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Research article

Estimation of the general population parameter in single- and two-phase sampling

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Abstract: Estimation of population characteristics has been an area of interest for many years. Various estimators of the population mean and the population variance have been proposed from time-to-time with a view to improve efficiency of the estimates. In this paper, we have proposed some estimators for estimation of the general population parameters. The estimators have been proposed for single-phase and two-phase sampling using information of single and multiple auxiliary variables. The bias and mean square errors of the proposed estimators have been obtained. Some comparison of the proposed estimators has been done with some existing estimators of mean and variance. Some specific cases of the proposed estimators have been discussed. Simulation and numerical study have also been conducted to see the performance of the proposed estimators.

Keywords: general population parameter; regression estimator; auxiliary variable; bias; mean square error
Mathematics Subject Classification: 62D05, 62F10, 62J05

1. Introduction

Estimation of some population parameters, using a specific sampling design, has been an interesting area of research. The popular parameters which have been an area of interest, in simple random sampling, are the population mean and variance. The basic estimators of the population mean and variance in simple random sampling are sample mean, $\bar{y}$, and sample variance, s^2. In certain situations, the information of some auxiliary variables is also available and can be used to obtain more efficient estimators for some population parameters. Several authors have proposed some improved estimators of the population mean and the population variance by using the information of the auxiliary variables. The popular estimators of population mean, using information of auxiliary variables, are the ratio and regression estimators given by [1]. The ratio and regression estimators have attracted several authors, and different modifications have been proposed from time to time. A class of estimators of population mean by using information of some auxiliary variables has been proposed by [2]. Another class of regression and ratio-product type estimators has been proposed by [3], and it performs better than the classical ratio estimator. Several estimators of population mean in cases of single- and two-phase sampling have been proposed by [4]. A general class of estimators of the population mean in single- and two-phase sampling has been proposed by [5]. More details on estimators of population mean can be found in [6,7], among others.

In recent years, the estimation of population variance has also attracted a lot of authors. Classical ratio and regression estimators of the population variance in single-phase sampling have been proposed by [8,9]. An improved ratio type estimator of the population variance has been proposed by [10]. Some ratio and regression type estimators of population variance in two-phase sampling have been proposed by [11]. The exponential type estimators have also attracted some authors in recent times and [12] have proposed an exponential estimator of population variance. Some general classes of exponential estimators have been proposed by [13,14]. An estimator of coefficient of variation in single-phase sampling has been proposed by [15]. Some other notable works on variance estimation are [16–21], among others.

Recently, [22] proposed an estimator of general population parameters in single-phase sampling. The estimator has been proposed by using information of a single auxiliary variable. The estimator provides a unified way to estimate the population mean, variance and coefficient of variation for specific values of the constants involved. In this paper, we have proposed some estimators of general population parameters in single- and two-phase sampling. The estimators have been proposed using information of a single and multiple auxiliary variables. The plan of the paper is as follows.

Some methodology and notations are given in Section 2. The new estimators of general population parameters for single phase sampling are proposed in Section 3. The estimators have been proposed by using information of single and multiple auxiliary variables. The expressions for the bias and the mean square error (*MSE*) of the proposed estimators are obtained. In Section 4, estimators for the general population parameters are proposed for two-phase sampling alongside the expressions for the bias and *MSE* of the proposed two-phase sampling estimators. In Section 5, the comparison of the estimators of specific parameters is given. Some numerical study of the proposed estimator is given in Section 6. The numerical study comprises simulation study and applications using some real populations, and the conclusions and recommendations are given in Section 7.

2. Methodology and notations

In this section, we have given some methodology and notations that will be used in this paper. Suppose that the units of a population are labeled as $U_1, U_2, \ldots, U_N$ while the values of some variable of interest are $Y_1, Y_2, \ldots, Y_N$. Suppose, further, that the estimation of some general population parameter

$$t_{(a,b)} = \bar{Y}^a \left(S_y^2\right)^{b/2}$$

is required, where

$$\bar{Y} = N^{-1}\sum_{i=1}^{N} Y_i$$

and

$$S_y^2 = (N-1)^{-1} \sum_{i=1}^{N} (Y_i - \bar{Y})^2$$

are, respectively, the population mean and variance of Y. It is to be noted that the general parameter $t_{(a,b)}$ reduces to the population mean for a=1 and b=0, and it reduces to the population variance for a=0 and b=2 and to the coefficient of variation for $a = -1$ and b=1. When information of some auxiliary variable is known, then the conventional regression estimator, using a sample of size n, is

$$\bar{y}_{lr} = \bar{y} + \beta(\bar{X} - \bar{x}), \tag{1}$$

where $\beta = S_{xy} / S_x^2$ is the population regression coefficient between X and Y, and

$$\bar{X} = N^{-1} \sum_{i=1}^{N} X_i$$

and

$$\bar{x} = n^{-1} \sum_{i=1}^{n} x_i$$

are the population and the sample mean of the auxiliary variable X. The mean square error of (1) is

$$MSE(\bar{y}_{lr}) = \theta S_y^2 (1 - \rho_{yx}^2), \tag{2}$$

where

$$\theta = n^{-1} - N^{1}$$

and

$$\rho = S_{yx} / \sqrt{S_x^2 S_y^2}$$

is the population correlation coefficient between X and Y.

In some situations, the population information of auxiliary variable is not available, and in such situations the regression estimator (1) cannot be used. The problem can be solved by using a two-phase sampling technique. In two-phase sampling, a first phase sample of size n_1 is drawn from a population of size N, and information of an auxiliary variable is recorded. A sub-sample of size $n_2 < n_1$ is drawn from the first-phase sample, and information of the auxiliary variable and the study variable is recorded. The conventional regression estimator, in two phase sampling, is given as

$$\bar{y}_{lr(2)} = \bar{y}_{(2)} + \beta \left[\bar{x}_{(1)} - \bar{x}_{(2)} \right], \tag{3}$$

where

$$\bar{y}_{(2)} = n_2^{-1} \sum_{i=1}^{n_2} y_i$$

is second phase sample mean of study variable Y,

$$\bar{x}_{(2)} = n_2^{-1} \sum_{i=1}^{n_2} x_i$$

is the second-phase sample mean of auxiliary variable X, and

$$\bar{x}_1 = n_{(1)}^{-1} \sum_{i=1}^{n_1} x_i$$

is the first-phase sample mean of auxiliary variable *X*. The *MSE* of two-phase sampling regression estimator is

$$MSE\left(\bar{y}_{lr(2)}\right) = \bar{Y}^2 C_y^2 \left[\theta_2 \left(1-\rho_{yx}^2\right) + \theta_1 \rho_{yx}^2\right], \quad (4)$$

where

$$\theta_2 = n_2^{-1} - N^{-1},$$

and

$$\theta_1 = n_1^{-1} - N^{-1}.$$

The regression estimator of population variance is given by [9] as

$$s_{y(lr)}^2 = s_y^2 + \gamma\left(S_x^2 - s_x^2\right), \quad (5)$$

where γ is a constant, S_x^2 and s_x^2 are, respectively, the population and the sample variances of the auxiliary variable, and s_y^2 is the sample variance of *Y*. The estimator for two-phase sampling can be easily written. Several modifications of the two-phase sampling regression estimator of mean are given in [6].

The derivation of bias and *MSE* of the estimators of the mean and the variance require certain notations. In this paper, we will assume that the sample mean and the sample variance of study and auxiliary variable are connected with the population mean and the population variance as

$$\bar{y} = \bar{Y}\left(1+\varepsilon_y\right), \bar{x}_j = \bar{X}\left(1+\varepsilon_{x_j}\right), s_y^2 = S_y^2\left(1+e_y\right),$$

and

$$s_{x_j}^2 = S_x^2\left(1+e_{x_j}\right).$$

The relation between sample estimates and the population parameters in case of two-phase sampling is

$$\bar{y}_{(2)} = \bar{Y}\left(1+\varepsilon_{y(2)}\right), \bar{x}_{j(2)} = \bar{X}\left(1+\varepsilon_{x_j(2)}\right), s_{y(2)}^2 = S_y^2\left(1+e_{y(2)}\right),$$

and

$$s_{x_j(2)}^2 = S_x^2\left(1+e_{x_j(2)}\right).$$

The expected values of error terms $\varepsilon' s$ and $e' s$ are all zero. Some additional expectations for single- and two-phase sampling and for single auxiliary variable, are

$$\left.\begin{aligned} &E\left(\varepsilon_y^2\right) = \theta C_y^2, E\left(\varepsilon_x^2\right) = \theta C_x^2,\ E\left(e_y^2\right) = \theta \varphi_{40}^*, E\left(e_x^2\right) = \theta \varphi_{04}^*, E\left(\varepsilon_y \varepsilon_x\right) = \theta \rho_{yx} C_x C_y, \\ &E\left(\varepsilon_x e_x\right) = \theta \varphi_{03} C_x, E\left(\varepsilon_y e_y\right) = \theta \varphi_{30} C_y, E\left(e_y \varepsilon_x\right) = \theta \varphi_{21} C_x, E\left(\varepsilon_y e_x\right) = \theta \varphi_{12} C_y, E\left(e_y e_x\right) = \theta \varphi_{22}^*. \end{aligned}\right\} \quad (6)$$

$$\left.\begin{aligned}&E\left(\varepsilon_{y(2)}^2\right)=\theta_2 C_y^2, E\left(e_{y(2)}^2\right)=\theta_2\varphi_{40}^*, E\left(\varepsilon_{y(2)}e_{y(2)}\right)=\theta_2\varphi_{30}C_y, E\left[\left(\varepsilon_{x(2)}-\varepsilon_{x(1)}\right)^2\right]=\left(\theta_2-\theta_1\right)C_x^2,\\&E\left[\left(e_{x(2)}-e_{x(1)}\right)^2\right]=\left(\theta_2-\theta_1\right)\varphi_{04}^*, E\left[\varepsilon_{y(2)}\left(\varepsilon_{x(2)}-\varepsilon_{x(1)}\right)\right]=\left(\theta_2-\theta_1\right)\rho_{yx}C_xC_y,\\&E\left[e_{y(2)}\left(e_{x(2)}-e_{x(1)}\right)\right]=\left(\theta_2-\theta_1\right)\varphi_{22}^*, E\left[\varepsilon_{y(2)}\left(e_{x(2)}-e_{x(1)}\right)\right]=\left(\theta_2-\theta_1\right)\varphi_{12}C_y,\\&E\left[e_{y(2)}\left(\varepsilon_{x(2)}-\varepsilon_{x(1)}\right)\right]=\left(\theta_2-\theta_1\right)\varphi_{21}C_x, E\left[\varepsilon_{x(2)}\left(e_{x(2)}-e_{x(1)}\right)\right]=\left(\theta_2-\theta_1\right)\varphi_{03}C_x.\end{aligned}\right\} \quad (7)$$

In case of multiple auxiliary variables, we will use the following results for single- and two-phase sampling:

$$\left.\begin{aligned}&E\left(\varepsilon_y\boldsymbol{\varepsilon}_x\right)=\theta C_y\mathbf{R}\mathbf{c}_x, E\left(\boldsymbol{\varepsilon}_x\boldsymbol{\varepsilon}_x^{/}\right)=\theta\mathbf{C}_x, E\left(e_y\boldsymbol{\varepsilon}_x\right)=\theta\boldsymbol{\Phi}_{21}\mathbf{c}_x, E\left(\boldsymbol{\varepsilon}_x\mathbf{e}_x^{/}\right)=\theta\boldsymbol{\Phi}_{012},\\&E\left(\varepsilon_y\mathbf{e}_x\right)=\theta C_y\boldsymbol{\varphi}_{12}, E\left(e_y\mathbf{e}_x\right)=\theta\boldsymbol{\varphi}_{22}^*, E\left(\mathbf{e}_x\mathbf{e}_x^{/}\right)=\theta\boldsymbol{\Phi}_x^*, E\left(\mathbf{e}_{x(2)}\mathbf{e}_{x(2)}^{/}\right)=\theta_2\boldsymbol{\Phi}_x^*,\\&E\left[\boldsymbol{\varepsilon}_{x(2)}\left(\boldsymbol{\varepsilon}_{x(2)}-\boldsymbol{\varepsilon}_{x(1)}\right)^{/}\right]=\left(\theta_2-\theta_1\right)\mathbf{C}_x, E\left[\boldsymbol{\varepsilon}_{x(2)}\left(\mathbf{e}_{x(2)}-\mathbf{e}_{x(1)}\right)^{/}\right]=\left(\theta_2-\theta_1\right)\boldsymbol{\Phi}_{012},\\&E\left[\varepsilon_{y(2)}\left(\boldsymbol{\varepsilon}_{x(2)}-\boldsymbol{\varepsilon}_{x(1)}\right)\right]=\left(\theta_2-\theta_1\right)C_y\mathbf{R}\mathbf{c}_x, E\left[\varepsilon_{y(2)}\left(\mathbf{e}_{x(2)}-\mathbf{e}_{x(1)}\right)^{/}\right]=\left(\theta_2-\theta_1\right)C_y\boldsymbol{\varphi}_{12},\\&E\left[e_{y(2)}\left(\boldsymbol{\varepsilon}_{x(2)}-\boldsymbol{\varepsilon}_{x(1)}\right)\right]=\left(\theta_2-\theta_1\right)\boldsymbol{\Phi}_{21}\mathbf{c}_x, E\left[e_{y(2)}\left(\mathbf{e}_{x(2)}-\mathbf{e}_{x(1)}\right)\right]=\left(\theta_2-\theta_1\right)\boldsymbol{\varphi}_{22}^*.\end{aligned}\right\} \quad (8)$$

where $\boldsymbol{\varepsilon}_x=\begin{bmatrix}\varepsilon_{x_1} & \cdots & \varepsilon_{x_q}\end{bmatrix}^{/}$, $\mathbf{e}_x=\begin{bmatrix}e_{x_1} & \cdots & e_{x_q}\end{bmatrix}^{/}$, $\mathbf{R}=diag\left(\rho_{yx_j}\right)$, $\boldsymbol{\Phi}_{21}=diag\left(\varphi_{21_j}\right)$, $\mathbf{C}_x^*=diag\left(C_{x_j}\right)$ and

$$\mathbf{c}_x=\begin{bmatrix}C_{x_1}\\C_{x_2}\\\vdots\\C_{x_q}\end{bmatrix}, \boldsymbol{\varphi}_{12}=\begin{bmatrix}\varphi_{12_1}\\\varphi_{12_2}\\\vdots\\\varphi_{12_q}\end{bmatrix}, \boldsymbol{\varphi}_{22}^*=\begin{bmatrix}\varphi_{22_1}^*\\\varphi_{22_2}^*\\\vdots\\\varphi_{22_q}^*\end{bmatrix}, \mathbf{C}_x=\begin{bmatrix}C_{x_1}^2 & \rho_{12}C_{x_1}C_{x_2} & \cdots & \rho_{1q}C_{x_1}C_{x_q}\\\rho_{21}C_{x_2}C_{x_1} & C_{x_2}^2 & \cdots & \rho_{2q}C_{x_2}C_{x_q}\\\vdots & \vdots & \ddots & \vdots\\\rho_{q1}C_{x_q}C_{x_1} & \rho_{q2}C_{x_q}C_{x_2} & \cdots & C_{x_q}^2\end{bmatrix},$$

$$\boldsymbol{\Phi}_x^*=\begin{bmatrix}\varphi_{04_j}^* & \varphi_{02_12_2}^* & \cdots & \varphi_{02_12_q}^*\\\varphi_{02_22_1}^* & \varphi_{04_2}^* & \cdots & \varphi_{02_22_q}^*\\\vdots & \vdots & \ddots & \vdots\\\varphi_{02_q2_1}^* & \varphi_{02_q2_2}^* & \cdots & \varphi_{04_q}^*\end{bmatrix},$$

and

$$\boldsymbol{\Phi}_{012}=\begin{bmatrix}\varphi_{03_1}C_{x_1} & \varphi_{01_12_2}C_{x_1} & \cdots & \varphi_{01_12_q}C_{x_1}\\\varphi_{01_22_1}C_{x_2} & \varphi_{03_2}C_{x_2} & \cdots & \varphi_{01_22_q}C_{x_2}\\\vdots & \vdots & \ddots & \vdots\\\varphi_{01_q2_1}C_{x_q} & \varphi_{01_q2_2}C_{x_q} & \cdots & \varphi_{03_q}C_{x_q}\end{bmatrix}.$$

Also,

$$\varphi_{rs_j}^*=\left(\varphi_{rs_j}-1\right), \varphi_{rs_j}=\mu_{rs_j}\Big/\left(\mu_{20}^{r/2}\mu_{02_j}^{s/2}\right),$$

$$\varphi^*_{0s_jt_h}=\left(\varphi_{0s_jt_h}-1\right),\ \varphi_{0s_jt_h}=\mu_{0s_jt_h}\Big/\left(\mu^{s/2}_{02_j0_h}\mu^{t/2}_{00_j2_h}\right),$$

$$\mu_{rs_j}=\left(N-1\right)^{-1}\sum\nolimits_{i=1}^{N}\left(y_i-\bar{Y}\right)^r\left(x_{ij}-\bar{X}_j\right)^s,$$

and

$$\mu_{0s_jt_h}=\left(N-1\right)^{-1}\sum\nolimits_{i=1}^{N}\left(x_{ij}-\bar{X}_j\right)^s\left(x_{ih}-\bar{X}_h\right)^t.$$

We will, now, propose some new estimators for single-phase sampling.

3. Estimators for single-phase sampling

In this section, we have proposed some new estimators of the general population parameter for single-phase sampling. These estimators have been proposed using information of a single and several auxiliary variables. These estimators are proposed in the following.

3.1. Estimator with single auxiliary variable

In the following, we have proposed a new estimator of general population parameter using information of a single auxiliary variable. The proposed estimator is

$$t_1=\bar{y}^a\left(s_y^2\right)^{b/2}\left[1+\alpha\left(\bar{X}-\bar{x}\right)+\beta\left(S_x^2-s_x^2\right)\right]. \tag{9}$$

It is easy to see that the proposed estimator reduces to the classical estimator of mean for

$$\left(a,b,\alpha,\beta\right)=\left(1,0,0,0\right).$$

Also, for

$$\left(a,b,\alpha,\beta\right)=\left(0,2,0,0\right),$$

the estimator (9) reduces to the classical estimator of variance. For

$$\left(a,b,\alpha,\beta\right)=\left(1,0,\alpha,\beta\right),$$

the estimator (9) becomes a regression type estimator of the population mean, and for

$$\left(a,b,\alpha,\beta\right)=\left(0,2,\alpha,\beta\right),$$

we have a regression type estimator of the population variance. Further, for

$$\left(a,b,\alpha,\beta\right)=\left(-1,1,\alpha,\beta\right),$$

the estimator (9) becomes a regression type estimator of coefficient of variation. Now, to obtain the bias and *MSE* of (9), we write the estimator using the error notations as

$$t_1=\left[\bar{Y}^a\left(1+\varepsilon_y\right)^a S_y^b\left(1+e_y\right)^{b/2}\right]\left(1-\alpha\bar{X}\varepsilon_x-\beta S_x^2e_x\right)=t_{(a,b)}\left(1+\varepsilon_y\right)^a\left(1+e_y\right)^{b/2}\left(1-\alpha\bar{X}\varepsilon_x-\beta S_x^2e_x\right).$$

Expanding, and retaining only the linear terms, we have

$$t_1 = t_{(a,b)}\left(1+a\varepsilon_y\right)\left(1+\frac{b}{2}e_y\right)\left(1-\alpha\bar{X}\varepsilon_x-\beta S_x^2 e_x\right)$$
$$= t_{(a,b)}\left[1+a\varepsilon_y+\frac{b}{2}e_y+\frac{ab}{2}\varepsilon_y e_y-\alpha\bar{X}\varepsilon_x-a\alpha\bar{X}\varepsilon_y\varepsilon_x-\frac{\alpha b}{2}\bar{X}e_y\varepsilon_x-\beta S_x^2 e_x-a\beta S_x^2\varepsilon_y e_x-\frac{b\beta}{2}S_x^2 e_y e_x\right],$$

or

$$t_1 - t_{(a,b)} = t_{(a,b)}\left[a\varepsilon_y+\frac{b}{2}e_y+\frac{ab}{2}\varepsilon_y e_y-\alpha\bar{X}\varepsilon_x-a\alpha\bar{X}\varepsilon_y\varepsilon_x -\frac{\alpha b}{2}\bar{X}e_y\varepsilon_x-\beta S_x^2 e_x-a\beta S_x^2\varepsilon_y e_x-\frac{b\beta}{2}S_x^2 e_y e_x\right]. \tag{10}$$

Applying expectation and simplifying, the bias of the proposed estimator (9) is

$$Bias\left(t_1\right) = \theta t_{(a,b)}\left[aC_y\left(\frac{b}{2}\theta\varphi_{30}-\alpha\bar{X}\rho_{yx}C_x-\beta S_x^2\varphi_{12}\right)-\frac{b}{2}\left(\alpha\bar{X}\varphi_{21}C_x-\beta S_x^2\varphi_{22}^*\right)\right]. \tag{11}$$

Again, squaring (10) and retaining only the quadratic terms, we have

$$\left(t_1-t_{(a,b)}\right)^2 = t_{(a,b)}^2\left[a^2\varepsilon_y^2+\frac{b^2}{4}e_y^2+\alpha^2\bar{X}^2\varepsilon_x^2+\beta^2S_x^4e_x^2+ab\varepsilon_y e_y-2a\alpha\bar{X}\varepsilon_y\varepsilon_x -2a\beta S_x^2\varepsilon_y e_x-\alpha b\bar{X}e_y\varepsilon_x-b\beta S_x^2 e_y e_x+2\alpha\beta\bar{X}S_x^2\varepsilon_x e_x\right].$$

Applying expectation and using (6), the *MSE* of (9) is

$$MSE\left(t_1\right) = \theta t_{(a,b)}^2\left[a^2C_y^2+\alpha^2\bar{X}^2C_x^2+\frac{b^2}{4}\varphi_{40}^*+\beta^2S_x^4\varphi_{04}^*-\left(2a\beta S_x^2\varphi_{12}-ab\varphi_{30}\right)C_y -2a\alpha\bar{X}\rho_{yx}C_xC_y-\left(\alpha b\bar{X}\varphi_{21}-2\alpha\beta\bar{X}S_x^2\varphi_{03}\right)C_x-b\beta S_x^2\varphi_{22}^*\right]. \tag{12}$$

We will, now, obtain the optimum values of α and β which minimize (12). For this, we differentiate (12) with respect to α and β, equate the derivatives to zero and solve the resulting equations simultaneously. Now, the derivatives of (12) with respect to α and β are

$$\frac{\partial}{\partial\alpha}MSE\left(t_1\right) = \theta t_{(a,b)}^2\bar{X}C_x\left(2\bar{X}C_x\alpha+2S_x^2\varphi_{03}\beta-2a\rho_{yx}C_y-b\varphi_{21}\right),$$

and

$$\frac{\partial}{\partial\beta}MSE\left(t_1\right) = \theta t_{(a,b)}^2S_x^2\left(2\bar{X}C_x\varphi_{03}\alpha+2S_x^2\varphi_{04}^*\beta-2aC_y\varphi_{12}-b\varphi_{22}^*\right).$$

Equating the above derivatives to zero and simultaneously solving the resulting equations, the optimum values of α and β which minimizes (9) are

$$\alpha = \frac{2aC_y\left(\rho_{yx}\varphi_{04}^*-\varphi_{03}\varphi_{12}\right)+b\left(\varphi_{04}^*\varphi_{21}-\varphi_{03}\varphi_{22}^*\right)}{2\bar{X}C_x\left(\varphi_{04}^*-\varphi_{03}^2\right)}, \tag{13}$$

and

$$\beta = \frac{2aC_y\left(\varphi_{12} - \rho_{yx}\varphi_{03}\right) + b\left(\varphi_{22}^* - \varphi_{03}\varphi_{21}\right)}{2S_x^2\left(\varphi_{04}^* - \varphi_{03}^2\right)}. \tag{14}$$

Using these values in (12), the minimum *MSE* of estimator given in (9) is

$$MSE_{\min}\left(t_1\right) = \frac{\theta t_{(a,b)}^2}{\left(\varphi_{04}^* - \varphi_{03}^2\right)}\left(a^2C_y^2 f_1^* + abC_y f_2^* + \frac{b^2}{4} f_3^*\right), \tag{15}$$

where

$$f_1^* = \varphi_{04}^*\left(1 - \rho_{yx}^2\right) - \varphi_{03}^2 + 2\rho_{yx}\varphi_{03}\varphi_{12} - \varphi_{12}^2,$$
$$f_2^* = \varphi_{30}\left(\varphi_{04}^* - \varphi_{03}^2\right) - \rho_{yx}\left(\varphi_{04}^*\varphi_{21} - \varphi_{03}\varphi_{22}^*\right) - \varphi_{12}\left(\varphi_{22}^* - \varphi_{03}\varphi_{21}\right),$$

and

$$f_3^* = \varphi_{40}^*\left(\varphi_{04}^* - \varphi_{03}^2\right) + \varphi_{21}\left(2\varphi_{03}\varphi_{22}^* - \varphi_{04}^*\varphi_{21}\right) - \varphi_{22}^{*2}.$$

The *MSE* for specific cases of (9) are readily obtained. For example, if

$$\left(a, b, \alpha, \beta\right) = \left(1, 0, \alpha_{opt}, \beta_{opt}\right),$$

then the *MSE* of a regression type estimator of population mean is obtained as

$$MSE_{\min}\left(t_1\right) = \theta\bar{Y}^2 C_y^2 f_1^*\left(\varphi_{04}^* - \varphi_{03}^2\right)^{-1}. \tag{16}$$

Further, if

$$\left(a, b, \alpha, \beta\right) = \left(0, 2, \alpha_{opt}, \beta_{opt}\right),$$

the expression for *MSE* of a regression type estimator of variance is obtained as

$$MSE_{\min}\left(t_1\right) = \theta S_y^4 f_3^*\left(\varphi_{04}^* - \varphi_{03}^2\right)^{-1}. \tag{17}$$

Again, if

$$\left(a, b, \alpha, \beta\right) = \left(-1, 1, \alpha_{opt}, \beta_{opt}\right),$$

the *MSE* of a regression type estimator of coefficient of variation is obtained as

$$MSE_{\min}\left(t_1\right) = \frac{\theta t_{(a,b)}^2}{\left(\varphi_{04}^* - \varphi_{03}^2\right)}\left(C_y^2 f_1^* - C_y f_2^* + \frac{1}{4} f_3^*\right). \tag{18}$$

It is interesting to note that for

$$\left(a, b, \alpha, \beta\right) = \left(1, 0, \alpha, 0\right),$$

the optimum *MSE* of (9) reduces to the *MSE* of classical regression estimator given in (2). Also, for

$$\left(a, b, \alpha, \beta\right) = \left(0, 2, 0, \beta\right),$$

the optimum *MSE* of (9) reduces to the classical regression type estimator of variance as given by [9].

3.2. Estimator with several auxiliary variables

In this section, we will give an estimator of general population parameter in single-phase sampling using the information of several auxiliary variables. The proposed estimator is

$$t_2 = \bar{y}^a \left(s_y^2\right)^{b/2} \left[1 + \sum_{j=1}^{q} \alpha_j \left(\bar{X}_j - \bar{x}_j\right) + \sum_{j=1}^{q} \beta_j \left(S_{x_j}^2 - s_{x_j}^2\right)\right]. \tag{19}$$

Again, it is easy to see that the proposed estimator (19) provides certain estimators as a special case for different values of $\left(a, b, \alpha_j, \beta_j\right)$. Using error notations, the estimator (19) can be written as

$$\begin{aligned} t_2 &= \left[\bar{Y}^a \left(1+\varepsilon_y\right)^a\right] \left[S_y^2 \left(1+e_y\right)\right]^{b/2} \left[1 - \sum_{j=1}^{q} \alpha_j \bar{X}_j \varepsilon_{x_j} - \sum_{j=1}^{q} \beta_j S_j^2 e_{x_j}\right] \\ &= \bar{Y}^a \left(S_y^2\right)^{b/2} \left(1+\varepsilon_y\right)^a \left(1+e_y\right)^{b/2} \left(1 - \boldsymbol{\alpha}'\bar{\mathbf{X}}\boldsymbol{\varepsilon}_x - \boldsymbol{\beta}'\mathbf{S}_x\mathbf{e}_x\right), \end{aligned}$$

where

$$\boldsymbol{\alpha}' = \left[\alpha_1 \quad \alpha_2 \quad \cdots \quad \alpha_q\right],\ \boldsymbol{\beta} = \left[\beta_1 \quad \beta_2 \quad \cdots \quad \beta_q\right],\ \bar{\mathbf{X}} = diag\left(\bar{X}_j\right),\ \mathbf{S}_x = diag\left(S_{x_j}^2\right).$$

Expanding, and retaining only the linear terms, we have

$$t_2 = t_{(a,b)} \left(1 + a\varepsilon_y + \frac{b}{2} e_y + \frac{ab}{2} \varepsilon_y e_y\right) \left(1 - \boldsymbol{\alpha}'\bar{\mathbf{X}}\boldsymbol{\varepsilon}_x - \boldsymbol{\beta}'\mathbf{S}_x\mathbf{e}_x\right),$$

or

$$\begin{aligned} t_2 - t_{(a,b)} = \Bigg[& a\varepsilon_y + \frac{b}{2} e_y + \frac{ab}{2} \varepsilon_y e_y - \boldsymbol{\alpha}'\bar{\mathbf{X}}\boldsymbol{\varepsilon}_x - a\boldsymbol{\alpha}'\bar{\mathbf{X}}\varepsilon_y\boldsymbol{\varepsilon}_x - \frac{b}{2}\boldsymbol{\alpha}'\bar{\mathbf{X}} e_y \boldsymbol{\varepsilon}_x \\ & - \boldsymbol{\beta}'\mathbf{S}_x\mathbf{e}_x - a\boldsymbol{\beta}'\mathbf{S}_x\varepsilon_y\mathbf{e}_x - \frac{b}{2}\boldsymbol{\beta}'\mathbf{S}_x e_y \mathbf{e}_x\Bigg]. \end{aligned} \tag{20}$$

Taking expectation on both sides, the bias of the proposed estimator (18) is

$$Bias\left(t_2\right) = \theta t_{(a,b)} \left[\left(\frac{ab}{2}\varphi_{30} - a\boldsymbol{\alpha}'\bar{\mathbf{X}}\mathbf{R}\mathbf{c}_x - a\boldsymbol{\beta}'\mathbf{S}_x\boldsymbol{\varphi}_{12}\right) - \frac{b}{2}\boldsymbol{\alpha}'\bar{\mathbf{X}}\boldsymbol{\Phi}_{21}\mathbf{c}_x - \frac{b}{2}\boldsymbol{\beta}'\mathbf{S}_x\boldsymbol{\varphi}_{22}^*\right]. \tag{21}$$

Again, squaring (20) and retaining only the quadratic terms, we have

$$\begin{aligned} \left(t_2 - t_{(a,b)}\right)^2 = t_{(a,b)}^2 \Bigg[& a^2\varepsilon_y^2 + \frac{b^2}{4} e_y^2 + \boldsymbol{\alpha}'\bar{\mathbf{X}}\boldsymbol{\varepsilon}_x\boldsymbol{\varepsilon}_x'\bar{\mathbf{X}}\boldsymbol{\alpha} + \boldsymbol{\beta}'\mathbf{S}_x\mathbf{e}_x\mathbf{e}_x'\mathbf{S}_x\boldsymbol{\beta} + ab\varepsilon_y e_y - 2a\boldsymbol{\alpha}'\bar{\mathbf{X}}\varepsilon_y\boldsymbol{\varepsilon}_x \\ & - 2a\boldsymbol{\beta}'\mathbf{S}_x\varepsilon_y\mathbf{e}_x - b\boldsymbol{\alpha}'\bar{\mathbf{X}} e_y\boldsymbol{\varepsilon}_x - b\boldsymbol{\beta}'\mathbf{S}_x e_y\mathbf{e}_x + 2\boldsymbol{\alpha}'\bar{\mathbf{X}}\boldsymbol{\varepsilon}_x\mathbf{e}_x'\mathbf{S}_x\boldsymbol{\beta}\Bigg]. \end{aligned}$$

Taking expectation of the above equation and using (8), the *MSE* of (19) is

$$MSE\left(t_2\right)=\theta t_{(a,b)}^2\Bigg[a^2C_y^2+\frac{b^2}{4}\varphi_{\mathbf{40}}^*+\boldsymbol{\alpha}'\bar{\mathbf{X}}\mathbf{C}_x\bar{\mathbf{X}}\boldsymbol{\alpha}+\boldsymbol{\beta}'\mathbf{S}_x\boldsymbol{\Phi}_x\mathbf{S}_x\boldsymbol{\beta}+ab\varphi_{\mathbf{30}}C_y-2aC_y\boldsymbol{\alpha}'\bar{\mathbf{X}}\mathbf{R}\mathbf{c}_x$$
$$-2aC_y\boldsymbol{\beta}'\mathbf{S}_x\boldsymbol{\varphi}_{12}-b\boldsymbol{\alpha}'\bar{\mathbf{X}}\boldsymbol{\Phi}_{21}\mathbf{c}_x-b\boldsymbol{\beta}'\mathbf{S}_x\boldsymbol{\varphi}_{22}^*+2\boldsymbol{\alpha}'\bar{\mathbf{X}}\boldsymbol{\Phi}_{012}\mathbf{S}_x\boldsymbol{\beta}\Bigg]. \tag{22}$$

We will, now, obtain the optimum values of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ which minimizes (22). For this, we will first differentiate (22) with respect to $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. The derivatives are

$$\frac{\partial}{\partial\boldsymbol{\alpha}}MSE\left(t_2\right)=\theta t_{(a,b)}^2\left(2\bar{\mathbf{X}}\mathbf{C}_x\bar{\mathbf{X}}\boldsymbol{\alpha}-2aC_y\bar{\mathbf{X}}\mathbf{R}\mathbf{c}_x-b\bar{\mathbf{X}}\boldsymbol{\Phi}_{21}\mathbf{c}_x+2\bar{\mathbf{X}}\boldsymbol{\Phi}_{012}\mathbf{S}_x\boldsymbol{\beta}\right),$$

and

$$\frac{\partial}{\partial\boldsymbol{\beta}}MSE\left(t_2\right)=\theta t_{(a,b)}^2\left(2\mathbf{S}_x\boldsymbol{\Phi}_x\mathbf{S}_x\boldsymbol{\beta}-2aC_y\mathbf{S}_x\boldsymbol{\varphi}_{12}-b\mathbf{S}_x\boldsymbol{\varphi}_{22}^*+2\mathbf{S}_x\boldsymbol{\Phi}_{012}'\bar{\mathbf{X}}\boldsymbol{\alpha}\right).$$

Equating the derivatives to zero, the normal equations are

$$\bar{\mathbf{X}}\mathbf{C}_x\bar{\mathbf{X}}\boldsymbol{\alpha}+\bar{\mathbf{X}}\boldsymbol{\Phi}_{012}\mathbf{S}_x\boldsymbol{\beta}=aC_y\bar{\mathbf{X}}\mathbf{R}\mathbf{c}_x+\frac{b}{2}\bar{\mathbf{X}}\boldsymbol{\Phi}_{21}\mathbf{c}_x,$$

and

$$\mathbf{S}_x\boldsymbol{\Phi}_{012}'\bar{\mathbf{X}}\boldsymbol{\alpha}+\mathbf{S}_x\boldsymbol{\Phi}_x\mathbf{S}_x\boldsymbol{\beta}=aC_y\mathbf{S}_x\boldsymbol{\varphi}_{12}+\frac{b}{2}\mathbf{S}_x\boldsymbol{\varphi}_{22}^*.$$

Writing the above equations in matrix form, we have

$$\begin{bmatrix}\bar{\mathbf{X}}\mathbf{C}_x\bar{\mathbf{X}} & \bar{\mathbf{X}}\boldsymbol{\Phi}_{012}\mathbf{S}_x\\ \mathbf{S}_x\boldsymbol{\Phi}_{012}'\bar{\mathbf{X}} & \mathbf{S}_x\boldsymbol{\Phi}_x\mathbf{S}_x\end{bmatrix}\begin{bmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{bmatrix}=\begin{bmatrix}aC_y\bar{\mathbf{X}}\mathbf{R}\mathbf{c}_x+\left(b/2\right)\bar{\mathbf{X}}\boldsymbol{\Phi}_{21}\mathbf{c}_x\\ aC_y\mathbf{S}_x\boldsymbol{\varphi}_{12}+\left(b/2\right)\mathbf{S}_x\boldsymbol{\varphi}_{22}^*\end{bmatrix}.$$

Solving the above matrix equations, the optimum values of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are given as the solution of

$$\begin{bmatrix}\boldsymbol{\alpha}_{opt}\\ \boldsymbol{\beta}_{opt}\end{bmatrix}=\begin{bmatrix}\bar{\mathbf{X}}\mathbf{C}_x\bar{\mathbf{X}} & \bar{\mathbf{X}}\boldsymbol{\Phi}_{012}\mathbf{S}_x\\ \mathbf{S}_x\boldsymbol{\Phi}_{012}'\bar{\mathbf{X}} & \mathbf{S}_x\boldsymbol{\Phi}_x\mathbf{S}_x\end{bmatrix}^{-1}\begin{bmatrix}\bar{\mathbf{X}}\left\{aC_y\mathbf{R}+\left(b/2\right)\boldsymbol{\Phi}_{21}\right\}\mathbf{c}_x\\ \mathbf{S}_x\left\{aC_y\boldsymbol{\varphi}_{12}+\left(b/2\right)\boldsymbol{\varphi}_{22}^*\right\}\end{bmatrix}. \tag{23}$$

Now, we invert the above partitioned matrix as below. Let

$$\boldsymbol{\Sigma}=\begin{bmatrix}\bar{\mathbf{X}}\mathbf{C}_x\bar{\mathbf{X}} & \bar{\mathbf{X}}\boldsymbol{\Phi}_{012}\mathbf{S}_x\\ \mathbf{S}_x\boldsymbol{\Phi}_{012}'\bar{\mathbf{X}} & \mathbf{S}_x\boldsymbol{\Phi}_x\mathbf{S}_x\end{bmatrix}=\begin{bmatrix}\mathbf{A}_{11} & \mathbf{A}_{12}\\ \mathbf{A}_{21} & \mathbf{A}_{22}\end{bmatrix},$$

and then

$$\boldsymbol{\Sigma}^{-1}=\begin{bmatrix}\mathbf{B}_{11} & \mathbf{B}_{12}\\ \mathbf{B}_{21} & \mathbf{B}_{22}\end{bmatrix},$$

where

$$\mathbf{B}_{11} = \left(\mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21}\right)^{-1} = \left[\bar{\mathbf{X}}\left\{\mathbf{C}_x - \mathbf{\Phi}_{012}\mathbf{S}_x\left(\mathbf{S}_x\mathbf{\Phi}_x\mathbf{S}_x\right)^{-1}\mathbf{S}_x\mathbf{\Phi}_{012}^{/}\right\}\bar{\mathbf{X}}\right]^{-1}$$
$$= \left[\bar{\mathbf{X}}\left(\mathbf{C}_x - \mathbf{\Phi}_{012}\mathbf{\Phi}_x^{-1}\mathbf{\Phi}_{012}^{/}\right)\bar{\mathbf{X}}\right]^{-1},$$
$$\mathbf{B}_{12} = -\mathbf{B}_{11}\mathbf{A}_{12}\mathbf{A}_{22}^{-1} = -\mathbf{B}_{11}\left(\bar{\mathbf{X}}\mathbf{\Phi}_{012}\mathbf{S}_x\right)\left(\mathbf{S}_x\mathbf{\Phi}_x\mathbf{S}_x\right)^{-1} = -\bar{\mathbf{X}}^{-1}\left(\mathbf{C}_x - \mathbf{\Phi}_{012}\mathbf{\Phi}_x^{-1}\mathbf{\Phi}_{012}^{/}\right)^{-1}\mathbf{\Phi}_{012}\mathbf{\Phi}_x^{-1}\mathbf{S}_x^{-1},$$
$$\mathbf{B}_{21} = -\mathbf{A}_{22}^{-1}\mathbf{A}_{21}\mathbf{B}_{11} = -\left(\mathbf{S}_x\mathbf{\Phi}_x\mathbf{S}_x\right)^{-1}\left(\mathbf{S}_x\mathbf{\Phi}_{012}^{/}\bar{\mathbf{X}}\right)\mathbf{B}_{11} = -\mathbf{S}_x^{-1}\mathbf{\Phi}_x^{-1}\mathbf{\Phi}_{012}^{/}\left(\mathbf{C}_x - \mathbf{\Phi}_{012}\mathbf{\Phi}_x^{-1}\mathbf{\Phi}_{012}^{/}\right)^{-1}\bar{\mathbf{X}}^{-1},$$

and

$$\mathbf{B}_{22} = \mathbf{A}_{22}^{-1}\left(\mathbf{I} + \mathbf{A}_{21}\mathbf{B}_{11}\mathbf{A}_{12}\mathbf{A}_{22}^{-1}\right) = \left(\mathbf{S}_x\mathbf{\Phi}_x\mathbf{S}_x\right)^{-1}\left[\mathbf{I} + \left(\mathbf{S}_x\mathbf{\Phi}_{012}^{/}\bar{\mathbf{X}}\right)\mathbf{B}_{11}\left(\bar{\mathbf{X}}\mathbf{\Phi}_{012}\mathbf{S}_x\right)\left(\mathbf{S}_x\mathbf{\Phi}_x\mathbf{S}_x\right)^{-1}\right]$$
$$= \left(\mathbf{S}_x\mathbf{\Phi}_x\mathbf{S}_x\right)^{-1} + \mathbf{S}_x^{-1}\mathbf{\Phi}_x^{-1}\mathbf{\Phi}_{012}^{/}\left(\mathbf{C}_x - \mathbf{\Phi}_{012}\mathbf{\Phi}_x^{-1}\mathbf{\Phi}_{012}^{/}\right)^{-1}\mathbf{\Phi}_{012}\mathbf{\Phi}_x^{-1}\mathbf{S}_x^{-1}.$$

Using the values of the inverted matrix in (22), the optimum values of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are

$$\boldsymbol{\alpha}_{opt} = \mathbf{B}_{11}\left[\bar{\mathbf{X}}\left\{aC_y\mathbf{R} + (b/2)\mathbf{\Phi}_{21}\right\}\mathbf{c}_x\right] + \mathbf{B}_{12}\left[\mathbf{S}_x\left\{aC_y\boldsymbol{\varphi}_{12} + (b/2)\boldsymbol{\varphi}_{22}^*\right\}\right] \tag{24}$$

and

$$\boldsymbol{\beta}_{opt} = \mathbf{B}_{21}\left[\bar{\mathbf{X}}\left\{aC_y\mathbf{R} + (b/2)\mathbf{\Phi}_{21}\right\}\mathbf{c}_x\right] + \mathbf{B}_{22}\left[\mathbf{S}_x\left\{aC_y\boldsymbol{\varphi}_{12} + (b/2)\boldsymbol{\varphi}_{22}^*\right\}\right]. \tag{25}$$

Using these optimum values of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ in (21), the minimum *MSE* of (18) is

$$\begin{aligned} MSE_{\min}\left(t_2\right) = \theta t_{(a,b)}^2 \Bigg[& a^2C_y^2 + ab\varphi_{30}C_y + \frac{b^2}{4}\varphi_{40}^* + \boldsymbol{\alpha}_{opt}^{/}\bar{\mathbf{X}}\mathbf{C}_x\bar{\mathbf{X}}\boldsymbol{\alpha}_{opt} + \boldsymbol{\beta}_{opt}^{/}\mathbf{S}_x\mathbf{\Phi}_x\mathbf{S}_x\boldsymbol{\beta}_{opt} \\ & -\boldsymbol{\alpha}_{opt}^{/}\bar{\mathbf{X}}\left(2aC_y\mathbf{R} + b\mathbf{\Phi}_{21}\right)\mathbf{c}_x - \boldsymbol{\beta}_{opt}^{/}\mathbf{S}_x\left(2aC_y\boldsymbol{\varphi}_{12} + b\boldsymbol{\varphi}_{22}^*\right) + 2\boldsymbol{\alpha}_{opt}^{/}\bar{\mathbf{X}}\mathbf{\Phi}_{012}\mathbf{S}_x\boldsymbol{\beta}_{opt}\Bigg]. \end{aligned} \tag{26}$$

It is interesting to note that, for

$$\left(a, b, \boldsymbol{\alpha}, \boldsymbol{\beta}\right) = \left(1, 0, \boldsymbol{\alpha}_{opt}, \mathbf{0}\right),$$

the minimum *MSE*, given in (26), reduces to the minimum mean square error of the classical regression estimator of mean with several auxiliary variables; see [6]. Also, for

$$\left(a, b, \boldsymbol{\alpha}, \boldsymbol{\beta}\right) = \left(0, 2, \mathbf{0}, \boldsymbol{\beta}_{opt}\right),$$

the minimum *MSE*, given in (26), reduces to the minimum *MSE* of a general estimator of variance given by [19].

4. Estimators for two-phase sampling

In this section, we have proposed some new estimators of the general population parameter for two-phase sampling. The estimators have been proposed using information of a single and several auxiliary variables.

4.1. Two-phase sampling estimator with single auxiliary variable

In the following, we have proposed a new estimator of general population parameter for two-phase sampling using information of a single auxiliary variable. The proposed estimator is

$$t_{1(2)} = \bar{y}_{(2)}^{a}\left[s_{y(2)}^{2}\right]^{b/2}\left[1+\alpha_{(2)}\left(\bar{x}_{(1)}-\bar{x}_{(2)}\right)+\beta_{(2)}\left(s_{x(1)}^{2}-s_{x(2)}^{2}\right)\right]. \quad (27)$$

It is easy to see that the proposed estimator (27) reduces to the regression type estimator of mean in two-phase sampling for

$$\left(a,b,\alpha_{(2)},\beta_{(2)}\right)=\left(1,0,\alpha_{(2)},0\right).$$

The estimator (27) reduces to the regression type estimator of variance in two-phase sampling for

$$\left(a,b,\alpha_{(2)},\beta_{(2)}\right)=\left(0,2,0,\beta_{(2)}\right).$$

Now, to derive the bias and *MSE* of (27), we write the estimator (27), using error notations, as

$$t_{1(2)} = \bar{Y}^{a}\left(1+\varepsilon_{y(2)}\right)^{a}\left(S_{y}^{2}\right)^{b/2}\left(1+e_{y(2)}\right)^{b/2}\left[1+\alpha_{(2)}\bar{X}\left(\varepsilon_{x(1)}-\varepsilon_{x(2)}\right)+\beta_{(2)}S_{x}^{2}\left(e_{x(1)}-e_{x(2)}\right)\right].$$

Now, expanding the power series and retaining only the linear terms, we have

$$t_{1(2)} = t_{(a,b)}\left(1+a\varepsilon_{y(2)}+\frac{b}{2}e_{y(2)}+\frac{ab}{2}\varepsilon_{y(2)}e_{y(2)}\right)\left[1-\alpha_{(2)}\bar{X}\left(\varepsilon_{x(2)}-\varepsilon_{x(1)}\right)-\beta_{(2)}S_{x}^{2}\left(e_{x(2)}-e_{x(1)}\right)\right],$$

or

$$\begin{aligned} t_{1(2)}-t_{(a,b)} = t_{(a,b)}\Bigg[& a\varepsilon_{y(2)}+\frac{b}{2}e_{y(2)}+\frac{ab}{2}\varepsilon_{y(2)}e_{y(2)}-\alpha_{(2)}\bar{X}\left(\varepsilon_{x(2)}-\varepsilon_{x(1)}\right) \\ & -a\alpha_{(2)}\bar{X}\varepsilon_{y(2)}\left(\varepsilon_{x(2)}-\varepsilon_{x(1)}\right)-\frac{b}{2}\alpha_{(2)}\bar{X}e_{y(2)}\left(\varepsilon_{x(2)}-\varepsilon_{x(1)}\right)-\beta_{(2)}S_{x}^{2}\left(e_{x(2)}-e_{x(1)}\right) \\ & -a\beta_{2}S_{x}^{2}\varepsilon_{y(2)}\left(e_{x(2)}-e_{x(1)}\right)-\frac{b}{2}\beta_{(2)}S_{x}^{2}e_{y(2)}\left(e_{x(2)}-e_{x(1)}\right)\Bigg]. \end{aligned} \quad (28)$$

Applying expectation on (28) and using (7), the bias of (27) is

$$\begin{aligned} Bias\left(t_{1(2)}\right) = t_{(a,b)}\Bigg[& \theta_{2}\frac{ab}{2}\varphi_{30}C_{y}-\left(\theta_{2}-\theta_{1}\right)aC_{y}\left\{\alpha_{(2)}\bar{X}\rho_{yx}C_{x}-\beta_{2}S_{x}^{2}\varphi_{12}\right\} \\ & -\left(\theta_{2}-\theta_{1}\right)\frac{b}{2}\left\{\alpha_{(2)}\bar{X}\varphi_{21}C_{x}-\beta_{(2)}S_{x}^{2}\varphi_{22}^{*}\right\}\Bigg]. \end{aligned} \quad (29)$$

Again, squaring (29) and retaining only the terms whose powers add up to 2, we have

$$\left(t_{1(2)}-t_{(a,b)}\right)^2=t_{(a,b)}^2\left[a^2\varepsilon_{y(2)}^2+\frac{b^2}{4}e_{y(2)}^2+\alpha_{(2)}^2\bar{X}^2\left(\varepsilon_{x(2)}-\varepsilon_{x(1)}\right)^2+\beta_{(2)}^2S_x^4\left(e_{x(2)}-e_{x(1)}\right)^2\right.$$
$$+ab\varepsilon_{y(2)}e_{y(2)}-2a\alpha_2\varepsilon_{y(2)}\left(\varepsilon_{x(2)}-\varepsilon_{x(1)}\right)-2a\beta_2\varepsilon_{y(2)}\left(e_{x(2)}-e_{x(1)}\right)$$
$$-\alpha_{(2)}b\bar{X}e_{y(2)}\left(\varepsilon_{x(2)}-\varepsilon_{x(1)}\right)-b\beta_{(2)}S_x^2e_{y(2)}\left(e_{x(2)}-e_{x(1)}\right)$$
$$\left.+2\alpha_{(2)}\beta_{(2)}\bar{X}S_x^2\left(\varepsilon_{x(2)}-\varepsilon_{x(1)}\right)\left(e_{x(2)}-e_{x(1)}\right)\right].$$

Applying expectation, and using (7), the mean square error of (29) is

$$MSE\left(t_{1(2)}\right)=t_{(a,b)}^2\left[\theta_2a^2C_y^2+\theta_2\frac{b^2}{4}\varphi_{40}^*+\left(\theta_2-\theta_1\right)\alpha_{(2)}^2\bar{X}^2C_x^2+\left(\theta_2-\theta_1\right)\beta_{(2)}^2S_x^4\varphi_{04}^*+\theta_2ab\varphi_{30}C_y\right.$$
$$-\left(\theta_2-\theta_1\right)2a\alpha_2\bar{X}\rho_{yx}C_yC_x-2\left(\theta_2-\theta_1\right)a\beta_{(2)}S_x^2\varphi_{12}C_y-\left(\theta_2-\theta_1\right)\alpha_{(2)}b\bar{X}\varphi_{21}C_x \tag{30}$$
$$\left.-\left(\theta_2-\theta_1\right)b\beta_{(2)}S_x^2\varphi_{22}^*+2\left(\theta_2-\theta_1\right)\alpha_{(2)}\beta_{(2)}\bar{X}S_x^2\varphi_{03}C_x\right].$$

The optimum values of α and β which minimize (30) are the same as given in (13) and (14). The minimum mean square error is obtained by using the optimum values of α and β in (28) and is

$$MSE_{\min}\left(t_{1(2)}\right)=\frac{t_{(a,b)}^2}{\left(\varphi_{04}^*-\varphi_{03}^2\right)}\left(a^2C_y^2f_{1(2)}^*+abC_yf_{2(2)}^*+\frac{b^2}{4}f_{3(2)}^*\right), \tag{31}$$

where

$$f_{1(2)}^*=\theta_2\varphi_{04}^*\left(1-\rho_{yx}^2\right)+\theta_1\rho_{yx}^2\varphi_{04}^*-\theta_2\varphi_{03}^2-\left(\theta_2-\theta_1\right)\left(\varphi_{12}^2-2\rho\varphi_{03}\varphi_{12}\right),$$
$$f_{2(2)}^*=\theta_2\varphi_{30}\left(\varphi_{04}^*-\varphi_{03}^2\right)-\left(\theta_2-\theta_1\right)\left[\rho_{xy}\left(\varphi_{04}^*\varphi_{21}-\varphi_{03}\varphi_{22}^*\right)+\varphi_{12}\left(\varphi_{22}^*-\varphi_{03}\varphi_{21}\right)\right],$$

and

$$f_{3(2)}^*=\theta_2\varphi_{40}^*\left(\varphi_{04}^*-\varphi_{03}^2\right)-\left(\theta_2-\theta_1\right)\left(\varphi_{04}^*\varphi_{21}^2-2\varphi_{03}\varphi_{21}\varphi_{22}^*+\varphi_{22}^{*2}\right).$$

It is to be noted that the minimum *MSE*, given in (31), reduces to (15) for $\theta_1=0$. Further, for

$$\left(a,b,\alpha_{(2)},\beta_{(2)}\right)=\left(1,0,\alpha_{(2)}^{opt},0\right),$$

the minimum *MSE*, given in (31), reduces to the *MSE* of the two-phase sampling regression estimator of the population mean. Also, for

$$\left(a,b,\alpha_{(2)},\beta_{(2)}\right)=\left(0,2,0,\beta_{(2)}^{opt}\right),$$

the minimum *MSE*, given in (31), reduces to the *MSE* of the two-phase sampling regression estimator of the population variance; see, for example, [19]. Further, for

$$\left(a,b,\alpha_{(2)},\beta_{(2)}\right)=\left(-1,1,\alpha_{(2)}^{opt},\beta_{(2)}^{opt}\right),$$

the minimum *MSE*, given in (31), reduces to the *MSE* of the two-phase sampling estimator of coefficient of variation and is given as

$$MSE_{\min}\left(t_{1(2)}\right)=\frac{S_y^2}{\bar{Y}^2\left(\varphi_{04}^*-\varphi_{03}^2\right)}\left(C_y^2 f_{1(2)}^*-C_y f_{2(2)}^*+\frac{1}{4}f_{3(2)}^*\right). \tag{32}$$

We will, now, propose a new estimator of general population parameter in two-phase sampling using information of several auxiliary variables.

4.2. Two-phase sampling estimator with several auxiliary variables

The proposed estimator of general population parameter in two-phase sampling with multiple auxiliary variables is

$$t_{2(2)}=\bar{y}_{(2)}^a\left(s_{y(2)}^2\right)^{b/2}\left[1+\sum\nolimits_{j=1}^q\alpha_{j(2)}\left(\bar{x}_{j(1)}-\bar{x}_{j(2)}\right)+\sum\nolimits_{j=1}^q\beta_{j(2)}\left(s_{x_j(1)}^2-s_{x_j(2)}^2\right)\right]. \tag{33}$$

The estimator (33) provides various estimators as special cases for specific choices of the parameters involved. For example, if

$$\left(a,b,\alpha_{j(2)},\beta_{j(2)}\right)=\left(1,0,\alpha_{j(2)},0\right),$$

then we have a regression type estimator of the population mean for two-phase sampling with multiple auxiliary variables. Again, if

$$\left(a,b,\alpha_{j(2)},\beta_{j(2)}\right)=\left(0,2,0,\beta_{j(2)}\right),$$

then we have a regression type estimator of the population variance in two-phase sampling with multiple auxiliary variables. Further, if

$$\left(a,b,\alpha_{j(2)},\beta_{j(2)}\right)=\left(-1,1,\alpha_{j(2)},\beta_{j(2)}\right),$$

then we have a two-phase sampling estimator of the coefficient of variation with multiple auxiliary variables. Now, to derive the bias and *MSE* of the proposed two-phase sampling estimator, we write it as

$$t_{2(2)}=\bar{Y}^a\left(1+\varepsilon_{y(2)}\right)^a S_y^b\left(1+e_{y(2)}\right)^{b/2}\left[1+\sum\nolimits_{j=1}^q\alpha_{(2)j}\bar{X}_j\left(\varepsilon_{x_j(1)}-\varepsilon_{x_j(2)}\right)+\sum\nolimits_{j=1}^q\beta_{(2)j}S_{x_j}^2\left(e_{x_j(1)}-e_{x_j(2)}\right)\right].$$

Expanding the powers and retaining only the linear terms, we have

$$t_{2(2)}=\bar{Y}^{a}S_{y}^{b}\left(1+a\varepsilon_{y(2)}\right)\left(1+\frac{b}{2}e_{y(2)}\right)\left[1+\boldsymbol{\alpha}_{(2)}^{/}\bar{\mathbf{X}}\left(\boldsymbol{\varepsilon}_{x(1)}-\boldsymbol{\varepsilon}_{x(2)}\right)+\boldsymbol{\beta}_{(2)}^{/}\mathbf{S}_{x}\left(\mathbf{e}_{x(1)}-\mathbf{e}_{x(2)}\right)\right]$$
$$=t_{(a,b)}\left[1+a\varepsilon_{y(2)}+\frac{b}{2}e_{y(2)}+\frac{ab}{2}\varepsilon_{y(2)}e_{y(2)}-\boldsymbol{\alpha}_{(2)}^{/}\bar{\mathbf{X}}\left(\boldsymbol{\varepsilon}_{x(2)}-\boldsymbol{\varepsilon}_{x(1)}\right)-a\boldsymbol{\alpha}_{(2)}^{/}\bar{\mathbf{X}}\varepsilon_{y(2)}\left(\boldsymbol{\varepsilon}_{x(2)}-\boldsymbol{\varepsilon}_{x(1)}\right)\right.$$
$$-\frac{b}{2}\boldsymbol{\alpha}_{(2)}^{/}\bar{\mathbf{X}}e_{y(2)}\left(\boldsymbol{\varepsilon}_{x(2)}-\boldsymbol{\varepsilon}_{x(1)}\right)-\boldsymbol{\beta}_{(2)}^{/}\mathbf{S}_{x}\left(\mathbf{e}_{x(2)}-\mathbf{e}_{x(1)}\right)-a\boldsymbol{\beta}_{(2)}^{/}\mathbf{S}_{x}\varepsilon_{y(2)}\left(\mathbf{e}_{x(2)}-\mathbf{e}_{x(1)}\right)$$
$$\left.-\frac{b}{2}\boldsymbol{\beta}_{(2)}^{/}\mathbf{S}_{x}e_{y(2)}\left(\mathbf{e}_{x(2)}-\mathbf{e}_{x(1)}\right)\right],$$

or

$$t_{2(2)}-t_{(a,b)}=\left[a\varepsilon_{y(2)}+\frac{b}{2}e_{y(2)}+\frac{ab}{2}\varepsilon_{y(2)}e_{y(2)}-\boldsymbol{\alpha}_{(2)}^{/}\bar{\mathbf{X}}\left(\boldsymbol{\varepsilon}_{x(2)}-\boldsymbol{\varepsilon}_{x(1)}\right)-a\boldsymbol{\alpha}_{(2)}^{/}\bar{\mathbf{X}}\varepsilon_{y(2)}\left(\boldsymbol{\varepsilon}_{x(2)}-\boldsymbol{\varepsilon}_{x(1)}\right)\right.$$
$$-\frac{b}{2}\boldsymbol{\alpha}_{(2)}^{/}\bar{\mathbf{X}}e_{y(2)}\left(\boldsymbol{\varepsilon}_{x(2)}-\boldsymbol{\varepsilon}_{x(1)}\right)-\boldsymbol{\beta}_{(2)}^{/}\mathbf{S}_{x}\left(\mathbf{e}_{x(2)}-\mathbf{e}_{x(1)}\right)-a\boldsymbol{\beta}_{(2)}^{/}\mathbf{S}_{x}\varepsilon_{y(2)}\left(\mathbf{e}_{x(2)}-\mathbf{e}_{x(1)}\right) \quad (34)$$
$$\left.-\frac{b}{2}\boldsymbol{\beta}_{(2)}^{/}\mathbf{S}_{x}e_{y(2)}\left(\mathbf{e}_{x(2)}-\mathbf{e}_{x(1)}\right)\right].$$

Applying expectations, and using (8), the bias of the proposed estimator is

$$Bias\left(t_{2(2)}\right)=\left[\theta_{2}\frac{ab}{2}C_{y}\varphi_{03}-\left(\theta_{2}-\theta_{1}\right)\left\{a\boldsymbol{\alpha}_{(2)}^{/}\bar{\mathbf{X}}C_{y}\mathbf{R}\mathbf{c}_{x}-\frac{b}{2}\boldsymbol{\alpha}_{(2)}^{/}\bar{\mathbf{X}}\boldsymbol{\Phi}_{21}\mathbf{c}_{x}\right.\right.$$
$$\left.\left.-aC_{y}\boldsymbol{\beta}_{(2)}^{/}\mathbf{S}_{x}\boldsymbol{\varphi}_{12}-\frac{b}{2}\boldsymbol{\beta}_{(2)}^{/}\mathbf{S}_{x}\boldsymbol{\varphi}_{22}^{*}\right\}\right]. \quad (35)$$

Again, squaring (34), applying expectation and using (8), the *MSE* of (35) is

$$MSE\left(t_{2(2)}\right)=t_{(a,b)}^{2}\left[a^{2}\theta_{2}C_{y}^{2}+\frac{b^{2}}{4}\theta_{2}\varphi_{\mathbf{40}}^{*}+abC_{y}\theta_{2}\varphi_{\mathbf{30}}+\left(\theta_{2}-\theta_{1}\right)\left\{\boldsymbol{\alpha}_{(2)}^{/}\bar{\mathbf{X}}\mathbf{C}_{x}\bar{\mathbf{X}}\boldsymbol{\alpha}_{(2)}\right.\right.$$
$$+\boldsymbol{\beta}_{(2)}^{/}\mathbf{S}_{x}\boldsymbol{\Phi}_{x}\mathbf{S}_{x}\boldsymbol{\beta}_{(2)}-2aC_{y}\boldsymbol{\alpha}_{(2)}^{/}\bar{\mathbf{X}}\mathbf{R}\mathbf{c}_{x}-2aC_{y}\boldsymbol{\beta}_{(2)}^{/}\mathbf{S}_{x}\boldsymbol{\varphi}_{12}-b\boldsymbol{\alpha}_{(2)}^{/}\bar{\mathbf{X}}\boldsymbol{\Phi}_{21}\mathbf{c}_{x} \quad (36)$$
$$\left.\left.-b\boldsymbol{\beta}_{(2)}^{/}\mathbf{S}_{x}\boldsymbol{\varphi}_{22}^{*}+2\boldsymbol{\alpha}_{(2)}^{/}\bar{\mathbf{X}}\boldsymbol{\Phi}_{012}\mathbf{S}_{x}\boldsymbol{\beta}_{(2)}\right\}\right].$$

The optimum values of $\boldsymbol{\alpha}_{(2)}$ and $\boldsymbol{\beta}_{(2)}$ which minimizes (36) are the same as given in (24) and (25). Using the optimum values $\boldsymbol{\alpha}_{(2)}$ and $\boldsymbol{\beta}_{(2)}$ in (36), the minimum *MSE* is

$$MSE_{\min}\left(t_{2(2)}\right)=t_{(a,b)}^{2}\left[\theta_{2}a^{2}C_{y}^{2}+\theta_{2}\frac{b^{2}}{4}\varphi_{\mathbf{40}}^{*}+\theta_{2}abC_{y}\varphi_{\mathbf{30}}+\left(\theta_{2}-\theta_{1}\right)\left\{\boldsymbol{\alpha}_{(2)}^{/*}\bar{\mathbf{X}}\mathbf{C}_{x}\bar{\mathbf{X}}\boldsymbol{\alpha}_{(2)}^{*}\right.\right.$$
$$+\boldsymbol{\beta}_{(2)}^{/*}\mathbf{S}_{x}\boldsymbol{\Phi}_{x}\mathbf{S}_{x}\boldsymbol{\beta}_{(2)}^{*}-\boldsymbol{\alpha}_{(2)}^{/*}\bar{\mathbf{X}}\left(2aC_{y}\mathbf{R}+b\boldsymbol{\Phi}_{21}\right)\mathbf{c}_{x} \quad (37)$$
$$\left.\left.-\boldsymbol{\beta}_{(2)}^{/*}\mathbf{S}_{x}\left(2aC_{y}\boldsymbol{\varphi}_{12}+\boldsymbol{\varphi}_{22}^{*}\right)+2\boldsymbol{\alpha}_{(2)}^{/*}\bar{\mathbf{X}}\boldsymbol{\Phi}_{012}\mathbf{S}_{x}\boldsymbol{\beta}_{(2)}^{*}\right\}\right].$$

The mean square error of specific cases of (33) can be easily obtained from (37) by using the specific values of the parameters.

5. Comparison of the proposed estimators

In this section we have given some comparison of the proposed estimators with some existing estimators. The comparison will be given in case of a single auxiliary variable. The comparison for the multiple auxiliary variables case is analogous.

We will first give a comparison of the proposed estimators with the general estimator of population parameter suggested by [22]. The estimator is

$$\hat{t}_{(a,b)}=\left[\bar{y}^a s_y^b+k\left(\bar{X}-\bar{x}\right)\right]\exp\left[\frac{w_1\left(\bar{X}-\bar{x}\right)}{\bar{X}+(\alpha-1)\bar{x}}\right]\exp\left[\frac{w_2\left(S_x^2-s_x^2\right)}{S_x^2+(\beta-1)s_x^2}\right], \tag{38}$$

with mean square error

$$MSE\left(\hat{t}_{(a,b)}\right)=\theta t_{(a,b)}^2\left[f_{1(a,b)}-\left(\varphi_{04}^*-\varphi_{03}^2\right)^{-1}\left\{f_{3(a,b)}^2-2f_{2(a,b)}f_{3(a,b)}\varphi_{03}+\varphi_{04}^*f_{2(a,b)}^2\right\}\right], \tag{39}$$

where

$$f_{1(a,b)}=a^2C_y^2+abC_y\varphi_{03}+\frac{b^2}{4}\varphi_{40}^*,\ f_{2(a,b)}=a\rho_{yx}C_y+\frac{b}{2}\varphi_{21},$$

and

$$f_{3(a,b)}=aC_y\varphi_{12}+\frac{b}{2}\varphi_{22}^*.$$

A close comparison of (39) with (15) indicates that the *MSE*s of the two estimators are equal. It is interesting to note that our proposed estimator (9) is much simpler in application than (38). We will, now, give a comparison of the estimators of specific population parameters.

5.1. Comparison with estimators of the population mean

In the following, we will give a comparison of estimators for estimation of the mean. We know that the proposed estimator reduces to the estimator of mean for $(a,b,\alpha,\beta)=\left(1,0,\alpha_{opt},\beta_{opt}\right)$ and is given as

$$t_{1(M)}=\bar{y}\left[1+\alpha\left(\bar{X}-\bar{x}\right)+\beta\left(S_x^2-s_x^2\right)\right]. \tag{40}$$

The *MSE* of the above estimator is given in (16) and can also be written as

$$MSE_{\min}\left(t_{1(M)}\right)=\theta\bar{Y}^2C_y^2\left(\varphi_{04}^*-\varphi_{03}^2\right)^{-1}\left[\varphi_{04}^*\left(1-\rho_{yx}^2\right)-\varphi_{03}^2+2\rho_{yx}\varphi_{03}\varphi_{12}-\varphi_{12}^2\right],$$

or

$$MSE_{\min}\left(t_{1(M)}\right)=Var\left(\bar{y}\right)\left[1-\frac{\left(\varphi_{04}^*\rho_{yx}^2-2\rho_{yx}\varphi_{03}\varphi_{12}+\varphi_{12}^2\right)}{\left(\varphi_{04}^*-\varphi_{03}^2\right)}\right], \tag{41}$$

where $Var(\bar{y}) = \theta \bar{Y}^2 C_y^2$ is the variance of the mean per unit estimator. From above, we can see that the proposed estimator of the mean is always more efficient than the mean per unit estimator. Again, the *MSE* of the proposed estimator of the mean can be written as

$$MSE_{\min}\left(t_{1(M)}\right) = Var\left(\bar{y}_{lr}\right)\left[1 - \frac{\left(\rho_{yx}\varphi_{03} - \varphi_{12}\right)^2}{\left(\varphi_{04}^* - \varphi_{03}^2\right)\left(1 - \rho_{yx}^2\right)}\right], \tag{42}$$

where

$$Var\left(\bar{y}_{lr}\right) = \theta \bar{Y}^2 C_y^2 \left(1 - \rho_{yx}^2\right)$$

is variance of the classical regression estimator of the mean. It is clear that the proposed estimator will be more efficient than the classical regression estimator of the mean if $\rho_{yx} \geq \varphi_{03}^{-1}\varphi_{12}$. Since the *MSE* of the estimators of the mean proposed by [23] is the same as the *MSE* of the classical regression estimator, the proposed estimator of the mean, (40), is more efficient than the estimator proposed by [23] if $\rho_{yx} \geq \varphi_{03}^{-1}\varphi_{12}$.

Further, the estimators proposed by [24,25] are less efficient than the classical regression estimator; therefore, they are less efficient than the proposed estimator of the mean, given in (40).

5.2. Comparison with estimators of the population variance

It is easy to see that the proposed estimator reduces to the regression type estimator of variance for

$$(a, b, \alpha, \beta) = \left(0, 2, \alpha_{opt}, \beta_{opt}\right)$$

and is given as

$$t_{1(V)} = s_y^2\left[1 + \alpha\left(\bar{X} - \bar{x}\right) + \beta\left(S_x^2 - s_x^2\right)\right]. \tag{43}$$

The *MSE* of the above estimator is given in (17) and can also be written as

$$MSE\left(t_{1(V)}\right) = \theta S_y^4\left[\varphi_{40}^* - \varphi_{21}^2 - \left(\varphi_{22}^* - \varphi_{03}\varphi_{21}\right)^2\left(\varphi_{04}^* - \varphi_{03}^2\right)^{-1}\right], \tag{44}$$

or

$$MSE\left(t_{1(V)}\right) = MSE\left(s_y^2\right) - \theta S_y^2\left[\varphi_{21}^2 + \left(\varphi_{22}^* - \varphi_{03}\varphi_{21}\right)^2\left(\varphi_{04}^* - \varphi_{03}^2\right)^{-1}\right],$$

where

$$MSE\left(s_y^2\right) = \theta S_y^4 \varphi_{40}^*$$

is the *MSE* of the classical estimator of the variance. The expression of *MSE*, (44), is same as the expression of the *MSE* of the variance estimator proposed by [14], but the construction of our proposed estimator of the variance, (42), is much simpler as compared with the variance estimator given by [14]. Further, it is easy to show that our proposed estimator, (42), is more efficient than the classical estimator of variance, s_y^2, and the estimator proposed by [13].

We will, now, compare our proposed estimator of variance with the estimators proposed by [18,19]. For this, we first see that the *MSE* of estimators proposed by [18,19] is the same and is given as

$$MSE_{\min}\left(\hat{S}^2_{MHS1}\right)=\theta S_Y^4\left(\varphi^*_{40}-\varphi^{*2}_{22}\varphi^{*-1}_{04}\right).$$

Now, our proposed estimator of variance will be more efficient than the estimators proposed by [18,19], if

$$\left(\varphi^*_{22}-\varphi_{03}\varphi_{21}\right)^2\geq\left(\varphi^*_{04}-\varphi^2_{03}\right)\left(\varphi^*_{22}\varphi^{*-1}_{04}-\varphi^2_{21}\right).$$

6. Numerical study

In this section, we have conducted numerical study of the specific cases of the proposed estimator of general population parameter. The numerical study has been conducted in two ways: simulation and study using real population. These numerical studies are given in the following sub-sections.

6.1. Simulation

In this section, the comparison of the proposed estimator is done with some existing estimators through simulation. The simulation has been done using some popular single- and two-phase sampling estimators of the mean and the variance. We have used two-phase versions of some of the estimators of mean and variance which are not available in the literature. The estimators used in the simulation, in addition to classical ratio and regression estimators of the mean, are given in Tables 1 and 2 below. The simulation algorithm for single-phase sampling is as below:

1) Generate an artificial population of size 5000 from a bivariate normal distribution $N_2\left(60,45,5^2,4^2,\rho\right)$ by using different values of the correlation coefficient.
2) Generate random samples of sizes 50, 100, 200 and 500 from the generated population.
3) Compute different estimators by using the generated samples.
4) Repeat steps 2 and 3 for 20000 times for each sample size.
5) Compute mean square error of each estimator of mean and variance at different sample sizes by using

$$MSE\left(t_i\right)=\frac{1}{20000}\sum_{j=1}^{20000}\left(t_{ij}-\overline{t}_i\right)^2;\ \overline{t}_i=\frac{1}{20000}\sum_{j=1}^{20000}t_{ij};\ i=C,R,BT,S,KC,AR,1(M);$$

$$MSE\left(t^*_k\right)=\frac{1}{20000}\sum_{j=1}^{20000}\left(t^*_{kj}-\overline{t}^*_k\right)^2;\ \overline{t}^*_k=\frac{1}{20000}\sum_{j=1}^{20000}t^*_{kj};\ k=C,R,YK,MHS,AR,1(V).$$

Table 1. Estimators of the mean.

Estimator	Single-Phase	Two-Phase
Bhal and Tuteja [23]	$t_{BT}=\bar{y}\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right)$	$t_{BT(2)}=\bar{y}_{(2)}\exp\left[\frac{\bar{x}_{(1)}-\bar{x}_{(2)}}{\bar{x}_{(1)}+\bar{x}_{(2)}}\right]$
Singh [26]	$t_S=\bar{y}\left(\frac{\bar{X}+S_x}{\bar{x}+S_x}\right)$	$t_{S(2)}=\bar{y}_{(2)}\left(\frac{\bar{x}_{(1)}+s_{x(1)}}{\bar{x}_{(2)}+s_{x(1)}}\right)$
Kadilar and Cingi [24]	$t_{KC}=\left[\bar{y}+b\left(\bar{X}-\bar{x}\right)\right]\frac{\bar{X}}{\bar{x}}$	$t_{KC(2)}=\left[\bar{y}_{(2)}+b\left(\bar{x}_{(1)}-\bar{x}_{(2)}\right)\right]\frac{\bar{x}_{(1)}}{\bar{x}_{(2)}}$
Adichwal et al. [21]	$t_{AR}=t_{(R)}\exp\left[\frac{\bar{X}-\bar{x}}{\bar{X}+(\alpha-1)\bar{x}}\right]\times\exp\left[\frac{S_x^2-s_x^2}{S_x^2+(\beta-1)s_x^2}\right]$	$t_{AR(2)}=t_{R(2)}\exp\left[\frac{\bar{x}_{(1)}-\bar{x}_{(2)}}{\bar{x}_{(1)}+(\alpha-1)\bar{x}_{(2)}}\right]\times\exp\left[\frac{s_{x(1)}^2-s_{x(2)}^2}{s_{x(1)}^2+(\beta-1)s_{x(2)}^2}\right]$
Proposed	$t_{1(M)}=\bar{y}\left[1+\alpha\left(\bar{X}-\bar{x}\right)+\beta\left(S_x^2-s_x^2\right)\right]$	$t_{1(M)(2)}=\bar{y}\left[1+\alpha\left(\bar{x}_1-\bar{x}_2\right)+\beta\left(s_{x(1)}^2-s_{x(2)}^2\right)\right]$

Table 2. Estimators of the variance.

Estimator	Single-Phase	Two-Phase
Isaki [9]	$t_C^*=s_y^2S_x^2/s_x^2$	$t_{C(2)}^*=s_{y(2)}^2s_{x(1)}^2/s_{x(2)}^2$
Isaki [9]	$t_R^*=s_y^2+\gamma\left(S_x^2-s_x^2\right)$	$t_{R(2)}^*=s_{y(2)}^2+\gamma\left(s_{x(1)}^2-s_{x(2)}^2\right)$
Yadav and Kadilar [18]	$t_{YK}^*=s_y^2\exp\left[\frac{S_x^2-s_x^2}{S_x^2+(a-1)s_x^2}\right]$	$t_{YK(2)}^*=s_{y(2)}^2\exp\left[\frac{s_{x(1)}^2-s_{x(2)}^2}{s_{x(1)}^2+(a-1)s_{x(2)}^2}\right]$
Al-Marshadi [19]	$t_{MHS}^*=s_y^2+\ln\left(s_x^2/S_x^2\right)^\alpha$	$t_{MHS(2)}^*=s_{y(2)}^2+\ln\left(s_{x(2)}^2/s_{x(1)}^2\right)^\alpha$
Adichwal et al. [21]	$t_{AR}=\left[s_y^2+k\left(\bar{X}-\bar{x}\right)\right]\exp\left[\frac{\bar{X}-\bar{x}}{\bar{X}+(\alpha-1)\bar{x}}\right]\times\exp\left[\frac{S_x^2-s_x^2}{S_x^2+(\beta-1)s_x^2}\right]$	$t_{AR}=\left[s_{y(2)}^2+k\left(\bar{x}_{(1)}-\bar{x}_{(2)}\right)\right]\exp\left[\frac{\bar{x}_{(1)}-\bar{x}_{(2)}}{\bar{x}_{(1)}+(\alpha-1)\bar{x}_2}\right]\times\exp\left[\frac{s_{x(1)}^2-s_{x(2)}^2}{s_{x(1)}^2+(\beta-1)s_{x(2)}^2}\right]$
Proposed	$t_{1(V)}=s_y^2\left[1+\alpha\left(\bar{X}-\bar{x}\right)+\beta\left(S_x^2-s_x^2\right)\right]$	$t_{1(V)(2)}=s_{y(2)}^2\left[1+\alpha\left(\bar{x}_{(1)}-\bar{x}_{(2)}\right)+\beta\left(s_{x(1)}^2-s_{x(2)}^2\right)\right]$

In the above tables $\bar{y}_{(2)}$ and $s_{y(2)}^2$ are the second phase mean and variance of the study variable. Similar notations hold for the auxiliary variable.

The simulation algorithm for two-phase sampling is as below:

1) Generate an artificial population of size 5000 from a bivariate normal distribution $N_2\left(60,45,5^2,4^2,\rho\right)$ by using different values of the correlation coefficient.

2) Generate first phase random samples of sizes 500 and 1000 from the generated population.
3) Generate second phase random samples of sizes 5%, 10% and 20% of the first phase sample.
4) Compute different estimators by using the second phase sample mean of *Y*, first and second phase sample means of auxiliary variable *X* and some population measures of auxiliary variable *X*.
5) Repeat steps 2–4 for 20000 times for each combination of first and second phase sample size.
6) Compute bias and mean square error of each estimator at different sample sizes as given in step 5 for the single-phase case above.

The results of the simulation study are given in Tables 3–6 below.

Table 3. Mean square error of estimators of mean in single-phase sampling.

ρ_{xy}	n	$t_{(C)}$	$t_{(R)}$	$t_{(BT)}$	$t_{(S)}$	$t_{(KC)}$	$t_{(AAR)}$	$t_{1(M)}$
–0.9	50	1.0427	0.6061	0.6287	0.9567	1.0519	0.7035	0.4151
	100	0.5201	0.2969	0.3133	0.4771	0.5227	0.3304	0.2006
	200	0.2571	0.1479	0.1566	0.2363	0.2575	0.1605	0.0991
	500	0.0959	0.0535	0.0575	0.0879	0.0960	0.0592	0.0357
–0.5	50	1.0844	0.6233	0.6561	0.9966	1.0931	0.7246	0.4283
	100	0.5172	0.3000	0.3141	0.4753	0.5198	0.3395	0.2030
	200	0.2526	0.1442	0.1526	0.2319	0.2533	0.1587	0.0966
	500	0.0945	0.0541	0.0572	0.0868	0.0945	0.0607	0.0361
0.5	50	1.0713	0.6151	0.6479	0.9843	1.0826	0.7256	0.4232
	100	0.5140	0.2954	0.3116	0.4724	0.5157	0.3342	0.1997
	200	0.2577	0.1475	0.1566	0.2369	0.2581	0.1619	0.0989
	500	0.0950	0.0541	0.0578	0.0873	0.0949	0.0598	0.0361
0.9	50	1.0681	0.6224	0.6477	0.9812	1.0797	0.7154	0.4279
	100	0.5247	0.2952	0.3150	0.4814	0.5272	0.3317	0.1990
	200	0.2560	0.1435	0.1534	0.2347	0.2560	0.1600	0.0962
	500	0.0972	0.0540	0.0583	0.0892	0.0973	0.0599	0.0360

Table 4. Mean square error of estimators of variance in single-phase sampling.

ρ_{xy}	n	t^*_C	t^*_R	t^*_{YK}	t^*_{MHS}	t^*_{AAR}	$t_{1(V)}$
–0.9	50	61.6715	26.9648	26.9308	26.8381	27.7020	22.0301
	100	27.7252	13.1662	13.1631	13.1524	13.3192	10.6404
	200	12.8528	6.3378	6.3375	6.3360	6.3789	5.1016
	500	4.6520	2.3183	2.3183	2.3183	2.3230	1.8582
–0.5	50	59.5725	27.2676	27.2198	27.1438	27.9374	22.2485
	100	27.4137	13.0218	13.0167	13.0075	13.1495	10.5085
	200	12.8765	6.2683	6.2680	6.2685	6.2991	5.0372
	500	4.6790	2.3569	2.3569	2.3568	2.3614	1.8891
0.5	50	58.6802	26.2101	26.1634	26.0787	26.9010	21.4129
	100	26.9036	12.7161	12.7129	12.7077	12.8867	10.2948
	200	12.6693	6.0603	6.0600	6.0589	6.0901	4.8706
	500	4.5555	2.2384	2.2384	2.2383	2.2453	1.7961
0.9	50	56.5997	24.9762	24.9441	24.8933	25.6947	20.4355
	100	26.0339	12.1086	12.1053	12.1030	12.2614	9.7939
	200	12.4012	5.8972	5.8967	5.8956	5.9264	4.7395
	500	4.5886	2.2476	2.2476	2.2476	2.2505	1.8003

Table 5. Mean square error of estimators of mean in two-phase sampling.

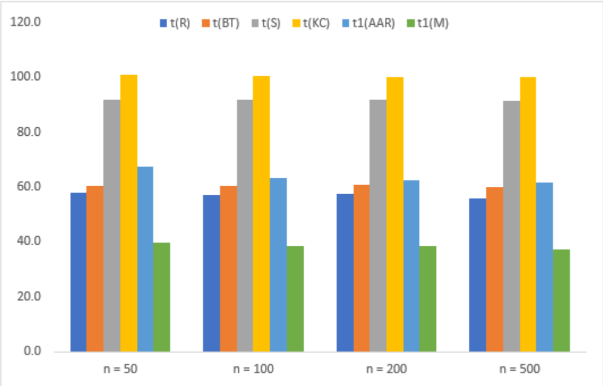
ρ_{yx}	n_1	n_2	$t_{(C)}$	$t_{(R)}$	$t_{(BT)}$	$t_{(S)}$	$t_{(KC)}$	$t_{(AAR)}$	$t_{1(M)}$
–0.9	500	25	2.0890	1.0373	1.9112	1.9194	2.1234	1.1317	0.8782
		50	0.9999	0.5051	0.9329	0.9213	1.0102	0.5195	0.4134
		100	0.4722	0.2473	0.4534	0.4369	0.4746	0.2507	0.2003
	1000	50	1.0267	0.5036	0.9417	0.9438	1.0365	0.5205	0.4134
		100	0.5100	0.2496	0.4710	0.4694	0.5129	0.2521	0.2015
		200	0.2321	0.1216	0.2230	0.2148	0.2328	0.1221	0.0977
–0.5	500	25	2.0671	1.0531	1.9128	1.9030	2.1063	1.1463	0.8937
		50	1.0167	0.5058	0.9433	0.9365	1.0255	0.5211	0.4152
		100	0.4712	0.2495	0.4559	0.4366	0.4739	0.2525	0.2018
	1000	50	1.0273	0.4998	0.9395	0.9445	1.0371	0.5154	0.4099
		100	0.4932	0.2450	0.4587	0.4545	0.4954	0.2479	0.1982
		200	0.2315	0.1211	0.2222	0.2142	0.2317	0.1217	0.0973
0.5	500	25	2.0570	1.0334	1.8796	1.8898	2.0893	1.1305	0.8792
		50	1.0152	0.5163	0.9491	0.9359	1.0239	0.5310	0.4228
		100	0.4744	0.2517	0.4595	0.4397	0.4760	0.2542	0.2031
	1000	50	1.0414	0.5136	0.9596	0.9584	1.0559	0.5271	0.4196
		100	0.4927	0.2492	0.4619	0.4544	0.4948	0.2527	0.2017
		200	0.2330	0.1212	0.2237	0.2157	0.2334	0.1217	0.0973
0.9	500	25	2.0973	1.0670	1.9361	1.9292	2.1516	1.1602	0.9049
		50	1.0017	0.5139	0.9407	0.9236	1.0123	0.5300	0.4219
		100	0.4711	0.2470	0.4530	0.4360	0.4733	0.2488	0.1989
	1000	50	1.0307	0.5008	0.9377	0.9464	1.0417	0.5150	0.4101
		100	0.4979	0.2491	0.4637	0.4586	0.5004	0.2520	0.2014
		200	0.2337	0.1219	0.2236	0.2160	0.2346	0.1224	0.0979

Table 6. Mean square error of estimators of variance in two-phase sampling.

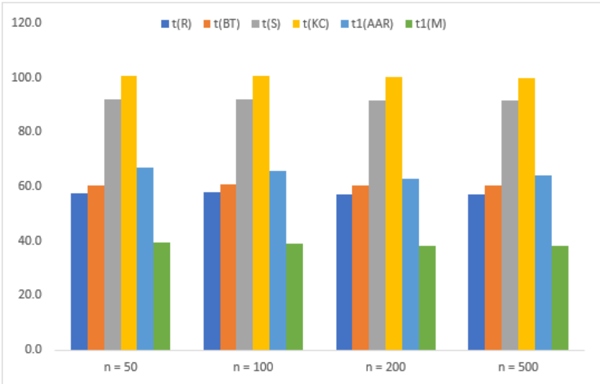
ρ_{yx}	n_1	n_2	t^*_C	t^*_R	t^*_{YK}	t^*_{MHS}	t^*_{AAR}	$t_{1(V)}$
–0.9	500	25	140.5358	57.3500	56.9743	56.3199	61.4633	48.0799
		50	57.6091	26.9982	26.9350	26.8751	27.7169	22.0829
		100	24.3699	13.1517	13.1486	13.1413	13.2844	10.6166
	1000	50	58.5709	26.8444	26.7882	26.6943	27.5534	21.9302
		100	26.1174	12.8386	12.8346	12.8230	12.9687	10.3608
		200	11.7293	6.3042	6.3039	6.3032	6.3292	5.0615
–0.5	500	25	138.3087	57.6001	57.1780	56.5402	61.8000	48.1299
		50	57.1008	26.5565	26.4872	26.4285	27.1874	21.6863
		100	24.3847	12.9400	12.9366	12.9321	13.0727	10.4453
	1000	50	59.4134	27.1282	27.0971	27.0369	27.9262	22.2336
		100	26.4397	13.0561	13.0523	13.0450	13.2347	10.5728
		200	11.5907	6.2638	6.2633	6.2621	6.2878	5.0286
0.5	500	25	136.2103	54.8615	54.5167	53.8812	58.8619	45.7701
		50	56.2874	26.2092	26.1650	26.1164	26.8153	21.3371
		100	23.7153	12.4007	12.3988	12.3934	12.4807	9.9721
	1000	50	56.8057	25.9255	25.8763	25.7851	26.5693	21.1616
		100	25.5146	12.5213	12.5186	12.5121	12.6728	10.1253
		200	11.3721	6.0205	6.0202	6.0187	6.0462	4.8353
0.9	500	25	129.8467	54.2527	53.9338	53.3785	57.7821	45.2401
		50	54.9371	25.4004	25.3675	25.3054	26.0071	20.6980
		100	23.6268	12.3209	12.3179	12.3106	12.4336	9.9337
	1000	50	57.6789	25.1016	25.0690	25.0215	25.8700	20.5610
		100	24.9836	12.3588	12.3558	12.3502	12.4728	9.9643
		200	11.3057	5.8657	5.8654	5.8639	5.8861	4.7076

We can see, from the above tables, that our proposed estimators of the mean and the variance outperform other competing estimators. The results given in the above tables also indicate that the mean square error of all of the estimators decreases with the increase in the sample size.

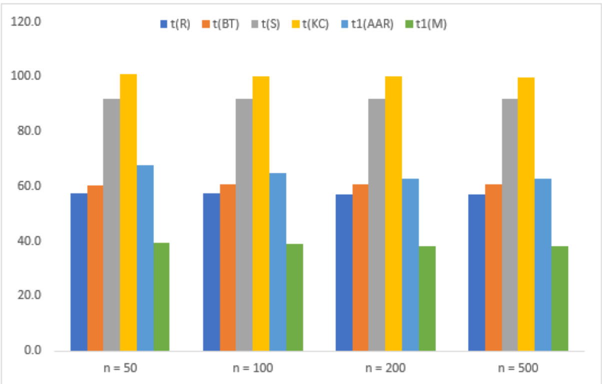
The graphs of relative efficiency of various estimators of the mean and the variance, relative to the ratio estimators of mean and variance, are given in Figures 1 and 2 below. The graphs also show that our proposed estimators of the mean and the variance have the best efficiency as compared with the competing estimators. We can also see, from the figures, that the estimator proposed by [25] is the worst estimator to estimate the population mean. This estimator is even worse than the ratio estimator. The derived estimator of the mean by [22] is better than some of the estimators used in the study, but still this estimator performs worse than the classical regression estimator of the mean and the estimator proposed by [24]. Similar conclusions can be drawn from the comparison of estimators of the variance, and we can see that our derived estimator of variance outperforms all other estimators used in the study. The relative efficiencies of the estimators of the variance show that all of the estimators used in the study perform better than the classical ratio estimator of variance proposed by [9].



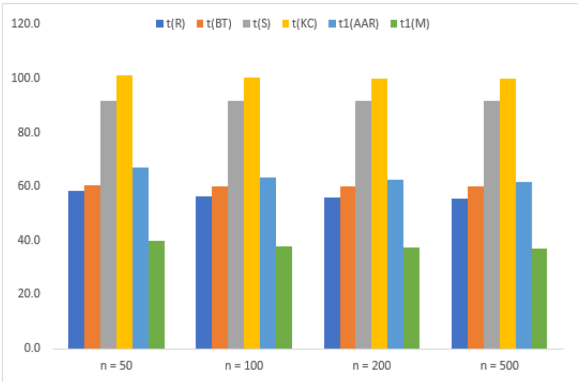
$\rho_{yx} = -0.9$



$\rho_{yx} = -0.5$

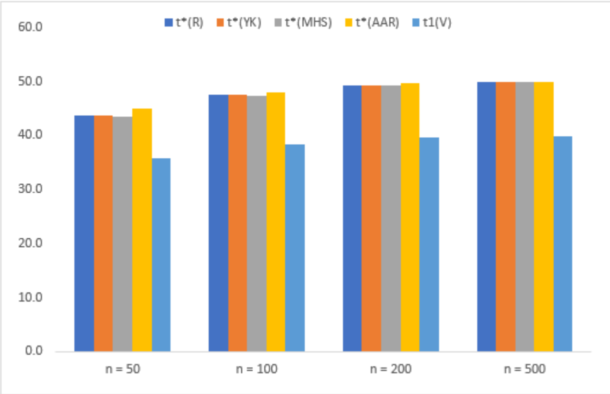


$\rho_{yx} = 0.5$

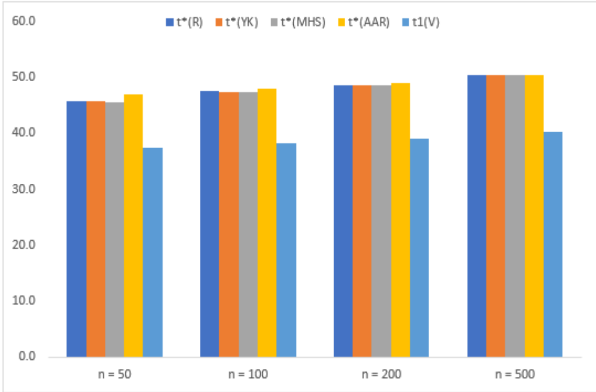


$\rho_{yx} = 0.9$

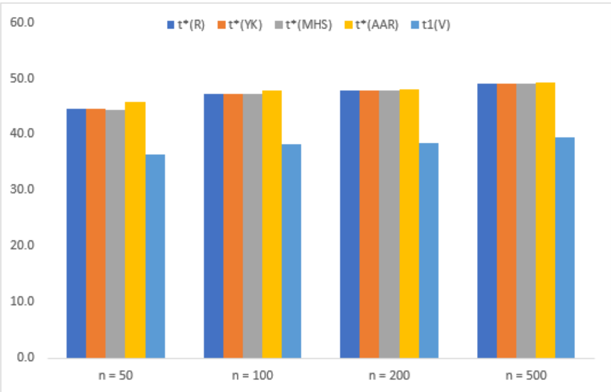
Figure 1. Relative efficiency of various estimators of mean.



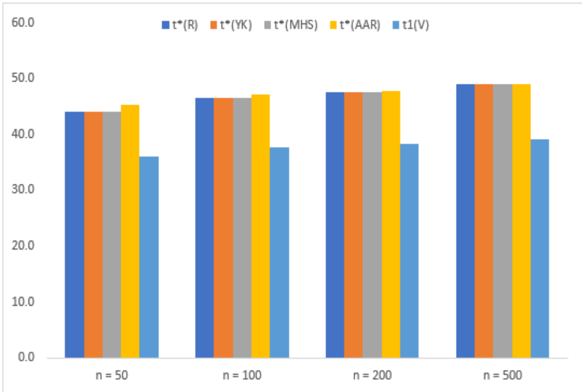
$\rho_{yx} = -0.9$



$\rho_{yx} = -0.5$



$\rho_{yx} = 0.5$



$\rho_{yx} = 0.9$

Figure 2. Relative efficiency of various estimators of variance.

6.2. Empirical study using real populations

In this section, we have conducted an empirical study of some popular estimators of the mean and the variance by using some real populations. We have used five populations for this empirical study. The first three populations are taken from [27], and the last two are taken from [28]. Summary measures of the populations are given in Table 7 below.

Table 7. Summary measures of the populations.

Measures	Pop-I	Pop-II	Pop-III	Pop-IV	Pop-V
N	17	58	32	23	110
$\bar{Y}$	202.9529	13.1879	55.9062	61.3478	6.8317
$\bar{X}$	25.0588	31.8207	4.4222	39.6087	27.4273
S_y^2	33.1739	2.4702	247.5071	279.3281	5.2488
S_x^2	9.1211	24.4701	2.1090	71.7036	278.3754
ρ_{yx}	0.9972	0.5557	0.7815	-0.7737	-0.0645
φ_{40}^*	0.9469	2.0227	2.2414	1.4656	1.6078
φ_{04}^*	0.9062	1.7776	1.8657	0.9113	0.7985
φ_{22}^*	0.9199	0.2282	1.2776	0.5534	-0.0469
φ_{03}	0.3713	0.4208	0.9532	0.2330	0.0280
φ_{21}	0.3175	0.0146	0.5743	-0.1069	0.1094
φ_{12}	0.3438	-0.0931	0.6016	0.0190	0.0139

The empirical study has been conducted by using a 25% sample from each of the populations. We have used six estimators of the mean and five estimators of the variance in this empirical study. The estimators of the mean that we have used are given in Table 1 above, excluding the estimator by [22], as it has the same mean square error as the mean square error of our proposed estimator. The estimators of variance that we have used in this empirical study are classical ratio and regression estimators by [9], estimator by [12], estimator by [13] and our derived estimator of variance, given in Table 2 above. The mean square error of various estimators is computed for each population. The results are given in Tables 8 and 9 below.

Table 8. Mean square error of selected estimators of mean.

Estimator	Population-1	Population-2	Population-3	Population-4	Population-5
t_C	67.0059	0.1675	12.4789	123.7334	0.6637
t_R	0.0353	0.0925	9.0312	17.5502	0.1461
t_{BT}	8.0789	0.0938	9.9412	76.9963	0.2845
t_S	49.5017	0.1365	9.2494	105.7338	0.3546
t_{KC}	114.4157	0.3203	40.6317	44.4738	0.6287
$t_{1(M)}$	0.0296	0.0836	8.5327	15.5233	0.1460

Table 9. Mean square error of selected estimators of variance.

Estimator	Population-1	Population-2	Population-3	Population-4	Population-5
t_C^*	2.8368	1.1056	8912.8527	15510.8332	1.9248
t_R^*	2.7938	0.6591	7848.1442	13794.3352	1.2358
t_{SC}^*	53.3671	0.7403	8213.9491	13922.3775	1.4277
t_{AA}^*	191.9420	0.6700	11944.0214	18316.7933	1.2579
$t_{1(V)}$	1.9203	0.6585	7779.2856	12993.2609	1.2263

From the above tables, we can see that our proposed estimators of the mean and the variance perform better than all other competing estimators. We can also see that the estimator of the mean proposed by [25] and the estimator of the variance proposed by [13] are the worst estimators. The performance of these estimators increases where population variance of the study variable is much smaller as compared with the population variance of the auxiliary variable.

7. Conclusions

In this paper, we have proposed some estimators of the general population parameters for single- and two-phase sampling. These estimators have been proposed by using information of a single and several auxiliary variables. The proposed estimators can be used to obtain estimators of population mean, population variance and population coefficient of variation. The expressions for the mean square error of the proposed estimators have been obtained for single- and two-phase sampling. We have seen that our proposed estimators have smaller mean square error as compared with some of the existing estimators. We have conducted extensive simulation study of the proposed estimator for single- and two-phase sampling. Several available estimators are compared in the simulation study. We have seen that our proposed estimators of the mean and the variance perform better than the competing estimators used in the study. We have also seen that the simulated mean square errors of various estimators decrease with increase in the sample size. We have also conducted an empirical study using some real populations. The empirical study has been conducted by computing the analytical mean square error of various estimators. The empirical study shows that our proposed estimators of the mean and the variance are better than the other estimators used in the study. It is, therefore, recommended that the proposed estimators are better choices for estimation of population mean and population variance as compared with the existing estimators.

Conflict of interest

The authors declare no conflicts of interest.

References

1. W. G. Cochran, The estimation of the yields of cereal experiments by sampling for the ratio gain to total produce, *J. Agric. Soc.*, **30** (1940), 262–275. https://doi.org/10.1017/S0021859600048012

2. S. K. Srivastava, H. S. Jhajj, A class of estimators of the population mean in survey sampling using auxiliary information, *Biometrika*, **68** (1981), 341–343. https://doi.org/10.1093/biomet/68.1.341
3. H. P. Singh, M. R. Espejo, On linear regression and ratio-product estimation of a finite population mean, *J. R. Stat. Soc.*, **52** (2003), 59–67. https://doi.org/10.1111/1467-9884.00341
4. M. Samiuddin, M. Hanif, Estimation of population mean in single- and two-phase sampling with or without additional information, *Pak. J. Stat.*, **23** (2007), 1–9.
5. A. Y. Dar, N. Saeed, M. O. A. Abu-Shawiesh, S. H. Shahbaz, M. Q. Shahbaz, A new class of ratio estimator in single- and two-phase sampling, *AIMS Math.*, **7** (2022), 14208–14226. https://doi.org/10.3934/math.2022783
6. Z. Ahmad, M. Q. Shahbaz, M. Hanif, *Two phase sampling*, Cambridge Scholars Publishing, 2013. https://doi.org/10.13140/2.1.4488.7042
7. M. Hanif, M. Q. Shahbaz, M. Ahmed, *Sampling techniques: methods and applications*, Nova Science Publishers, 2018.
8. S. K. Srivastava, H. S. Jhajj, A class of estimators using auxiliary information for estimating finite population variance, *Sankhya*, **42** (1980), 87–96.
9. C. Isaki, Variance estimation using auxiliary information, *J. Am. Stat. Assoc.*, **78** (1983), 117–123. https://doi.org/10.2307/2287117
10. C. Kadilar, H. Cingi, Ratio estimators for the population variance in simple and stratified random sampling, *Appl. Math. Comput.*, **173** (2006), 1047–1059. https://doi.org/10.1016/j.amc.2005.04.032
11. W. Abu-Dayyeh, M. S. Ahmed, Ratio and regression estimator for the variance under two-phase sampling, *Int. J. Stat. Sci.*, **4** (2005), 49–56.
12. R. Singh, P. Chauhan, N. Sawan, F. Smarandache, Improved exponential estimator for population variance using two auxiliary variables, *ArXiv*, 2009. https://doi.org/10.48550/arXiv.0902.0126
13. A. Asghar, A. Sanaullah, M. Hanif, Generalized exponential type estimator for population variance in survey sampling, *Rev. Colomb. Estad.*, **37** (2014), 211–222. https://doi.org/10.15446/rce.v37n1.44368
14. J. Shabbir, S. Gupta, A note on generalized exponential type estimator of population variance in survey sampling, *Rev. Colomb. Estad.*, **38** (2015), 385–397. https://doi.org/10.15446/rce.v38n2.51667
15. R. Singh, M. Mishra, B. P. Singh, P. Singh, N. K. Adichwal, Improved estimators for population coefficient of variation using auxiliary variable, *J. Stat. Manage. Syst.*, **21** (2018), 1335–1355. https://doi.org/10.1080/09720510.2018.1503405
16. S. Gupta, J. Shabbir, Variance estimation in simple random sampling using auxiliary information, *Hacettepe J. Math. Stat.*, **37** (2008), 57–67.
17. J. Subramani, G. Kumarapandiyan, Variance estimation using median of the auxiliary variable, *Int. J. Prob. Stat.*, **1** (2012), 62–66. https://doi.org/10.5923/j.ijps.20120103.02
18. S. K. Yadav, C. Kadilar, Improved exponential type ratio estimator of population variance, *Rev. Colomb. Estad.*, **36** (2013), 145–152.
19. A. H. Al-Marshadi, A. H. Alharby, M. Q. Shahbaz, On some new estimators of population variance in single and two-phase sampling, *Maejo Int. J. Sci. Technol.*, **12** (2018), 272–281.
20. T. Akhlaq, M. Ismail, M. Q. Shahbaz, On efficient estimation of process variability, *Symmetry*, **11** (2019), 554. https://doi.org/10.3390/sym11040554

21. C. Long, W. Chen, R. Yang, D. Yao, Ratio estimation of the population mean using auxiliary information under the optimal sampling design, *Probab. Eng. Inf. Sci.*, **36** (2022), 449–460. https://doi.org/10.1017/S0269964820000625
22. N. K. Adichwal, A. A. H. Ahmadini, Y. S. Raghav, R. Singh, I. Ali, Estimation of general parameters using auxiliary information in simple random sampling without replacement, *J. King Saud Univ. Sci.*, **34** (2022), 101754. https://doi.org/10.1016/j.jksus.2021.101754
23. L. N. Upadhyaya, H. P. Singh, S. Chatterjee, R. Yadav, Improved ratio and product exponential type estimators, *J. Stat. Theory Pract.*, **5** (2011), 285–302. https://doi.org/10.1080/15598608.2011.10412029
24. S. Bahl, R. K. Tuteja, Ratio and product type exponential estimators, *J. Inf. Optim. Sci.*, **12** (1991), 159–164. https://doi.org/10.1080/02522667.1991.10699058
25. C. Kadilar, H. Cingi, Ratio estimators in simple random sampling, *Appl. Math. Comput.*, **151** (2004), 893–902. https://doi.org/10.1016/S0096-3003(03)00803-8
26. G. N. Singh, On the improvement of product method of estimation in sample surveys, *J. Indian Soc. Agrci. Stat.*, **56** (2003), 267–275.
27. S. Weisberg, *Applied linear regression*, 2 Eds., John Wiley, 1987. https://doi.org/10.2307/2531984
28. M. H. Kutner, *Applied linear statistical models*, 5 Eds., McGraw Hill Irwin, 2005.