$, then $\beta_4^{a^m} = \beta_5$ for some a^m, where $0 < m < k$. It follows that there has $a^m : \beta_4 \mapsto \beta_5 \mapsto \beta_6 \mapsto \cdots \mapsto \mathbf{1}$. So $G_{\mathbf{0}}$ acting on $\Gamma(\mathbf{0})$ has one orbit which is a contradiction as $r \geq 5$. Thus $\beta_5 \notin \Delta_4$.

If $\beta_5 \in \Delta_3$, then $\beta_3^{a^{m'}} = \beta_5$ for some $a^{m'}$, where $0 < m' < k$. Correspondingly, there have $a^{m'} : \beta_3 \mapsto \beta_5 \mapsto \beta_7 \mapsto \cdots \mapsto \mathbf{1}$ and $a^{m'} : \beta_2 \mapsto \beta_4 \mapsto \beta_6 \mapsto \cdots \mapsto \beta_2$. It is easy to see that $G_{\mathbf{0}}$ acting on $\Gamma(\mathbf{0})$ has two orbits which is a contradiction as $r \geq 5$. Thus $\beta_5 \notin \Delta_3$.

If $\beta_5 \in \Delta_2$, then $\beta_2^{a^{m''}} = \beta_5$ for some $a^{m''}$, where $0 < m'' < k$. Correspondingly, there have $a^{m''} : \beta_2 \mapsto \beta_5 \mapsto \beta_8 \mapsto \cdots \mapsto \beta_2$, $a^{m''} : \beta_3 \mapsto \beta_6 \mapsto \beta_9 \mapsto \cdots \mapsto \beta_3$ and $a^{m''} : \beta_4 \mapsto \beta_7 \mapsto \beta_{10} \mapsto \cdots \mapsto \mathbf{1}$. Then we have that $G_{\mathbf{0}}$ acting on $\Gamma(\mathbf{0})$ has three orbits which is a contradiction as $r \geq 5$. Thus $\beta_5 \notin \Delta_2$.

If $\beta_5 \in \Delta_1$, then $\mathbf{1}^{a^s} = \beta_5$ for some a^s, where $0 < s < k$. Correspondingly, there have $a^s : \mathbf{1} \mapsto \beta_5 \mapsto \beta_9 \mapsto \cdots \mapsto \mathbf{1}$, $a^s : \beta_2 \mapsto \beta_6 \mapsto \beta_{10} \mapsto \cdots \mapsto \beta_2$, $a^s : \beta_3 \mapsto \beta_7 \mapsto \beta_{11} \mapsto \cdots \mapsto \beta_3$ and $a^s : \beta_4 \mapsto \beta_8 \mapsto \beta_{12} \mapsto \cdots \mapsto \beta_4$. Thus $G_{\mathbf{0}}$ acting on $\Gamma(\mathbf{0})$ has four orbits, equivalently, $r = 4$. So

$$\rho|_{r=4} := (\mathbf{1}, \beta_2, \beta_3, \beta_4, \mathbf{1}^a, \beta_2^a, \beta_3^a, \beta_4^a, \cdots, \mathbf{1}^{a^{k-1}}, \beta_2^{a^{k-1}}, \beta_3^{a^{k-1}}, \beta_4^{a^{k-1}})$$

is a circular permutation of $F^{\#}$, and the number of $\rho|_{r=4}$ is determined by the choices of $\mathbf{1}$, β_2, β_3, β_4 and a. It follows that the number of $\rho|_{r=4}$ equals

$$(r-1)k \cdot (r-2)k \cdot (r-3)k \cdot \phi(k) = 3!k^3\phi(k).$$

Let the corresponding maps generated by $\rho|_{r=4}$ be

$$\mathcal{M}_4(\mathbf{1}, \beta_2, \beta_3, \beta_4, a).$$

Thus the number of orientable vertex transitive complete maps of K_p equals $3!k^3\phi(k)$ if $r = 4$.

Here r can be generalized to any integer. Similarly, if $G_\alpha = \langle a \rangle$ acting on $\Gamma(\alpha)$ has r orbits, then $\beta_i \in \Delta_i$ such that $\beta_1 = \mathbf{1}$ and $1 \leq i \leq r$, and

$$\rho|_r := (\mathbf{1}, \beta_2, \cdots, \beta_r, \mathbf{1}^a, \beta_2^a, \cdots, \beta_r^a, \cdots, \mathbf{1}^{a^{k-1}}, \beta_2^{a^{k-1}}, \cdots, \beta_r^{a^{k-1}})$$

is a circular permutation of $F^{\#}$. Since β_i has $(r-i+1)k$ different choices with $2 \leq i \leq r$, and a has $\phi(k)$ different choices, it follows that the number of $\rho|_r$ equals

$$(r-1)k \cdot (r-2)k \cdots (r-r+1)k \cdot \phi(k) = (r-1)!k^{r-1}\phi(k).$$

Let the corresponding maps generated by $\rho|_r$ be

$$\mathcal{M}_r(\mathbf{1}, \beta_2, \beta_3, \cdots, \beta_r, a).$$

Hence if $G_\alpha \cong \mathbb{Z}_k$, then the number of different orientable vertex transitive maps of K_p equals $(r-1)!k^{r-1}\phi(k)$. $\square$

Recall that a Cayley map $\mathsf{CayM}(G, S)$ is called *balanced* if s and $-s$ are placed on the antipodal points for all elements $s \in S$, see [21]. The map $\mathcal{M}_r(\mathbf{1}, \beta_2, \beta_3, \cdots, \beta_r, a)$ is a Cayley map of the group F^+. Let η be the unique involution of $\mathsf{GL}(1, p) \cong \mathbb{Z}_{p-1}$. Then

$$\eta : \ x \mapsto -x, \ \text{for all } x \in F^+$$

is an automorphism of $\mathcal{M}_r(\mathbf{1}, \beta_2, \beta_3, \cdots, \beta_r, a)$.

Lemma 3.4. *A map $\mathcal{M}_r(\mathbf{1}, \beta_2, \beta_3, \cdots, \beta_r, a)$ is balanced if and only if $\beta_i^{-1} = \beta_{i+\frac{p-1}{2}}$ with $1 \leq i \leq r$.*

Proof. Assume that $\mathcal{M}_r(\mathbf{1}, \beta_2, \beta_3, \cdots, \beta_r, a)$ is balanced. Then the vertex β_i^{-1} is placed at the antipodal position of the vertex β_i with p an odd prime and $1 \leq i \leq \frac{p-1}{2}$. Thus $\beta_i^{-1} = \beta_{i+\frac{p-1}{2}}$ with $1 \leq i \leq r$.

Conversely, assume that $\beta_i^{-1} = \beta_{i+\frac{p-1}{2}}$ with p odd prime and $1 \leq i \leq r$. Then for any $1 \leq l \leq k-1$, we have

$$\beta_{i+lr}^{-1} = (\beta_i^{-1})^{a^l} = (\beta_{i+\frac{p-1}{2}})^{a^l} = \beta_{\frac{p-1}{2}+i+lr},$$

reading the subscripts modulo $(p-1)$. So $\beta_j^{-1} = \beta_{\frac{p-1}{2}+j}$ is at the antipodal position of β_j for all j with $1 \leq j \leq \frac{p-1}{2}$, and therefore $\mathcal{M}_r(\mathbf{1}, \beta_2, \beta_3, \cdots, \beta_r, a)$ is balanced. $\square$

4. Enumeration of non-isomorphic embeddings

We notice that many different orientable vertex transitive maps of K_p may be isomorphic. The complete Cayley maps (that is, complete map and its automorphism group is regular on the vertices) of non-isomorphic groups were not isomorphic. However, to determine the number of non-isomorphic maps, we need the following lemma.

Lemma 4.1. *Let $\mathcal{M}$ be an orientable vertex transitive map with underlying graph K_p and $p \geq 5$ a prime. Let $G = \mathsf{Aut}(\mathcal{M})$. If $G \cong \mathbb{Z}_p{:}G_\alpha$, where G_α is a cyclic group for any $\alpha \in V$, then $\mathcal{M}^\sigma \cong \mathcal{M}$ for each $\sigma \in \mathsf{Aut}(G)$. On the contrary, $\sigma \in \mathsf{Aut}(G)$ if $\mathcal{M}^\sigma \cong \mathcal{M}$.*

Proof. Since $G \cong \mathbb{Z}_p{:}G_\alpha$ such that G_α is a cyclic group for any $\alpha \in V$, it follows that by [4, Lemma 4.5],

$$\mathsf{Aut}(G) = \mathbb{Z}_p{:}N_{\mathsf{Aut}(\mathbb{Z}_p)}(G_\alpha) \cong \mathbb{Z}_p{:}N_{\mathbb{Z}_{p-1}}(G_\alpha) = \mathbb{Z}_p{:}\mathbb{Z}_{p-1}.$$

Suppose that σ fixes α for each $\sigma \in \mathsf{Aut}(G)$. Since $\mathbb{Z}_p$ is a regular and normal subgroup of $\mathsf{Aut}(G)$, we have that for any $1 \neq x \in \mathbb{Z}_p$,

$$\alpha^x = \begin{cases} \alpha x, & \text{if } x \text{ is the right multiplication.} \\ x^{-1}\alpha, & \text{if } x \text{ is the left multiplication.} \end{cases}$$

So $\alpha^x \neq \alpha$, namely, x does not fix α. It follows that $\sigma \in \mathbb{Z}_{p-1}$, and then $G_\alpha^\sigma = G_\alpha$ for each $\sigma \in \mathbb{Z}_{p-1}$. Further, for each $\tau \in \mathsf{Aut}(G)$, since

$$\mathcal{M}^\tau = \mathcal{M}^{x\sigma} = \mathcal{M}^\sigma \cong \mathcal{M}$$

such that $\tau = x\sigma$, where $x \in \mathbb{Z}_p$ and $\sigma \in \mathbb{Z}_{p-1}$. Hence $\mathcal{M}^\sigma \cong \mathcal{M}$ for each $\sigma \in \mathsf{Aut}(G)$ by arbitrariness of x.

On the contrary, if $\mathcal{M}^\sigma \cong \mathcal{M}$, then $\mathsf{Aut}(\mathcal{M}^\sigma) \cong \mathsf{Aut}(\mathcal{M}) = G$. It follows that $(\mathsf{Aut}(\mathcal{M}))^\sigma = G^\sigma = G \cong \mathsf{Aut}(\mathcal{M}^\sigma)$ for each $\sigma \in \mathsf{Aut}(\mathcal{M})$. Note that $\mathsf{Z}(G) = 1$, then $G = G/\mathsf{Z}(G) \cong \mathsf{Inn}(G) \lhd \mathsf{Aut}(G)$. Hence $\sigma \in \mathsf{Aut}(G)$. □

Now, we determine the number of non-isomorphic vertex transitive embeddings of K_p if $G_\alpha \cong \mathbb{Z}_k$. Let

$$\mathcal{A}_r = \{\mathcal{M}_r := \mathcal{M}_r(\mathbf{1}, \beta_2, \beta_3, \cdots, \beta_r, a) | \beta_1 = \mathbf{1},\ \beta_i \in \Delta_i,\ \beta_i^a = \beta_{i+r},$$
$$\textit{where } 1 \leq i \leq r,\ o(a) = k \geq 2, \textit{ and read the subscripts modulo } p-1\}.$$

Then $\mathcal{A}_r$ is a finite non-empty set, and $|\mathcal{A}_r| = (r-1)!k^{r-1}\phi(k)$. Let $X = \mathsf{Aut}(G)$. Then

$$X \cong \mathbb{Z}_p{:}N_{\mathsf{Aut}(\mathbb{Z}_p)}(\mathbb{Z}_k) \cong \mathbb{Z}_p{:}\mathbb{Z}_{p-1}.$$

Let $\langle z \rangle = \mathbb{Z}_{p-1}$. Since $\mathbb{Z}_p \lhd X$, and $\mathbb{Z}_p$ acting on V is regular, we by the 'Frattini Argument' (or [5, Exercise 1.4.1]) have that $X_\alpha \cong \mathbb{Z}_{p-1} = \langle z \rangle$, and further $G \lhd X$.

Lemma 4.2. *If r is a prime, then the number of non-isomorphic orientable vertex transitive maps of K_p equals*

$$\frac{(r-1)!k^{r-1}\phi(k) - \phi(p-1)}{r}.$$

Proof. Since $o(a) = k = \frac{p-1}{r}$, it follows that $z^r = a$. Let

$$(\mathbf{1}, \beta_2, \beta_3, \cdots, \beta_r, \mathbf{1}^a, \beta_2^a, \beta_3^a, \cdots, \beta_r^a, \cdots, \mathbf{1}^{a^{k-1}}, \beta_2^{a^{k-1}}, \beta_3^{a^{k-1}}, \cdots, \beta_r^{a^{k-1}})$$

be a circular permutation of $F^\#$ such that $\beta_i = \mathbf{1}^{z^i}$ and $2 \leq i \leq r$. Then there have $z^i : \mathbf{1} \mapsto \beta_i \mapsto \beta_{2i} \cdots \mapsto \mathbf{1}$, $z^i : \beta_2 \mapsto \beta_{i+2} \mapsto \beta_{2i+2} \cdots \mapsto \beta_2, \cdots, z^i : \beta_{i-1} \mapsto \beta_{2i-1} \mapsto \beta_{3i-1} \cdots \mapsto \beta_{i-1}$. Furthermore, z^i can be identified with the permutation

$$z^i = (\mathbf{1}\beta_i\beta_{2i} \cdots \beta_{p-i})(\beta_2\beta_{i+2}\beta_{2i+2} \cdots \beta_{p-i+1}) \cdots (\beta_{i-1}\beta_{2i-1}\beta_{3i-1} \cdots \beta_{p-1}).$$

Thus $G_\alpha = \langle z^i \rangle$ and $a = z^r \in \langle z^i \rangle$, which is a contradiction as $i \nmid r$.

Let $(\mathbf{1}, \beta_2, \beta_3, \cdots, \beta_r, \mathbf{1}^a, \beta_2^a, \beta_3^a, \cdots, \beta_r^a, \cdots, \mathbf{1}^{a^{k-1}}, \beta_2^{a^{k-1}}, \beta_3^{a^{k-1}}, \cdots, \beta_r^{a^{k-1}})$ be a circular permutation of $F^{\#}$ such that $\beta_2 = \mathbf{1}^z$, and gives rise to a unique Cayley embedding $\mathcal{M}_1$ of K_p. Then there have

$$z : \mathbf{1} \mapsto \beta_2 \mapsto \beta_3 \mapsto \cdots \mapsto \beta_r \mapsto \mathbf{1}^a \mapsto \beta_2^a \mapsto \cdots \mapsto \beta_r^a \mapsto \cdots \mapsto \mathbf{1}.$$

and z^i can be identified with the permutation

$$z = (\mathbf{1}, \beta_2, \cdots \beta_r, \mathbf{1}^a, \beta_2^a, \cdots \beta_r^a, \mathbf{1}^{a^{k-1}}, \beta_2^{a^{k-1}}, \cdots \beta_r^{a^{k-1}}).$$

It follows that $\mathsf{Aut}(\mathcal{M}_1) = F^+{:}\langle z \rangle = G.r > G$, and $\mathcal{M}_1$ is arc transitive. Thus $\mathcal{M}_1 \in \mathcal{A}_1$, $\mathcal{A}_1 \subset \mathcal{A}_r$ and

$$|\mathcal{A}_r \setminus \mathcal{A}_1| = |\mathcal{A}_r| - |\mathcal{A}_1| = (r-1)!k^{r-1}\phi(k) - \phi(p-1).$$

So $X \setminus G$ contains no element which is an automorphism of $\mathcal{M}'_r$ for $\mathcal{M}'_r \in \mathcal{A}_r \setminus \mathcal{A}_1$. Since $G \lhd X$ and

$$(X/G)_{\mathcal{M}'_r} = \{xG \in X/G | (\mathcal{M}'_r)^{xG} = (\mathcal{M}'_r)^{Gx} = (\mathcal{M}'_r)^x = \mathcal{M}'_r\} = G,$$

we have that $X/G \cong \mathbb{Z}_r$ acting on $\mathcal{A}_r \setminus \mathcal{A}_1$ is semiregular. Let X act on $\mathcal{A}_r \setminus \mathcal{A}_1$. Then $(\mathcal{M}'_r)^X$ is an orbit of this action, and the length of this orbit equals

$$|(\mathcal{M}'_r)^X| = \frac{|X|}{|X_{\mathcal{M}'_r}|} = \frac{|X|}{|\mathsf{Aut}(\mathcal{M}'_r)|} = \frac{|X|}{|G|} = r.$$

It follows that by Lemma 4.1, there are

$$\frac{|\mathcal{A}_r \setminus \mathcal{A}_1|}{r} = \frac{|\mathcal{A}_r| - |\mathcal{A}_1|}{r} = \frac{(r-1)!k^{r-1}\phi(k) - \phi(p-1)}{r}$$

non-isomorphic orientable vertex transitive maps of K_p. □

Furthermore, we can obtain the following results by the proof of Lemma 4.2.

Lemma 4.3. *If $r = p_1p_2$ with p_i different primes and $i = 1, 2$, then the number of non-isomorphic vertex-transitive maps of K_p equals*

$$\frac{|\mathcal{A}_{p_1p_2} \setminus (\mathcal{A}_{p_1} \cup \mathcal{A}_{p_2})|}{p_1p_2} = \frac{|\mathcal{A}_{p_1p_2}| - |\mathcal{A}_{p_1}| - |\mathcal{A}_{p_2}| + |\mathcal{A}_1|}{p_1p_2}.$$

For example, when $r = 15 = 3 \cdot 5$, then the number of non-isomorphic vertex-transitive complete maps equals $\frac{|\mathcal{A}_{15} \setminus (\mathcal{A}_3 \cup \mathcal{A}_5)|}{15} = \frac{|\mathcal{A}_{15}| - |\mathcal{A}_3| - |\mathcal{A}_5| + |\mathcal{A}_1|}{15}$.

Lemma 4.4. *If $r = p_1^2p_2$ with p_i different prime and $i = 1, 2$, then the number of non-isomorphic vertex-transitive maps of K_p equals*

$$\frac{|\mathcal{A}_{p_1^2p_2} \setminus (\mathcal{A}_{p_1p_2} \cup \mathcal{A}_{p_1^2})|}{p_1^2p_2} = \frac{|\mathcal{A}_{p_1^2p_2}| - |\mathcal{A}_{p_1p_2}| - |\mathcal{A}_{p_1^2}| + |\mathcal{A}_{p_1}|}{p_1^2p_2}.$$

For example, when $r = 28 = 2^2 \cdot 7$, then the number of non-isomorphic vertex-transitive complete maps equals $\frac{|\mathcal{A}_{28} \setminus (\mathcal{A}_{14} \cup \mathcal{A}_4)|}{28} = \frac{|\mathcal{A}_{28}| - |\mathcal{A}_{14}| - |\mathcal{A}_4| + |\mathcal{A}_2|}{28}$.

Furthermore, if $r = p_1^{l_1}p_2^{l_2} \cdots p_t^{l_t}$, where p_i are different from each other primes, $1 \leq l_i$ and $1 \leq i \leq t$, then here can obtain generalization of the above results.

5. Proof of Theorem

In this section, we complete the proof of Theorem 1.1 in view of the above series of results.

Proof of Theorem 1.1.

Let $\mathcal{M} = (V, E, F)$ be an orientable vertex transitive map with underlying graph $\mathsf{K}_p = (V, E)$, where $p \geq 5$ is a prime. Let $G = \mathsf{Aut}(\mathcal{M})$. We by Lemma 3.2 have that $\mathcal{M}$ is a Cayley map of $\mathbb{Z}_p$, and $G = \mathbb{Z}_p{:}G_\alpha$ is a Frobenius group, where G_α is a cyclic group for each $\alpha \in V$. Further, if $G_\alpha \cong \mathbb{Z}_k$ acting on the neighborhood of α has r orbits with $(k, p) = 1$, $rk = p - 1$ and $r \geq 2$ a prime, then by Lemma 3.3 and Lemma 4.2 there are exactly

$$[(r-1)!k^{r-1}\phi(k) - \phi(p-1)]/r$$

non-isomorphic orientable vertex transitive maps of K_p. □

6. Conclusions

Determining and enumerating all the 2-cell embeddings of a given class of graphs is one of the main research topics in topological graph theory. Complete maps have always been a focus of attention for many scholars. Li characterized the classification of vertex-transitive embeddings of complete graphs in [17]. The manuscript obtained accurate counting results of non-isomorphic orientable vertex-transitive complete maps with p vertices, where $p \geq 5$ is a prime.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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