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On enhanced general linear groups: nilpotent orbits and support variety for Weyl module

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Abstract: Associated with a reductive algebraic group G and its rational representation (ρ, M) over an algebraically closed filed $\mathbf{k}$, the authors define the enhanced reductive algebraic group $\underline{G} := G \ltimes_\rho M$, which is a product variety $G \times M$ and endowed with an enhanced cross product in [5]. If $\underline{G} = GL(V) \ltimes_\eta V$ with the natural representation (η, V) of $\mathrm{GL}(V)$, it is called an enhanced general linear algebraic group. And the authors give a precise classification of finite nilpotent orbits via a finite set of so-called enhanced partitions of $n = \dim V$ for the enhanced group $\underline{G} = \mathrm{GL}(V) \ltimes_\eta V$ in [6, Theorem 3.5]. We will give another way to prove this classification theorem in this paper. Then we focus on the support variety of the Weyl module for $\underline{G} = \mathrm{GL}(V) \ltimes_\eta V$ in characteristic p, and obtain that it coinsides with the closure of an enhanced nilpotent orbit under some mild condition.

Keywords: enhanced nilpotent orbits; enhanced general linear algebric groups; support variety; Weyl module
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1. Introduction

This is a sequel to [5, 6]. The authors introduced the semi-reductive algebraic group in [5]. In [6], the authors studied on the nilpotent orbit theory for the enhanced general linear algebraic group. They gave the finiteness criterion of nilpotent orbits under the enhanced group action and decribed the precise indexing for the enhanced nilpotent cone $\mathcal{N}(\underline{\mathfrak{g}})$ under the adjoint action of $\underline{G} = \mathrm{GL}(V) \ltimes_\eta V$ based on G_J-conjugacy classes in $\widetilde{V} = V/\mathrm{im}J$, where G_J is the centralizer of nipotent element J in $G = \mathrm{GL}(V)$ and imJ is the image of J on V. They made a research about the related intersection cohomology. In this paper, we will give another way to classify the $\underline{G}$-orbits on $\mathcal{N}(\underline{\mathfrak{g}})$. Our work is based on the results of $G = \mathrm{GL}(V)$-orbits on $\mathcal{N}(\underline{\mathfrak{g}})$ in the paper [1]. It proved that G-orbits in $\mathcal{N}(\underline{\mathfrak{g}})$ are parametrized by the bipartitions $(\mu; \nu)$ of n, where $n = \dim V$. We define an equivalence relation on the set Q_n of bipartitions $(\mu; \nu)$ of n and there exists an unique maximal element in every equivalence class 3.6

under the well-defined partial order on Q_n. On the other hand, the main classification problem about the $\underline{G} = \mathrm{GL}(V) \ltimes_\eta V$-orbits in $\mathcal{N}(\underline{\mathfrak{g}})$ are one-to-one correspondence to the equivalent class on Q_n (Lemma 3.7). Hence we get the main classification Theorem 3.8.

Jantzen proposed in [2, 2.7(1)] a conjecture for a reductive algebraic group G over $\mathbf{k}$ with $char(\mathbf{k}) = p$ good, which says that the variety of an induced module must be the closure of a certain Richardson orbit. He verified this is true for type A (the conjecture for any case is proved by Nakano-Parshall-Vella in [3]). We repeat the same story in §4.1 for the enhanced case, and find that it still true under the mild condition $char(\mathbf{k}) > dim(V)$.

2. Semi-reductive groups and semi-reductive Lie algebras

In this section, all vector spaces and varieties are over a field $\mathbf{k}$ which stands for either the complex number field $\mathbb{C}$, or an algebraically closed field of characteristic $p > 0$.

Definition 2.1. *An algebraic group G over $\mathbf{k}$ is called semi-reductive if $G = G_0 \ltimes U$ with G_0 being a reductive subgroup, and U the unipotent radical. Let $\mathfrak{g} = \mathsf{Lie}(G)$, and $\mathfrak{g}_0 = \mathsf{Lie}(G_0)$ and $\mathfrak{u} = \mathsf{Lie}(U)$, then $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{u}$.*

Example 2.2. *(Enhanced reductive algebraic groups) Let G_0 be a connected reductive algebraic group over $\mathbf{k}$, and (M, ρ) be a finite-dimensional rational representation of G_0 with representation space M over $\mathbf{k}$. Consider the product variety $G_0 \times M$. Regard M as an additive algebraic group. The variety $G_0 \times M$ is endowed with an enhanced cross product structure denoted by $G_0 \times_\rho M$, by defining for any $(g_1, v_1), (g_2, v_2) \in G_0 \times M$*

$$(g_1, v_1) \cdot (g_2, v_2) := (g_1 g_2, \rho(g_1) v_2 + v_1). \tag{2.1}$$

Then it's easy to check that $\underline{G_0} := G_0 \times_\rho M$ becomes a group with identity $(e, 0)$ for the identity $e \in G_0$, and $(g, v)^{-1} = (g^{-1}, -\rho(g)^{-1} v)$ by a straightforward computation. And $G_0 \times_\rho M$ has a subgroup G_0 identified with $(G_0, 0)$ and a subgroup M identified with (e, M). Furthermore, $\underline{G_0}$ is connected since G_0 and M are irreducible varieties. We call $\underline{G_0}$ an enhanced reductive algebraic group associated with the representation space *M. What is more, G_0 and M are closed subgroups of $\underline{G_0}$, and M is a normal closed subgroup. Actually, we have $(g, w)(e, v)(g, w)^{-1} = (e, \rho(g)v)$ for any $(g, w) \in \underline{G_0}$. From now on, we will write down $\dot{g}$ for $(g, 0)$ and e^v for (e, v) unless other conventions. It is clear that $e^v \cdot e^w = e^{v+w}$ for $v, w \in V$.*

Suppose $\underline{\mathfrak{g}_0} = \mathsf{Lie}(\underline{G_0})$. Then $(M, \mathsf{d}(\rho))$ becomes a representation of $\mathfrak{g}_0$. Naturally, $\mathsf{Lie}(\underline{G_0}) = \mathfrak{g}_0 \oplus M$, with Lie bracket

$$[(X_1, v_1), (X_2, v_2)] := ([X_1, X_2], \mathsf{d}(\rho)(X_1) v_2 - \mathsf{d}(\rho)(X_2) v_1),$$

which is called an enhanced reductive Lie algebra.

Clearly, $\underline{G_0}$ is a semi-reductive group with M being the unipotent radical.

In fact, the enhanced reductive algebraic group $\underline{G_0}$ can be realized as an subgroup of $GL(\underline{V \oplus M})$ in the above Example 2.2, where $G_0 \subset GL(V)$ and $\underline{V \oplus M}$ is one dimensional extension of $V \oplus M$. For saving the notations, we still write G_0 to represent the subgroup its realization in $GL(V)$. So we claim that the $\underline{G_0}$ have the block matrix form as follows

$$\begin{pmatrix} G_0 & 0 & 0 \\ 0 & \rho(G_0) & M \\ 0 & 0 & 1 \end{pmatrix}.$$

The element $(g, v) \in \underline{G_0}$ have the form as follows

$$\begin{pmatrix} g & 0 & 0 \\ 0 & \rho(g) & v \\ 0 & 0 & 1 \end{pmatrix}.$$

Let $\mathcal{I}_{\underline{G_0}} \subset \mathbf{k}[GL(\underline{V \oplus M})]$ be the ideal of regular fuctions that vanish on $\underline{G_0}$. Similarly, we have the ideals $\mathcal{I}_{G_0}, \mathcal{I}_M \subset \mathbf{k}[GL(\underline{V \oplus M})]$. Then we $\mathcal{I}_{\underline{G_0}} \simeq \mathcal{I}_{G_0} \otimes 1 + 1 \otimes \mathcal{I}_M$. By the definition of Lie algebra, we can have $\mathsf{Lie}(\underline{G_0}) = \mathfrak{g}_0 \oplus M$, where $\mathfrak{g}_0 = \mathsf{Lie}(G_0)$. By the communication of d_ρ with the Lie bracket on $\mathfrak{g}_0 = \mathsf{Lie}(G_0)$, we get the Lie bracket on $\mathsf{Lie}(\underline{G_0}) = \mathfrak{g}_0 \oplus M$ are as follows

$$[(X_1, v_1), (X_2, v_2)] := ([X_1, X_2], \mathsf{d}(\rho)(X_1)v_2 - \mathsf{d}(\rho)(X_2)v_1).$$

Since ρ is the rational reperesentation, we can write the block matrix form

$$\begin{pmatrix} g & v \\ 0 & 1 \end{pmatrix}$$

for the $(g, v) \in \underline{G_0}$ throughout the article.

3. Nilpotent orbits in general linear semi-reductive Lie algebras

Keep the same notations and convention as before. In particular, V is an n-dimensional vector space over $\mathbf{k}$, $\mathfrak{g} = \mathfrak{gl}_n(V)$, and $\underline{\mathfrak{g}} = \mathfrak{g} \oplus V$ is a general linear semi-reductive Lie algebra. In this section, we classify nilpotent orbits in $\underline{\mathfrak{g}}$ under the action of $\underline{G} = G \ltimes V$, where $G = GL_n(V)$, i.e., we determine $\underline{G}$-orbits of $\underline{\mathcal{N}} = \mathcal{N} \times V$, where $\underline{\mathcal{N}}$ and $\mathcal{N}$ are the nilpotent cones of $\underline{\mathfrak{g}}$ and $\mathfrak{g}$, respectively.

3.1. The classification of $\underline{G}$-orbits in $\underline{\mathcal{N}}$

3.1.1. The $GL(V)$-orbits in $\underline{\mathcal{N}}$

A partition of n is a nonincreasing sequence $\lambda = (\lambda_1, \lambda_2, \cdots)$ of nonnegative integers such that $\sum \lambda_i = n$. The set of all partitions of size n is denoted by $\mathcal{P}_n$. Its length, denoted $l(\lambda)$, is the number of nonzero terms. The transpose partition λ^t is defined by $\lambda_i^t = |\{j|\lambda_j \geq i\}|$.

It is well known that G-orbits in $\mathcal{N}$ are in bijection with $\mathcal{P}_n$, via the Jordan normal form. Explicitly, the G-orbit O_λ consists of the following elements $X \in \mathcal{N}$. For $X \in O_\lambda$, there exist positive integers $r = l(\lambda)$ and vectors $v_1, v_2, \cdots, v_r$ such that all $X^j v_i$ with $1 \leq i \leq r$ and $1 \leq j \leq \lambda_i$ are a basis for V and such that $X_i^\lambda v_i = 0$ for all i.

Let $v_{i,j} = X^{\lambda_i - j} v_i$, then

$$Xv_{i,j} = \begin{cases} v_{i,j-1}, & if\ j > 1 \\ 0, & if\ j = 1 \end{cases},$$

this basis of V is called the Jordan basis with X and λ is the Jordan type of X.

Following [1], we have the following definition and two conclusions.

Definition 3.1. (1) *A bipartition* $(\mu;\nu)$ *of* n *is an ordered pair of partitions such that* $\sum \mu_i + \sum \nu_i = n$. *The set of bipartitions of* n *is denoted by* $\mathcal{Q}_n$.

(2) *A normal basis of an element* $(X, v) \in \underline{\mathcal{N}} = \mathcal{N} \times V$ *is a Jordan basis* $\{v_{ij}\}(1 \le i \le l(\lambda),\ 1 \le j \le \lambda_i)$ *in* V *for* X *of Jordan type* $\lambda = (\lambda_1, \lambda_2, \cdots)$ *such that* $v = \sum_{i}^{l(\lambda)} v_{i,\mu_i}$ *and* $\mu = (\mu_1, \mu_2, \cdots), \nu = (\nu_1, \nu_2, \cdots) = \lambda - \mu = (\lambda_1 - \mu_1, \lambda_2 - \mu_2, \cdots)$ *are partitions. The bipartition* $(\mu;\nu)$ *is called the type of the normal basis* $\{v_{ij}\}$ *or of the element* (X, v).

Lemma 3.2. *For any* $(X, v) \in \mathcal{N} \times V$, *there exists a normal basis for* (X, v) *of some type* $(\mu;\nu) \in \mathcal{Q}_n$.

Proof. Let $\{v_{ij}\}(1 \le i \le l(\lambda),\ 1 \le j \le \lambda_i)$ be the Jordan basis for X such that $v = \sum_{i=1}^{l(\lambda)} \sum_{j=1}^{\lambda_i} c_{i,j} v_{i,j}$.

Let $\mu_i \in \{0, 1, ..., \lambda_i\}$ be minimal such that $c_{i,j} = 0$ if $\mu_i < j \le \lambda_i$ and $\nu_i = \lambda_i - \mu_i$. If $\mu_i \neq 0$, we change basis of the ith Jordan block as follows such that the decomposition component is v_{i,μ_i}.

$$v'_{i,\lambda_i} = \sum_{j=1}^{\mu_i} c_{i,j} v_{i,j+\nu_i} \text{ and } v'_{i,j} = X^{\lambda_i - j} v'_{i,\lambda_i} \text{ for } 1 \le j \le \lambda_i - 1,$$

and then redefine $v_{i,j}$ to be $v'_{i,j}$. Then we have

$$v = \sum_{i=1,\mu_i \neq 0}^{l(\lambda)} v_{i,\mu_i} \ (*).$$

If $(\mu_1, \mu_2, ...)$ and $(\nu_1, \nu_2, ...)$ are partitions, we have done. If not, we need to choose the first position between $(\mu_1, \mu_2, ...)$ and $(\nu_1, \nu_2, ...)$ that does not conform to the size order relation. Since $\lambda_i \ge \lambda_{i+1}$, $\mu_i < \mu_{i+1}$ and $\nu_i < \nu_{i+1}$ can't both exist. According to these two situations, we take different adjustment operations. Let's take $\mu_1 < \mu_2$ and $\nu_1 < \nu_2$ as two examples.

Case (I). If $\mu_1 < \mu_2$, we redefine $\mu = (\mu_2, \mu_2, ...)$ and $\nu = (\lambda_1 - \mu_2, \nu_2, ...)$ $(\lambda_1 - \mu_2 \ge \lambda_2 - \mu_2 = \nu_2)$ by the two following actions. We adjust the basis of the second Jordan block first. Refine $v_{21}, v_{22}, ..., v_{2\lambda_2}$ to be

$$v_{21} - v_{11}, v_{22} - v_{12}, ..., v_{2\lambda_2} - v_{1\lambda_2}.$$

But the equation $(*)$ no longer holds, we should repeat the first operation the first Jordan block to change $v_{1\mu_1} + v_{1\mu_2}$ (or $v_{1\mu_2}$ if $\mu_1 = 0$) and to recovery equation $(*)$.

Case (II). If $\nu_1 < \nu_2$, we redefine $\nu = (\nu_1, \nu_1, ...)$ and $\mu = (\mu_1, \lambda_2 - \nu_1, ...)$ $(\mu_1 = \lambda_1 - \nu_1 \ge \lambda_2 - \nu_1)$ by the two following actions. We adjust the basis of the first Jordan block first. Refine $v_{11}, v_{12}, ..., v_{1\lambda_1}$ to be

$$v_{11}, ..., v_{1,\lambda_1-\lambda_2};\ v_{1,\lambda_1-\lambda_2+1} - v_{21}, ..., v_{1,\mu_1} - v_{2,\lambda_2-\nu_1}, ..., v_{1\lambda_1} - v_{2\lambda_2}.$$

The equation $(*)$ no longer holds after this change, we should also repeat the first operation in the second Jordan block to change $v_{2\mu_2} + v_{2,\lambda_2-\nu_1}$ (or $v_{2,\lambda_2-\nu_1}$ if $\mu_2 = 0$) and to recovery equation $(*)$.

Arguing by induction on the number $l(\lambda)$ and Repeating the above operations, we can draw the desired conclusion. □

Proposition 3.3. *The set of G-orbits in* $\mathcal{N} \times V$ *is in one-to-one correspondence with* $\mathcal{Q}_n$. *The orbit corresponding to* $(\mu;\nu)$, *denoted* $\mathcal{O}_{\mu;\nu}$, *consists of pairs* (X, v) *for which there exists normal basis of type* $(\mu;\nu)$.

In addition to the Lemma 3.2, we also need to prove the type of the normal basis is detemined uniquely by (X, v) for this proposition. But that's not the point of this article and interested readers refer to [1].

3.1.2. The clasification of the $\underline{G}$-orbits in $\underline{\mathcal{N}}$

We now give the definition of the partial order on G-orbits in $\mathcal{N} \times V = \bigcup_{(\mu;\nu)\in Q_n} O_{\mu;\nu}$.

Definition 3.4. *(1) For $(\rho;\sigma), (\mu;\nu) \in Q_n$, we say that $(\rho;\sigma) \leq (\mu;\nu)$ if and only if the following inequalities hold for all $k \geq 0$:*

$$\rho_1 + \sigma_1 + \rho_2 + \sigma_2 + \cdots + \rho_k + \sigma_k \leq \mu_1 + \nu_1 + \mu_2 + \nu_2 + \cdots + \mu_k + \nu_k, \text{ and}$$
$$\rho_1 + \sigma_1 + \cdots + \rho_k + \sigma_k + \rho_{k+1} \leq \mu_1 + \nu_1 + \cdots + \mu_k + \nu_k + \mu_{k+1}.$$

(2) If $(\rho;\sigma) < (\mu;\nu)$ and there is no $(\tau;\upsilon) \in Q_n$ such that $(\rho;\sigma) < (\tau;\upsilon) < (\mu;\nu)$ for $(\rho;\sigma), (\mu;\nu) \in Q_n$, then we say that $(\mu;\nu)$ dominates $(\rho;\sigma)$.

Note that the inequalities of the first kind simply say that $\rho + \sigma \leq \mu + \nu$ for the dominant order. Obviously $\rho \leq \mu$ and $\sigma \leq \nu$ together imply $(\rho;\sigma) \leq (\mu;\nu)$, but the converse is false.

For convenience, assume that $(1)_k = \underbrace{(1, 1, \cdots, 1)}_{k}$. Denote $(\lambda_1 + 1, \lambda_2 + 1, \cdots, \lambda_k + 1, \lambda_{k+1}, \cdots)$ by $\lambda + (1)_k$ and $(\lambda_1 - 1, \lambda_2 - 1, \cdots, \lambda_k - 1, \lambda_{k+1}, \cdots)$ by $\lambda - (1)_k$ for $\lambda \in \mathcal{P}_n$. It's worth noting that $\lambda - (1)_k$ may be not a partition if $\lambda_k = \lambda_{k+1}$ on here.

Definition 3.5. *Two bipartitions $(\mu;\nu), (\varsigma;\tau) \in Q_n$ are said to be equivalent, denoted by $(\mu;\nu) \sim (\varsigma;\tau)$, if $\mu + \nu = \varsigma + \tau$ and $l(\nu) = l(\tau)$.*

Lemma 3.6. *Under the above definition, there exists an unique maximal element in every equivalent class and its form is $(\lambda - (1)_k; (1)_k) = (\lambda_1 - 1, \lambda_2 - 1, \cdots, \lambda_k - 1, \lambda_{k+1}, \cdots; \underbrace{1, 1, \cdots, 1}_{k}) \in Q_n$ for some $\lambda \in \mathcal{P}_n$.*

Proof. Assume that $(\rho;\sigma), (\mu;\nu) \in Q_n$ and $(\rho;\sigma) \sim (\mu;\nu)$. Then $\mu + \nu = \varsigma + \tau$ and $l(\nu) = l(\tau)$. Obviously, there is a partial order relationship between them. Let $\lambda = \mu + \nu$, $k = l(\nu)$ and $r(\nu)$ be the maximal integer satisfing $\nu_{r(\nu)} > 1$ in the sequence $\nu = (\nu_1, \cdots, \nu_k)$ ($r(\nu) = 0$ if $\nu_1 \leq 1$). Then $\nu - (1)_{r(\nu)}$ is also a partition and $l(\nu - (1)_{r(\nu)}) = l(\nu)$. Denote $(\rho;\sigma) = (\mu + (1)_{r(\nu)}; \nu - (1)_{r(\nu)})$, then $(\rho;\sigma) \sim (\mu;\nu)$ and $(\rho;\sigma) \geq (\mu;\nu)$. Certainly, we have $r(\sigma) \leq r(\nu)$. By the mathematical induction, we can obtain a bipartition $(\lambda - (1)_k; (1)_k) \in Q_n$ which is equivalent to the given partition $(\mu;\nu)$. It's easy to check that $(\lambda - (1)_k; (1)_k) \in Q_n$ is the maximal element in the equivalent class of $(\mu;\nu)$. □

Lemma 3.7. *For any $(X, v), (Y, w) \in \underline{\mathcal{N}}$ with that their normal types are respectively $(\mu;\nu)$ and $(\rho;\tau)$, then $(X, v), (Y, w)$ belong to a common $\underline{G}$-orbit if and only if $(\rho;\sigma) \sim (\mu;\nu)$.*

Proof. Since G is a subgroup of $\underline{G}$, it follows from the group homomorphism $G \to \underline{G}$ with $g \to (g, 0)$. Note that we have $(\sigma, w) = (\sigma, 0) \cdot (1, \sigma^{-1}w)$ for any $(\sigma, w) \in \underline{G}$. Beside, $Ad(G, 0)((X, v)) = O_{\mu;\nu}$ ($GL(V)$-orbit). So we only need to consider the action $Ad(1, V)((X, v))$. We may assume that $(Y, w) \in Ad(1, V)((X, v))$, there exists a vector $u \in V$ such that $(Y, w) = Ad(1, -u)((X, v)) = (X, Xu + v)$.

Suppose that $\{v_{i,j}\}$ is the normal basis of type $(\mu;\nu)$ for $(X,v) \in \underline{\mathcal{N}}$ and $\lambda = \mu+\nu$. For the k-th Jordan block of rank λ_k of X

$$J_k = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{\lambda_k\times\lambda_k}$$

and the component coefficient of vectors $v,u \in V$ corresponding to this block on this basis are as follows:

$$u_k = \begin{pmatrix} v_{k,1} & v_{k,2} & \cdots & v_{k,\lambda_k} \end{pmatrix}\begin{pmatrix} a_{k,1} \\ a_{k,2} \\ \vdots \\ a_{k,\lambda_k} \end{pmatrix},$$

$$v_k = \begin{pmatrix} v_{k,1} & v_{k,2} & \cdots & v_{k,\lambda_k} \end{pmatrix}\begin{pmatrix} b_{k,1} \\ b_{k,2} \\ \vdots \\ b_{k,\lambda_k} \end{pmatrix},$$

i.e., $u_k = \sum\limits_{j=1}^{\lambda_k} a_{k,j}v_{k,j}$, $v_k = \sum\limits_{j=1}^{\lambda_k} b_{k,j}v_{k,j}$ with $a_{k,j}, b_{k,j} \in \mathbf{k}$.

($\Rightarrow$) (Proof by contradiction): If $(X,v),(Y,w)$ are belong to a common $\underline{G}$-orbit, the normal type $(\rho;\sigma)$ of (Y,w) must be satisfied with $\rho+\tau = \mu+\nu = \lambda$. Assume that $(\rho;\tau) \nsim (\mu;\nu)$, then we have $l(\nu) \neq l(\tau)$. If $l(\nu) < l(\tau)$, we have $\mu_{l(\nu)+1} = \lambda_{l(\nu)+1} > \rho_{l(\nu)+1}$. Note that the component $v_{l(\nu)+1,\mu_{l(\nu)+1}} \neq 0$ of $v = \sum\limits_{i}^{l(\mu)} v_{i,\mu_i}$, then the element $Xu+v$ still contains this part and Xu never offer this part, in this particular position for any $u \in V$. If $e_{i,j}$ is the normal basis of type (ρ,τ) for $(X, Xu+v)$, we have that $Xu+v = \sum\limits_{i}^{l(\rho)} e_{i,\rho_i}$ and $e_{l(\nu)+1,\lambda_{l(\nu)+1}}$ does not exist here for $\rho_{l(\nu)+1} < \lambda_{l(\nu)+1}$. There exists an reversible linear transformation $\sigma \in GL(V)_X$ between $v_{i,j}$ and $e_{i,j}$. The component of $Xu+v$ on last position of $(l(\nu)+1)$-th Jordan block does not disappear under the transformation $\sigma \in GL(V)_X$. It contradicts what we know that $e_{l(\nu)+1,\lambda_{l(\nu)+1}}$ does not exist in $Xu+v = \sum\limits_{i}^{l(\rho)} e_{i,\rho_i}$. In a similar way, $l(\nu) > l(\tau)$ is also impossible. Then the assumption about $(\rho;\tau) \nsim (\mu;\nu)$ is incorrect. So we have that $(\rho;\tau) \sim (\mu;\nu)$ if $(X,v),(Y,w)$ are belong to a same $\underline{G}$-orbit.

($\Leftarrow$): For any $(\rho;\sigma) \sim (\mu;\nu)$, and denote $t = l(\nu) = l(\sigma)$. Let $\{v_{i,j}\}$ be the normal basis of V for (X,v), then we have that $v = \sum\limits_{i}^{l(\mu)} v_{i,\mu_i}$. We only need to choose a vector $u \in V$ such that the normal type of $Ad(1,-u)((X,v)) = (X, Xu+v)$ is just $(\rho;\sigma)$. Let $\lambda = \mu+\nu$ and $u = \sum\limits_{i=1}^{l(\lambda)}\sum\limits_{j=1}^{\lambda_i} a_{i,j}v_{i,j} = \sum\limits_{i=1}^{l(\lambda)} u_i$, where $u_i = \sum\limits_{j=1}^{\lambda_i} a_{i,j}v_{i,j}$. Denote $t = l(\nu) = l(\sigma)$:

(1) If $1 \le i \le t$ and $\rho_i = \mu_i$, take $a_{i,j} = 0$ $(2 \le j \le \lambda_i)$; if $1 \le i \le t$ and $\rho_i = \mu_i, \rho_i \neq \mu_i$, take $a_{i,\rho_i+1} = 1$, $a_{i,\mu_i+1} = -1$, and $a_{i,j} = 0$ $(2 \le j \le \lambda_i, j \neq \rho_i+1, \mu_i+1)$.

(2) If $i > t$, take $a_{i,j} = 0$ $(2 \le j \le \lambda_i)$. Then we have that $Xu + v = \sum_i^{l(\rho)} v_{i,\rho_i}$ and the normal type of $Ad(1, -u)((X, v)) = (X, Xu + v)$ is $(\rho; \sigma)$, and $\{v_{i,j}\}$ is also the normal basis for $(X, Xu + v)$ in V.

As a result, the lemma is proved and the $\underline{G}$-orbit of (X, v) is in one-to-one correspondence with the equivalent class of $(\mu; \nu)$ in Q_n by the Definition 3.5. □

Theorem 3.8. *The set of $\underline{G}$-orbits in $\underline{\mathcal{N}} = \mathcal{N} \times V$ is in one-to-one correspondence with $Q_n/_\sim = \{(\lambda_1 - 1, \lambda_2 - 1, \cdots, \lambda_k - 1, \lambda_{k+1}, \cdots ; 1, 1, \cdots, 1, 0, \cdots) \in Q_n | \lambda \in \mathcal{P}_n, k \in \mathbb{Z}_{\ge 0}\} = \{(\lambda - (1)_k; (1)_k) \in Q_n | k \in \mathbb{Z}_{\ge 0}\}$. The $\underline{G}$-orbits corresponding to $(\lambda_1 - 1, \lambda_2 - 1, \cdots, \lambda_k - 1, \lambda_{k+1}, \cdots ; 1, 1, \cdots, 1, 0, \cdots) = (\lambda - (1)_k; (1)_k) \in Q_n$, denoted by $O_{(\lambda-(1)_k;(1)_k)}$, consists of pairs $(x, v) \in \mathcal{N} \times V$ such that a normal basis of type $(\mu; \nu) \sim (\lambda - (1)_k; (1)_k)$ exists, i.e., there is a Jordan basis v_{ij} for x such that $v = \sum v_{i,\mu_i}$.*

Proof. The Lemma 3.6 implies that there is an bijection map between $\{(\lambda - (1)_k; (1)_k) \in Q_n \,|\, k \in \mathbb{Z}_{\ge 0}\}$ and the set of equivalent class in the sense of 3.5, denote $Q_n/_\sim$ by this set. On the other hand, the set of $\underline{G}$-orbits in $\underline{\mathcal{N}} = \mathcal{N} \times V$ is in one-to-one correspondence with $Q_n/_\sim$. Therefore, the theorem is proved. □

Remark 3.9. *Since $(\lambda - (1)_k; (1)_k) \in Q_n$, the number k does not have to take all the numbers in $\{1, 2, ..., l(\lambda)\}$.*

Certainly, the partial order in the paper [1] is still valid in here. Furthermore, we still have the definition of covering relations.

3.2. The dimension of $\underline{G}$-orbits

Lemma 3.10. *Keep the notations and we have the following conclusions.*

(1) *Assume that $\lambda \in \mathcal{P}_n$ and $N(\lambda) = \{k \in \mathbb{N} \,|\, (\lambda - (1)_k; (1)_k) \in Q_n\}$, then*

$$\bigcup_{k \in N(\lambda)} O_{(\lambda-(1)_k;(1)_k)} = O_\lambda \times V.$$

(2) *For any $\lambda \in \mathcal{P}_n$, $\overline{O_{(\lambda-(1)_k;(1)_k)}} \subseteq \overline{O_\lambda} \times V$.*

Proof. In fact, $O_{(\lambda-(1)_k;(1)_k)} \subseteq O_\lambda \times V$. Conversely, for any $(X, v) \in O_\lambda \times V$, its normal type (μ, ν) must be satisfied with the condition $\mu + \nu = \lambda$, then the $\underline{G}$-orbit of the element (X, v) is $O_{(\lambda-(1)_r;(1)_r)}$ for some nonnegative integer r by the Theroem 3.8. So the conclusion (1) is satisfied.

Firstly, we have $\overline{O_\lambda \times V} \subset \overline{O_\lambda} \times V$ by $O_\lambda \times V \subset \overline{O_\lambda} \times V$. On the other hand, they are irreducible and share the common dimension, so $\overline{O_\lambda \times V} = \overline{O_\lambda} \times V$. Hence the equation

$$\bigcup_{k \in N(\lambda)} \overline{O_{(\lambda-(1)_k;(1)_k)}} = \overline{O_\lambda \times V}$$

is true by (1). So

$$\overline{O_{(\lambda-(1)_k;(1)_k)}} \subseteq \overline{O_\lambda} \times V.$$

□

Definition 3.11. (1) *For $(x, v) \in \underline{\mathcal{N}}$, define*

$$A^{(X,v)} = \{(Y, w) \in \underline{\mathfrak{g}} \,|\, XY = YX, -Xw + Yv = 0\};$$

$$B^{(X,v)} = \{(Y, w) \in \underline{\mathfrak{g}} \,|\, XY = YX, -Xw + Yv = v\};$$

$$\underline{G}^{(X,v)} = \{(\sigma, w) \in \underline{G} \,|\, X\sigma = \sigma X, -Xw + \sigma v = v\}.$$

(2) *For any* $\lambda \in \mathcal{P}_n$, *define* $n(\lambda) = \sum(i-1)\lambda_i$.

The following result determines dimensions of stablizers of nilpotent elements in a general linear semi-reductive Lie algebra, so that we obtain the dimensions of nilpotent orbits.

Theorem 3.12. *Let* $(\lambda - (1)_k; (1)_k) \in \mathcal{Q}_n$ *and* $(X, v) \in \mathcal{O}_{(\lambda-(1)_k;(1)_k)}$. *Then,*

(1) *both* $A^{(X,v)}$ *and* $B^{(X,v)}$ *are irreducible affine varieties, and* $\underline{G}^{(X,v)}$ *is a principal open subvarieties of* $B^{(X,v)}$*;*

(2) $\underline{G}^{(X,v)}$ *is a connected algebraic group of dimension* $n + 2n(\lambda) + k$*;*

(3) $dim\mathcal{O}_{(\lambda-(1)_k;(1)_k)} = n^2 - 2n(\lambda) - k$.

Proof. (1) is obvious.

(2) Let $\mathfrak{g}_X = \{Y \in \mathfrak{g} | XY = YX\}$, and $W = \{-Xw + Yv \,|\, (Y, w) \in \mathfrak{g}_X \times V\}$. Then W is a subspaces of V. Moreover, it follows from Proposition 2.8(5) in [1] that $dimW = n - k$.

Let

$$\begin{aligned} \psi : \mathfrak{g}_X \times V &\longrightarrow W, \\ (Y, w) &\longrightarrow -Xw + Yv, \end{aligned}$$

which is a surjective morphism with kernel $A^{(x,v)}$. Hence

$$\begin{aligned} dimA^{(X,v)} &= dim(\mathfrak{g}_X \times V) - dimW \\ &= dim\mathfrak{g}_X + dimV - dimW \\ &= n + 2n(\lambda) + k, \end{aligned}$$

where the last equality hold by Proposition 2.8(2) in [1]. Cosequently, (2) follows from (1). (3) follows (2), since

$$\begin{aligned} dim\mathcal{O}_{(\lambda-(1)_k;(1)_k)} &= dim\underline{G} - dim\underline{G}^{(X,v)} \\ &= n^2 + n - (n + 2n(\lambda) + k) \\ &= n^2 - 2n(\lambda) - k. \end{aligned}$$

□

Corollary 3.13. *Let* $(\lambda - (1)_k; (1)_k) \in \mathcal{Q}_n$. *Then* $\overline{\mathcal{O}_{(\lambda-(1)_k;(1)_k)}} = \overline{\mathcal{O}_\lambda} \times V$ *if only if* $k = 0$.

Proof. The inclusion ⊆ is obvious, so the equation holds when the right-hand side is the same dimension as the left-hand side. □

4. Enhanced nilpotent cones and support varieties

In this section, we always assume **k** be an algebraically closed field of positive characteristic $p > 0$.

4.1. Equivariant line bundles on $\mathcal{B}$

G is a connected reductive group over $\mathbf{k}$, and T a maximal torus. B is the Borel subgroups of G that contain T. Let $X(T)$ be the set of rational characters.

Let $\lambda \in X(T)$ be a character of T. The composite of λ and the homomorphism $B \to B/B_u \to T$ defines a character of B. Let $V = \mathbf{k}v_\lambda$ be the one dimensional B-module with underling vector space over $\mathbf{k}$ and action $b \cdot v_\lambda = \lambda(b)^{-1}v_\lambda$ for any $b \in B$. We can write the fiber bundle

$$\mathscr{L}(\lambda) = G \times^B V.$$

This is a G-variety, on which the action comes from left translations in G. The natural G-morphism $\rho : \mathscr{L}(\lambda) \to \mathcal{B}$ has local sections. The fibers of ρ is just $\mathbf{k}$. So $\mathscr{L}(\lambda)$ is an equivariant line bundle on $\mathcal{B}$ defined by λ. Denote by $\Gamma(\lambda)$ the global section $\Gamma(\mathscr{L}(\lambda), \mathcal{B})$. By [4, Proposition I.5.12 and §I.5.15(1)], $\Gamma(\lambda)$ can be regarded as $H^0(\mathcal{B}, \mathscr{L}(\lambda))$, denoted by $H^0_G(\lambda)$. Furthermore, $H^0(\lambda)$ coincides with the induced G-module $\mathbf{Ind}_B^G(\lambda)$ from the one-dimensional representation given by λ of the Borel subgroup B. We have an analogue to the classical result on equivariant line bundles on the flag varieties of reductive groups (see [4, Proposition II.2.6]).

Lemma 4.1. *The global section $\Gamma(\lambda) = \Gamma(\mathscr{L}(\lambda), \mathcal{B})$ is a finite dimensional vector space, which is non-zero if and only if $\lambda \in X(T)^+$.*

4.2. Weyl modules

Let G_0 be a connected reductive group over $\mathbf{k}$ and $G = G_0 \ltimes V$ be the corresponding semi-reductive group. Let $X(T)^+$ be the set of dominant weights. For each $\lambda \in X(T)$, denote by $\mathbf{Ind}_{B^-}^G(\lambda)$ the G-modules induced from the one-dimensional representation given by λ of the Borel subgroup B^- of G generated by all subgroup U_α with $\alpha \in \Phi(G, T)^-$ and the unipotent radical V.

By the arguments as in §4.1, we know $\Gamma(\lambda)$ for $\lambda \in X(T)^+$. Furthermore, in the enhanced case, as a G-module $\Gamma(\lambda)$ coincides with the dual Weyl module $H^0_G(\lambda)$.

Lemma 4.2. *(1) As a G_0-module, $H^0_G(\lambda)$ coincides with $H^0_{G_0}(\lambda) := \mathbf{Ind}_{B_0}^{G_0}\lambda$.*

(2) The action of the unipotent radical V of G on the induced modules $H^0_G(\lambda)$ is identical, i.e. $H^0_G(\lambda)^V = H^0_G(\lambda)$.

Proof. (1) Note that we have algebraic group isomorphism $G_0 \cong G/V$ and $B_0 \cong B/V$. On the other hand, the one-dimensional B-module λ is endowed with identical action of V. Hence by the definition we have the first statement.

(2) Recall that

$$H^0_G(\lambda) = \{f \in \mathbf{k}[G] \mid f(gb) = \lambda(b)^{-1} f(g) \ \ \forall g \in G(A), b \in B(A), \text{ for all } A\}.$$

Here A stands for any commutative $\mathbf{k}$-algebra. The action of G is given by left translation. For any $f \in H^0(\lambda)$ we want to prove

$$v \cdot f = f \ \ \forall v \in V(A) := V \otimes_{\mathbf{k}} A.$$

Actually, for any $g \in G(A)$ we can write $g = (u_1 t u_2, v')$ for $u_1 \in U^+(A)$, $t \in T(A)$ and $u_2 \in U(A)$ and $v' \in V(A)$. Then we have

$$v \cdot f(g) = v \cdot f((u_1 t u_2, v'))$$

$$= f(v^{-1}(u_1tu_2, v')$$
$$= f((u_1tu_2, -\rho(u_1tu_2)^{-1}v + v')$$
$$= \lambda(t)^{-1}f(u_1).$$

On the other hand,

$$f(g) = f((u_1tu_2, v')) = \lambda(t)^{-1}f(u_1).$$

Hence $f \in H^0_G(\lambda)^V$. We complete the proof. □

Note that $\mathsf{Lie}(V) = V$. We have the following corollary.

Corollary 4.3. *As a* $\mathfrak{g}$*-module,* $H^0_G(\lambda)$ *is annihilated by* V.

4.3. Support varieties for enhanced general linear group

Let $\mathfrak{g}$ be a finite dimensional restricted Lie algebra over $\mathbf{k}$ (with p-th power operation denoted by $x \to x^{[p]}$) One can associate to each finite-dimensional restricted -module M a subvariety of $\mathfrak{g}$ which is defined using cohomology theory (compare [7]), but has the following more elementary description (see [8]). It consists of 0 and of all nonzero elements $X \in \mathfrak{g}$, with $X^{[p]} = 0$ such that M is not injective (= projective) as a restricted module for the one dimensional p-Lie algebra $\mathbf{k}[X]$. Hence we have

$$\mathcal{V}_{\underline{\mathfrak{g}}}(H^0_{\underline{G}}(\lambda)) = \{(X, v) \in \underline{\mathfrak{g}} : (X, v)^{[p]} = 0$$

and $H^0_{\underline{G}}(\lambda)$ is not projective as a restricted

$$\mathbf{k}[(X, v)] - \text{module}\} \cup \{(0, 0)\}.$$

Theorem 4.4. *Let* $G_0 = \mathrm{GL}(V)$ *over* $\mathbf{k}$*,* $G = G_0 \ltimes V$ *and* $char(\mathbf{k}) = p > dim(V)$*. Then for any Weyl-module* $H^0_G(\lambda)$ *of the enhanced general linear group G, there exists an G-orbit* $\mathcal{O}_{\lambda-(1)_k,(1)_k}$ *in the sense of Theorem* 3.8*, such that the support variety of* $H^0_G(\lambda)$ *coincides with* $\overline{\mathcal{O}_{\lambda-(1)_k,(1)_k}}$.

Proof. According to Corollary 3.13, $\overline{\mathcal{O}_{(\lambda-(1)_k;(1)_k)}} = \overline{\mathcal{O}_\lambda} \times V$ if only if $k = 0$. On the one hand, by [2] there exists a unique dominant integer weight λ such that the support variety $\mathcal{V}_{\mathfrak{g}_0}(H^0_{G_0}(\lambda))$ of $H^0_{G_0}(\lambda)$ is just $\overline{\mathcal{O}_\lambda}$. By the realization of matrix form for (X, v), we have $X^{[p]} = 0$ if $(X, v)^{[p]} = 0$ and the reverse is true when $char(\mathbf{k}) = p > dim(V)$.

On the other hand, the second condition of the description of $\mathcal{V}_{\mathfrak{g}_0}(H^0_{G_0}(\lambda))$ and $\mathcal{V}_{\mathfrak{g}}(H^0_G(\lambda))$ is also equivalent by Corollary 4.3. So we claim $\mathcal{V}_{\mathfrak{g}}(H^0_G(\lambda)) = \overline{\mathcal{O}_\lambda} \times V = \overline{\mathcal{O}_{(\lambda;(0))}}$. The proof is completed. □

5. Conclusions

As we all know that the nilpotent orbital theory of reductive Lie algebras over algebraically closed fields is quite perfect. But there are relatively few theories for the non-reductive case. This article is a discussion of a special non-reductive case (the Enhanced general linear Lie algebra) and it guarantees that the number of nilpotent orbits is finite. It is difficult to ensure the finite condition of the number of nilpotent orbits for other semi-reductive Lie algebras, which makes our further study more difficult. These challenging issues will be our future research topics.

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Conflict of interest

The author declares no conflict of interest.

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