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Research article

Linear Rayleigh-Taylor instability for compressible viscoelastic fluids

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Abstract: In this paper, we consider the linear Rayleigh-Taylor instability of an equilibrium state of 3D gravity-driven compressible viscoelastic fluid with the elasticity coefficient κ is less than a critical number κ_c in a moving horizontal periodic domain. We first construct the maximal growing mode solutions to the linearized equations by studying a family of modified variational problems, and then we prove an estimate for arbitrary solutions to the linearized equations.

Keywords: linear Rayleigh-Taylor instability; compressible viscoelastic fluid; moving domain
Mathematics Subject Classification: 35R35, 76E17

1. Introduction

In this paper we study the compressible viscoelastic fluids in a three-dimensional moving horizontal periodic domain $\Omega(t)$ with an upper free surface $\Sigma_F(t)$ and a fixed bottom Σ_B (see Figure 1)

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho v) = 0, & \text{in } \Omega(t), \\ \rho(\partial_t v + (v\cdot\nabla)v) + \operatorname{div}\mathbb{S} = -g\rho e_3, & \text{in } \Omega(t), \\ \partial_t U + (v\cdot\nabla)U = \nabla v\, U, & \text{in } \Omega(t), \\ \mathbb{S}n(t) = P_{\text{atm}}n(t) - \sigma H n(t), & \text{on } \Sigma_F(t), \\ v = 0, & \text{on } \Sigma_B, \\ v|_{t=0} = v_0, \quad U|_{t=0} = U_0, & \text{in } \Omega(0), \end{cases} \tag{1.1}$$

where the viscoelastic stress tensor is given by

$$\mathbb{S} = P(\rho)\mathbb{I} - \varepsilon(\rho)(\mathbb{D}(v) - \frac{2}{3}\operatorname{div} v\mathbb{I}) - \delta(\rho)(\operatorname{div} v)\mathbb{I} - \kappa\rho UU^T,$$

The fluid is described by density, velocity and deformation tensor function, which are given for each $t \geq 0$ by

$$\rho(\cdot,t) : \Omega(t) \to \mathbb{R}, \; v(\cdot,t) : \Omega(t) \to \mathbb{R}^3 \text{ and } U(\cdot,t) : \Omega(t) \to \mathbb{R}^3 \times \mathbb{R}^3$$

respectively.

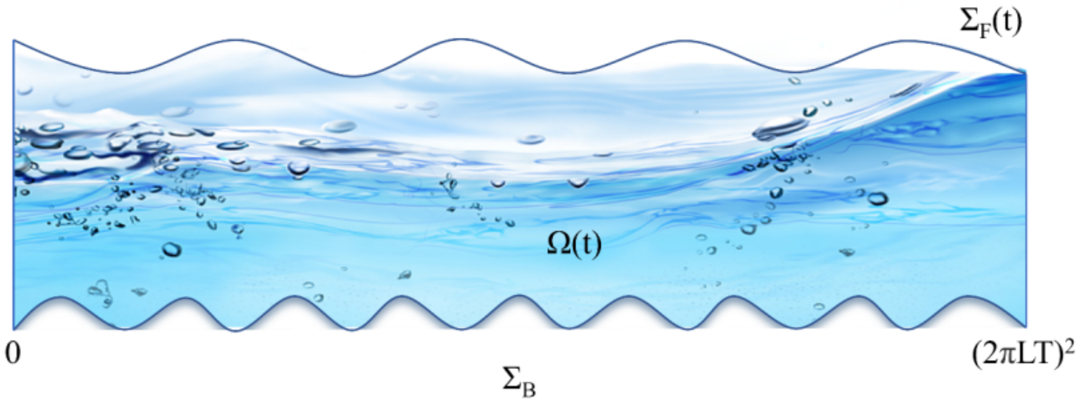

Figure 1. The plan view of $\Omega(t)$.

In this expression the superscript T means matrix transposition and $\mathbb{I}$ is the 3×3 identity matrix. The scalar function P is the pressure which is a function of density $P = P(\rho) > 0$, and the pressure function is assumed to be smooth, positive and strictly increasing, and P_{atm} stands for the atmospheric pressure, assumed to be constant. We denote $\mathcal{V}(\Sigma_F(t))$ the outer-normal velocity of the free surface $\Sigma_F(t)$, and $\mathcal{V}(\Sigma_F(t)) = v \cdot n(t)$. The elasticity coefficient κ denotes the constant elasticity coefficient of the fluid, and ε, δ denote the shear viscosity, the bulk viscosity, respectively, and we assume $\varepsilon > 0$, $\delta \geq 0$, $\varepsilon(\rho)$, $\delta(\rho) \in C^\infty((0,\infty))$. we denote $n(t)$ the outward-pointing unit normal on $\Sigma_F(t)$, H twice the mean curvature of the surface $\Sigma_F(t)$ and the surface tension to be a constant $\sigma > 0$, $(\mathbb{D}(v))_{ij} = (\nabla v + \nabla v^T)_{ij} = \partial_j v^i + \partial_i v^j$ twice the symmetric gradient of the velocity. The constant $g > 0$ stands for the strength of gravity, $e_3 = (0,0,1)$ is the vertical unit vector, and $-g\rho e_3$ is known as the gravitational force. We write div$\mathbb{S}$ for the vector with i^{th} component $\partial_j \mathbb{S}_{ij}$.

The equilibrium in a uniform gravitational field, in which a heavy fluid is on top of a light fluid, is unstable. This phenomenon was first studied by Rayleigh [20] and then Taylor [21], and is called therefore the Rayleigh-Taylor instability. In the last decades, this phenomenon has been extensively investigated from both physical and numerical aspects, see [1, 10, 13, 15] for examples. It has been also widely investigated how the Rayleigh-Taylor instability evolves under the effects of other physical factors, internal surface tension [6, 24], magnetic fields [2, 9, 11, 12, 14], and so on.

We mention some previous mathematical results concerning the Rayleigh-Taylor instability. For the inviscid Rayleigh-Taylor problem without surface tension, Ebin [3] proved the nonlinear ill-posedness of the problem for incompressible fluids. Guo and Tice [5] showed an analogous result for compressible fluids. Hwang and Guo [8] obtained the nonlinear instability of the incompressible problem with a continuous density distribution. For the viscous Rayleigh-Taylor problem, Prüss and Simonett [19] proved the nonlinear instability for incompressible fluids with surface tension in an L^p setting by using Henry's instability theorem. Wang and Tice [23] established the sharp nonlinear instability criteria for the incompressible surface-internal wave problem with or without surface tension. Jang, Tice and Wang [17] proved the nonlinear instability of the dynamics of two layers of compressible, barotropic, viscous fluid lying atop one another via an argument from Jang and Tice [16], in which the authors utilized the linear growing mode to construct initial data for the nonlinear problem.

For the viscoelastic Rayleigh-Taylor problem, Huang, Jiang and Wang [7] obtained that the nonhomogeneous incompressible viscoelastic Rayleigh-Taylor equilibrium state is unstable in L^2-norm based on a bootstrap instability method. Wang and Zhao [22] proved the instability of compressible viscoelastic Rayleigh-Taylor problem in the sense of Hadamard. There are also a lot of great papers for studying the viscoelastic Rayleigh-Taylor problem, such as [18, 25] and their references.

In this paper, we investigate the linear Rayleigh-Taylor instability for the compressible viscoelastic fluid around a steady-state profile with heavier fluid lying above lighter fluid. We consider the equations with surface tension and the viscosity allowed to depend on the density. The linear instability analysis of this paper comprises the first step in an analysis of the nonlinear instability of the compressible viscoelastic fluids, which will be left for our future work.

Formulation in Lagrangian Coordinates We use Lagrange transformation to change the free boundary into a fixed boundary. Firstly, we define the fixed Lagrangian domain to be the horizontal periodic slab

$$\Omega = \{x = (x_h, x_3) | -b < x_3 < 0,\ x_h = (x_1, x_2) \in (2\pi L\mathbb{T})^2\}$$

with the bottom $\Sigma_b = \{x_3 = -b\}$ and the top surface $\Sigma_0 = \{x_3 = 0\}$, where the positive constant b is the depth of the fluid at infinity, and $2\pi L\mathbb{T}$ stands for the 1D-torus of length $2\pi L$.

We assume that there exists a mapping

$$\eta_0 : \Omega \to \Omega(0)$$

that is invertible, and satisfies $\Sigma_F(0) = \eta_0(\Sigma_0)$, and $\Sigma_B = \eta_0(\Sigma_b)$.

We define the flow map η, as the solution to

$$\begin{cases} \partial_t \eta(t, x) = v(t, \eta(x, t)), & t > 0,\ x \in \Omega, \\ \eta(0, x) = \eta_0(x), & x \in \Omega. \end{cases}$$

This implies that $\Omega(t) = \eta(t, \Omega)$, $\Sigma_F(t) = \eta(\Sigma_0)$, and $\Sigma_B = \eta(\Sigma_b)$. In order to switch back and forth from Lagrangian to Eulerian coordinates we assume that $\eta(t, x)$ is invertible.

We define the Lagrangian unknowns

$$u(t, x) \overset{\text{def}}{=} v(t, \eta(t, x)),\ q(t, x) \overset{\text{def}}{=} \rho(t, \eta(t, x)),\ V(t, x) \overset{\text{def}}{=} U(t, \eta(t, x)),$$

which are defined for $(t, x) \in \mathbb{R}^+ \times \Omega$.

Define the matrix $\mathcal{A}$ via $\mathcal{A} = (D\eta)^{-T}$, where

$$D\eta = \begin{pmatrix} \partial_1\eta^1 & \partial_2\eta^1 & \partial_3\eta^1 \\ \partial_1\eta^2 & \partial_2\eta^2 & \partial_3\eta^2 \\ \partial_1\eta^3 & \partial_2\eta^3 & \partial_3\eta^3 \end{pmatrix}.$$

By using the chain rule, we can directly derive

$$
\begin{aligned}
&\partial_t(\rho(t,\eta(t,x))) = (\partial_t\rho)(t,\eta(t,x)) + (v\cdot\nabla\rho)(t,\eta(t,x)),\\
&\partial_t(v(t,\eta(t,x))) = (\partial_t v)(t,\eta(t,x)) + (v\cdot\nabla v)(t,\eta(t,x)),\\
&\partial_t(U(t,\eta(t,x))) = (\partial_t U)(t,\eta(t,x)) + (v\cdot\nabla U)(t,\eta(t,x)),\\
&(\partial_j v_i)(t,\eta(t,x)) = (\partial_k\eta_j)^{-1}\partial_k(v_i(t,\eta(t,x))) = \mathcal{A}_{jk}\partial_k(v_i(t,\eta(t,x))),\\
&(\partial_i v_i)(t,\eta(t,x)) = (\partial_j\eta_i)^{-1}\partial_j(v_i(t,\eta(t,x))) = \mathcal{A}_{ij}\partial_j(v_i(t,\eta(t,x))),
\end{aligned}
\tag{1.2}
$$

where we used the fact that $\mathcal{A}(D\eta)^T = \mathbb{I}$, i.e. $\mathcal{A}_{lk}\partial_k\eta_j = \delta_{lj}$. Using (1.2), we can derive the Lagrangian form of (1.1). Writing $\partial_j = \partial/\partial_{x_j}$, the evolution equations for u, q, V, η in Lagrangian coordinates are,

$$
\begin{cases}
\partial_t\eta_i = u_i, & \text{in } \Omega,\\
\partial_t q + q\mathcal{A}_{ij}\partial_j u_i = 0, & \text{in } \Omega,\\
q\partial_t u_i + \mathcal{A}_{jk}\partial_k T_{ij} = -gq\mathcal{A}_{ij}\partial_j\eta_3, & \text{in } \Omega,\\
\partial_t V_{km} = \mathcal{A}_{ij}\partial_j u_k V_{im}, & \text{in } \Omega,\\
Tn = P_{\text{atm}}n - \sigma Hn, & \text{on } \Sigma_0,\\
u|_{\Sigma_b} = 0,\\
\eta|_{t=0} = \eta_0,\ \ u|_{t=0} = u_0,\ \ V|_{t=0} = V_0
\end{cases}
\tag{1.3}
$$

where

$$
T_{ij} = P(q)\mathbb{I}_{ij} - \varepsilon(q)(\mathcal{A}_{jk}\partial_k u_i + \mathcal{A}_{ik}\partial_k u_j - \frac{2}{3}(\mathcal{A}_{lk}\partial_k u_l)\mathbb{I}_{ij}) - \delta(q)(\mathcal{A}_{lk}\partial_k u_l)\mathbb{I}_{ij} - \kappa q V_{ik}V_{jk}.
$$

Here we have employed the Einstein convention of summing over repeated indices and written

$$
n := \frac{\partial_1\eta\times\partial_2\eta}{|\partial_1\eta\times\partial_2\eta|}\Big|_{x_3=0}
$$

in Lagrangian coordinates and

$$
H = \Big(\frac{|\partial_1\eta|^2\partial_2^2\eta - 2(\partial_1\eta\cdot\partial_2\eta)\partial_1\partial_2\eta + |\partial_2\eta|^2\partial_1^2\eta}{|\partial_1\eta|^2|\partial_2\eta|^2 - |\partial_1\eta\cdot\partial_2\eta|^2}\Big)\cdot n.
$$

Steady-state solution We now seek a steady-state equilibrium solution to (1.3) with $u = 0,\ \eta = Id,\ q(t,x) = \rho_0(x_3),\ V = \bar{U}$, where

$$
\bar{U} \overset{\text{def}}{=} \begin{pmatrix} \bar{u} & 0 & 0\\ 0 & \bar{u} & 0\\ 0 & 0 & \bar{u} \end{pmatrix}
$$

with

$$
\bar{u} \equiv \bar{u}(x_3) \overset{\text{def}}{=} \pm\sqrt{\frac{F(P'(\rho_0)\rho_0' + g\rho_0) + C}{\kappa\rho_0}},
$$

here $F(P'(\rho_0)\rho_0' + g\rho_0)$ denotes a primitive function of $P'(\rho_0)\rho_0' + g\rho_0$ and C is a positive constant satisfying

$$\inf_{x_3\in[-b,0]}\{F(P'(\rho_0)\rho_0' + g\rho_0) + C\} > 0.$$

By a direct calculation, we can find that $\bar{U}\bar{U}^T = \bar{u}^2\mathbb{I}$. Then the system (1.3) reduces to the following ODE with the equilibrium density $\rho_0 = \rho_0(x_3)$

$$\begin{cases} \frac{d(P(\rho_0)-\kappa\rho_0\bar{u}^2)}{dx_3} = -g\rho_0 \quad \forall\ x_3 \in (-b, 0), \\ P(\rho_0) - \kappa\rho_0\bar{u}^2 = P_{\text{atm}}, \quad \text{at } x_3 = 0. \end{cases} \tag{1.4}$$

Noticed that, $\kappa\rho_0\bar{u}^2$ can be expressed as a function of variable ρ_0 by $G(\rho_0)$, i.e., $G(\rho_0) \overset{\text{def}}{=} \kappa\rho_0\bar{u}^2$, and we define $M(\rho_0) \overset{\text{def}}{=} P(\rho_0)-\kappa\rho_0\bar{u}^2$. We consider the case that the density satisfies the following conditions:

$$\rho_0 \in C^\infty(\bar{\Omega}), \quad \inf_{x_3\in[-b,0]}\rho_0(x_3) > 0, \tag{1.5}$$

and the Rayleigh-Taylor condition

$$\inf_{x_3\in[-b,0]}\rho_0'(x_3) > 0. \tag{1.6}$$

The Rayleigh-Taylor condition assures that the density has larger density with increasing height x_3, thus leading to the classical Rayleigh-Taylor instability. From (1.4) and (1.5), we can get $M'(\rho_0)\rho_0' = -g\rho_0 < 0$, which along with (1.6) implies $M'(\rho_0) < 0$, $\forall\ \rho_0 \in [\bar{\rho}_2, \bar{\rho}_1]$, i.e. $M(\rho_0)$ is strictly decreasing with respect to ρ_0 on $[\bar{\rho}_2, \bar{\rho}_1]$, where $P'(\rho_0) = P'(s)|_{s=\rho_0}$, $M'(\rho_0) = M'(s)|_{s=\rho_0}$ and $\rho_0'(x_3) = \frac{d\rho_0}{dx_3}$, $\bar{\rho}_1 \overset{\text{def}}{=} M^{-1}(P_{\text{atm}})$, $\bar{\rho}_2 \overset{\text{def}}{=} \rho_0(-b)$.

We may claim that the necessary and sufficient for the existence of an equilibrium to (1.4) are as follows:

$$(1)\ P_{\text{atm}} \in M(\mathbb{R}^+), \qquad (2)\ 0 < b \le \frac{1}{g}\int_{\bar{\rho}_1}^{\bar{\rho}_2} \frac{M'(s)}{s}ds. \tag{1.7}$$

In fact, since the function $M = M(\rho_0)$ is smooth and strictly decreasing with respect to ρ_0 on $[\bar{\rho}_2, \bar{\rho}_1]$, the second equation in (1.4) holds if and only if $P_{\text{atm}} \in M(\mathbb{R}^+)$, which defines $\bar{\rho}_1 = M^{-1}(P_{\text{atm}})$, that is, the first condition in (1.7) holds. On the other hand, we introduce the function $h : (0, +\infty) \to \mathbb{R}$ given by

$$h(z) \overset{\text{def}}{=} \int_{\bar{\rho}_1}^{z} \frac{M'(s)}{s}ds,$$

which is smooth, strictly decreasing and positive on $[\bar{\rho}_2, \bar{\rho}_1]$. From (1.4), we get

$$\begin{cases} \frac{dh(\rho_0)}{dx_3} = -g, \quad \forall\ x_3 \in (-b, 0), \\ \rho_0(0) = \bar{\rho}_1. \end{cases}$$

Solve this ODE to find $\rho_0(x_3) = h^{-1}(-gx_3)$, which gives a well-defined, smooth and increasing function $\rho_0 : [-b, 0] \to [\bar{\rho}_2, \bar{\rho}_1]$ if and only if

$$gb \in h([\bar{\rho}_2, \bar{\rho}_1]),$$

that is, the second condition in (1.7) holds.

Linearization around the steady-state We now linearize the Eq (1.3) around the steady-state solution $u = 0,\ \eta = Id,\ q(t, x) = \rho_0(x_3),\ V = \bar{U}$. Then the resulting linearized viscoelastic equations are

$$\begin{cases} \partial_t \eta = u & \text{in } \Omega, \\ \partial_t q + \rho_0 \operatorname{div} u = 0 & \text{in } \Omega, \\ \rho_0 \partial_t u + \nabla(P'(\rho_0)q) + gqe_3 + g\rho_0 \nabla\eta_3 = \operatorname{div}(\varepsilon_0(\mathbb{D}(u) - \frac{2}{3}(\operatorname{div} u)\mathbb{I}) & \\ \qquad\qquad + \delta_0(\operatorname{div} u)\mathbb{I} + \kappa\rho_0(V\bar{U}^T + \bar{U}V^T) + \kappa q\bar{u}^2\mathbb{I}) & \text{in } \Omega, \\ \partial_t V = \nabla u \bar{U} & \text{in } \Omega, \\ (P'(\rho_0)q\mathbb{I} - \varepsilon_0\mathbb{D}(u) - (\delta_0 - \frac{2}{3}\varepsilon_0) \operatorname{div} u\mathbb{I} & \\ \qquad -\kappa\rho_0(V\bar{U}^T + \bar{U}V^T) - \kappa q\bar{u}^2\mathbb{I})e_3 = -\sigma\Delta_{x_1,x_2}\eta_3 e_3 & \text{on } \Sigma_0, \\ u|_{\Sigma_b} = 0, & \end{cases} \tag{1.8}$$

where $\varepsilon_0 = \varepsilon(\rho_0)$ and $\delta_0 = \delta(\rho_0)$.

Before further stating our result, we shall introduce some notations used throughout this paper. We first define the weighted L^2 norm and the viscosity seminorm by

$$\|u\|_*^2 = \int_\Omega \rho_0 |u|^2 dx \ \text{ and } \ \|u\|_{**}^2 = \int_\Omega \frac{\varepsilon_0}{2}|\mathbb{D}(u) - \frac{2}{3}(\operatorname{div} u)\mathbb{I}|^2 + \delta_0 |\operatorname{div} u|^2 dx. \tag{1.9}$$

And we denote

$$\begin{aligned} &H_0^1 := \{w \in H^1(\Omega)|\ w|_{\Sigma_b} = 0\},\ \ \Psi(w) = \int_\Omega \rho_0\bar{u}^2(\frac{1}{2}|\mathbb{D}(w)|^2 - |\operatorname{div} w|^2)dx, \\ &\Theta_1(w) := \int_\Omega (g\rho_0 w_3 \operatorname{div} w - g\rho_0 w \cdot \nabla w_3)dx - \int_\Omega P'(\rho_0)\rho_0 |\operatorname{div} w|^2 dx \\ &\qquad\qquad - \int_{\Sigma_0} \sigma |\nabla_{x_1,x_2} w_3|^2 dS_0,\ \ \Theta(w) = \Theta_1(w) - \kappa\Psi(w), \\ &\widetilde{\kappa_c} := \sup_{w \in H_0^1} \frac{\Theta(w)}{\Psi(w)},\ \ a \lesssim b \text{ means that } a \le Cb,\ \text{for some constant } C > 0. \end{aligned}$$

Remark 1.1. *We mention that*

$$\Psi(w) \ge 0, \tag{1.10}$$

for all $w \in H_0^1$. In fact, by a direct calculation,

$$\begin{aligned} &\int_\Omega \rho_0\bar{u}^2(\frac{1}{2}|\mathbb{D}(w)|^2 - |\operatorname{div} w|^2)dx \\ &= \int_\Omega \rho_0\bar{u}^2((\partial_1 w_2 + \partial_2 w_1)^2 + (\partial_1 w_3 + \partial_3 w_1)^2 + (\partial_2 w_3 + \partial_3 w_2)^2 \\ &\qquad + |\partial_1 w_1|^2 + |\partial_2 w_2|^2 + |\partial_3 w_3|^2)dx. \end{aligned}$$

Hence (1.10) *holds. Moreover, we can see that, if* $\Psi(w) = 0$*, for some* $w \in H_0^1$*, then* $w = 0$*. This implies that* $\widetilde{\kappa}_c > 0$ *is equivalent to the condition:*

$$\text{there exists a function } \widetilde{w} \in H_0^1 \text{ such that } \Theta(\widetilde{w}) > 0,$$

and $\widetilde{\kappa}_c > 0$ *is also equivalent to the following condition:*

$$\kappa < \kappa_c := \sup_{w \in H_0^1} \frac{\Theta_1(w)}{\Psi(w)}. \tag{1.11}$$

Now we state our main result.

Theorem 1.1. *Let* (u, η, q, V) *be a solution to* (1.8)*. Then*

$$\begin{aligned} &\|u(t)\|_*^2 + \|u(t)\|_{**}^2 + \|\partial_t u(t)\|_*^2 \le Ce^{2\Lambda t} I_0, \ \ \|\eta(t)\|_{H^1} \le Ce^{\Lambda t}(\|\eta_0\|_{H^1} + \sqrt{I_0}), \\ &\|q(t)\|_{L^2} \le Ce^{\Lambda t}(\|q_0\|_{L^2} + \sqrt{I_0}), \ \ \|V(t)\|_{L^2} \le Ce^{\Lambda t}(\|V_0\|_{L^2} + \sqrt{I_0}), \end{aligned}$$

for a constant $0 < C = C(\rho_0, P, \Lambda, \varepsilon, \delta, \kappa, \sigma, g, m, l)$*, where* Λ *is defined in* (2.37) *below and*

$$I_0 = \|\partial_t u(0)\|_*^2 + \|u(0)\|_*^2 + \|u(0)\|_{**}^2 + \sigma \int_{\Sigma_0} |\nabla_{x_1,x_2} u_3(0)|^2.$$

The rest of the paper is organized as follows. In section 2, we construct a growing mode solution to (1.8) by assuming an ansatz, and then we study a family of modified variational problems in order to produce maximal growing modes. In section 3, we firstly take the preliminary estimates, and then prove our main result.

2. Growing mode solution

We want to construct a growing mode solution to (1.8) by assuming an ansatz

$$u(t,x) = w(x)e^{\lambda t}, \ V(t,x) = \widetilde{V}(x)e^{\lambda t}, \ \eta(t,x) = \widetilde{\eta}(x)e^{\lambda t}, \ q(t,x) = \widetilde{q}(x)e^{\lambda t}, \tag{2.1}$$

for some $\lambda > 0$. Substituting this ansatz into $(1.8)_1$, $(1.8)_2$ and $(1.8)_4$, respectively, we find that $\widetilde{\eta} = \lambda^{-1} w$, $\widetilde{q} = -\lambda^{-1}\rho_0 \mathrm{div}\, w$, $\widetilde{V} = \lambda^{-1}\nabla w \bar{U}$. By this fact we can eliminate $\widetilde{\eta}, \widetilde{q}, \widetilde{V}$ from (1.8) and then arrive at the following time-invariant system for w:

$$\begin{aligned} &\lambda^2\rho_0 w - \nabla(P'(\rho_0)\rho_0 \mathrm{div}\, w) - g\rho_0 \mathrm{div}\, w e_3 + g\rho_0 \nabla w_3 \\ &= \mathrm{div}(\lambda\varepsilon_0(\mathbb{D}(w) - \frac{2}{3}(\mathrm{div}\, w)\mathbb{I}) + \lambda\delta_0(\mathrm{div}\, w)\mathbb{I} - \kappa\rho_0\bar{u}^2(\mathrm{div}\, w)\mathbb{I} + \kappa\rho_0\bar{u}^2\mathbb{D}(w)). \end{aligned} \tag{2.2}$$

The boundary conditions linearize to: on Σ_0

$$((\lambda\delta_0 - 2\lambda\varepsilon_0/3 + P'(\rho_0)\rho_0 - \kappa\rho_0\bar{u}^2)\mathrm{div}\, w\mathbb{I} + (\lambda\varepsilon_0 + \kappa\rho_0\bar{u}^2)\mathbb{D}(w))e_3 = \sigma\Delta_{x_1,x_2} w_3 e_3,$$

and $w|_{\Sigma_b} = 0$. Since the domain is horizontally flat, we are free to make the further structural assumption that the x' dependence of w is given as a Fourier mode $e^{ix'\cdot\xi}$, where $x' \cdot \xi = x_1\xi_1 + x_2\xi_2$ for $\xi \in$

$(L^{-1}\mathbb{Z})^2$, $x' = (x_1, x_2)$. Together with the growing mode ansatz (2.1), this constitutes a "normal mode" ansatz (see [1]). We define the new unknowns $\varphi, \theta, \psi : (-b, 0) \to \mathbb{R}$ according to

$$w_1(x) = -i\varphi(x_3)e^{ix'\cdot\xi},\ w_2(x) = -i\theta(x_3)e^{ix'\cdot\xi},\ \text{and } w_3(x) = \psi(x_3)e^{ix'\cdot\xi}.$$

By a direct calculation, we can get

$$\operatorname{div} w = (\xi_1\varphi + \xi_2\theta + \psi')e^{ix'\cdot\xi},$$

and

$$\nabla w + \nabla w^T = \begin{pmatrix} 2\xi_1\varphi & \xi_1\theta + \xi_2\varphi & i(\xi_1\psi - \varphi') \\ \xi_1\theta + \xi_2\varphi & 2\xi_2\theta & i(\xi_2\psi - \theta') \\ i(\xi_1\psi - \varphi') & i(\xi_2\psi - \theta') & 2\psi' \end{pmatrix} e^{ix'\cdot\xi}. \tag{2.3}$$

For each fixed nonzero spatial frequency ξ, we deduce from the Eq (2.2) a system of ODEs for $\varphi, \theta, \psi, \lambda$, denoting $' = d/dx_3$: in $(-b, 0)$

$$\begin{aligned} &-(\lambda\varepsilon_0\varphi' + \kappa\rho_0\bar{u}^2\varphi')' + [\lambda^2\rho_0 + (\lambda\varepsilon_0 + \kappa\rho_0\bar{u}^2)|\xi|^2 + \xi_1^2(\lambda\delta_0 + \frac{1}{3}\lambda\varepsilon_0 + P'(\rho_0)\rho_0)]\varphi \\ &= -\xi_1[(\lambda\delta_0 + \frac{1}{3}\lambda\varepsilon_0 + P'(\rho_0)\rho_0)\psi' + (\lambda\varepsilon_0' + (\kappa\rho_0\bar{u}^2)' - g\rho_0)\psi] - \xi_1\xi_2[\lambda\delta_0 + \frac{1}{3}\lambda\varepsilon_0 + P'(\rho_0)\rho_0]\theta, \end{aligned} \tag{2.4}$$

$$\begin{aligned} &-(\lambda\varepsilon_0\theta' + \kappa\rho_0\bar{u}^2\theta')' + [\lambda^2\rho_0 + (\lambda\varepsilon_0 + \kappa\rho_0\bar{u}^2)|\xi|^2 + \xi_2^2(\lambda\delta_0 + \frac{1}{3}\lambda\varepsilon_0 + P'(\rho_0)\rho_0)]\theta \\ &= -\xi_2[(\lambda\delta_0 + \frac{1}{3}\lambda\varepsilon_0 + P'(\rho_0)\rho_0)\psi' + (\lambda\varepsilon_0' + (\kappa\rho_0\bar{u}^2)' - g\rho_0)\psi] - \xi_1\xi_2[\lambda\delta_0 + \frac{1}{3}\lambda\varepsilon_0 + P'(\rho_0)\rho_0]\varphi, \end{aligned} \tag{2.5}$$

$$\begin{aligned} &-[(4\lambda\varepsilon_0/3 + \lambda\delta_0 + P'(\rho_0)\rho_0 + \kappa\rho_0\bar{u}^2)\psi']' + (\lambda^2\rho_0 + \lambda\varepsilon_0|\xi|^2 + \kappa\rho_0\bar{u}^2|\xi|^2)\psi \\ &= [(\lambda\delta_0 + \frac{1}{3}\lambda\varepsilon_0 + P'(\rho_0)\rho_0)(\xi_1\varphi + \xi_2\theta)]' + (g\rho_0 - \lambda\varepsilon_0' - (\kappa\rho_0\bar{u}^2)')(\xi_1\varphi + \xi_2\theta). \end{aligned} \tag{2.6}$$

The upper boundary condition for the new unknowns is:

$$\begin{aligned} \Big((\lambda\delta_0 - 2\lambda\varepsilon_0/3 + P'(\rho_0)\rho_0 - \kappa\rho_0\bar{u}^2)(\xi_1\varphi + \xi_2\theta + \psi')e_3 \\ + (\lambda\varepsilon_0 + \kappa\rho_0\bar{u}^2)\begin{pmatrix} i(\xi_1\psi - \varphi') \\ i(\xi_2\psi - \theta') \\ 2\psi' \end{pmatrix}\Big) = -\sigma|\xi|^2\psi e_3, \quad \text{at } x_3 = 0, \end{aligned}$$

which follows that at $x_3 = 0$

$$(\lambda\varepsilon_0 + \kappa\rho_0\bar{u}^2)(\varphi' - \xi_1\psi) = (\lambda\varepsilon_0 + \kappa\rho_0\bar{u}^2)(\theta' - \xi_2\psi) = 0,$$

and

$$(\lambda\delta_0 + \lambda\varepsilon_0/3 + P'(\rho_0)\rho_0)(\psi' + \xi_1\varphi + \xi_2\theta) + (\lambda\varepsilon_0 + \kappa\rho_0\bar{u}^2)(\psi' - \xi_1\varphi - \xi_2\theta) = -\sigma|\xi|^2\psi.$$

and the bottom boundary conditions become

$$\varphi(-b) = \theta(-b) = \psi(-b) = 0.$$

Note that if φ, θ, ψ solve the Eqs (2.4)–(2.6) for $\xi \in (L^{-1}\mathbb{Z})^2$ and λ, then for any rotation operator $R \in SO(2)$, $(\widetilde{\varphi}, \widetilde{\theta}) := R(\varphi, \theta)$ solve the same equations for $\widetilde{\xi} := R\xi$ with ψ, λ unchanged. Then, we choose a rotation operator R so that $R\xi = (|\xi|, 0)$ to see that θ solves

$$\begin{cases} -(\lambda\varepsilon_0\theta' + \kappa\rho_0\bar{u}^2\theta')' + (\lambda^2\rho_0 + (\lambda\varepsilon_0 + \kappa\rho_0\bar{u}^2)|\xi|^2)\theta = 0, & \text{in } (-b, 0), \\ (\lambda\varepsilon_0 + \kappa\rho_0\bar{u}^2)\theta' = 0, & \text{at } x_3 = 0, \\ \theta(-b) = 0. \end{cases} \tag{2.7}$$

Multiplying (2.7) by θ, integrating over $(-b, 0)$, integrating by parts, and using the boundary conditions then yields

$$\int_{-b}^{0} (\lambda\varepsilon_0 + \kappa\rho_0\bar{u}^2)|\theta'|^2 + (\lambda^2\rho_0 + (\lambda\varepsilon_0 + \kappa\rho_0\bar{u}^2)|\xi|^2)\theta^2 dx_3 = 0,$$

which implies that $\theta = 0$. Then (2.4)–(2.6) reduce to the pair of equations in $(-b, 0)$ for φ, ψ

$$\begin{aligned} -\lambda^2\rho_0\varphi = & -(\lambda\varepsilon_0\varphi' + \kappa\rho_0\bar{u}^2\varphi')' + |\xi|^2(4\lambda\varepsilon_0/3 + \lambda\delta_0 + P'(\rho_0)\rho_0 + \kappa\rho_0\bar{u}^2)\varphi \\ & + |\xi|[(\lambda\delta_0 + \lambda\varepsilon_0/3 + P'(\rho_0)\rho_0)\psi' + (\lambda\varepsilon_0' + (\kappa\rho_0\bar{u}^2)' - g\rho_0)\psi], \end{aligned} \tag{2.8}$$

$$\begin{aligned} -\lambda^2\rho_0\psi = & -[(4\lambda\varepsilon_0/3 + \lambda\delta_0 + P'(\rho_0)\rho_0 + \kappa\rho_0\bar{u}^2)\psi']' + (\lambda\varepsilon_0 + \kappa\rho_0\bar{u}^2)|\xi|^2\psi \\ & - |\xi|[((\lambda\delta_0 + \lambda\varepsilon_0/3 + P'(\rho_0)\rho_0)\varphi)' + (g\rho_0 - \lambda\varepsilon_0' - (\kappa\rho_0\bar{u}^2)')\varphi], \end{aligned} \tag{2.9}$$

along with the boundary conditions: at $x_3 = 0$

$$(\lambda\varepsilon_0 + \kappa\rho_0\bar{u}^2)(\varphi' - |\xi|\psi) = 0, \tag{2.10}$$

$$(\lambda\delta_0 + \lambda\varepsilon_0/3 + P'(\rho_0)\rho_0)(\psi' + |\xi|\varphi) + (\lambda\varepsilon_0 + \kappa\rho_0\bar{u}^2)(\psi' - |\xi|\varphi) = -\sigma|\xi|^2\psi, \tag{2.11}$$

and at $x_3 = -b$

$$\varphi = \psi = 0. \tag{2.12}$$

Applying a modified variational method to construct a solution of (2.8)–(2.12), we modify (2.8)–(2.12) as follows: in $(-b, 0)$

$$\begin{aligned} -\lambda^2\rho_0\varphi = & -(s\varepsilon_0\varphi' + \kappa\rho_0\bar{u}^2\varphi')' + |\xi|^2(\frac{4}{3}s\varepsilon_0 + s\delta_0 + P'(\rho_0)\rho_0 + \kappa\rho_0\bar{u}^2)\varphi \\ & + |\xi|[(s\delta_0 + \frac{1}{3}s\varepsilon_0 + P'(\rho_0)\rho_0)\psi' + (s\varepsilon_0' + (\kappa\rho_0\bar{u}^2)' - g\rho_0)\psi], \end{aligned} \tag{2.13}$$

$$\begin{aligned} -\lambda^2\rho_0\psi = & -[(\frac{4}{3}s\varepsilon_0 + s\delta_0 + P'(\rho_0)\rho_0 + \kappa\rho_0\bar{u}^2)\psi']' + (s\varepsilon_0 + \kappa\rho_0\bar{u}^2)|\xi|^2\psi \\ & - |\xi|[((s\delta_0 + \frac{1}{3}s\varepsilon_0 + P'(\rho_0)\rho_0)\varphi)' + (g\rho_0 - s\varepsilon_0' - (\kappa\rho_0\bar{u}^2)')\varphi], \end{aligned} \tag{2.14}$$

where $s > 0$ is an arbitrary parameter, along with the boundary conditions: at $x_3 = 0$

$$(s\varepsilon_0 + \kappa\rho_0\bar{u}^2)(\varphi' - |\xi|\psi) = 0, \tag{2.15}$$

$$(s\delta_0 + \frac{1}{3}s\varepsilon_0 + P'(\rho_0)\rho_0)(\psi' + |\xi|\varphi) + (s\varepsilon_0 + \kappa\rho_0\bar{u}^2)(\psi' - |\xi|\varphi) = -\sigma|\xi|^2\psi, \tag{2.16}$$

and at $x_3 = -b$

$$\varphi = \psi = 0. \tag{2.17}$$

Note that for any fixed $s > 0$ and ξ, (2.13) and (2.14) is a standard eigenvalue problem for $-\lambda^2$, which has a natural variational structure that allows us to use variational methods to construct solutions. For this, we define the energies

$$E(\varphi, \psi; |\xi|, s) = E_0(\varphi, \psi; |\xi|) + sE_1(\varphi, \psi; |\xi|), \tag{2.18}$$

and

$$J(\varphi, \psi) = \frac{1}{2}\int_{-b}^{0} \rho_0(\varphi^2 + \psi^2)dx_3,$$

where

$$\begin{aligned} E_0(\varphi, \psi; |\xi|) :=& \frac{\sigma|\xi|^2}{2}(\psi(0))^2 + \frac{1}{2}\int_{-b}^{0} P'(\rho_0)\rho_0(\psi' + |\xi|\varphi)^2 - 2g\rho_0|\xi|\varphi\psi dx_3 \\ &+ \frac{1}{2}\int_{-b}^{0} \kappa\rho_0\bar{u}^2((\varphi' - |\xi|\psi)^2 + (\psi' - |\xi|\varphi)^2)dx_3, \\ E_1(\varphi, \psi; |\xi|) :=& \frac{1}{2}\int_{-b}^{0}(\delta_0 + \frac{1}{3}\varepsilon_0)(\psi' + |\xi|\varphi)^2 + \varepsilon_0((\varphi' - |\xi|\psi)^2 + (\psi' - |\xi|\varphi)^2)dx_3, \end{aligned}$$

which are both well-defined on the space $H_0^1((-b,0)) \times H_0^1((-b,0))$, where $H_0^1((-b,0)) := \{f \in H^1(-b,0) | f(-b) = 0\}$. Consider the set

$$C = \{(\varphi, \psi) \in H_0^1((-b,0)) \times H_0^1((-b,0)) | J(\varphi, \psi) = 1\}.$$

Notice that by employing the identity $-2ab = (a-b)^2 - (a^2+b^2)$ and the constraint on $J(\varphi, \psi)$ we may rewrite

$$\begin{aligned} E(\varphi, \psi; |\xi|, s) =& \frac{\sigma|\xi|^2}{2}(\psi(0))^2 + \frac{1}{2}\int_{-b}^{0} P'(\rho_0)\rho_0(\psi' + |\xi|\varphi)^2 + g\rho_0|\xi|(\varphi - \psi)^2 dx_3 \\ &- \frac{1}{2}\int_{-b}^{0} g\rho_0|\xi|(\varphi^2 + \psi^2)dx_3 + \frac{1}{2}\int_{-b}^{0} \kappa\rho_0\bar{u}^2((\varphi' - |\xi|\psi)^2 + (\psi' - |\xi|\varphi)^2)dx_3 \\ &+ \frac{s}{2}\int_{-b}^{0}(\delta_0 + \frac{1}{3}\varepsilon_0)(\psi' + |\xi|\varphi)^2 + \varepsilon_0((\varphi' - |\xi|\psi)^2 + (\psi' - |\xi|\varphi)^2)dx_3 \geq -g|\xi| \end{aligned} \tag{2.19}$$

for any $(\varphi, \psi) \in C$. Then we want to find the smallest $-\lambda^2$ by minimizing

$$-\lambda^2(|\xi|; s) = \alpha(|\xi|; s) := \inf_{(\varphi,\psi)\in C} E(\varphi, \psi; |\xi|, s). \tag{2.20}$$

The first Lemma asserts that a minimizer of $E(\varphi, \psi; |\xi|, s)$ in (2.18) over C exists and the minimizer solves (2.13)–(2.17).

Lemma 2.1. *Let ξ and $s>0$ be fixed. Then the following hold:*
(1) $E(\varphi,\psi;|\xi|,s)$ achieves its infimum over C.
(2) Let $(\widetilde{\varphi},\widetilde{\psi})\in C$ be the minimizers of E constructed in (1). Let $\alpha:=E(\widetilde{\varphi},\widetilde{\psi};|\xi|,s)$. Then $(\widetilde{\varphi},\widetilde{\psi})$ are smooth in $(-b,0)$ and satisfy

$$\begin{aligned}\alpha\rho_0\widetilde{\varphi}=&-(s\varepsilon_0\widetilde{\varphi}'+\kappa\rho_0\bar{u}^2\widetilde{\varphi}')'+|\xi|^2(\frac{4}{3}s\varepsilon_0+s\delta_0+P'(\rho_0)\rho_0+\kappa\rho_0\bar{u}^2)\widetilde{\varphi}\\&+|\xi|[(s\delta_0+\frac{1}{3}s\varepsilon_0+P'(\rho_0)\rho_0)\widetilde{\psi}'+(s\varepsilon_0'+(\kappa\rho_0\bar{u}^2)'-g\rho_0)\widetilde{\psi}],\end{aligned}\tag{2.21}$$

$$\begin{aligned}\alpha\rho_0\widetilde{\psi}=&-[(\frac{4}{3}s\varepsilon_0+s\delta_0+P'(\rho_0)\rho_0+\kappa\rho_0\bar{u}^2)\widetilde{\psi}']'+(s\varepsilon_0+\kappa\rho_0\bar{u}^2)|\xi|^2\widetilde{\psi}\\&-|\xi|[((s\delta_0+\frac{1}{3}s\varepsilon_0+P'(\rho_0)\rho_0)\widetilde{\varphi})'+(g\rho_0-s\varepsilon_0'-(\kappa\rho_0\bar{u}^2)')\widetilde{\varphi}],\end{aligned}\tag{2.22}$$

along with the boundary conditions: at $x_3=0$

$$(s\varepsilon_0+\kappa\rho_0\bar{u}^2)(\widetilde{\varphi}'-|\xi|\widetilde{\psi})=0,$$

$$(s\delta_0+\frac{1}{3}s\varepsilon_0+P'(\rho_0)\rho_0)(\widetilde{\psi}'+|\xi|\widetilde{\varphi})+(s\varepsilon_0+\kappa\rho_0\bar{u}^2)(\widetilde{\psi}'-|\xi|\widetilde{\varphi})=-\sigma|\xi|^2\widetilde{\psi},$$

and $\widetilde{\varphi}(-b)=\widetilde{\psi}(-b)=0$.

Proof. (1) Let $(\varphi_n,\psi_n)\in C$ be a minimizing sequence. By (2.19), one can see

$$E(\varphi_n,\psi_n;|\xi|,s)\geq -g|\xi|,$$

which shows that $E(\varphi_n,\psi_n;|\xi|,s)$ is bounded below on C. Then, there exists a pair $(\widetilde{\varphi},\widetilde{\psi})\in C$ and a subsequence (still denoted by (φ_n,ψ_n) for simplicity), such that $(\varphi_n,\psi_n)\to(\widetilde{\varphi},\widetilde{\psi})$ weakly in H_0^1 and strongly in L^2. Moreover, by the lower semi-continuity, we find

$$\begin{aligned}&\inf_{(\varphi,\psi)\in C}E(\varphi,\psi;|\xi|,s)=\liminf_{n\to\infty}E(\varphi_n,\psi_n;|\xi|,s)\\&=\lim_{n\to\infty}\Big(\frac{\sigma|\xi|^2}{2}(\psi_n(0))^2+\frac{1}{2}\int_{-b}^0P'(\rho_0)\rho_0(\psi_n'+|\xi|\varphi_n)^2+g\rho_0|\xi|(\varphi_n-\psi_n)^2dx_3\\&+\frac{s}{2}\int_{-b}^0(\delta_0+\frac{1}{3}\varepsilon_0)(\psi_n'+|\xi|\varphi_n)^2dx_3+\varepsilon_0((\varphi_n'-|\xi|\psi_n)^2+(\psi_n'-|\xi|\varphi_n)^2)dx_3\\&+\frac{1}{2}\int_{-b}^0\kappa\rho_0\bar{u}^2((\varphi_n'-|\xi|\psi_n)^2+(\psi_n'-|\xi|\varphi_n)^2)dx_3\Big)\\&-\frac{1}{2}\limsup_{n\to\infty}\int_{-b}^0g\rho_0|\xi|(\varphi_n^2+\psi_n^2)dx_3\geq E(\widetilde{\varphi},\widetilde{\psi};|\xi|,s)\geq\inf_{(\varphi,\psi)\in C}E(\varphi,\psi;|\xi|,s).\end{aligned}$$

This implies that $E(\varphi,\psi;|\xi|,s)$ achieves its infimum over C.

(2) We refer to Propositions 3.2 of [6], just only need to replace $s\varepsilon_0$ by $s\varepsilon_0+\kappa\rho_0\bar{u}^2$ and $s\delta_0$ by $s\delta_0-\frac{1}{3}\kappa\rho_0\bar{u}^2$ in this paper, applying a bootstrap argument to prove the smoothness. Here we omit the specific details.

Remark 2.1. *Compared to the paper [6], the new term $\kappa\rho_0\bar{u}^2$ in the energy equality does not have a factor s, so we can not directly refer to the construction of φ, ψ in [6] to prove that the infimum of E over C is negative for s sufficiently small. In this paper, we use the definition of κ_c (see* (1.11)*) and the condition $\kappa < \kappa_c$ to prove the following Lemma.*

Lemma 2.2. *If $\kappa < \kappa_c$, there exists a $\xi \in (L^{-1}\mathbb{Z})^2\backslash\{0\}$ and a pair of functions $(\widetilde{\varphi}, \widetilde{\psi}) \in C$, such that $E_0(\widetilde{\varphi}, \widetilde{\psi}; |\xi|) < 0$.*

Proof. From the definition (1.11) of κ_c, we find that there exists a function $\widetilde{w}(x) = (\widetilde{w}_1(x), \widetilde{w}_2(x), \widetilde{w}_3(x)) \in H^1_0(\Omega)$ such that $\Theta_1(\widetilde{w}) > \kappa\Psi(\widetilde{w})$, i.e.,

$$\begin{aligned} &\int_\Omega (g\rho_0\widetilde{w}_3\mathrm{div}\widetilde{w} - g\rho_0\widetilde{w}\cdot\nabla\widetilde{w}_3)dx - \int_\Omega P'(\rho_0)\rho_0|\mathrm{div}\widetilde{w}|^2dx \\ &\quad - \int_{\Sigma_0}\sigma|\nabla_{x_1,x_2}\widetilde{w}_3|^2dS_0 > \int_\Omega \kappa\rho_0\bar{u}^2(\frac{1}{2}|\mathbb{D}(\widetilde{w})|^2 - |\mathrm{div}\widetilde{w}|^2)dx. \end{aligned} \tag{2.23}$$

Let $\widehat{w}(\xi, x_3)$ be the horizontal Fourier transform of $w(x)$, i.e.,

$$\widehat{w}(\xi, x_3) = \int_{(2\pi L\mathbb{T})^2} w(x_h, x_3)e^{-ix_h\cdot\xi}dx_h,$$

and the functions φ, θ and ψ are defined by the relations

$$\widehat{w}_1(\xi, x_3) = -i\varphi(\xi, x_3),\ \widehat{w}_2(\xi, x_3) = -i\theta(\xi, x_3),\ \widehat{w}_3(\xi, x_3) = \psi(\xi, x_3),$$

where $(\widehat{w}_1, \widehat{w}_2, \widehat{w}_3) = \widehat{w}$ and $\xi \in (L^{-1}\mathbb{Z})^2$. Then

$$\widehat{\nabla w} = \begin{pmatrix} \xi_1\varphi & \xi_2\varphi & -i\varphi' \\ \xi_1\theta & \xi_2\theta & -i\theta' \\ i\xi_1\psi & i\xi_2\psi & \psi' \end{pmatrix}, \tag{2.24}$$

and

$$\widehat{\mathrm{div}w} = \xi_1\varphi + \xi_2\theta + \psi'. \tag{2.25}$$

Recalling the Fubini and Parseval theorems: in the periodic case, if $f \in L^2(\Omega)$, we have that $\widehat{f} \in L^2(\Omega)$, and

$$\int_\Omega |f(x)|^2dx = \frac{1}{4\pi^2L^2}\sum_{\xi\in(L^{-1}\mathbb{Z})^2}\int_{-b}^0 |\widehat{f}(\xi, x_3)|^2dx_3. \tag{2.26}$$

Using (2.24)–(2.26) and the identity $2ab = (a^2 + b^2) - (a - b)^2$, we have

$$\begin{aligned}
&\int_\Omega (g\rho_0\widetilde{w}_3\mathrm{div}\widetilde{w} - g\rho_0\widetilde{w}\cdot\nabla\widetilde{w}_3)dx = \frac{1}{8\pi^2L^2}\sum_{\xi\in(L^{-1}\mathbb{Z})^2}\int_{-b}^0 g\rho_0\Big(|\widehat{\widetilde{w}_3}|^2 + |\widehat{\mathrm{div}\widetilde{w}}|^2 \\
&\quad - |\widehat{\widetilde{w}_3} - \widehat{\mathrm{div}\widetilde{w}}|^2 - \sum_{i=1}^3(|\widehat{\widetilde{w}_i}|^2 + |\widehat{\partial_i\widetilde{w}_3}|^2 - |\widehat{\widetilde{w}_i} - \widehat{\partial_i\widetilde{w}_3}|^2)\Big)dx_3 \\
&= \frac{1}{8\pi^2L^2}\sum_{\xi\in(L^{-1}\mathbb{Z})^2}\int_{-b}^0 g\rho_0(|\widetilde{\psi}|^2 + |\xi_1\widetilde{\varphi} + \xi_2\widetilde{\theta} + \widetilde{\psi}'|^2 - |\widetilde{\psi} - \xi_1\widetilde{\varphi} - \xi_2\widetilde{\theta} - \widetilde{\psi}'| - |\widetilde{\varphi}|^2 \\
&\quad - |\widetilde{\theta}|^2 - |\widetilde{\psi}|^2 - |\xi|^2|\widetilde{\psi}|^2 - |\widetilde{\psi}'|^2 + |\widetilde{\varphi} + \xi_1\widetilde{\psi}|^2 - |\widetilde{\theta} + \xi_2\widetilde{\psi}|^2 + |\widetilde{\psi} - \widetilde{\psi}'|^2)dx_3 \\
&= \frac{1}{4\pi^2L^2}\sum_{\xi\in(L^{-1}\mathbb{Z})^2\setminus\{0\}}\int_{-b}^0 g\rho_0(2\xi_1\Re\widetilde{\varphi}\Re\widetilde{\psi} + 2\xi_1\Im\widetilde{\varphi}\Im\widetilde{\psi} + 2\xi_2\Re\widetilde{\theta}\Re\widetilde{\psi} + 2\xi_2\Im\widetilde{\theta}\Im\widetilde{\psi})dx_3,
\end{aligned} \tag{2.27}$$

here $\Re(\widetilde{\varphi}, \widetilde{\theta}, \widetilde{\psi})$ and $\Im(\widetilde{\varphi}, \widetilde{\theta}, \widetilde{\psi})$ denote the real and imaginary parts of $(\widetilde{\varphi}, \widetilde{\theta}, \widetilde{\psi})$, respectively.

By the similar argument, we can get

$$\begin{aligned}
&\int_\Omega P'(\rho_0)\rho_0|\mathrm{div}\widetilde{w}|^2dx + \int_{\Sigma_0}\sigma|\nabla_{x_1,x_2}\widetilde{w}_3|^2dS_0 + \int_\Omega \kappa\rho_0\bar{u}^2(\frac{1}{2}|\mathbb{D}(\widetilde{w})|^2 - |\mathrm{div}\widetilde{w}|^2)dx \\
&= \frac{1}{4\pi^2L^2}\sum_{\xi\in(L^{-1}\mathbb{Z})^2\setminus\{0\}}\Big(\sigma|\xi|^2|\widetilde{\psi}(\xi,0)|^2 + \int_{-b}^0 P'(\rho_0)\rho_0|\xi_1\widetilde{\varphi} + \xi_2\widetilde{\theta} + \widetilde{\psi}'|^2dx_3 \\
&\quad + \int_{-b}^0 \kappa\rho_0\bar{u}^2(|\widetilde{\varphi}' - \xi_1\widetilde{\psi}|^2 + |\widetilde{\psi}' - \xi_1\widetilde{\varphi}|^2 + \xi_2^2(|\widetilde{\varphi}|^2 + |\widetilde{\theta}|^2 + |\widetilde{\psi}|^2) + \xi_1^2|\widetilde{\theta}|^2 + |\widetilde{\theta}'|^2 \\
&\quad - 2\xi_2\Re\widetilde{\theta}'\Re\widetilde{\psi} - 2\xi_2\Im\widetilde{\theta}'\Im\widetilde{\psi} - 2\xi_2\Re\widetilde{\theta}\Re\widetilde{\psi}' - 2\xi_2\Im\widetilde{\theta}\Im\widetilde{\psi}')dx_3\Big) \\
&\quad + \frac{1}{4\pi^2L^2}\int_{-b}^0 P'(\rho_0)\rho_0|\widetilde{\psi}'(0,x_3)|^2dx_3 + \kappa\rho_0\bar{u}^2(|\widetilde{\varphi}'(0,x_3)|^2 + |\widetilde{\psi}'(0,x_3)|^2 + |\widetilde{\theta}'(0,x_3)|^2)dx_3.
\end{aligned} \tag{2.28}$$

Because the above equalities are invariant under simultaneous rotations of ξ and $(\widetilde{\varphi}, \widetilde{\theta})$, without loss of generality we may assume that $\xi = (|\xi|, 0)$ with $|\xi| > 0$ and $\widetilde{\theta} = 0$. Then from (2.23), combining with (2.27)–(2.28), one can see

$$\begin{aligned}
&\sum_{\xi\in(L^{-1}\mathbb{Z})^2\setminus\{0\}}\int_{-b}^0 2g\rho_0|\xi|(\Re\widetilde{\varphi}\Re\widetilde{\psi} + \Im\widetilde{\varphi}\Im\widetilde{\psi})dx_3 \\
&> \sum_{\xi\in(L^{-1}\mathbb{Z})^2\setminus\{0\}}\Big(\sigma|\xi|^2|\widetilde{\psi}(\xi,0)|^2 + \int_{-b}^0 P'(\rho_0)\rho_0||\xi|\widetilde{\varphi} + \widetilde{\psi}'|^2dx_3 \\
&\qquad + \int_{-b}^0 \kappa\rho_0\bar{u}^2(|\widetilde{\varphi}' - |\xi|\widetilde{\psi}|^2 + |\widetilde{\psi}' - |\xi|\widetilde{\varphi}|^2)dx_3\Big) \\
&\quad + \int_{-b}^0 P'(\rho_0)\rho_0|\widetilde{\psi}'(0,x_3)|^2dx_3 + \kappa\rho_0\bar{u}^2(|\widetilde{\varphi}'(0,x_3)|^2 + |\widetilde{\psi}'(0,x_3)|^2)dx_3,
\end{aligned}$$

which follows that

$$\sum_{\xi\in(L^{-1}\mathbb{Z})^2\backslash\{0\}}\int_{-b}^{0} g\rho_0|\xi|(|\widetilde{\varphi}|^2+|\widetilde{\psi}|^2-|\widetilde{\varphi}-\widetilde{\psi}|^2)dx_3$$
$$>\sum_{\xi\in(L^{-1}\mathbb{Z})^2\backslash\{0\}}\Big(\sigma|\xi|^2|\widetilde{\psi}(\xi,0)|^2+\int_{-b}^{0}P'(\rho_0)\rho_0||\xi|\widetilde{\varphi}+\widetilde{\psi}'|^2dx_3 \tag{2.29}$$
$$+\int_{-b}^{0}\kappa\rho_0\bar{u}^2(|\widetilde{\varphi}'-|\xi|\widetilde{\psi}|^2+|\widetilde{\psi}'-|\xi|\widetilde{\varphi}|^2)dx_3\Big).$$

(2.29) implies that there is a $\xi\in(L^{-1}\mathbb{Z})^2\backslash\{0\}$, such that

$$\sigma|\xi|^2|\widetilde{\psi}(\xi,0)|^2+\int_{-b}^{0}P'(\rho_0)\rho_0||\xi|\widetilde{\varphi}+\widetilde{\psi}'|^2-g\rho_0|\xi|(|\widetilde{\varphi}|^2+|\widetilde{\psi}|^2-|\widetilde{\varphi}-\widetilde{\psi}|^2)dx_3$$
$$+\int_{-b}^{0}\kappa\rho_0\bar{u}^2(|\widetilde{\varphi}'-|\xi|\widetilde{\psi}|^2+|\widetilde{\psi}'-|\xi|\widetilde{\varphi}|^2)dx_3<0.$$

Then we complete the proof of Lemma 2.2.

Remark 2.2. *Owning to Lemma 2.2, there is an unstable frequency $\xi\in(L^{-1}\mathbb{Z})^2\backslash\{0\}$ and a pair of functions $(\widetilde{\varphi},\widetilde{\psi})\in H_0^1((-b,0))\times H_0^1((-b,0))$ satisfying $E_0(\widetilde{\varphi},\widetilde{\psi};|\xi|)<0$, if $\kappa<\kappa_c$. Thus, the unstable frequency-set $\mathbb{F}$ consisted of all unstable frequencies is not empty for $\kappa<\kappa_c$.*

We want to show that there is a fixed point s so that $\lambda(|\xi|,s)=s$, which will then allow us to construct a solution to the original problem (2.8)–(2.9). To this end, we study the behavior of $\alpha(s):=\alpha(|\xi|;s)$ as a function of $s>0$.

Lemma 2.3. *Let $\alpha(s):(0,\infty)\to\mathbb{R}$ be defined by* (2.20)*. Then the following hold.*
(1) $\alpha(s)\in C^{0,1}_{loc}((0,\infty))$ and $\alpha(s)$ is strictly increasing in s.
(2) For any $\xi\in\mathbb{F}$, there exist constants $c_2,c_3>0$ depending on $g,\kappa,\rho_0,\varepsilon_0,\delta_0,\sigma$ and $|\xi|$, such that

$$\alpha(s)\le -c_2+sc_3,\ \ \text{for any } s\in(0,\infty). \tag{2.30}$$

Proof. (1) Fix a compact interval $Q=[a,b]\subset\subset(0,\infty)$. From (2.18), $E_1(\varphi,\psi;|\xi|)\ge 0$ implies that $E(\varphi,\psi;|\xi|,s)$ is non-decreasing in s with $(\varphi,\psi)\in C$ kept fixed. And, from Lemma 2.1, we can find a pair $(\varphi_s,\psi_s)\in C$ so that

$$E(\varphi_s,\psi_s;|\xi|,s)=\inf_{(\varphi,\psi)\in C}E(\varphi,\psi;|\xi|,s)=\alpha(s),\ \text{for fixed } s>0. \tag{2.31}$$

Note that if $0<s_1<s_2<\infty$, then the decomposition (2.18) and (2.31) implies that

$$\alpha(s_1)=E(\varphi_{s_1},\psi_{s_1};|\xi|,s_1)\le E(\varphi_{s_2},\psi_{s_2};|\xi|,s_1)\le E(\varphi_{s_2},\psi_{s_2};|\xi|,s_2)=\alpha(s_2), \tag{2.32}$$

which shows that $\alpha(s)$ is non-decreasing in s. Supposed that $\alpha(s_1)=\alpha(s_2)$, from (2.32), we can obtain

$$s_1E_1(\varphi_{s_2},\psi_{s_2};|\xi|)=s_2E_1(\varphi_{s_2},\psi_{s_2};|\xi|),$$

which yields that $E_1(\varphi_{s_2}, \psi_{s_2}; |\xi|) = 0$. This means that $\varphi_{s_2} = \psi_{s_2} = 0$, which contradicts the fact that $(\varphi_{s_2}, \psi_{s_2}) \in C$. Then we can get that $\alpha(s_1) < \alpha(s_2)$, i.e., $\alpha(s)$ is strictly increasing in s.

Nextly, we show the continuity of $\alpha(s)$. Fixed any pair $(\varphi_0, \psi_0) \in C$, the fact that $E(\varphi, \psi; |\xi|, s)$ is non-decreasing in s, the minimality of (φ_s, ψ_s), and the equality (2.19) ensure that

$$E(\varphi_0, \psi_0; |\xi|, b) \geq E(\varphi_0, \psi_0; |\xi|, s) \geq sE_1(\varphi_s, \psi_s; |\xi|) - g|\xi|, \quad \forall\, s \in Q,$$

which implies that there exists a constant $0 < K = K(a, b, \varphi_0, \psi_0, g, |\xi|) < \infty$ so that

$$\sup_{s \in Q} E_1(\varphi_s, \psi_s; |\xi|) \leq K. \tag{2.33}$$

Let $s_i \in Q$ for $i = 1, 2$. Using (2.31) and (2.33), we can see

$$\begin{aligned}\alpha(s_1) \leq E(\varphi_{s_2}, \psi_{s_2}; |\xi|, s_1) &\leq E(\varphi_{s_2}, \psi_{s_2}; |\xi|, s_2) + |s_1 - s_2| E(\varphi_{s_2}, \psi_{s_2}; |\xi|) \\ &\leq \alpha(s_2) + K|s_1 - s_2|.\end{aligned}$$

Reversing the role of the indices 1 and 2 in the derivation of this inequality gives the same bound with the indices switched, i.e.,

$$\alpha(s_2) - \alpha(s_1) \leq K|s_1 - s_2|.$$

The above two inequalities ensure that

$$|\alpha(s_1) - \alpha(s_2)| \leq K|s_1 - s_2|,$$

which proves $\alpha(s) \in C^{0,1}_{\mathrm{loc}}((0, \infty))$.

(2) Since $\xi \in \mathbb{F}$, by the definition of $\mathbb{F}$ and Lemma 2.2, there exists $(\widetilde{\varphi}, \widetilde{\psi}) \in H^1_0((-b, 0)) \times H^1_0((-b, 0))$, such that

$$\begin{aligned}E_1(\widetilde{\varphi}, \widetilde{\psi}; |\xi|) = \frac{\sigma|\xi|^2 |\widetilde{\psi}(\xi, 0)|^2}{2} &+ \frac{1}{2}\int_{-b}^{0} P'(\rho_0)\rho_0 ||\xi|\widetilde{\varphi} + \widetilde{\psi}'|^2 - g\rho_0|\xi|(|\widetilde{\varphi}|^2 + |\widetilde{\psi}|^2 - |\widetilde{\varphi} - \widetilde{\psi}|^2) dx_3 \\ &+ \frac{1}{2}\int_{-b}^{0} \kappa\rho_0 \bar{u}^2 (|\widetilde{\varphi}' - |\xi|\widetilde{\psi}|^2 + |\widetilde{\psi}' - |\xi|\widetilde{\varphi}|^2) dx_3 < 0.\end{aligned}$$

Thus, one can see

$$\begin{aligned}\alpha(s) = \inf_{(\varphi,\psi) \in C} E(\varphi, \psi; |\xi|, s) &= \inf_{(\varphi,\psi) \in H^1_0((-b,0)) \times H^1_0((-b,0))} \frac{E(\varphi, \psi; |\xi|, s)}{J(\varphi, \psi)} \\ &\leq \frac{E(\widetilde{\varphi}, \widetilde{\psi}; |\xi|, s)}{J(\widetilde{\varphi}, \widetilde{\psi})} = \frac{E_1(\widetilde{\varphi}, \widetilde{\psi}; |\xi|)}{J(\widetilde{\varphi}, \widetilde{\psi})} + s\frac{E_0(\widetilde{\varphi}, \widetilde{\psi}; |\xi|)}{J(\widetilde{\varphi}, \widetilde{\psi})} =: -c_2 + sc_3.\end{aligned} \tag{2.34}$$

Then, we complete the proof of Lemma 2.3.

Given $\xi \in \mathbb{F}$, from (2.30), there exists a $s_0 > 0$ depending on the quantities $g, \kappa, \rho_0, \varepsilon_0, \delta_0, |\xi|$ so that $\alpha(|\xi|; s) < 0$ for any $s \in (0, s_0]$. Now, we define

$$S \overset{\mathrm{def}}{=} \sup\{s |\, \alpha(\tau) < 0 \text{ for any } \tau \in (0, s)\}.$$

Then $S > 0$. Applying the monotonicity of $\alpha(s)$ and the fact $\alpha(s) = \inf_{(\varphi,\psi)\in C} E(\varphi, \psi; |\xi|, s) > -\infty$, one can see that

$$\lim_{s\to 0^+} \alpha(s) \text{ exists and the limit is a negative constant.}$$

Using (2.19), we find that there exists a constant c_4 depends on $\varepsilon_0, \delta_0, |\xi|$, such that

$$\alpha(s) \geq -g|\xi| + sc_4,$$

where $\xi \in \mathbb{F}$. Thus, if $s \geq \frac{g|\xi|}{c_4}$, then $\alpha(s) \geq 0$. Hence, $S < +\infty$, and $\lim_{s\to S^-} \alpha(s) = 0$.

Therefore, combining with the above argument, we can employ a fixed-point argument to obtain the following Lemma.

Lemma 2.4. *Let $\xi \in \mathbb{F}$. Then there exists a unique $s \in (0, S)$ so that $\lambda(|\xi|; s) = \sqrt{-\alpha(|\xi|; s)} > 0$ and*

$$s = \lambda(|\xi|; s). \tag{2.35}$$

Remark 2.3. *By Lemma 2.4, for each fixed $\xi \in \mathbb{F}$, we can find a unique $s \in (0, S)$ so that $s = \lambda(|\xi|; s)$. Then, we can write $s = s(|\xi|)$ and $\lambda = \lambda(|\xi|)$.*

Proposition 2.1. *For $\xi \in (L^{-1}\mathbb{Z})^2$, if $\kappa < \kappa_c$, there exists a solution $\varphi = \varphi(\xi, x_3)$, $\theta = \theta(\xi, x_3)$, $\psi = \psi(\xi, x_3)$, and $\lambda = \lambda(|\xi|) > 0$ to (2.4)–(2.6) satisfying the boundary conditions. The solutions are smooth in $(-b, 0)$, and they are equivariant in ξ in the sense that if $R \in SO(2)$ is a rotation operator, then*

$$\begin{pmatrix} \varphi(R\xi, x_3) \\ \theta(R\xi, x_3) \\ \psi(R\xi, x_3) \end{pmatrix} = \begin{pmatrix} R_{11} & R_{12} & 0 \\ R_{21} & R_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \varphi(\xi, x_3) \\ \theta(\xi, x_3) \\ \psi(\xi, x_3) \end{pmatrix}.$$

Proof. We may find a rotation operator $R \in SO(2)$ so that $R\xi = (|\xi|, 0)$. For $s = s(|\xi|)$ given in (2.35), we define $(\varphi(\xi, x_3), \theta(\xi, x_3)) = R^{-1}(\varphi(|\xi|, x_3), 0)$ and $\psi(\xi, x_3)) = \psi(|\xi|, x_3)$, where the functions $\varphi(|\xi|, x_3)$ and $\psi(|\xi|, x_3)$ are the solutions to (2.8) and (2.9) from Lemma 2.1. This gives a solution to (2.4)–(2.6). The equivariance in ξ follows from the definition.

To obtain a largest growth rate, we next show that $\lambda(|\xi|)$ is a bounded, continuous function of $|\xi|$. We assume throughout that $\xi \in \mathbb{F}$.

Lemma 2.5. *The function $\xi \in \mathbb{F} \longmapsto \lambda(|\xi|) \in (0, \infty)$ is bounded, continuous, and satisfies*

$$\lim_{|\xi|\to 0} \lambda(|\xi|) = 0. \tag{2.36}$$

Proof. Since $\lambda = \sqrt{-\alpha}$, it suffices to prove the continuity of $\alpha(|\xi|)$. By Lemma 2.1, for every $\xi \in \mathbb{F}$, there exist functions $(\varphi_{|\xi|}, \psi_{|\xi|}) \in C$ satisfying (2.13) and (2.14) so that $\alpha(|\xi|; s) = E(\varphi_{|\xi|}, \psi_{|\xi|}; |\xi|, s)$. From Lemma 2.2, we know that $\alpha(|\xi|; s) < 0$, which along with (2.19), yields the bound

$$-g|\xi| \leq \alpha(|\xi|; s) < 0,$$

which ensures that $\lambda(|\xi|) \in (0,\infty)(\xi \in \mathbb{F})$ is bounded and satisfies (2.36). Now suppose $|\xi|_n$ is a sequence so that $|\xi|_n \to |\xi|(\xi \in \mathbb{F})$. By (2.34), there exist positive constants C_0, C_1 such that $\alpha(|\xi|_n) \le -C_0 + s(|\xi|_n)C_1$, combining with $-\alpha(|\xi|_n) = \lambda^2(|\xi|_n) = s^2(|\xi|_n)$, so

$$s^2(|\xi|_n) + C_1 s(|\xi|_n) - C_0 \ge 0,$$

which implies $s(|\xi|_n)$ is bounded below by a positive constant. Then $(\varphi_{|\xi|_n}, \psi_{|\xi|_n}) \in C$ and the fact that

$$\begin{aligned} - g|\xi|_n + \frac{s(|\xi|_n)}{2} \int_{-b}^{0} \Big(& (\delta_0 + \frac{1}{3}\varepsilon_0)(\psi'_{|\xi|_n} + |\xi|_n\varphi_{|\xi|_n})^2 \\ & + \varepsilon_0((\varphi'_{|\xi|_n} - |\xi|_n\psi_{|\xi|_n})^2 + (\psi'_{|\xi|_n} - |\xi|_n\varphi_{|\xi|_n})^2)\Big)dx_3 \le \alpha(|\xi|_n) < 0 \end{aligned}$$

imply that $\varphi_{|\xi|_n}$ and $\psi_{|\xi|_n}$ are uniformly bounded in $H^1((-b,0))$. Plugging into the ODE (2.21)–(2.22) in $(-b,0)$, we find that $\varphi_{|\xi|_n}$ and $\psi_{|\xi|_n}$ are uniformly bounded in $H^2((-b,0))$. Then there exists a subsequence(still denoted by $|\xi|_n$) such that

$$(\varphi_{|\xi|_n}, \psi_{|\xi|_n}) \to (\varphi_{|\xi|}, \psi_{|\xi|}) \text{ strongly in } H^1((-b,0)),$$

which implies

$$\alpha(|\xi|_n) = E(\varphi_{|\xi|_n}, \psi_{|\xi|_n}) \to E(\varphi_{|\xi|}, \psi_{|\xi|}) = \alpha(|\xi|).$$

i.e., $\alpha(|\xi|_n) \to \alpha(|\xi|)$, hence $\alpha(|\xi|)$ is continuous.

Lemma 2.5 then allows us to define

$$0 < \Lambda := \sup_{\xi\in\mathbb{F}} \lambda(|\xi|) < \infty. \tag{2.37}$$

Lemma 2.6. *Suppose $\xi \in \mathbb{F}$. Let (φ,θ,ψ) be the solutions constructed in Proposition 2.1. Then for each $k \ge 0$ there exists a constant $A_k > 0$ depending on the parameters $\rho_0, P, g, \varepsilon_0, \delta_0, \sigma, b$,*

$$\|\varphi(\xi,\cdot)\|_{H^k((-b,0))} + \|\theta(\xi,\cdot)\|_{H^k((-b,0))} + \|\psi(\xi,\cdot)\|_{H^k((-b,0))} \le A_k.$$

Proof. From Proposition 2.1, it suffices to prove

$$\|\varphi(|\xi|,\cdot)\|_{H^k((-b,0))} + \|\psi(|\xi|,\cdot)\|_{H^k((-b,0))} \le A_k.$$

for the solutions $\varphi = \varphi(|\xi|, x_3)$, $\psi = \psi(|\xi|, x_3)$ constructed in Proposition 2.1. By (1.5), we can see that ρ_0, $P'(\rho_0)$, $\kappa\rho_0\bar{u}^2$, ε_0 and δ_0 are smooth in $(-b,0)$ and bounded above and below.

We prove this lemma by induction on k. For $k = 0$, the fact that $(\varphi(|\xi|,\cdot), \psi(|\xi|,\cdot)) \in C$ implies that there exists a constant $A_0 > 0$ depending on the various parameters so that

$$\|\varphi(|\xi|,\cdot)\|_{L^2((-b,0))} + \|\psi(|\xi|,\cdot)\|_{L^2((-b,0))} \le A_0.$$

Suppose now that the bound holds some $k \ge 0$, i.e.,

$$\|\varphi(|\xi|,\cdot)\|_{H^k((-b,0))} + \|\psi(|\xi|,\cdot)\|_{H^k((-b,0))} \le A_k.$$

From Lemmas 2.4 and 2.5, $\lambda(|\xi|) = s(|\xi|)$ is bounded above and below by positive quantities as functions of $|\xi|$. Then, we differentiate the Eqs (2.13) and (2.14) to get that there exists a constant $C > 0$ depending on the various parameters so that

$$\begin{aligned}&\|\varphi(|\xi|,\cdot)\|_{H^{k+1}((-b,0))} + \|\psi(|\xi|,\cdot)\|_{H^{k+1}((-b,0))}\\&\qquad\le C(\|\varphi(|\xi|,\cdot)\|_{H^{k}((-b,0))} + \|\psi(|\xi|,\cdot)\|_{H^{k}((-b,0))}) \le CA_k := A_{k+1},\end{aligned}$$

which shows that the bound holds for $k+1$. Then, by induction the bound holds for all $k \ge 0$.

We may now construct a growing mode solution to the linearized problem (1.8).

Proposition 2.2. *Let $\xi_1, \xi_2 \in (L^{-1}\mathbb{Z})^2$ be lattice points such that $\xi_1 = -\xi_2$ and $\lambda(|\xi_j|) = \Lambda$, for $j = 1, 2$, where Λ is defined by* (2.37). *Define*

$$\widehat{w}(\xi, x_3) = -i\varphi(\xi, x_3)e_1 - i\theta(\xi, x_3)e_2 + \psi(\xi, x_3)e_3,$$

where φ, θ, ψ are the solutions provided by Proposition 2.1. Writing $x' = x_1e_1 + x_2e_2$, we define

$$\eta(x,t) = e^{\Lambda t}\sum_{j=1}^{2}\widehat{w}(\xi_j, x_3)e^{ix'\cdot\xi_j},$$

$$u(x,t) = \Lambda e^{\Lambda t}\sum_{j=1}^{2}\widehat{w}(\xi_j, x_3)e^{ix'\cdot\xi_j},$$

$$q(x,t) = -e^{\Lambda t}\rho_0(x_3)\sum_{j=1}^{2}(\xi_1\varphi(\xi_j,x_3) + \xi_2\theta(\xi_j,x_3) + \partial_3\psi(\xi_j,x_3))e^{ix'\cdot\xi_j},$$

and

$$V(x,t) = e^{\Lambda t}\sum_{j=1}^{2}B(\xi_j, x_3)\bar{U}e^{ix'\cdot\xi_j},$$

where

$$B(\xi_j, x_3) = \begin{pmatrix} \xi_1\varphi & \xi_2\varphi & -i\partial_3\varphi \\ \xi_1\theta & \xi_2\theta & -i\partial_3\theta \\ i\xi_1\psi & i\xi_2\psi & \partial_3\psi \end{pmatrix}.$$

Then η, u, q, V are real solutions to (1.8). *For every $t \ge 0$, we have $\eta(t), u(t), q(t), V(t) \in H^k(\Omega)$ and*

$$\begin{aligned}&\|q(t)\|_{H^k(\Omega)} = e^{\Lambda t}\|q(0)\|_{H^k(\Omega)}, \quad \|u(t)\|_{H^k(\Omega)} = e^{\Lambda t}\|u(0)\|_{H^k(\Omega)},\\&\|V(t)\|_{H^k(\Omega)} = e^{\Lambda t}\|V(0)\|_{H^k(\Omega)}, \quad \|\eta(t)\|_{H^k(\Omega)} = e^{\Lambda t}\|\eta(0)\|_{H^k(\Omega)}.\end{aligned} \tag{2.38}$$

Proof. By direct calculation, η, u, q, V defined in this way are solutions to (1.8). That they are real-valued follows from the equivariance in ξ stated in Proposition 2.1. From Lemma 2.6, the solutions are in $H^k(\Omega)$ at $t = 0$. The definitions of η, u, q, V ensure the growth in time stated in (2.38).

3. Growth of solutions to the linearized problem

3.1. Preliminary estimates

In this section we will prove estimates for the growth in time of arbitrary solutions to (1.8) in terms of the largest growing mode Λ defined by (2.37). Firstly, we differentiate $(1.8)_3$ in time and eliminate the q, η and V terms using the other equations. This yields the equation

$$\begin{aligned}&\rho_0\partial_{tt}u-\nabla(P'(\rho_0)\rho_0\mathrm{div}u)+g\rho_0\nabla u_3-g\rho_0\mathrm{div}ue_3\\&=\mathrm{div}(\varepsilon_0(\mathbb{D}(\partial_tu)-\frac{2}{3}(\mathrm{div}\partial_tu)\mathbb{I})+\delta_0(\mathrm{div}\partial_tu)\mathbb{I}+\kappa\rho_0\bar{u}^2(\mathbb{D}(u)-(\mathrm{div}u)\mathbb{I}),\end{aligned}\tag{3.1}$$

coupled to the boundary conditions:

$$\begin{aligned}(P'(\rho_0)\rho_0\mathrm{div}u\mathbb{I}+&\varepsilon_0\mathbb{D}(\partial_tu)+(\delta_0-\frac{2}{3}\varepsilon_0)\mathrm{div}\partial_tu\mathbb{I}\\&+\kappa\rho_0\bar{u}^2(\mathbb{D}(u)-\mathrm{div}u\mathbb{I}))e_3=\sigma\Delta_{x_1,x_2}u_3e_3,\ \text{on }\Sigma_0,\end{aligned}$$

and $\partial_tu(x_1,x_2,-b,t)=0$.

First, we state the following energy identity.

Lemma 3.1. *Let u solve* (3.1) *and the corresponding boundary conditions. Then*

$$\begin{aligned}&\partial_t\int_\Omega\rho_0\frac{|\partial_tu|^2}{2}dx+\int_\Omega\frac{\varepsilon_0}{2}\left|\mathbb{D}(\partial_tu)-\frac{2}{3}(\mathrm{div}\partial_tu)\mathbb{I}\right|^2+\delta_0|\mathrm{div}\partial_tu|^2dx\\&=\partial_t\int_\Omega(-\frac{P'(\rho_0)\rho_0}{2}|\mathrm{div}u|^2+g\rho_0u_3\mathrm{div}u+\frac{g\rho_0'}{2}|u_3|^2-\frac{\kappa\rho_0\bar{u}^2}{2}(\frac{1}{2}|\mathbb{D}(u)|^2-|\mathrm{div}u|^2))dx\\&\quad-\partial_t\int_{\Sigma_0}\frac{g\rho_0}{2}|u_3|^2dS_0-\partial_t\int_{\Sigma_0}\frac{\sigma}{2}|\nabla_{x_1,x_2}u_3|^2dS_0.\end{aligned}\tag{3.2}$$

Proof. Take the dot product of (3.1) with ∂_tu and integrate over Ω. After integrating by parts, we get

$$\begin{aligned}&\int_\Omega\rho_0\partial_tu\cdot\partial_{tt}u+P'(\rho_0)\rho_0(\mathrm{div}\,u)(\mathrm{div}\,\partial_tu)-g\rho_0(u_3\mathrm{div}\,\partial_tu+\partial_tu_3\mathrm{div}\,u)\\&+\int_\Omega\frac{\varepsilon_0}{2}\left|\mathbb{D}(\partial_tu)-\frac{2}{3}dx(\mathrm{div}\partial_tu)\mathbb{I}\right|^2+\delta_0|\mathrm{div}\partial_tu|^2dx-\int_\Omega g\rho_0'u_3\partial_tu_3dx\\&+\int_\Omega\kappa\rho_0\bar{u}^2(\mathbb{D}(u):\nabla\partial_tu-(\mathrm{div}\,u)(\mathrm{div}\,\partial_tu))=-\int_{\Sigma_0}g\rho_0u_3\partial_tu_3dS_0+\int_{\Sigma_0}Te_3\cdot\partial_tudS_0\end{aligned}$$

where

$$T=(P'(\rho_0)\rho_0\mathrm{div}\,u)\mathbb{I}+\varepsilon_0(\mathbb{D}(\partial_tu)-\frac{2}{3}(\mathrm{div}\,\partial_tu)\mathbb{I})+\delta_0\mathrm{div}\,\partial_tu\mathbb{I}+\kappa\rho_0\bar{u}^2(\mathbb{D}(u)-\mathrm{div}\,u\mathbb{I}).$$

We may pull time derivatives out of the first integrals on each side of the equation to arrive at the equality

$$\begin{aligned}&\partial_t\int_\Omega\rho_0\frac{|\partial_tu|^2}{2}dx+\int_\Omega\frac{\varepsilon_0}{2}\left|\mathbb{D}(\partial_tu)-\frac{2}{3}(\mathrm{div}\partial_tu)\mathbb{I}\right|^2+\delta_0|\mathrm{div}\partial_tu|^2dx\\&=\partial_t\int_\Omega(-\frac{P'(\rho_0)\rho_0}{2}|\mathrm{div}u|^2+g\rho_0u_3\mathrm{div}u+\frac{g\rho_0'}{2}|u_3|^2-\frac{\kappa\rho_0\bar{u}^2}{2}(\frac{1}{2}|\mathbb{D}(u)|^2-|\mathrm{div}u|^2))dx\\&\quad-\partial_t\int_{\Sigma_0}\frac{g\rho_0}{2}|u_3|^2dS_0+\int_{\Sigma_0}Te_3\cdot\partial_tudS_0.\end{aligned}\tag{3.3}$$

Using the upper boundary condition, one can see

$$\begin{aligned}\int_{\Sigma_0} Te_3\cdot\partial_t u dS_0 &= \int_{\Sigma_0}\sigma\Delta_{x_1,x_2}u_3\partial_t u_3 dS_0\\ &= -\sigma\int_{\Sigma_0}\nabla_{x_1,x_2}u_3\cdot\nabla_{x_1,x_2}\partial_t u_3 dS_0 = -\partial_t\int_{\Sigma_0}\frac{\sigma}{2}|\nabla_{x_1,x_2}u_3|^2 dS_0.\end{aligned} \tag{3.4}$$

Combining with (3.3) and (3.4), we can obtain (3.2).

Lemma 3.2. *Let $v\in H_0^1(\Omega)$ be arbitrary. We have the inequality*

$$\begin{aligned}&\int_\Omega\Big(-\frac{P'(\rho_0)\rho_0}{2}|\mathrm{div}\, v|^2+g\rho_0 v_3\,\mathrm{div}\, v+\frac{g\rho_0'}{2}|v_3|^2-\frac{\kappa\rho_0\bar u^2}{2}\big(\frac{1}{2}|\mathbb{D}(v)|^2-|\mathrm{div}\, v|^2\big)\Big)dx\\ &\quad-\int_{\Sigma_0}\frac{g\rho_0}{2}|v_3|^2dS_0-\int_{\Sigma_0}\frac{\sigma}{2}|\nabla_{x_1,x_2}v_3|^2dS_0\\ &\le\frac{\Lambda^2}{2}\int_\Omega\rho_0|v|^2dx+\frac{\Lambda}{2}\int_\Omega\frac{\varepsilon_0}{2}\Big|\nabla v+\nabla v^T-\frac{2}{3}(\mathrm{div}\, v)\mathbb{I}\Big|^2+\delta_0|\mathrm{div}\, v|^2dx.\end{aligned} \tag{3.5}$$

Proof. Integrating by parts, we can get

$$\begin{aligned}&\int_\Omega\Big(-\frac{P'(\rho_0)\rho_0}{2}|\mathrm{div}\, v|^2+g\rho_0 v_3\mathrm{div}\, v+\frac{g\rho_0'}{2}|v_3|^2-\frac{\kappa\rho_0\bar u^2}{2}\big(\frac{1}{2}|\mathbb{D}(v)|^2-|\mathrm{div}\, v|^2\big)\Big)dx\\ &\quad-\int_{\Sigma_0}\frac{g\rho_0}{2}|v_3|^2dS_0-\int_{\Sigma_0}\frac{\sigma}{2}|\nabla_{x_1,x_2}v_3|^2dS_0\\ &=\frac{1}{2}\int_\Omega g\rho_0(v_3\mathrm{div}\, v-\nabla v_3\cdot v)-P'(\rho_0)\rho_0|\mathrm{div}\, v|^2-\kappa\rho_0\bar u^2\big(\frac{1}{2}|\mathbb{D}(v)|^2-|\mathrm{div}\, v|^2\big)dx\\ &\quad-\int_{\Sigma_0}\frac{\sigma}{2}|\nabla_{x_1,x_2}v_3|^2dS_0.\end{aligned} \tag{3.6}$$

Firstly, we consider all $v\in H_0^1$ satisfying (2.23). Writing $\varphi(x_3)=i\widehat{v_1},\ \theta(x_3)=i\widehat{v_2},\ \psi(x_3)=\widehat{v_3}$, by the similar argument as the proof of Lemma 2.2, we take the horizontal Fourier transform to see that

$$\begin{aligned}&\frac{1}{2}\int_\Omega g\rho_0(v_3\mathrm{div}\, v-\nabla v_3\cdot v)-P'(\rho_0)\rho_0|\mathrm{div}\, v|^2-\kappa\rho_0\bar u^2\big(\frac{1}{2}|\mathbb{D}(v)|^2-|\mathrm{div}\, v|^2\big)dx\\ &\quad-\int_{\Sigma_0}\frac{\sigma}{2}|\nabla_{x_1,x_2}v_3|^2dS_0\\ &=\frac{1}{4\pi^2L^2}\sum_{\xi\in(L^{-1}\mathbb{Z})^2\setminus\{0\}}\frac{1}{2}\Big(\int_{-b}^0 g\rho_0(2\xi_1\Re\varphi\Re\psi+2\xi_1\Im\varphi\Im\psi+2\xi_2\Re\theta\Re\psi+2\xi_2\Im\theta\Im\psi)dx_3\\ &\quad-\sigma|\xi|^2|\psi(\xi,0)|^2-\int_{-b}^0P'(\rho_0)\rho_0|\xi_1\varphi+\xi_2\theta+\psi'|^2dx_3\\ &\quad-\int_{-b}^0\kappa\rho_0\bar u^2(|\varphi'-\xi_1\psi|^2+|\psi'-\xi_1\varphi|^2+\xi_2^2(|\varphi|^2+|\theta|^2+|\psi|^2)+\xi_1^2|\theta|^2+|\theta'|^2\\ &\quad-2\xi_2\Re\theta'\Re\psi-2\xi_2\Im\theta'\Im\psi-2\xi_2\Re\theta\Re\psi'-2\xi_2\Im\theta\Im\psi')dx_3\Big)\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{8\pi^2L^2}\int_{-b}^{0}P'(\rho_0)\rho_0|\psi'(0,x_3)|^2+\kappa\rho_0\bar{u}^2(|\varphi'(0,x_3)|^2+|\psi'(0,x_3)|^2+|\theta'(0,x_3)|^2)dx_3\\
&=:\frac{1}{4\pi^2L^2}\sum_{\xi\in(L^{-1}\mathbb{Z})^2\backslash\{0\}}Z(\varphi,\theta,\psi;\xi)\\
&\quad-\frac{1}{8\pi^2L^2}\int_{-b}^{0}P'(\rho_0)\rho_0|\psi'(0,x_3)|^2+\kappa\rho_0\bar{u}^2(|\varphi'(0,x_3)|^2+|\psi'(0,x_3)|^2+|\theta'(0,x_3)|^2)dx_3.
\end{aligned}
$$

Notice that $Z(\varphi,\theta,\psi;\xi)$ is obviously invariant under simultaneous rotations of ξ and (φ,θ), so without loss of generality we may assume that $\xi=(|\xi|,0)$ with $|\xi|>0$ and $\theta=0$. Then, for $\xi\in\mathbb{F}$, we have

$$
\begin{aligned}
Z(\varphi,\theta,\psi;\xi)&=\frac{1}{2}\int_{-b}^{0}g\rho_0|\xi|(|\varphi|^2+|\psi|^2-|\varphi-\psi|^2)dx_3-\frac{\sigma|\xi|^2|\psi(\xi,0)|^2}{2}\\
&\quad-\frac{1}{2}\int_{-b}^{0}P'(\rho_0)\rho_0||\xi|\varphi+\psi'|^2dx_3+\kappa\rho_0\bar{u}^2(|\varphi'-|\xi|\psi|^2+|\psi'-|\xi|\varphi|^2)dx_3\\
&=-E(\varphi,\psi;\lambda(|\xi|))+\frac{\lambda(|\xi|)}{2}\int_{-b}^{0}\delta_0(\psi'+|\xi|\varphi)^2dx_3\\
&\quad+\frac{\lambda(|\xi|)}{2}\int_{-b}^{0}\varepsilon_0((\varphi'-|\xi|\psi)^2+(\psi'-|\xi|\varphi)^2+\frac{1}{3}(\psi'+|\xi|\varphi)^2)dx_3,
\end{aligned}
$$

and hence

$$
\begin{aligned}
Z(\varphi,\theta,\psi;\xi)&\le\frac{\Lambda^2}{2}\int_{-b}^{0}\rho_0(|\varphi|^2+|\psi|^2)+\frac{\Lambda}{2}\int_{-b}^{0}\delta_0(\psi'+|\xi|\varphi)^2dx_3\\
&\quad+\frac{\Lambda}{2}\int_{-b}^{0}\varepsilon_0((\varphi'-|\xi|\psi)^2+(\psi'-|\xi|\varphi)^2+\frac{1}{3}(\psi'+|\xi|\varphi)^2)dx_3,
\end{aligned}
\tag{3.7}
$$

where we have used the following variational characterization for Λ, which follows from the definitions (2.20) and (2.37),

$$
0>E(\varphi,\psi;\lambda(|\xi|))\ge-\lambda(|\xi|)^2J(\varphi,\psi)\ge-\frac{\Lambda^2}{2}\int_{-b}^{0}\rho_0(|\varphi|^2+|\psi|^2).
$$

For $\xi\in(L^{-1}\mathbb{Z})^2\backslash\mathbb{F}$, owning to the definition of $\mathbb{F}$, one can see

$$
Z(\varphi,\theta,\psi;\xi)\le 0.
$$

Then, for all $\xi\in(L^{-1}\mathbb{Z})^2$, $Z(\varphi,\theta,\psi;\xi)$ satisfies (3.7). Combining with (3.6), and translating the resulting inequality back to the original notation for fixed ξ, by the Fubini and Parseval theorems, we find

$$
\begin{aligned}
&\frac{1}{2}\int_{\Omega}g\rho_0(v_3\mathrm{div}\,v-\nabla v_3\cdot v)-P'(\rho_0)\rho_0|\mathrm{div}\,v|^2-\kappa\rho_0\bar{u}^2(\frac{1}{2}|\mathbb{D}(v)|^2-|\mathrm{div}\,v|^2)dx\\
&\le\frac{1}{4\pi^2L^2}\sum_{\xi\in(L^{-1}\mathbb{Z})^2}\Big(\frac{\Lambda^2}{2}\int_{-b}^{0}\rho_0(|\varphi|^2+|\psi|^2)+\frac{\Lambda}{2}\int_{-b}^{0}\delta_0(\psi'+|\xi|\varphi)^2dx_3\\
&\quad+\frac{\Lambda}{2}\int_{-b}^{0}\varepsilon_0((\varphi'-|\xi|\psi)^2+(\psi'-|\xi|\varphi)^2+\frac{1}{3}(\psi'+|\xi|\varphi)^2)dx_3\Big)\\
&=\frac{\Lambda^2}{2}\int_{\Omega}\rho_0|v|^2dx+\frac{\Lambda}{2}\int_{\Omega}\frac{\varepsilon_0}{2}\Big|\nabla v+\nabla v^T-\frac{2}{3}(\mathrm{div}\,v)I\Big|^2+\delta_0|\mathrm{div}\,v|^2dx.
\end{aligned}
\tag{3.8}
$$

For any $v \in H_0^1$ not satisfying (2.23), we have

$$\begin{aligned}&\frac{1}{2}\int_\Omega g\rho_0(v_3\mathrm{div}\, v-\nabla v_3\cdot v)-P'(\rho_0)\rho_0|\mathrm{div}\, v|^2-\kappa\rho_0\bar{u}^2(\frac{1}{2}|\mathbb{D}(v)|^2-|\mathrm{div}\, v|^2)dx\\&\quad-\int_{\Sigma_0}\frac{\sigma}{2}|\nabla_{x_1,x_2}v_3|^2dS_0\leq 0,\end{aligned}\tag{3.9}$$

which implies that (3.5) holds trivially since the right of (3.5) is non-negative. Combining with (3.6), (3.8) and (3.9), we conclude Lemma 3.2.

3.2. Proof of Theorem 1.1

Now we can prove our main result.

Proof of Theorem 1.1. Integrating the result of Lemma 3.1 in time from 0 to t, by Lemma 3.2, we get

$$\begin{aligned}&\int_\Omega\rho_0\frac{|\partial_t u(t)|^2}{2}dx+\int_0^t\int_\Omega\frac{\varepsilon_0}{2}\left|\mathbb{D}(\partial_t u)(s)-\frac{2}{3}(\mathrm{div}\partial_t u(s))\mathbb{I}\right|^2+\delta_0|\mathrm{div}\partial_t u(s)|^2dxds\\&\leq K_0+\int_\Omega(-\frac{P'(\rho_0)\rho_0}{2}|\mathrm{div}u(t)|^2+g\rho_0u_3(t)\mathrm{div}u(t)+\frac{g\rho_0'}{2}|u_3(t)|^2\\&-\frac{\kappa\rho_0\bar{u}^2}{2}(\frac{1}{2}|\mathbb{D}(u)(t)|^2-|\mathrm{div}u(t)|^2))dx-\int_{\Sigma_0}\frac{g\rho_0}{2}|u_3(t)|^2dS_0-\int_{\Sigma_0}\frac{\sigma}{2}|\nabla_{x_1,x_2}u_3(t)|^2dS_0\\&\leq K_0+\frac{\Lambda^2}{2}\int_\Omega\rho_0|u(t)|^2dx+\frac{\Lambda}{2}\int_\Omega\frac{\varepsilon_0}{2}\left|\mathbb{D}(u(t))-\frac{2}{3}(\mathrm{div}u(t))I\right|^2+\delta_0|\mathrm{div}u(t)|^2dx.\end{aligned}\tag{3.10}$$

where

$$\begin{aligned}K_0=&\int_\Omega\rho_0\frac{|\partial_t u(0)|^2}{2}+\frac{\kappa\rho_0}{4}|\mathbb{D}(u(0))|^2+\frac{P'(\rho_0)\rho_0}{2}|\mathrm{div}u(0)|^2dx+\int_{\Sigma_0}\frac{g\rho_0}{2}|u_3(0)|^2dS_0\\&+\int_\Omega g\rho_0(|u_3(0)|^2+|\mathrm{div}u(0)|^2)dx+\int_{\Sigma_0}\frac{\sigma}{2}|\nabla_{x_1,x_2}u_3(0)|^2dS_0.\end{aligned}$$

Using the definitions of the norms $\|\cdot\|_*$, $\|\cdot\|_{**}$ given in (1.9), we may compactly rewrite (3.10) as

$$\frac{1}{2}\|\partial_t u(t)\|_*^2+\int_0^t\|\partial_t u(s)\|_{**}^2\,ds\leq K_0+\frac{\Lambda^2}{2}\|u(t)\|_*^2+\frac{\Lambda}{2}\|u(t)\|_{**}^2.\tag{3.11}$$

Integrating in time and using Cauchy's inequality, we may bound

$$\begin{aligned}\Lambda\|u(t)\|_{**}^2&=\Lambda\|u(0)\|_{**}^2+\Lambda\int_0^t 2\langle u(s),\partial_t u(s)\rangle_{**}\,ds\\&\leq\Lambda\|u(0)\|_{**}^2+\int_0^t\|\partial_t u(s)\|_{**}^2\,ds+\Lambda^2\int_0^t\|u(s)\|_{**}^2\,ds.\end{aligned}\tag{3.12}$$

On the other hand

$$\Lambda\partial_t\|u(t)\|_*^2=2\Lambda\langle\partial_t u(t),u(t)\rangle_*\leq\Lambda^2\|u(t)\|_*^2+\|\partial_t u(t)\|_*^2.\tag{3.13}$$

Hence, combining (3.12) and (3.13) with (3.11), we derive the differential inequality

$$\partial_t \|u(t)\|_*^2 + \|u(t)\|_{**}^2 \le K_1 + 2\Lambda(\|u(t)\|_*^2 + \int_0^t \|u(s)\|_{**}^2\, ds) \tag{3.14}$$

for $K_1 = 2K_0/\Lambda + 2\|u(0)\|_{**}^2$. Applying Gronwall's inequality to (3.14), we can get

$$\|u(t)\|_*^2 + \int_0^t \|u(s)\|_{**}^2\, ds \le e^{2\Lambda t}\|u(0)\|_*^2 + \frac{K_1}{2\Lambda}(e^{2\Lambda t} - 1) \tag{3.15}$$

for all $t \ge 0$. Now plugging (3.15) and (3.12) into (3.11), we find that

$$\frac{1}{\Lambda}\|\partial_t u(t)\|_*^2 + \|u(t)\|_{**}^2 \le K_1 + \Lambda\|u(t)\|_*^2 + 2\Lambda \int_0^t \|u(s)\|_{**}^2\, ds \le e^{2\Lambda t}(2\Lambda\|u(0)\|_*^2 + K_1). \tag{3.16}$$

By the trace theorem, we have

$$K_0, K_1 \le C(\|\partial_t u(0)\|_*^2 + \|u(0)\|_*^2 + \|u(0)\|_{**}^2 + \sigma \int_{\Sigma_0} |\nabla_{x_1,x_1} u_3(0)|^2)$$

for a constant $C > 0$ depending on $\rho_0, P, \Lambda, \varepsilon, \delta, \kappa, \sigma, g, b$.

For the estimates for η, q, V, we can get from $(1.8)_1$, $(1.8)_2$, $(1.8)_4$, (3.16) and the Korn's lemma (seen Lemma 3.6 in [4]) that

$$\begin{aligned}\|\eta(t)\|_{H^1} &\le \|\eta_0\|_{H^1} + \int_0^t \|\partial_s \eta(s)\|_{H^1} ds \le \|\eta_0\|_{H^1} + \int_0^t \|u(s)\|_{H^1} ds \\ &\le Ce^{\Lambda t}(\|\eta_0\|_{H^1} + \sqrt{I_0}),\end{aligned}$$

$$\begin{aligned}\|q(t)\|_{L^2} &\le \|q_0\|_{L^2} + \int_0^t \|\partial_s q(s)\|_{L^2} ds \le \|q_0\|_{L^2} + \int_0^t \|u(s)\|_{H^1} ds \\ &\le Ce^{\Lambda t}(\|q_0\|_{L^2} + \sqrt{I_0}),\end{aligned}$$

and

$$\begin{aligned}\|V(t)\|_{L^2} &\le \|V_0\|_{L^2} + \int_0^t \|\partial_s V(s)\|_{L^2} ds \le \|V_0\|_{L^2} + \int_0^t \|u(s)\|_{H^1} ds \\ &\le Ce^{\Lambda t}(\|V_0\|_{L^2} + \sqrt{I_0}),\end{aligned}$$

where $I_0 = \|\partial_t u(0)\|_*^2 + \|u(0)\|_*^2 + \|u(0)\|_{**}^2 + \sigma \int_{\Sigma_0} |\nabla_{x_1,x_2} u_3(0)|^2$. Then we complete the proof of Theorem 1.1.

Conclusions

This paper considers the linear Rayleigh-Taylor instability of an equilibrium state of 3D gravity-driven compressible viscoelastic fluid with the elasticity coefficient κ is less than a critical number κ_c in a moving horizontal periodic domain. We apply a method of studying a family of modified variational problems in order to produce the maximal growing mode solutions to the linearized equations, and then prove an estimate for arbitrary solutions to the linearized equations in terms of the fastest possible growth rate for the growing modes.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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