AIMS ***Mathematics***

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(6): 14867–14893.
DOI: 10.3934/math.2023760
Received: 22 February 2023
Revised: 27 March 2023
Accepted: 05 April 2023
Published: 21 April 2023

Research article

Weighted pseudo almost periodic solutions of octonion-valued neural networks with mixed time-varying delays and leakage delays

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Abstract: In this paper, we propose a class of octonion-valued neural networks with leakage delays and mixed delays. Considering that the multiplication of octonion algebras does not satisfy the associativity and commutativity, we can obtain the existence and global exponential stability of weighted pseudo almost periodic solutions for octonion-valued neural networks with leakage delays and mixed delays by using the Banach fixed point theorem, the proof by contradiction and the non-decomposition method. Finally, we will give one example to illustrate the feasibility and effectiveness of the main results.

Keywords: octonion algebra; neural networks; stability; weighted pseudo almost periodic solutions; delays
Mathematics Subject Classification: 34A34, 34C25, 34D23, 34K20

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1. Introduction

During the past decades, neural networks have attracted the attention of researchers and have been extensively applied, such as pattern recognition, associative memory, signal processing and so on. There are many good results about exponential stability and synchronization of the equilibrium point, periodic or anti-periodic solutions, almost periodic solutions and weighted pseudo almost periodic solutions for neural networks (see [1–10]).

Leakage delay is the time delay in the leakage term of the systems and a considerable factor affecting dynamics in the systems. Leakage delay has a great impact on the dynamic behavior of neural networks. Some good results of neural networks with leakage delay have been studied. For example, some authors have studied the periodic (or anti-periodic solutions) for neural networks with leakage terms (see [11–13]), some authors have studied almost periodic solutions for neural networks with leakage delays (see [14–16]), some authors have studied the almost sure stability of stochastic neural networks with time delays in the leakage terms (see [17]) and some authors have studied the fractional-order neural networks with leakage delays (see [18–22]).

Octonion-valued neural networks, which were first proposed by Popa in [23], represent a generalization of real-valued neural networks, complex-valued neural networks and quaternion-valued neural networks. The octonions are the largest of the four normed division algebras. While somewhat neglected due to their nonassociativity, they stand at the crossroads of many interesting fields of mathematics [24, 25]. Recently, some authors have studied the equilibrium point for octonion-valued neural networks (see [26–30]).

As is well known, the properties of weighted pseudo almost periodic solutions have been successfully applied in many neural networks with delays. The stability analysis of weighted pseudo-almost periodic solutions is more general and interesting than that of equilibrium points. Recently, some authors have studied the existence and global exponential stability of weighted pseudo almost periodic solutions for neural networks with delays (see [31–36]).

With inspiration from previous research, to fill the gap in the research field of octonion-valued neural networks, the work of this article comes from two main motivations. (1) In practical applications, a weighted pseudo almost periodic motion is an interesting and significant dynamical property for differential equations. (2) Recently, in [26–29], Popa has studied the global exponential stability of the equilibrium point for octonion-valued neural networks. Therefore, it is worth studying the weighted pseudo almost periodic motion of octonion-valued neural network models via a non-decomposition method.

Compared with the previous kinds of literature, the main contributions of this paper are listed as follows. (1) First, to the best of our knowledge, this is the first time study on the weighted pseudo almost periodic solutions for octonion-valued neural networks. (2) Second, without separating the octonion-valued neural networks into real-valued neural networks (or complex-valued neural networks), the results are less conservative and more general. (3) Third, in [26–30], some authors studied octonion-valued neural network systems by using the decomposition method. Therefore, to avoid the complexity of the calculation, this paper discusses octonion-valued neural network systems by using the non-decomposition method, the Banach fixed point theorem and the proof by contradiction. (4) Fourth, our method in this paper can be used to study the weighted pseudo almost periodic solutions for other types of octonion-valued neural networks. (5) Fifth, examples and numerical simulations are given to verify the effectiveness of the conclusion.

Motivated by the above statement, in this paper, we will study the following octonion-valued neural networks with leakage delays and mixed delays:

$$\begin{aligned} x_i'(t) \;=\; & -c_i(t)x_i(t-\eta_i(t)) + \sum_{j=1}^{n} a_{ij}(t)f_j(x_j(t)) \\ & + \sum_{j=1}^{n} b_{ij}(t)g_j(x_j(t-\tau_{ij}(t))) \\ & + \sum_{j=1}^{n} d_{ij}(t)\int_{t-\delta_{ij}(t)}^{t} h_j(x_j(s))ds + I_i(t), \end{aligned} \tag{1.1}$$

where $i = 1, 2, \ldots, n$, $x_i(t) \in \mathbb{O}$ is the state vector of the ith unit at time t, $c_i(t) > 0$ represents the rate which the ith unit will reset its potential to the resting state in isolation when disconnected from the network and external inputs, $a_{ij}, b_{ij}, d_{ij} \in \mathbb{O}$ denote the strength of connectivity between unit i and j at time t, the activation functions $f_j, g_j, h_j \in \mathbb{O}$ show how the jth neuron reacts to input, $I_i \in \mathbb{O}$ denotes the ith component of an external input source introduced from outside the network to the unit i at time t, $\eta_i(t) : \mathbb{R} \longrightarrow \mathbb{R}^+$ denote the leakage delay, $\tau_{ij}(t) : \mathbb{R} \longrightarrow \mathbb{R}$ are the time-varying delays and $\delta_{ij}(t) : \mathbb{R} \longrightarrow \mathbb{R}$ are the distributed delays.

The initial conditions of the system (1.1) are of the form

$$x_i(s) = \varphi_i(s), \quad s \in [-\theta, 0], \tag{1.2}$$

where $i = 1, 2, \cdots, n$, $\varphi_i \in \mathbb{O}$, $\theta = \max\{\eta^+, \tau, \delta^+\}$, $\eta^+ = \max\limits_{1\le i,j\le n} \left\{\sup\limits_{t\in\mathbb{R}} \eta_i(t)\right\}$, $\tau = \max\limits_{1\le i,j\le n} \left\{\sup\limits_{t\in\mathbb{R}} \tau_{ij}(t)\right\}$, $\delta^+ = \max\limits_{1\le i,j\le n} \left\{\sup\limits_{t\in\mathbb{R}} \delta_{ij}(t)\right\}$.

This paper is organized as follows: In Section 2, we introduce some definitions and lemmas. In Section 3, we establish some sufficient conditions for the existence and global exponential stability of weighted pseudo almost periodic solutions for system (1.1). In Section 4, one numerical example is provided to verify the effectiveness of the theoretical results. Finally, we draw a conclusion in Section 5.

Notations: $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}_+ = [0, +\infty)$ denotes the set of non-negative real numbers, $\mathbb{O}$ denotes the set of octonion numbers, $\mathbb{O}^8$ denotes the 8 dimensional octonion numbers, $\|\cdot\|_{\mathbb{O}}$ represents the vector octonion norm. For $x \in \mathbb{O}$, we define $\|x\|_{\mathbb{O}} = |x|$ and for $x = (x_1, x_2, \cdots, x_n)^T \in \mathbb{O}^n$, we define $\|x\|_{\mathbb{O}^n} = \sum\limits_{i=1}^{n} \|x_i\|_{\mathbb{O}}$.

2. Preliminaries

In this section, we will introduce some basic definitions and lemmas.

The algebra of octonion is defined as

$$\mathbb{O} = \left\{x = \sum_{p=0}^{7} [x]_p e_p : [x]_0, [x]_1, \cdots, [x]_7 \in \mathbb{R}\right\},$$

where e_p are the octonion units, $0 \le p \le 7$, and when $p = 0$, we have $e_0 = 1$. The octonion units obey the octonion multiplication rules: $e_p e_q = -e_q e_p \neq e_q e_p, \forall 0 < p \neq q \le 7$, from which we deduce that $\mathbb{O}$ is not commutative, and that $(e_p e_q)e_k = -e_p(e_q e_k) \neq e_p(e_q e_k)$, for k, p, q distinct, $0 < k, p, q \le 7$, or $e_p e_q \neq \pm e_k$, thus $\mathbb{O}$ is also not associative.

Octonion addition is defined by $x + y = \sum\limits_{p=0}^{7} ([x]_p + [y]_p)e_p$, scalar multiplication is given by $\alpha x = \sum\limits_{p=0}^{7} (\alpha[x]_p)e_p$, and octonion multiplication is given by the multiplication of the octonion units (see Table 1):

Table 1. The multiplication of the octonion units.

$\times$	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_0	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	$-e_0$	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	e_2	$-e_3$	$-e_0$	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_3	e_2	$-e_1$	$-e_0$	e_7	$-e_6$	e_5	$-e_4$
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	$-e_0$	e_1	e_2	e_3
e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	$-e_0$	$-e_3$	e_2
e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	$-e_0$	$-e_1$
e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	$-e_0$

The conjugate of an octonion x is defined as $\bar{x} = [x]_0 e_0 - \sum\limits_{p=1}^{7} [x]_p e_p$, its norm as $|x| = \sqrt{x\bar{x}} = \sqrt{\sum_{p=0}^{7} [x]_p^2}$, and its inverse as $x^{-1} = \frac{\bar{x}}{|x|^2}$. We can now see that $\mathbb{O}$ is a normed division algebra, and it can be proved that the only three division algebras that can be defined over the reals are the complex, quaternion and octonion algebras.

Definition 2.1. *([37]) Let $f \in BC(\mathbb{R}, \mathbb{R}^n)$. Function f is said to be almost periodic if, for any $\epsilon > 0$, it is possible to find a real number $l = l(\epsilon) > 0$, for any interval with length $l(\epsilon)$, there exists a number $\tau = \tau(\epsilon)$ in this interval such that*

$$| f(t + \tau) - f(t) | < \epsilon, \quad \forall t \in \mathbb{R}.$$

We denote by $AP(\mathbb{R}, \mathbb{R}^n)$ the set of all almost periodic functions from $\mathbb{R}$ to $\mathbb{R}^n$, $AP^1(\mathbb{R}, \mathbb{R}^n)$ the set of all continuously differentiable functions $f : \mathbb{R} \to \mathbb{R}^n$ satisfying $f, f' \in AP(\mathbb{R}, \mathbb{R}^n)$.

Definition 2.2. *Let $f \in BC(\mathbb{R}, \mathbb{O}^n)$. Function f is said to be almost periodic if, for any $\epsilon > 0$, it is possible to find a real number $l = l(\epsilon) > 0$, for any interval with length $l(\epsilon)$, there exists a number $\tau = \tau(\epsilon)$ in this interval such that*

$$\|f(t + \tau) - f(t)\|_{\mathbb{O}^n} < \epsilon, \quad \forall t \in \mathbb{R}.$$

We denote by $AP(\mathbb{R}, \mathbb{O}^n)$ the set of all almost periodic functions from $\mathbb{R}$ to $\mathbb{O}^n$, $AP^1(\mathbb{R}, \mathbb{O}^n)$ the set of all continuously differentiable functions $f : \mathbb{R} \to \mathbb{O}^n$ satisfying $f, f' \in AP(\mathbb{R}, \mathbb{O}^n)$.

Lemma 2.1. *Suppose that $\alpha \in \mathbb{R}$, $f, \ g \in AP(\mathbb{R}, \mathbb{O})$, then $\alpha f, f + g, fg \in AP(\mathbb{R}, \mathbb{O})$.*

Proof. Since $f, \ g \in AP(\mathbb{R}, \mathbb{O})$. Therefore, $f, \ g \in BC(\mathbb{R}, \mathbb{O})$, namely, there exist two positive constants M_1, M_2 such that

$$\|f\|_{\mathbb{O}} \leq M_1, \ \ \|g\|_{\mathbb{O}} \leq M_2.$$

For any $\epsilon > 0$, we have

$$\|f(t + \tau) - f(t)\|_{\mathbb{O}} < \frac{\epsilon}{2M_2}, \ \ \|g(t + \tau) - g(t)\|_{\mathbb{O}} < \frac{\epsilon}{2M_1}.$$

Hence, we have

$$\begin{aligned}
& \|f(t+\tau)g(t+\tau)-f(t)g(t)\|_{\mathbb{O}} \\
\leq\ & \|f(t+\tau)g(t+\tau)-f(t)g(t+\tau)\|_{\mathbb{O}}+\|f(t)g(t+\tau)-f(t)g(t)\|_{\mathbb{O}} \\
\leq\ & \|f(t+\tau)-f(t)\|_{\mathbb{O}}\|g(t+\tau)\|_{\mathbb{O}}+\|f(t)\|_{\mathbb{O}}\|g(t+\tau)-g(t)\|_{\mathbb{O}} \\
<\ & \frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon,
\end{aligned}$$

which implies that $fg \in AP(\mathbb{R}, \mathbb{O})$.

Similarly, we can show that $\alpha f, f+g \in AP(\mathbb{R}, \mathbb{O})$. The proof is complete. □

Lemma 2.2. *If $f \in C(\mathbb{O}, \mathbb{O})$ satisfies the Lipschitz condition, $x \in AP(\mathbb{R}, \mathbb{O})$, then $f(x(\cdot)) \in AP(\mathbb{O}, \mathbb{O})$.*

Proof. Since $f \in C(\mathbb{O}, \mathbb{O})$ satisfies the Lipschitz condition, $x \in AP(\mathbb{R}, \mathbb{O})$. Let $u, v \in \mathbb{O}$, for any $\epsilon > 0$, there exists a positive constant L such that

$$\|x(t+\tau)-x(t)\|_{\mathbb{O}} < \frac{\epsilon}{L},\ \ \|f(u)-f(v)\|_{\mathbb{O}} \leq L\|u-v\|_{\mathbb{O}}.$$

Hence, we have

$$\begin{aligned}
\|f(x(t+\tau))-f(x(t))\|_{\mathbb{O}} &\leq L\|x(t+\tau)-x(t)\|_{\mathbb{O}} \\
&< \epsilon,
\end{aligned}$$

which implies that $f(x(\cdot)) \in AP(\mathbb{O}, \mathbb{O})$. The proof is complete. □

Lemma 2.3. *If $x \in AP(\mathbb{R}, \mathbb{O})$, $\rho \in AP(\mathbb{R}, \mathbb{R})$, then $x(\cdot - \rho(\cdot)) \in AP(\mathbb{R}, \mathbb{O})$.*

Proof. Since $x \in AP(\mathbb{R}, \mathbb{O})$, it follows that x is uniformly continuous. For any $\epsilon > 0$, there exists a constant $0 < \delta = \delta(\epsilon) < \frac{\epsilon}{2}$ such that

$$\|x(t_1)-x(t_2)\|_{\mathbb{O}} < \frac{\epsilon}{2},\ \ \forall t_1,\ \ t_2 \in \mathbb{R},\ \ |t_1-t_2| < \delta. \tag{2.1}$$

For this $\delta > 0$, there exists a $l = l(\delta) = l(\delta(\epsilon)) > 0$, for any interval with length $l(\delta)$, there exists a number $\tau = \tau(\epsilon)$ in this interval such that

$$|\rho(t+\tau)-\rho(t)| < \delta,\ \ \|x(t+\tau)-x(t)\|_{\mathbb{O}} < \delta < \frac{\epsilon}{2},\ \ \forall t \in \mathbb{R}. \tag{2.2}$$

From (2.1) and (2.2), we have

$$\begin{aligned}
& \|x(t+\tau-\rho(t+\tau))-x(t-\rho(t))\|_{\mathbb{O}} \\
\leq\ & \|x(t+\tau-\rho(t+\tau))-x(t+\tau-\rho(t))\|_{\mathbb{O}} \\
& +\|x(t+\tau-\rho(t))-x(t-\rho(t))\|_{\mathbb{O}} \\
<\ & \frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon,
\end{aligned}$$

which implies that $x(\cdot - \rho(\cdot)) \in AP(\mathbb{R}, \mathbb{O})$. The proof is complete. □

Let $\mathbb{W}$ denote the collection of functions (weights) $\mu : \mathbb{R} \to (0, +\infty)$, which are locally integrable over $\mathbb{R}$ such that $\mu > 0$ almost everywhere. For $\mu \in \mathbb{W}$ and $r > 0$, we denote

$$\mu([-r, r]) := \int_{-r}^{r} \mu(x)dx.$$

The space of weights $\mathbb{W}_\infty$ is defined by

$$\mathbb{W}_\infty := \Big\{\mu \in \mathbb{W} : \inf_{t\in\mathbb{R}} \mu(t) = \mu_0 > 0, \lim_{r\to+\infty} \mu([-r, r]) = +\infty\Big\}.$$

Definition 2.3. *Fix $\mu \in \mathbb{W}_\infty$. Function $f \in BC(\mathbb{R}, \mathbb{O}^n)$ is said to be weighted pseudo almost periodic, if it can be written as $f = f_1 + f_2$ with $f_1 \in AP(\mathbb{R}, \mathbb{O}^n)$ and $f_2 \in PAP_0(\mathbb{R}, \mathbb{O}^n, \mu)$, where the space $PAP_0(\mathbb{R}, \mathbb{O}^n, \mu)$ is defined by*

$$PAP_0(\mathbb{R}, \mathbb{O}^n, \mu) := \Big\{f \in BC(\mathbb{R}, \mathbb{O}^n) : \lim_{r\to+\infty} \frac{1}{\mu([-r, r])} \int_{-r}^{r} \|f(t)\|_{\mathbb{O}^n}\mu(t)dt = 0\Big\}.$$

We denote by $PAP(\mathbb{R}, \mathbb{O}^n, \mu)$ the set of all weighted pseudo almost periodic functions from $\mathbb{R}$ to $\mathbb{O}^n$, $PAP^1(\mathbb{R}, \mathbb{O}^n, \mu)$ the set of all continuously differentiable functions $f : \mathbb{R} \to \mathbb{O}^n$ satisfying $f, f' \in PAP(\mathbb{R}, \mathbb{O}^n, \mu)$.

Lemma 2.4. *Suppose that $x \in PAP(\mathbb{R}, \mathbb{O}, \mu)$, $\tau \in AP^1(\mathbb{R}, \mathbb{R}_+)$ and $\beta := \inf_{t\in\mathbb{R}}(1-\dot\tau(t)) > 0$, then $x(t-\tau(t)) \in PAP(\mathbb{R}, \mathbb{O}, \mu)$.*

Proof. Since $x \in PAP(\mathbb{R}, \mathbb{O}, \mu)$, by Definition 2.3, we have $x = x_1 + x_2$, where $x_1 \in AP(\mathbb{R}, \mathbb{O})$ and $x_2 \in PAP_0(\mathbb{R}, \mathbb{O}, \mu)$. Clearly, $x_1(t - \tau(t)) \in AP(\mathbb{R}, \mathbb{O})$.

Let $\alpha = \frac{1}{\beta} \times \sup_{t\in\mathbb{R}} \frac{\mu(t)}{\mu(t-\tau(t))}$, $\tau = \sup_{t\in\mathbb{R}} \tau(t)$, $s = t - \tau(t)$, we have

$$\begin{aligned}
0 \ &\leq \ \frac{1}{\mu([-r, r])} \int_{-r}^{r} \|x_2(t - \tau(t))\|_{\mathbb{O}}\mu(t)dt \\
&\leq \ \frac{1}{\mu([-r, r])} \int_{-r}^{r} \|x_2(t - \tau(t))\|_{\mathbb{O}}\mu(t - \tau(t))dt \sup_{t\in\mathbb{R}} \frac{\mu(t)}{\mu(t - \tau(t))} \\
&\leq \ \frac{1}{\mu([-r, r])} \int_{-r-\tau(t)}^{r-\tau(t)} \frac{1}{1 - \dot\tau(s)}\|x_2(s)\|_{\mathbb{O}}\mu(s)ds \sup_{t\in\mathbb{R}} \frac{\mu(t)}{\mu(t - \tau(t))} \\
&\leq \ \alpha\frac{1}{\mu([-r, r])} \int_{-r-\tau(t)}^{r-\tau(t)} \|x_2(s)\|_{\mathbb{O}}\mu(s)ds \\
&\leq \ \alpha\frac{1}{\mu([-r, r])} \int_{-r-\tau}^{r+\tau} \|x_2(s)\|_{\mathbb{O}}\mu(s)ds \\
&\leq \ \alpha \sup_{r\geq1} \frac{\mu([-r - \tau, r + \tau]}{\mu([-r, r])} \frac{1}{\mu([-r - \tau, r + \tau])} \int_{-r-\tau}^{r+\tau} \|x_2(s)\|_{\mathbb{O}}\mu(s)ds,
\end{aligned}$$

together with the fact that

$$\lim_{r\to+\infty} \frac{1}{\mu([-r - \tau, r + \tau])} \int_{-r-\tau}^{r+\tau} \|x_2(s)\|_{\mathbb{O}}\mu(s)ds = 0,$$

which implies that
$$\lim_{r\to+\infty}\frac{1}{\mu([-r,r])}\int_{-r}^{r}\|x_2(t-\tau(t))\|_{\mathbb{O}}\mu(t)dt=0.$$
Hence, $x_2(t-\tau(t))\in PAP_0(\mathbb{R},\mathbb{O},\mu)$. The proof is completed. □

Lemma 2.5. *Suppose that $\alpha\in\mathbb{R}$, $f,\ g\in PAP(\mathbb{R},\mathbb{O},\mu)$, then $\alpha f, f+g, fg\in PAP(\mathbb{R},\mathbb{O},\mu)$.*

Proof. Since $f,\ g\in PAP(\mathbb{R},\mathbb{O},\mu)$, by Definition 2.3, we have $f=f_1+f_2$, $g=g_1+g_2$, where $f_1,g_1\in AP(\mathbb{R},\mathbb{O})$, $f_2,g_2\in PAP_0(\mathbb{R},\mathbb{O},\mu)$.

Therefore,
$$fg=(f_1+f_2)(g_1+g_2)=f_1g_1+f_1g_2+f_2(g_1+g_2)=f_1g_1+f_1g_2+f_2g.$$
Clearly, $f_1g_1\in AP(\mathbb{R},\mathbb{O})$.

Next, we will show $f_1g_2+f_2g\in PAP_0(\mathbb{R},\mathbb{O},\mu)$. Note that $f_1\in AP(\mathbb{R},\mathbb{O})$, $g\in PAP(\mathbb{R},\mathbb{O},\mu)$, we have that $f_1,g\in BC(\mathbb{R},\mathbb{O})$. There exist two positive constants L_1,L_2 such that
$$\|f_1(t)\|_{\mathbb{O}}\le L_1,\ \ \|g(t)\|_{\mathbb{O}}\le L_2,\forall t\in\mathbb{R}.$$
Hence, we have
$$\begin{aligned}0\ \le\ &\frac{1}{\mu([-r,r])}\int_{-r}^{r}\|f_1(t)g_2(t)+f_2(t)g(t)\|_{\mathbb{O}}\mu(t)dt\\ \le\ &\frac{1}{\mu([-r,r])}\int_{-r}^{r}\|f_1(t)g_2(t)\|_{\mathbb{O}}\mu(t)dt\\ &+\frac{1}{\mu([-r,r])}\int_{-r}^{r}\|f_2(t)g(t)\|_{\mathbb{O}}\mu(t)dt\\ \le\ &\frac{L_1}{\mu([-r,r])}\int_{-r}^{r}\|g_2(t)\|_{\mathbb{O}}\mu(t)dt\\ &+\frac{L_2}{\mu([-r,r])}\int_{-r}^{r}\|f_2(t)\|_{\mathbb{O}}\mu(t)dt,\end{aligned}$$
together with the fact that
$$\lim_{r\to+\infty}\frac{1}{\mu([-r,r])}\int_{-r}^{r}\|f_2(t)\|_{\mathbb{O}}\mu(t)dt=0,$$
and
$$\lim_{r\to+\infty}\frac{1}{\mu([-r,r])}\int_{-r}^{r}\|g_2(t)\|_{\mathbb{O}}\mu(t)dt=0,$$
which implies that
$$\lim_{r\to+\infty}\frac{1}{\mu([-r,r])}\int_{-r}^{r}\|f_1(t)g_2(t)+f_2(t)g(t)\|_{\mathbb{O}}\mu(t)dt=0.$$
Hence, $f_1g_2+f_2g\in PAP_0(\mathbb{R},\mathbb{O},\mu)$.

Similarly, we can show that $\alpha f, f+g\in PAP(\mathbb{R},\mathbb{O},\mu)$. The proof is completed. □

Lemma 2.6. *Suppose that $x\in PAP(\mathbb{R},\mathbb{O},\mu)$, $f\in C(\mathbb{O},\mathbb{O})$ satisfies the Lipschitz condition, then $f(x(\cdot))\in PAP(\mathbb{R},\mathbb{O},\mu)$.*

Proof. Since $x \in PAP(\mathbb{R}, \mathbb{O}, \mu)$, by Definition 2.3, we have $x = x_1 + x_2$, where $x_1 \in AP(\mathbb{R}, \mathbb{O})$ and $x_2 \in PAP_0(\mathbb{R}, \mathbb{O}, \mu)$. Let

$$f(x(\cdot)) = f(x_1(\cdot) + x_2(\cdot)) = f(x_1(\cdot)) + f(x_1(\cdot) + x_2(\cdot)) - f(x_1(\cdot)),$$

clearly, $f(x_1(\cdot)) \in AP(\mathbb{R}, \mathbb{O})$.

Next, we will show $f(x_1(\cdot) + x_2(\cdot)) - f(x_1(\cdot)) \in PAP_0(\mathbb{R}, \mathbb{O}, \mu)$. Since $f \in C(\mathbb{O}, \mathbb{O})$ satisfies the Lipschitz condition, for $u, v \in \mathbb{O}$, there exists a positive constant L such that

$$\|f(u) - f(v)\|_{\mathbb{O}} \leq L\|u - v\|_{\mathbb{O}}.$$

Hence, we have

$$\begin{aligned} 0 &\leq \frac{1}{\mu([-r, r])} \int_{-r}^{r} \|f(x_1(t) + x_2(t)) - f(x_1(t))\|_{\mathbb{O}} \mu(t) dt \\ &\leq \frac{1}{\mu([-r, r])} \int_{-r}^{r} L\|x_2(t)\|_{\mathbb{O}} \mu(t) dt, \end{aligned}$$

together with the fact that

$$\lim_{r \to +\infty} \frac{1}{\mu([-r, r])} \int_{-r}^{r} \|x_2(t)\|_{\mathbb{O}} \mu(t) dt = 0,$$

which implies that

$$\lim_{r \to +\infty} \frac{1}{\mu([-r, r])} \int_{-r}^{r} \|f(x_1(t) + x_2(t)) - f(x_1(t))\|_{\mathbb{O}} \mu(t) dt = 0.$$

Hence, $f(x_1(\cdot) + x_2(\cdot)) - f(x_1(\cdot)) \in PAP_0(\mathbb{R}, \mathbb{O}, \mu)$. The proof is completed. □

Let

$$\mathbb{X} = \Big\{\phi \in C^1(\mathbb{R}, \mathbb{O}^n) \mid \phi, \phi' \in PAP\Big(\mathbb{R}, \mathbb{O}^n, \mu\Big)\Big\}$$

be a Banach space equipped with the norm

$$\|\phi\|_{\mathbb{X}} = \sup_{t \in \mathbb{R}} \max \Big\{\|\phi(t)\|_{\mathbb{O}^n}, \|\phi'(t)\|_{\mathbb{O}^n}\Big\},$$

and

$$\phi_0(t) = \begin{pmatrix} \int_{-\infty}^{t} e^{\int_t^s c_1(v)dv} I_1(s) ds \\ \int_{-\infty}^{t} e^{\int_t^s c_2(v)dv} I_2(s) ds \\ \cdot \\ \cdot \\ \cdot \\ \int_{-\infty}^{t} e^{\int_t^s c_n(v)dv} I_n(s) ds \end{pmatrix}.$$

Definition 2.4. *Let* $x(t) = \big(x_1(t), x_2(t), \cdots, x_n(t)\big)^T$ *be a weighted pseudo almost periodic solution of system* (1.1) *with the initial value* $\varphi(s) = \big(\varphi_1(s), \varphi_2(s), \cdots, \varphi_n(s)\big)^T$ *and* $y(t) = \big(y_1(t), y_2(t), \cdots, y_n(t)\big)^T$

be arbitrary solution of system (1.1) *with the initial value* $\psi(s) = (\psi_1(s), \psi_2(s), \cdots, \psi_n(s))^T$, *where* $\varphi, \psi \in C([-\theta, 0], \mathbb{O}^n)$. *If there exist constants* $\lambda > 0$ *and* $M > 0$ *such that*

$$\|x(t) - y(t)\|_{\mathbb{X}} \leq M\|\varphi - \psi\|_{\mathbb{X}} e^{-\lambda t}, \forall t > 0,$$

then the weighted pseudo almost periodic solution of system (1.1) *is said to be globally exponentially stable, where*

$$\|x - y\|_{\mathbb{X}} = \sup_{t\in\mathbb{R}} \max\{\|x(t) - y(t)\|_{\mathbb{O}^n}, \|(x(t) - y(t))'\|_{\mathbb{O}^n}\}$$

and

$$\|\varphi - \psi\|_{\mathbb{X}} = \sup_{s\in[\theta,0]} \max\{\|\varphi(s) - \psi(s)\|_{\mathbb{O}^n}, \|(\varphi(s) - \psi(s))'\|_{\mathbb{O}^n}\}.$$

In order to study the existence of weighted pseudo almost periodic solutions for system (1.1), we need the following assumptions:

- Assumption 1: For $i, j = 1, 2, \cdots, n$, $c_i, \delta_{ij} \in AP(\mathbb{R}, \mathbb{R}_+)$, $\eta_i, \tau_{ij} \in AP^1(\mathbb{R}, \mathbb{R}_+)$, $a_{ij}, b_{ij}, d_{ij}, I_i \in C(\mathbb{R}, \mathbb{O})$ are weighted pseudo almost periodic.
- Assumption 2: For $j = 1, 2, \cdots, n$, there exist positive constants L_f, L_g, L_h such that

$$\|f_j(u) - f_j(v)\|_{\mathbb{O}} \leq L_f\|u - v\|_{\mathbb{O}},$$

$$\|g_j(u) - g_j(v)\|_{\mathbb{O}} \leq L_g\|u - v\|_{\mathbb{O}},$$

$$\|h_j(u) - h_j(v)\|_{\mathbb{O}} \leq L_h\|u - v\|_{\mathbb{O}}.$$

- Assumption 3: There exists a positive constant $\xi \in (0, 1)$ such that

$$0 < \max\left\{\frac{\Theta}{c^-}, \left(1 + \frac{c^+}{c^-}\right)\Theta\right\} < \xi < 1,$$

where

$$\begin{aligned}\Theta \;:=\; & c^+\eta^+ + \sum_{j=1}^n a^+L_f + \sum_{j=1}^n b^+L_g + \sum_{j=1}^n d^+\delta^+L_h \\ & + \sum_{j=1}^n \frac{a^+M_f(1-\xi)}{L} + \sum_{j=1}^n \frac{b^+M_g(1-\xi)}{L} \\ & + \sum_{j=1}^n \frac{d^+\delta^+M_h(1-\xi)}{L},\end{aligned}$$

$$c^- = \min_{1\leq i\leq n} \inf_{t\in\mathbb{R}} c_i(t), c^+ = \max_{1\leq i\leq n} \sup_{t\in\mathbb{R}} c_i(t), \|\phi_0\|_{\mathbb{X}} \leq L,$$

$$a^+ = \max_{1\leq i,j\leq n} \sup_{t\in\mathbb{R}} \|a_{ij}(t)\|_{\mathbb{O}}, b^+ = \max_{1\leq i,j\leq n} \sup_{t\in\mathbb{R}} \|b_{ij}(t)\|_{\mathbb{O}},$$

$$d^+ = \max_{1\leq i,j\leq n} \sup_{t\in\mathbb{R}} \|d_{ij}(t)\|_{\mathbb{O}}, M_f = \max_{1\leq j\leq n} \|f_j(0)\|_{\mathbb{O}},$$

$$M_g = \max_{1\leq j\leq n} \|g_j(0)\|_{\mathbb{O}}, M_h = \max_{1\leq j\leq n} \|h_j(0)\|_{\mathbb{O}}.$$

3. Main results

In this section, we will investigate the existence and global exponential stability of weighted pseudo almost periodic solutions for delayed octonion-valued neural networks (1.1) by applying the non-decomposition method, Banach fixed point theorem and the proof by contradiction.

Theorem 3.1. *Let $\mu \in \mathbb{W}_\infty$. Assume that Assumptions 1–3 hold. Then system* (1.1) *has a unique weighted pseudo almost periodic solution in the region* $\mathbb{X}^* = \left\{\phi \mid \phi \in \mathbb{X}, \|\phi - \phi_0\|_{\mathbb{X}} \le \frac{\xi L}{1-\xi}\right\}$.

Proof. System (1.1) can be transformed into the following system:

$$
\begin{aligned}
x_i'(t) \;=\; & -c_i(t)x_i(t) + c_i(t)\int_{t-\eta_i(t)}^{t} x_i'(s)ds + \sum_{j=1}^{n} a_{ij}(t) \\
& \times f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij}(t)g_j(x_j(t-\tau_{ij}(t))) \\
& + \sum_{j=1}^{n} d_{ij}(t)\int_{t-\delta_{ij}(t)}^{t} h_j(x_j(s))ds + I_i(t).
\end{aligned}
\tag{3.1}
$$

It is well known that a solution of system (3.1) is equivalent to find a solution of the integral equation:

$$
\begin{aligned}
x_i(t) \;=\; & \int_{-\infty}^{t} e^{\int_t^s c_i(\nu)d\nu}\Big[c_i(s)\int_{s-\eta_i(s)}^{s} x_i'(\nu)d\nu + \sum_{j=1}^{n} a_{ij}(s) \\
& \times f_j(x_j(s)) + \sum_{j=1}^{n} b_{ij}(s)g_j(x_j(s-\tau_{ij}(s))) \\
& + \sum_{j=1}^{n} d_{ij}(s)\int_{s-\delta_{ij}(s)}^{s} h_j(x_j(\nu))d\nu + I_i(s)\Big]ds,
\end{aligned}
\tag{3.2}
$$

where $i = 1, 2, \cdots, n$.

Now, we define a mapping $\Psi : \mathbb{X}^* \to \mathbb{X}$ as follows

$$
(\Psi\phi)(t) = \left(x_1^{\phi}(t), x_2^{\phi}(t), \cdots, x_n^{\phi}(t)\right)^T,
$$

where $i = 1, 2, \cdots, n$, $x_i^{\phi}(t) \in \mathbb{O}$ and

$$
\begin{aligned}
x_i^{\phi}(t) \;=\; & \int_{-\infty}^{t} e^{\int_t^s c_i(\nu)d\nu}\Big[c_i(s)\int_{s-\eta_i(s)}^{s} \phi_i'(\nu)d\nu + \sum_{j=1}^{n} a_{ij}(s) \\
& \times f_j(\phi_j(s)) + \sum_{j=1}^{n} b_{ij}(s)g_j(\phi_j(s-\tau_{ij}(s))) \\
& + \sum_{j=1}^{n} d_{ij}(s)\int_{s-\delta_{ij}(s)}^{s} h_j(\phi_j(\nu))d\nu + I_i(s)\Big]ds,
\end{aligned}
\tag{3.3}
$$

where $\phi_i \in \mathbb{X}$.

Let

$$\begin{aligned} F_i(s) &= c_i(s)\int_{s-\eta_i(s)}^{s}\phi_i'(v)dv+\sum_{j=1}^{n}a_{ij}(s) \\ &\quad \times f_j(\phi_j(s))+\sum_{j=1}^{n}b_{ij}(s)g_j(\phi_j(s-\tau_{ij}(s))) \\ &\quad +\sum_{j=1}^{n}d_{ij}(s)\int_{s-\delta_{ij}(s)}^{s}h_j(\phi_j(v))dv+I_i(s),\ \ i=1,2,\cdots,n. \end{aligned}$$

By Lemmas 2.4–2.6, for $i=1,2,\cdots,n$, we can get $F_i(t)\in PAP(\mathbb{R},\mathbb{O},\mu)$. Let $F_i=F_i^1+F_i^2$, where $F_i^1\in AP(\mathbb{R},\mathbb{O})$ and $F_i^2\in PAP_0(\mathbb{R},\mathbb{O},\mu)$. Then we have

$$\begin{aligned} x_i^{\phi}(t) &= \int_{-\infty}^{t}e^{\int_t^s c_i(v)dv}F_i^1(s)ds+\int_{-\infty}^{t}e^{\int_t^s c_i(v)dv}F_i^2(s)ds \\ &:= \Gamma_i^1(t)+\Gamma_i^2(t). \end{aligned}$$

First, we will show that $\Gamma_i^1\in AP(\mathbb{R},\mathbb{O})$ and $\Gamma_i^2\in PAP_0(\mathbb{R},\mathbb{O},\mu)$. Since $c_i,F_i^1\in AP(\mathbb{R},\mathbb{O})$, let $\alpha_i=\sup\limits_{t\in\mathbb{R}}\|F_i^1(t)\|_{\mathbb{O}}$, for any $\epsilon>0$, it is possible to find a real number $l=l(\epsilon)>0$, for any interval with length $l(\epsilon)$, there exists a number $\varrho=\varrho(\epsilon)$ in this interval such that

$$|\,c_i(t+\varrho)-c_i(t)\,|\le\frac{(c^-)^2}{2\alpha_i}\epsilon,\ \ \|F_i^1(t+\varrho)-F_i^1(t)\|_{\mathbb{O}}\le\frac{c^-}{2}\epsilon.$$

Hence, we have that

$$\begin{aligned} &\|\Gamma_i^1(t+\varrho)-\Gamma_i^1(t)\|_{\mathbb{O}} \\ =\ & \Big\|\int_{-\infty}^{t+\varrho}e^{\int_{t+\varrho}^{s}c_i(v)dv}F_i^1(s)ds-\int_{-\infty}^{t}e^{\int_t^s c_i(v)dv}F_i^1(s)ds\Big\|_{\mathbb{O}} \\ =\ & \Big\|\int_{-\infty}^{t}e^{\int_{t+\varrho}^{s+\varrho}c_i(v)dv}F_i^1(s+\varrho)ds-\int_{-\infty}^{t}e^{\int_t^s c_i(v)dv}F_i^1(s)ds\Big\|_{\mathbb{O}} \\ \le\ & \Big\|\int_{-\infty}^{t}e^{\int_{t+\varrho}^{s+\varrho}c_i(v)dv}F_i^1(s+\varrho)ds-\int_{-\infty}^{t}e^{\int_{t+\varrho}^{s+\varrho}c_i(v)dv}F_i^1(s)ds\Big\|_{\mathbb{O}} \\ &+\Big\|\int_{-\infty}^{t}e^{\int_{t+\varrho}^{s+\varrho}c_i(v)dv}F_i^1(s)ds-\int_{-\infty}^{t}e^{\int_t^s c_i(v)dv}F_i^1(s)ds\Big\|_{\mathbb{O}} \\ \le\ & \int_{-\infty}^{t}e^{\int_{t+\varrho}^{s+\varrho}c_i(v)dv}\|F_i^1(s+\varrho)-F_i^1(s)\|_{\mathbb{O}}ds \\ &+\int_{-\infty}^{t}|\,e^{\int_{t+\varrho}^{s+\varrho}c_i(v)dv}-e^{\int_t^s c_i(v)dv}\,|\cdot\|F_i^1(s)\|_{\mathbb{O}}ds \\ \le\ & \frac{\epsilon}{2}+\int_{-\infty}^{t}\int_{t}^{s}e^{\int_t^{\zeta}c_i(v)dv}\,|\,c_i(\zeta+\varrho)-c_i(\zeta)\,|\,d\zeta\|F_i^1(s)\|_{\mathbb{O}}ds \\ \le\ & \frac{\epsilon}{2}+\frac{(c^-)^2}{2}\epsilon\int_{-\infty}^{t}\int_{t}^{s}e^{\int_t^{\zeta}c_i(v)dv}d\zeta ds \end{aligned}$$

$$\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

which implies that $\Gamma_i^1 \in AP(\mathbb{R}, \mathbb{O})$, $i = 1, 2, \cdots, n$.

Since $F_i^2 \in PAP_0(\mathbb{R}, \mathbb{O}, \mu)$, let $\zeta = t - s$, we have

$$\begin{aligned}
0 &\leq \frac{1}{\mu([-r,r])} \int_{-r}^{r} \|\Gamma_i^2(t)\|_{\mathbb{O}} \mu(t) dt \\
&\leq \frac{1}{\mu([-r,r])} \int_{-r}^{r} \Big\| \int_{-\infty}^{t} e^{\int_t^s c_i(\nu) d\nu} F_i^2(s) ds \Big\|_{\mathbb{O}} \mu(t) dt \\
&\leq \frac{1}{\mu([-r,r])} \int_{-r}^{r} \Big(\int_{-\infty}^{t} e^{\int_t^s c_i(\nu) d\nu} \|F_i^2(s)\|_{\mathbb{O}} ds \Big) \mu(t) dt \\
&\leq \frac{1}{\mu([-r,r])} \int_{-r}^{r} \Big(\int_{0}^{+\infty} e^{-c^- \zeta} \|F_i^2(t-\zeta)\|_{\mathbb{O}} d\zeta \Big) \mu(t) dt \\
&\leq \int_{0}^{+\infty} e^{-c^- \zeta} \Big(\frac{1}{\mu([-r,r])} \int_{-r}^{r} \|F_i^2(t-\zeta)\|_{\mathbb{O}} \mu(t) dt \Big) d\zeta,
\end{aligned}$$

together with the fact that

$$\lim_{r \to +\infty} \frac{1}{\mu([-r,r])} \int_{-r}^{r} \|F_i^2(t-\zeta)\|_{\mathbb{O}} \mu(t) dt = 0,$$

which implies that

$$\lim_{r \to +\infty} \frac{1}{\mu([-r,r])} \int_{-r}^{r} \|\Gamma_i^2(t)\|_{\mathbb{O}} \mu(t) dt = 0.$$

Hence, $\Gamma_i^2 \in PAP_0(\mathbb{R}, \mathbb{O}, \mu)$, $x_i^\phi \in PAP(\mathbb{R}, \mathbb{O}, \mu)$, $i = 1, 2, \cdots, n$.

Second, we will show that $(x_i^\phi)' \in PAP(\mathbb{R}, \mathbb{O}, \mu)$. For $i = 1, 2, \cdots, n$, we have

$$\begin{aligned}
(x_i^\phi(t))' &= F_i(t) - c_i(t) \int_{-\infty}^{t} e^{\int_t^s c_i(\nu) d\nu} F_i(s) ds \\
&= -c_i(t) x_i^\phi(t) + F_i(t).
\end{aligned}$$

Since $c_i(t) \in AP(\mathbb{R}, \mathbb{O})$, $x_i^\phi, F_i \in PAP(\mathbb{R}, \mathbb{O}, \mu)$. Therefore, we can conclude that $(x_i^\phi)' \in PAP(\mathbb{R}, \mathbb{O}, \mu)$.

Third, we show that the mapping Ψ is a self-mapping from $\mathbb{X}^*$ to $\mathbb{X}^*$. By Assumptions 1–3, for $\forall \phi \in \mathbb{X}^*$, we have

$$\|\phi\|_{\mathbb{X}} \leq \|\phi - \phi_0\|_{\mathbb{X}} + \|\phi_0\|_{\mathbb{X}} \leq \frac{\xi L}{1-\xi} + L = \frac{L}{1-\xi}.$$

Hence,

$$\begin{aligned}
&\|(\Psi\phi)(t) - \phi_0(t)\|_{\mathbb{O}^n} \\
= &\sum_{i=1}^{n} \Big\| \int_{-\infty}^{t} e^{\int_t^s c_i(\nu) d\nu} \Big[c_i(s) \int_{s-\eta_i(s)}^{s} \phi_i'(\nu) d\nu \\
&+ \sum_{j=1}^{n} a_{ij}(s) f_j(\phi_j(s)) + \sum_{j=1}^{n} b_{ij}(s) g_j(\phi_j(s - \tau_{ij}(s)))
\end{aligned}$$

$$
\begin{aligned}
& + \sum_{j=1}^{n} d_{ij}(s) \int_{s-\delta_{ij}(s)}^{s} h_j(\phi_j(v))dv \Big] ds \Big\|_{\mathbb{O}} \\
\leq\ & \sum_{i=1}^{n} \int_{-\infty}^{t} e^{\int_t^s c_i(v)dv} \Big[c_i(s) \int_{s-\eta_i(s)}^{s} \|\phi_i'(v)\|_{\mathbb{O}} dv \\
& + \sum_{j=1}^{n} \|a_{ij}(s)\|_{\mathbb{O}} \|f_j(\phi_j(s))\|_{\mathbb{O}} + \sum_{j=1}^{n} \|b_{ij}(s)\|_{\mathbb{O}} \\
& \times \|g_j(\phi_j(s-\tau_{ij}(s)))\|_{\mathbb{O}} + \sum_{j=1}^{n} \|d_{ij}(s)\|_{\mathbb{O}} \\
& \times \int_{s-\delta_{ij}(s)}^{s} \|h_j(\phi_j(v))\|_{\mathbb{O}} dv \Big] ds \\
\leq\ & \sum_{i=1}^{n} \int_{-\infty}^{t} e^{\int_t^s c_i(v)dv} \Big[c^+ \eta^+ \|\phi_i'(s)\|_{\mathbb{O}} + \sum_{j=1}^{n} a^+ L_f \\
& \times \|\phi_j(s)\|_{\mathbb{O}} + \sum_{j=1}^{n} b^+ L_g \|\phi_j(s-\tau_{ij}(s))\|_{\mathbb{O}} \\
& + \sum_{j=1}^{n} d^+ \delta^+ L_h \|\phi_j(s)\|_{\mathbb{O}} + \sum_{j=1}^{n} a^+ M_f \\
& + \sum_{j=1}^{n} b^+ M_g + \sum_{j=1}^{n} d^+ \delta^+ M_h \Big] ds \\
\leq\ & \int_{-\infty}^{t} e^{\int_t^s c_i(v)dv} \Big[c^+ \eta^+ \|\phi\|_{\mathbb{X}} + \sum_{j=1}^{n} a^+ L_f \|\phi\|_{\mathbb{X}} \\
& + \sum_{j=1}^{n} b^+ L_g \|\phi\|_{\mathbb{X}} + \sum_{j=1}^{n} d^+ \delta^+ L_h \|\phi\|_{\mathbb{X}} + \sum_{j=1}^{n} a^+ \\
& \times M_f + \sum_{j=1}^{n} b^+ M_g + \sum_{j=1}^{n} d^+ \delta^+ M_h \Big] ds \\
\leq\ & \frac{1}{c^-} \Big[c^+ \eta^+ + \sum_{j=1}^{n} a^+ L_f + \sum_{j=1}^{n} b^+ L_g + \sum_{j=1}^{n} d^+ \\
& \times \delta^+ L_h + \sum_{j=1}^{n} \frac{a^+ M_f (1-\xi)}{L} + \sum_{j=1}^{n} \frac{b^+ M_g (1-\xi)}{L} \\
& + \sum_{j=1}^{n} \frac{d^+ \delta^+ M_h (1-\xi)}{L} \Big] \frac{L}{1-\xi} \\
\leq\ & \frac{\xi L}{1-\xi},
\end{aligned}
$$

and

$$
\begin{aligned}
& \|((\Psi\phi)(t)-\phi_0(t))'\|_{\mathbb{O}^n} \\
= & \sum_{i=1}^{n}\Bigg\|c_i(t)\int_{t-\eta_i(t)}^{t}\phi_i'(\nu)d\nu+\sum_{j=1}^{n}a_{ij}(t)f_j(\phi_j(t)) \\
& +\sum_{j=1}^{n}b_{ij}(t)g_j(\phi_j(t-\tau_{ij}(t)))+\sum_{j=1}^{n}d_{ij}(t) \\
& \times\int_{t-\delta_{ij}(t)}^{t}h_j(\phi_j(\nu))d\nu-c_i(t)\int_{-\infty}^{t}e^{\int_t^s c_i(\nu)d\nu} \\
& \times\Bigg[c_i(s)\int_{s-\eta_i(s)}^{s}\phi_i'(\nu)d\nu+\sum_{j=1}^{n}a_{ij}(s)f_j(\phi_j(s)) \\
& +\sum_{j=1}^{n}b_{ij}(s)g_j(\phi_j(s-\tau_{ij}(s)))+\sum_{j=1}^{n}d_{ij}(s) \\
& \times\int_{s-\delta_{ij}(s)}^{s}h_j(\phi_j(\nu))d\nu\Bigg]ds\Bigg\|_{\mathbb{O}} \\
\leq & \Bigg[c^+\eta^+ +\sum_{j=1}^{n}a^+L_f+\sum_{j=1}^{n}b^+L_g+\sum_{j=1}^{n}d^+\delta^+L_h \\
& +\sum_{j=1}^{n}\frac{a^+M_f(1-\xi)}{L}+\sum_{j=1}^{n}\frac{b^+M_g(1-\xi)}{L} \\
& +\sum_{j=1}^{n}\frac{d^+\delta^+M_h(1-\xi)}{L}\Bigg]\frac{L}{1-\xi}+\frac{c^+}{c^-}\Bigg[c^+\eta^+ \\
& +\sum_{j=1}^{n}a^+L_f+\sum_{j=1}^{n}b^+L_g+\sum_{j=1}^{n}d^+\delta^+L_h \\
& +\sum_{j=1}^{n}\frac{a^+M_f(1-\xi)}{L}+\sum_{j=1}^{n}\frac{b^+M_g(1-\xi)}{L} \\
& +\sum_{j=1}^{n}\frac{d^+\delta^+M_h(1-\xi)}{L}\Bigg]\frac{L}{1-\xi} \\
\leq & \left(1+\frac{c^+}{c^-}\right)\Bigg[c^+\eta^+ +\sum_{j=1}^{n}a^+L_f+\sum_{j=1}^{n}b^+L_g \\
& +\sum_{j=1}^{n}d^+\delta^+L_h+\sum_{j=1}^{n}\frac{a^+M_f(1-\xi)}{L}+\sum_{j=1}^{n}b^+ \\
& \times\frac{M_g(1-\xi)}{L}+\sum_{j=1}^{n}\frac{d^+\delta^+M_h(1-\xi)}{L}\Bigg]\frac{L}{1-\xi} \\
\leq & \frac{\xi L}{1-\xi}.
\end{aligned}
$$

Hence, we have

$$\|\Psi\phi - \phi_0\|_{\mathbb{X}} \leq \frac{\xi L}{1-\xi},$$

which implies that the mapping Ψ is a self-mapping from $\mathbb{X}^*$ to $\mathbb{X}^*$.

Finally, we show Ψ is a contraction mapping. By Assumption 2 and Assumption 3, for any $\phi, \chi \in \mathbb{X}^*$,

$$\begin{aligned}
&\|(\Psi\phi)(t) - (\Psi\chi)(t)\|_{\mathbb{O}^n} = \sum_{i=1}^{n} \|x_i^{\phi}(t) - x_i^{\chi}(t)\|_{\mathbb{O}} \\
= \ & \sum_{i=1}^{n} \Big\| \int_{-\infty}^{t} e^{\int_t^s c_i(v)dv} \Big[c_i(s) \int_{s-\eta_i(s)}^{s} (\phi_i'(v) - \chi_i'(v)) dv \\
& + \sum_{j=1}^{n} a_{ij}(s) \Big(f_j(\phi_j(s)) - f_j(\chi_j(s)) \Big) + \sum_{j=1}^{n} b_{ij}(s) \\
& \times \Big(g_j(\phi_j(s-\tau_{ij}(s))) - g_j(\chi_j(s-\tau_{ij}(s))) \Big) \\
& + \sum_{j=1}^{n} d_{ij}(s) \int_{s-\delta_{ij}(s)}^{s} \Big(h_j(\phi_j(v)) - h_j(\chi_j(v)) \Big) dv \Big] ds \Big\|_{\mathbb{O}} \\
\leq \ & \sum_{i=1}^{n} \int_{-\infty}^{t} e^{\int_t^s c_i(v)dv} \Big[c^+ \int_{s-\eta_i(s)}^{s} \|(\phi_i'(v) - \chi_i'(v))\|_{\mathbb{O}} dv \\
& + \sum_{j=1}^{n} a^+ \Big\| \Big(f_j(\phi_j(s)) - f_j(\chi_j(s)) \Big) \Big\|_{\mathbb{O}} + \sum_{j=1}^{n} b^+ \\
& \times \Big\| \Big(g_j(\phi_j(s-\tau_{ij}(s))) - g_j(\chi_j(s-\tau_{ij}(s))) \Big) \Big\|_{\mathbb{O}} \\
& + \sum_{j=1}^{n} d^+ \int_{s-\delta_{ij}(s)}^{s} \Big\| \Big(h_j(\phi_j(v)) - h_j(\chi_j(v)) \Big) \Big\|_{\mathbb{O}} dv \Big] ds \\
\leq \ & \sum_{i=1}^{n} \int_{-\infty}^{t} e^{\int_t^s c_i(v)dv} \Big[c^+ \eta^+ \|(\phi_i'(v) - \chi_i'(v))\|_{\mathbb{O}} \\
& + \sum_{j=1}^{n} a^+ L_f \|\phi_j(s) - \chi_j(s)\|_{\mathbb{O}} + \sum_{j=1}^{n} b^+ L_g \\
& \times \|\phi_j(s-\tau_{ij}(s)) - \chi_j(s-\tau_{ij}(s))\|_{\mathbb{O}} \\
& + \sum_{j=1}^{n} d^+ \delta^+ L_h \|\phi_j(v) - \chi_j(v)\|_{\mathbb{O}} \Big] ds \\
\leq \ & \int_{-\infty}^{t} e^{\int_t^s c_i(v)dv} \Big[c^+ \eta^+ + \sum_{j=1}^{n} a^+ L_f + \sum_{j=1}^{n} b^+ L_g \\
& + \sum_{j=1}^{n} d^+ \delta^+ L_h \Big] \|\phi - \chi\|_{\mathbb{X}} ds
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{c^-}\Big[c^+\eta^+ + \sum_{j=1}^n a^+L_f + \sum_{j=1}^n b^+L_g \\
&\quad + \sum_{j=1}^n d^+\delta^+L_h\Big]\|\phi-\chi\|_{\mathbb{X}} \\
&\leq \xi\|\phi-\chi\|_{\mathbb{X}},
\end{aligned}
$$

and

$$
\begin{aligned}
&\|((\Psi\phi)(t)-(\Psi\chi)(t))'\|_{\mathbb{O}^n} = \sum_{i=1}^n \Big\|(x_i^\phi(t)-x_i^\chi(t))'\Big\|_{\mathbb{O}} \\
= &\sum_{i=1}^n \Big\| c_i(t)\int_{t-\eta_i(t)}^t (\phi_i'(\nu)-\chi_i'(\nu))d\nu + \sum_{j=1}^n a_{ij}(t) \\
&\times\Big(f_j(\phi_j(t))-f_j(\chi_j(t))\Big) + \sum_{j=1}^n b_{ij}(t) \\
&\times\Big(g_j(\phi_j(t-\tau_{ij}(t)))-g_j(\chi_j(t-\tau_{ij}(t)))\Big) \\
&+\sum_{j=1}^n d_{ij}(t)\int_{t-\delta_{ij}(t)}^t \Big(h_j(\phi_j(\nu))-h_j(\chi_j(\nu))\Big)d\nu \\
&-c_i(t)\int_{-\infty}^t e^{\int_t^s c_i(\nu)d\nu}\Big[c_i(s)\int_{s-\eta_i(s)}^s (\phi_i'(\nu)-\chi_i'(\nu))d\nu \\
&+\sum_{j=1}^n a_{ij}(s)\Big(f_j(\phi_j(s))-f_j(\chi_j(s))\Big) + \sum_{j=1}^n b_{ij}(s) \\
&\times\Big(g_j(\phi_j(s-\tau_{ij}(s)))-g_j(\chi_j(s-\tau_{ij}(s)))\Big) \\
&+\sum_{j=1}^n d_{ij}(s)\int_{s-\delta_{ij}(s)}^s \Big(h_j(\phi_j(\nu))-h_j(\chi_j(\nu))\Big)d\nu\Big]ds\Big\|_{\mathbb{O}} \\
\leq &\Big(1+\frac{c^+}{c^-}\Big)\Big[c^+\eta^+ + \sum_{j=1}^n a^+L_f + \sum_{j=1}^n b^+L_g \\
&+\sum_{j=1}^n d^+\delta^+L_h\Big]\|\phi-\chi\|_{\mathbb{X}} \\
\leq &\ \xi\|\phi-\chi\|_{\mathbb{X}}.
\end{aligned}
$$

Hence, we have

$$\|\Psi\phi-\Psi\chi\|_{\mathbb{X}} \leq \xi\|\phi-\chi\|_{\mathbb{X}},$$

which implies that Ψ is a contraction mapping.

Therefore, by Banach fixed point theorem, system (1.1) has a unique weighted pseudo almost periodic solution. The proof is completed. □

Remark 3.1. *Compared with literature [26–30], this paper discusses the existence of weighted pseudo almost periodic solutions for octonion-valued neural networks with mixed time-varying delays and leakage delays via the non-decomposition method. Therefore, the results are less conservative and more general.*

Theorem 3.2. *Assume that Assumptions 1–3 hold. If the following condition is satisfied:*

- *Assumption 4: There exists a positive constant λ such that*

$$0 < \max\left\{\frac{\Pi}{c^- - \lambda}, \left(1 + \frac{c^+}{c^- - \lambda}\right)\Pi\right\} < 1,$$

where

$$\Pi := c^+\eta^+ + \sum_{j=1}^{n} a^+ L_f + \sum_{j=1}^{n} b^+ L_g + \sum_{j=1}^{n} d^+\delta^+ L_h.$$

Then system (1.1) *has a unique weighted pseudo almost periodic solution that is globally exponentially stable.*

Proof. By Theorem 3.1, system (1.1) has at least a weighted pseudo almost periodic solution. Let $x(t)$ be a weighted pseudo almost periodic solution of system (1.1) with the initial value $\varphi(t)$ and $y(t)$ be an arbitrary solution of system (1.1) with the initial value $\psi(t)$. Set $z(t) = \left(z_1(t), z_2(t), \cdots, z_n(t)\right)^T$, where $z_i(t) = x_i(t) - y_i(t)$ with the initial condition:

$$\phi_i(s) = \varphi_i(s) - \psi_i(s), \quad s \in [-\theta, 0], \tag{3.4}$$

where $i = 1, 2, \ldots, n$.

Let $M = \min\left\{c^-, \left(1 + \frac{c^+}{c^-}\right)^{-1}\right\}\Pi^{-1}$, by Assumption 4, we have $M > 1$,

$$\frac{1}{M} \leq \frac{1}{c^- - \lambda}\left(c^+\eta^+ + \sum_{j=1}^{n} a^+ L_f + \sum_{j=1}^{n} b^+ L_g + \sum_{j=1}^{n} d^+\delta^+ L_h\right),$$

and

$$\frac{1}{M} \leq \left(1 + \frac{c^+}{c^- - \lambda}\right)\left(c^+\eta^+ + \sum_{j=1}^{n} a^+ L_f + \sum_{j=1}^{n} b^+ L_g + \sum_{j=1}^{n} d^+\delta^+ L_h\right).$$

For any $t \geq 0$, we have that

$$\begin{aligned} z_i'(t) \quad = \quad & -c_i(t)z_i(t) + c_i(t)\int_{t-\eta_i(t)}^{t} z_i'(s)ds + \sum_{j=1}^{n} a_{ij}(t) \\ & \times\Big(f_j(x_j(t)) - f_j(y_j(t))\Big) + \sum_{j=1}^{n} b_{ij}(t) \\ & \times\Big(g_j(x_j(t - \tau_{ij}(t))) - g_j(y_j(t - \tau_{ij}(t)))\Big) \\ & + \sum_{j=1}^{n} d_{ij}(t)\int_{t-\delta_{ij}(t)}^{t}\Big(h_j(x_j(s)) - h_j(y_j(s))\Big)ds. \end{aligned} \tag{3.5}$$

Multiplying both sides of (3.5) by $e^{\int_0^t c_i(\xi)d\xi}$ and integrating on $[0,t]$, we have

$$\begin{aligned} z_i(t) &= \phi_i(0)e^{-\int_0^t c_i(\xi)d\xi} + \int_0^t e^{-\int_s^t c_i(\xi)d\xi}\Big[c_i(s) \\ &\quad \times \int_{s-\eta_i(s)}^s z_i'(u)du + \sum_{j=1}^n a_{ij}(s)\Big(f_j(x_j(s)) \\ &\quad - f_j(y_j(s))\Big) + \sum_{j=1}^n b_{ij}(s)\Big(g_j(x_j(s-\tau_{ij}(s))) \\ &\quad - g_j(y_j(s-\tau_{ij}(s)))\Big) + \sum_{j=1}^n d_{ij}(s) \\ &\quad \times \int_{s-\delta_{ij}(s)}^s \Big(h_j(x_j(u)) - h_j(y_j(u))\Big)du\Big]ds, \end{aligned}$$

where $i = 1, 2, \ldots, n$.

It is easy to see that

$$\|z(t)\|_{\mathbb{X}} = \|\phi(t)\|_{\mathbb{X}} \leq M\|\phi(t)\|_{\mathbb{X}} e^{-\lambda t}, \ \ t \in [-\theta, 0].$$

We claim that

$$\|z(t)\|_{\mathbb{X}} \leq M\|\phi(t)\|_{\mathbb{X}} e^{-\lambda t}, \ \ t \in [0, +\infty). \tag{3.6}$$

To prove (3.6) holds, we show that for any $\epsilon > 1$, the following inequality holds

$$\|z(t)\|_{\mathbb{X}} < \epsilon M\|\phi(t)\|_{\mathbb{X}} e^{-\lambda t}, \ \ t > 0. \tag{3.7}$$

If it is not true, then there must be some $t_1 > 0$ such that

$$\begin{aligned} \|z(t_1)\|_{\mathbb{X}} &= \max\{\|z(t_1)\|_{\mathbb{O}^n}, \|z'(t_1)\|_{\mathbb{O}^n}\} \\ &= \epsilon M\|\phi(t_1)\|_{\mathbb{X}} e^{-\lambda t_1} \end{aligned} \tag{3.8}$$

and

$$\|z(t)\|_{\mathbb{X}} < \epsilon M\|\phi(t)\|_{\mathbb{X}} e^{-\lambda t}, \ \ t \in [-\theta, t_1).$$

Hence, we have that

$$\begin{aligned} & \|z(t_1)\|_{\mathbb{O}^n} = \max_{1\leq i\leq n}\{\|z_i(t_1)\|_{\mathbb{O}}\} \\ \leq\ & \|\phi\|_{\mathbb{X}} e^{-c^- t_1} + \epsilon M\|\phi\|_{\mathbb{X}} \int_0^{t_1} e^{-(t_1-s)c^-} \\ & \times\Big[c^+\eta^+ + \sum_{j=1}^n a^+ L_f + \sum_{j=1}^n b^+ L_g e^{\lambda\tau} \end{aligned}$$

$$
\begin{aligned}
& +\sum_{j=1}^{n} d^{+}\delta^{+}L_{h}\Big]e^{-\lambda s}ds \\
\leq\ & \epsilon M\|\phi\|_{\mathbb{X}}e^{-\lambda t_1}\Big[\frac{e^{(\lambda-c^{-})t_1}}{\epsilon M}+\frac{1}{c^{-}-\lambda}\Big(c^{+}\eta^{+} \\
& +\sum_{j=1}^{n} a^{+}L_{f}+\sum_{j=1}^{n} b^{+}L_{g}e^{\lambda\tau}+\sum_{j=1}^{n} d^{+}\delta^{+}L_{h}\Big) \\
& \times(1-e^{(\lambda-\underline{c})t_1})\Big] \\
\leq\ & \epsilon M\|\phi\|_{\mathbb{X}}c^{-}e^{-\lambda t_1}\Big[e^{(\lambda-c^{-})t_1}\Big(\frac{1}{M}-\frac{1}{c^{-}-\lambda}\Big(c^{+}\eta^{+} \\
& +\sum_{j=1}^{n} a^{+}L_{f}+\sum_{j=1}^{n} b^{+}L_{g}e^{\lambda\tau}+\sum_{j=1}^{n} d^{+}\delta^{+}L_{h}\Big)\Big) \\
& +\frac{1}{c^{-}-\lambda}\Big(c^{+}\eta^{+}+\sum_{j=1}^{n} a^{+}L_{f}+\sum_{j=1}^{n} b^{+}L_{g}e^{\lambda\tau} \\
& +\sum_{j=1}^{n} d^{+}\delta^{+}L_{h}\Big)\Big] \\
\leq\ & \epsilon M\|\phi\|_{\mathbb{X}}e^{-\lambda t_1}\Big[\frac{1}{c^{-}-\lambda}\Big(c^{+}\eta^{+}+\sum_{j=1}^{n} a^{+}L_{f} \\
& +\sum_{j=1}^{n} b^{+}L_{g}e^{\lambda\tau}+\sum_{j=1}^{n} d^{+}\delta^{+}L_{h}\Big)\Big] \\
<\ & \epsilon M\|\phi\|_{\mathbb{X}}e^{-\lambda t_1},
\end{aligned}
$$

and

$$
\begin{aligned}
& \|(z(t_1))'\|_{\mathbb{O}^n}=\max_{1\leq i\leq n}\{\|(z_i(t_1))'\|_{\mathbb{O}}\} \\
\leq\ & c^{+}\|\phi\|_{\mathbb{X}}e^{-c^{-}t_1}+\epsilon M\|\phi\|_{\mathbb{X}}e^{-\lambda t_1}\Big[c^{+}\eta^{+} \\
& +\sum_{j=1}^{n} a^{+}L_{f}+\sum_{j=1}^{n} b^{+}L_{g}e^{\lambda\tau}+\sum_{j=1}^{n} d^{+}\delta^{+}L_{h}\Big] \\
& +\epsilon M\|\phi\|_{\mathbb{X}}\int_{0}^{t_1} c^{+}e^{-(t_1-s)c^{-}}\Big[c^{+}\eta+\sum_{j=1}^{n} a^{+}L_{f} \\
& +\sum_{j=1}^{n} b^{+}L_{g}e^{\lambda\tau}+\sum_{j=1}^{n} d^{+}\delta^{+}L_{h}\Big]e^{-\lambda s}ds \\
\leq\ & \epsilon M\|\phi\|_{\mathbb{X}}e^{-\lambda t_1}\Big[^{+}\eta^{+}+\sum_{j=1}^{n} a^{+}L_{f}+\sum_{j=1}^{n} b^{+}L_{g}e^{\lambda\tau}
\end{aligned}
$$

$$
\begin{aligned}
& +\sum_{j=1}^{n} d^+\delta^+L_h\Big] + \epsilon M\|\phi\|_{\mathbb{X}} e^{-\lambda t_1}\Big[\frac{c^+e^{(\lambda-c^-)t_1}}{M} + \frac{c^+}{c^- - \lambda} \\
& \times\Big(c^+\eta^+ + \sum_{j=1}^{n} a^+L_f + \sum_{j=1}^{n} b^+L_g e^{\lambda\tau} + \sum_{j=1}^{n} d^+\delta^+L_h\Big) \\
& \times(1-e^{(\lambda-c^-)t_1})\Big] \\
\leq\ & \epsilon M\|\phi\|_{\mathbb{X}} e^{-\lambda t_1}\Big[\Big(1+\frac{c^+}{c^- - \lambda}\Big)\Big(c^+\eta^+ + \sum_{j=1}^{n} a^+L_f \\
& +\sum_{j=1}^{n} b^+L_g e^{\lambda\tau} + \sum_{j=1}^{n} d^+\delta^+L_h\Big)\Big] \\
<\ & \epsilon M\|\phi\|_{\mathbb{X}} e^{-\lambda t_1}.
\end{aligned}
$$

Hence, we have

$$
\|z(t_1)\|_{\mathbb{X}} < \epsilon M\|\phi(t_1)\|_{\mathbb{X}} e^{-\lambda t_1},
$$

which contradicts the equality (3.8), and so (3.7) holds. Letting $\epsilon \longrightarrow 1$, then (3.6) holds.

Therefore, by Definition 2.4, the weighted pseudo almost periodic solution of system (1.1) is globally exponentially stable. The proof is completed. □

Remark 3.2. *In [26–30], some authors have shown stability of octonion-valued neural networks by using the Lyapunov function method. However, unlike the method of the above literature, we obtain the global exponential stability of weighted pseudo almost periodic solutions for octonion-valued neural networks with leakage delays and mixed delays by using the proof by contradiction.*

4. Illustrative example

In this section, we give one example to show the feasibility and effectiveness of main results.

Example 4.1. *Consider the following delayed octonion-valued neural networks with two neurons:*

$$
\begin{aligned}
x_i'(t) &= -c_i(t)x_i(t-\eta_i(t)) + \sum_{j=1}^{2} a_{ij}(t)f_j(x_j(t)) \\
&\quad + \sum_{j=1}^{2} b_{ij}(t)g_j\big(x_j(t-\tau_{ij}(t))\big) \\
&\quad + \sum_{j=1}^{n} d_{ij}(t)\int_{t-\delta_{ij}(t)}^{t} h_j(x_j(s))ds + I_i(t), \qquad (4.1)
\end{aligned}
$$

where $i = 1, 2$, $c_1(t) = 1.5 + 0.3\sin\sqrt{2}t$, $c_2(t) = 1.4 + 0.2\cos\sqrt{5}t$, $\eta_1(t) = \eta_2(t) = 0.02\sin 2t$, $\tau_{ij}(t) = \frac{1}{2}\sin\sqrt{5}t$, $\delta_{ij}(t) = 0.03\cos\sqrt{7}t$, *and*

$$a_{11}(t) = 0.1(\sqrt{6}\cos t, 2\sin\sqrt{2}t, 0, \sin t, \cos\sqrt{3}t, \sqrt{3}\cos t, 0, 2\sin t)^T,$$
$$a_{12}(t) = 0.1(0, \sqrt{2}\sin t, \sqrt{5}\cos t, \sin\sqrt{5}t, 2\cos t, 0, \sqrt{2}\cos t, \sin\sqrt{7}t)^T,$$
$$a_{21}(t) = 0.1(\sin 2t, \sqrt{2}\cos t, 0, \sin\sqrt{6}t, \sin\sqrt{2}t, 0, 2\sin\sqrt{3}t, \sqrt{3}\cos t)^T,$$
$$a_{22}(t) = 0.1(2\cos\sqrt{2}t, \sin\sqrt{3}t, 0, \sqrt{3}\sin\sqrt{2}t, \sqrt{6}\cos 2t, 0, \sin\sqrt{5}t, \sqrt{2}\cos 2t)^T,$$
$$b_{11}(t) = 0.1(0, \sqrt{2}\sin t, \sin\sqrt{7}t, \sqrt{5}\cos 2t, 0, \sin t, \sqrt{2}\cos\sqrt{3}t, \sqrt{3}\sin\sqrt{2}t)^T,$$
$$b_{12}(t) = 0.1(2\cos t, \sin\sqrt{5}t, 0, \sqrt{3}\cos 2t, \sqrt{2}\cos\sqrt{3}t, \sin\sqrt{6}t, \sqrt{3}\sin t, 0)^T,$$
$$b_{21}(t) = 0.1(\sin 2t, 0, \sqrt{2}\sin t, \sqrt{5}\cos t, \sin\sqrt{7}t, 2\cos\sqrt{3}t, 0, \sqrt{3}\cos\sqrt{6}t)^T,$$
$$b_{22}(t) = 0.1(\sqrt{2}\cos\sqrt{6}t, \sqrt{3}\cos 2t, 0, \sin\sqrt{5}t, \sqrt{5}\cos 2t, \sin\sqrt{2}t, 0, 2\sin\sqrt{3}t)^T,$$
$$d_{11}(t) = 0.1(\sqrt{3}\sin 2t, 0, \sin\sqrt{6}t, 0, 2\cos\sqrt{2}t, \cos\sqrt{5}t, \sqrt{2}\sin t, \sqrt{3}\cos 2t)^T,$$
$$d_{12}(t) = 0.1(\cos\sqrt{5}t, \sin 2t, 0, \sqrt{3}\cos\sqrt{6}t, \sqrt{2}\sin 3t, \cos t, 0, \sqrt{2}\sin\sqrt{3}t)^T,$$
$$d_{21}(t) = 0.1(\sqrt{6}\sin 2t, \sqrt{3}\sin 3t, 0, \cos\sqrt{5}t, \sqrt{2}\sin 2t, \cos\sqrt{7}t, \sqrt{2}\sin 3t, 0)^T,$$
$$d_{22}(t) = 0.1(0, \sqrt{3}\cos 3t, \cos t, \cos\sqrt{5}t, 0, \sqrt{3}\sin 2t, \sqrt{2}\cos t, 2\sin\sqrt{5}t)^T,$$
$$I_1(t) = \frac{1}{40}(\sin\sqrt{2}t, \cos 3t, 3\sin t, \cos\sqrt{5}t, \cos\sqrt{3}t, \sin 2t, \sin t, \sin\sqrt{6}t)^T,$$
$$I_2(t) = \frac{1}{35}(\cos\sqrt{3}t, \sin 2t, \sin\sqrt{6}t, \cos 3t, \cos\sqrt{5}t, \sin\sqrt{7}t, \cos 3t, \cos 4t)^T,$$
$$[f_j(x_j)]_p = \frac{1}{60}\sin([x_j]_p),\ \ [g_j(x_j)]_p = \frac{1}{50}\tanh([x_j]_p),$$
$$[h_j(x_j)]_p = \frac{1}{45}\cos([x_j]_p).$$

Let $\lambda = 0.5$, $\xi = 0.8$, *and by calculating, we have*

$$c^+ = 1.8,\ \ c^- = 1.2,\ \ \tau = \frac{1}{2},\ \ \eta^+ = 0.02,\ \ \delta^+ = 0.03,$$
$$a^+ = \frac{\sqrt{19}}{10},\ \ b^+ = \frac{2}{5},\ \ d^+ = \frac{\sqrt{15}}{10},\ \ M_f = \frac{\sqrt{2}}{30},\ \ M_g = \frac{\sqrt{2}}{25},$$
$$M_h = \frac{2\sqrt{2}}{45},\ \ L_f = \frac{1}{60},\ \ L_g = \frac{1}{50},\ \ L_h = \frac{1}{45},\ \ L = \frac{1}{10}.$$
$$0 < \max\left\{\frac{\Theta}{c^-}, \left(1 + \frac{c^+}{c^-}\right)\Theta\right\} \approx 0.6183 < \xi < 1,$$
$$0 < \max\left\{\frac{\Pi}{c^- - \lambda}, \left(1 + \frac{c^+}{c^- - \lambda}\right)\Pi\right\} \approx 0.2395 < 1,$$

where

$$\begin{aligned}\Theta \; &:= \; c^+\eta^+ + \sum_{j=1}^{n} a^+L_f + \sum_{j=1}^{n} b^+L_g + \sum_{j=1}^{n} d^+\delta^+L_h \\ &\quad + \sum_{j=1}^{n} \frac{a^+M_f(1-\xi)}{L} + \sum_{j=1}^{n} \frac{b^+M_g(1-\xi)}{L} \\ &\quad + \sum_{j=1}^{n} \frac{d^+\delta^+M_h(1-\xi)}{L}, \\ \Pi \; &:= \; c^+\eta^+ + \sum_{j=1}^{n} a^+L_f + \sum_{j=1}^{n} b^+L_g + \sum_{j=1}^{n} d^+\delta^+L_h.\end{aligned}$$

It is not difficult to verify that all conditions Assumptions 1–4 are satisfied. Therefore, by Theorem 1 *and Theorem* 2*, system* (4.1) *has a unique weighted pseudo almost periodic solution that is globally exponentially stable (see Figures 1–4).*

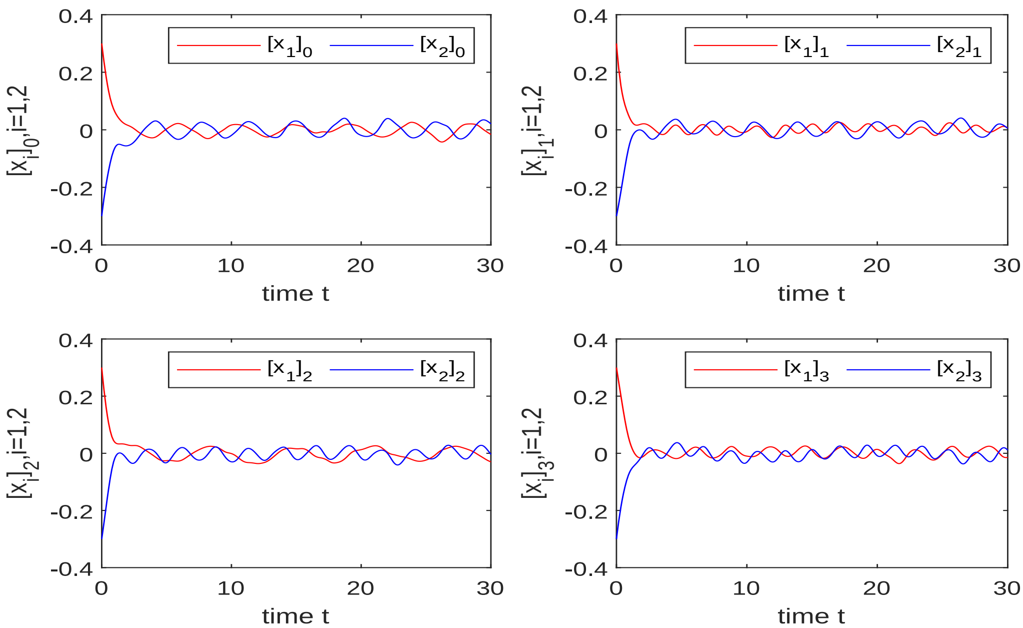

Figure 1. Transient states of the solutions $([x_1]_p, [x_2]_p)^T$ with the initial value $(0.3, -0.3)^T$, where $p = 0, 1, 2, 3$.

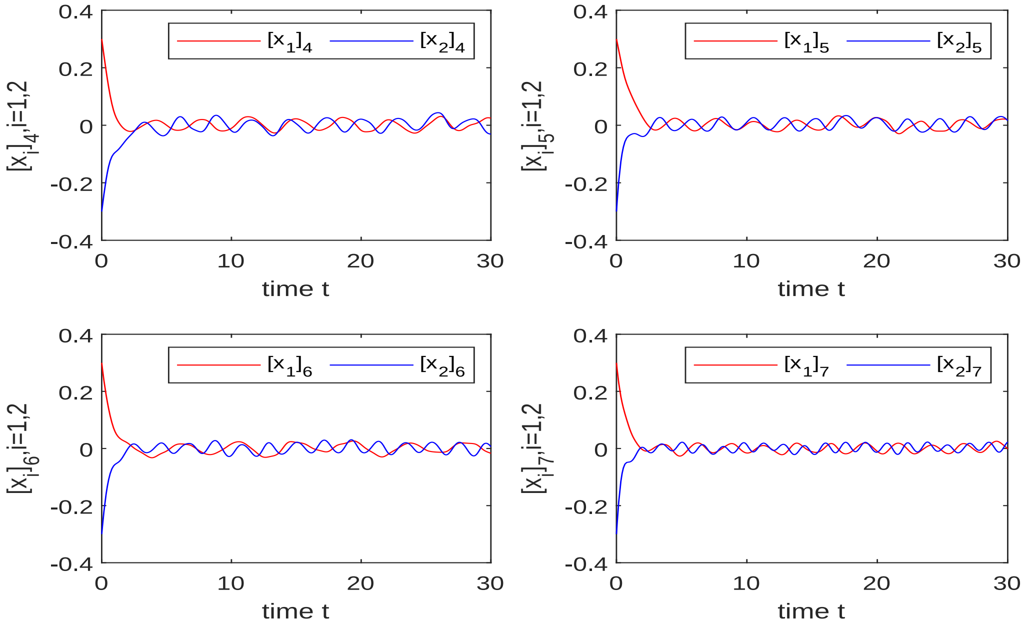

Figure 2. Transient states of the solutions $([x_1]_p, [x_2]_p)^T$ with the initial value $(0.3, -0.3)^T$, where $p = 4, 5, 6, 7$.

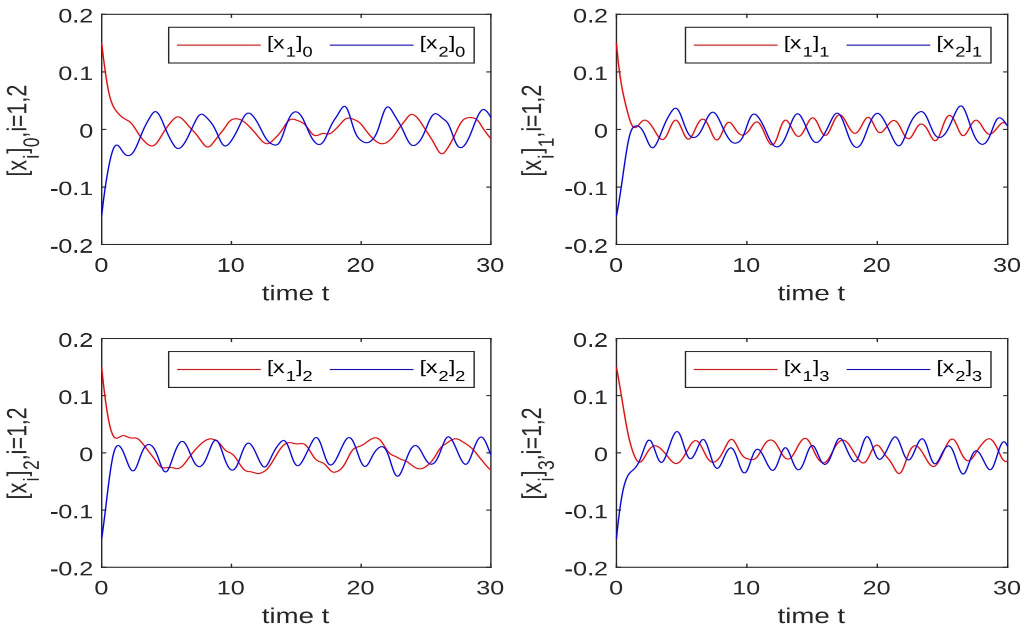

Figure 3. Transient states of the solutions $([x_1]_p, [x_2]_p)^T$ with the initial value $(0.15, -0.15)^T$, where $p = 0, 1, 2, 3$.

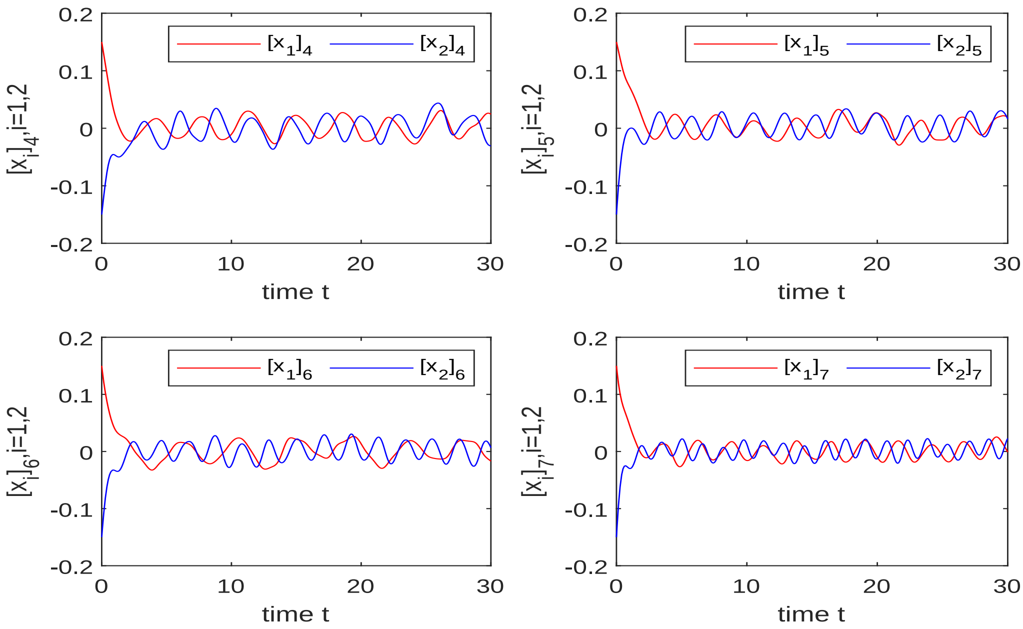

Figure 4. Transient states of the solutions $([x_1]_p, [x_2]_p)^T$ with the initial value $(0.15, -0.15)^T$, where $p = 4, 5, 6, 7$.

5. Conclusions

In this paper, we consider a class of octonion-valued neural networks with leakage delays and mixed delays. By using the Banach fixed point theorem, the proof by contradiction and the direct method, we obtain some sufficient conditions for the existence and global exponential stability of weighted pseudo almost periodic solutions for octonion-valued neural networks. To demonstrate the usefulness of the presented results, some examples are given. Our method can be extended to study the almost periodic solutions or anti-periodic solutions for other types of octonion-valued neural networks.

Meanwhile, future directions will include the study of octonion-valued neural network systems with impulses, reaction-diffusion terms, Markovian jump parameters and so on. We can specifically explore the stability and synchronization of the above systems, which will be a direction worth exploring.

Acknowledgments

This research is supported by the Science Research Fund of Education Department of Yunnan Province of China [grant number 2022J0986] and Youth academic and technical leader of Pu'er College (No. QNRC21-01).

Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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