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Research article

Attractor of the nonclassical diffusion equation with memory on time-dependent space

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Abstract: We consider the dynamic behavior of solutions for a nonclassical diffusion equation with memory

$$u_t - \varepsilon(t)\triangle u_t - \triangle u - \int_0^\infty \kappa(s)\triangle u(t-s)ds + f(u) = g(x)$$

on time-dependent space for which the norm of the space depends on the time t explicitly, and the nonlinear term satisfies the critical growth condition. First, based on the classical Faedo-Galerkin method, we obtain the well-posedness of the solution for the equation. Then, by using the contractive function method and establishing some delicate estimates along the trajectory of the solutions on the time-dependent space, we prove the existence of the time-dependent global attractor for the problem. Due to very general assumptions on memory kernel κ and the effect of time-dependent coefficient $\varepsilon(t)$, our result will include and generalize the existing results of such equations with constant coefficients. It is worth noting that the nonlinear term cannot be treated by the common decomposition techniques, and this paper overcomes the difficulty by dealing with it as a whole.

Keywords: nonclassical diffusion equation; time-dependent coefficient; contractive function; critical growth; time-dependent global attractor
Mathematics Subject Classification: 35B25, 35B40, 35B41, 35K57, 45K05

1. Introduction

In this paper, we are concerned with the following nonclassical diffusion equation with memory:

$$\begin{cases} u_t - \varepsilon(t)\triangle u_t - \triangle u - \int_0^\infty \kappa(s)\triangle u(t-s)ds + f(u) = g(x), & x \in \Omega,\ t \geq \tau, \\ u|_{\partial\Omega} = 0, \quad t \geq \tau, \\ u(x,t) = u_\tau(x), \quad x \in \Omega,\ t \leq \tau, \tau \in \mathbb{R}, \end{cases} \tag{1.1}$$

on time-dependent space, where $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) is a bounded smooth domain. The model (1.1) describes diffusion in solids of substances which behave as viscous fluid, where unknown function $u = u(x,t) : \Omega \times [\tau, \infty) \to \mathbb{R}$ represents the density of the fluid, $\tau \in \mathbb{R}$ is an initial time, $u_\tau(x, r)$: $\Omega \times (-\infty, \tau]$ is the initial value function that characterizes the past time, $g = g(\cdot) \in H^{-1}(\Omega)$ represents a forcing term, κ is a nonnegative non-increasing function describing memory effects in the material, the term $-\triangle u$ concerns linear diffusion processes, the term $-\varepsilon(t)\triangle u_t$ is used to model the effect of viscosity of the diffusing substance, and $\varepsilon(t)$ can be interpreted as the coefficient of viscosity. In addition, $\varepsilon(t) \in C^1(\mathbb{R})$ is a decreasing bounded function satisfying

$$\lim_{t\to+\infty} \varepsilon(t) = 0, \tag{1.2}$$

and there is a constant $L > 0$ such that

$$\sup_{t\in\mathbb{R}}(|\varepsilon(t)| + |\varepsilon'(t)|) \leq L. \tag{1.3}$$

The nonlinear function $f \in C^1(\mathbb{R})$ with $f(0) = 0$ satisfies the critical growth condition

$$|f'(s)| \leq C(1 + |s|^{\frac{4}{N-2}}), \;\; \forall\, s \in \mathbb{R},\; N \geq 3, \tag{1.4}$$

and the dissipation condition

$$\liminf_{|s|\to\infty} \frac{f(s)}{s} > -\lambda_1, \;\; \forall\, s \in \mathbb{R}, \tag{1.5}$$

where C is a positive constant and λ_1 is the first eigenvalue of $-\triangle$ with Dirichlet boundary condition in $H_0^1(\Omega)$.

The memory kernel κ is a nonnegative summable function satisfying $\int_0^\infty \kappa(s)ds = 1$ and having the following form:

$$\kappa(s) = \int_s^\infty \mu(r)dr, \tag{1.6}$$

where $\mu \in L^1(\mathbb{R}^+)$ is a decreasing piecewise absolutely continuous function and is allowed to have infinitely many discontinuity points. We assume

$$\kappa(s) \leq \Theta\mu(s), \;\; \forall\, s \in \mathbb{R}^+,\; \Theta > 0. \tag{1.7}$$

From [19], the above inequality (1.7) is equivalent to the following:

$$\mu(r + s) \leq Me^{-\delta r}\mu(s), \tag{1.8}$$

where $M \geq 1, \delta > 0$ and $r \geq 0$ are constants. As is well known, the nonclassical diffusion equation is an important mathematical model used to describe several physical phenomena, such as heat conduction, solid mechanics and non-Newtonian flows. In 1980, Aifantis [1] came up with a quite general approach for establishing such partial differential equation models describing different physical phenomena related to diffusion in solids. Among them, the author proposed a pseudo-parabolic equation

$$\varrho_t = D\nabla^2\varrho + \bar{D}\nabla^2\varrho_t$$

under a phenomenon, which is called the nonclassical diffusion equation, where $\bar{D}$ is a non-negative function. However, the research was focused on the nonclassical diffusion equation with constant coefficient early on. In 1990, the diffusion equation with memory was proposed by Jäckle ([21]) in the study of heat conduction and relaxation of high viscosity liquids. The convolution term represents the influence of past history on its future evolution and describes more accurately the diffusive process in certain materials, such as high viscosity liquids at low temperatures and polymers. Hence, it is necessary and scientifically significant to study the nonclassical diffusion equation with the time-dependent coefficient (i.e., variable coefficient) and memory.

Provided that the function $\varepsilon(t)$ is a positive constant in Eq (1.1), the long-time behavior of solutions for this kind of problem has been widely studied, and a lot of excellent results have been obtained when $\kappa(s) = 0$ or $\kappa(s) \neq 0$. Meanwhile, the research of the nonclassical diffusion equation with memory (i.e., $\kappa(s) \neq 0$) is relatively less. In 2009, Wang and Zhong [41] first considered the nonclassical diffusion equation with memory and applied the condition of memory kernel

$$\mu'(s) + \delta\mu(s) \leq 0, \ \ \delta, s \geq 0, \ \ \mu \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+), \tag{1.9}$$

introduced in [17]. They obtained the existence and regularity of a uniform attractor for the problem (1.1) with critical growth restriction in $H_0^1(\Omega) \times L_\mu^2(\mathbb{R}^+, H_0^1(\Omega))$ ($\Omega \in \mathbb{R}^N,\ N \geq 3$). Since then, the condition (1.9) has been used for this type of problem with memory; see [2–4, 6–10, 40, 42, 45]. Particularly, Wang et al. [42] proved the existence of a global attractor for the problem (1.1) with critical growth restriction in $H_0^1(\Omega) \times L_\mu^2(\mathbb{R}^+, H_0^1(\Omega))$ ($\Omega \in \mathbb{R}^N,\ N \geq 3$). In [3, 6], the authors obtained the existences of the uniform attractor and pullback attractors, respectively. The global attractors were obtained for nonclassical diffusion equations with memory and singularly oscillating external forces in [4, 7]. In [8], Conti et al. proved the existence of the global attractor for the problem (1.1) lacking instantaneous damping. In 2014, Conti et al. [9] applied the memory kernel condition (1.7) rather than (1.9) and proved the existence of a global attractor for the problem (1.1) in $H_0^1(\Omega) \times L_\mu^2(\mathbb{R}^+, H_0^1(\Omega))$ ($\Omega \in \mathbb{R}^3$). In 2016, the authors of [10] also got the existence of the exponential attractor for the above problem. Later, the authors of [2] obtained the existence of the global attractor for the nonclassical diffusion equations with memory and a new class of nonlinearity. It is worth noting that (1.9) is weaker than (1.8), which shows (1.7) is more general in [2, 9, 10]. It is also obvious that (1.8) with $M = 1$ boils down to (1.9). On the contrary, when (1.8) holds for some $M > 1$, then it is far more general that (1.9). In fact, any compactly supported decreasing function μ satisfies the condition (1.8) for some $M > 1$, but it does not satisfy (1.9).

When $\varepsilon(t)$ is a decreasing function and satisfies (1.2)–(1.3), the equation

$$u_t - \varepsilon(t)\triangle u_t - \triangle u + \lambda u + f(u) = g(x) \tag{1.10}$$

has been investigated by some authors. The characteristic of this kind of problem is that the phase space depends on time, that is, its norm depends on time t explicitly. So, the problem (1.10) is still non-autonomous although the forcing term g is independent of t. If not, the time-dependent coefficient leads to the loss of the dissipation of the natural energy as $t \to \pm\infty$, which affects the existence of an absorbing set in the general sense. Thereby, in order to overcome the above difficulty, the relevant definitions and theories of the time-dependent global attractor first were introduced in [35]. Then, Conti et al. [11] improved the work in [35] and proved the existence of the time-dependent global attractor for a wave equation. Moreover, Meng et al. [27, 28] obtained a new method (i.e., contractive function method) to verify compactness and also got a necessary and sufficient condition for the existence of time-dependent attractors. In recent years, some researchers have extensively studied this type of problem with time-dependent coefficient. In [12,29,36], the authors obtained the corresponding results for wave equations. Liu and Ma [23,24] proved the corresponding results of the time-dependent global attractor for plate equations on the bounded and unbounded domain, respectively. In [30, 39], the authors investigated the beam equation and a nonlinear viscoelastic equation with time-dependent memory kernel on time-dependent spaces, respectively. In addition, it should be mentioned that there is a class of studies on variable coefficient equations devoted to studying different solutions or dynamical behavior using numerical simulation methods, which makes the problem more intuitive by graphs. For example, the authors [22, 26, 32, 33] considered, respectively, multiple soliton and M-lump solutions of the variable coefficients Kadomtsev-Petviashvili equation, nonlinear dynamics of different non-autonomous wave structures solutions for a 3D variable-coefficient generalized shallow water wave equation, the stability of the corresponding dynamical system and invariant solutions for a (2+1)-dimensional Kadomtsev-Petviashvili equation with competing dispersion effect and novel multiple soliton solutions of (3+1)-dimensional generalized variable-coefficient B-type Kadomtsev-Petviashvili (VC B-type KP) equation. In [5, 18], the authors studied the soliton solutions of the nonlinear diffusive predator-prey system and the diffusion-reaction equations and the Tikhonov regularization method and the inverse source problem for time fractional heat equation, respectively. In [34], various solitons and solutions of the fractional fifth-order Korteweg-de Vries equations were realized. These articles mentioned above, further explained the solutions and dynamics of systems by depicting graphs. In short, it can be seen that the study of variable coefficient partial differential equations has attracted much attention in various fields.

Compared to the studies of the aforementioned other equations, the relevant results of the nonclassical diffusion equations on time-dependent spaces are not abundant. When the forcing term $g \in L^2(\Omega)$ ($\Omega \subset \mathbb{R}^3$) and the nonlinearity $f(u)$ satisfies $|f'(u)| \leq C(1+|u|)$, Ding and Liu [16] recognized the existence of a time-dependent global attractor for (1.10) by using the decomposition technique. Using the same method, Ma et al. [31] proved the existence, regularity and asymptotic structure of the time-dependent global attractor for (1.10), when the forcing term $g \in H^{-1}(\Omega)$ ($\Omega \subset \mathbb{R}^N, N \geq 3$) and the nonlinear term $f(u)$ satisfies the critical exponential growth condition. By applying the contractive function method, the authors of [43, 46] recognized almost simultaneously the existence of the time-dependent global attractor for (1.10), when the nonlinearity $f(u)$ satisfies a polynomial growth condition of arbitrary order. However, the authors also proved the regularity and asymptotic structure of the time-dependent global attractor in [43]. In [37], Qin and Yang proved the existence and regularity of time-dependent pullback attractors for non-autonomous nonclassical diffusion equations with nonlocal diffusion when the nonlinear term satisfies critical exponential growth and the external force term $g \in L^2_{loc}(\mathbb{R}, H^{-1}(\Omega))$. Recently, Xie et al. [44] recognized the existence and regularity of time-dependent pullback global attractors for the problem (1.10) with memory and lacking instantaneous damping by using the contractive process and new analytical technique, when the nonlinearity satisfies a polynomial growth condition of arbitrary order and the external force term $g \in L^2(\Omega)$.

However, we find that the studies of nonclassical diffusion equations with memory are extremely rare on time-dependent spaces. In this paper, based on the idea of [10, 27, 31] and using the theory framework provided in [27], we discuss the existence of the time-dependent global attractor in $\mathscr{U}_t$ for problem (1.1). When the nonlinearity and the external force term satisfy the same condition, our result will generalize the result obtained in [42] because of the generality of memory kernel condition and the time-dependent nature of ε.

In order to obtain the corresponding results for problem (1.1), we need to overcome two difficulties. On the one hand, the weaker memory kernel condition (1.7) makes the energy functional obtained unavailable. On the other hand, due to the influence of the nonlinear term with critical growth condition and the term $-\varepsilon(t)\triangle u_t$, it is not easy to get relative compactness of the solution in $L^2([\tau, T], H_0^1(\Omega))$ when we prove the asymptotic compactness by the contractive function method. To deal with above problems, we construct a new energy functional by introducing a new function related to the memory kernel, and obtain the existence of the time-dependent absorbing set. Then, when applying the contractive function method, we treat the nonlinear term as a whole, which will yield the asymptotic compactness for the corresponding process of the problem (1.1).

The paper is organized as follows. In Section 2, we introduce notations of function spaces involved, some abstract results for the time-dependent global attractor and important lemmas. In Section 3, we will prove the well-posedness of the solution. Based on the existence of the solution, we obtain the process generated by the weak solution. In Section 4, we investigate the existence of the time-dependent global attractor. In Section 5, conclusions and discussion are given.

2. Preliminaries

As in [17], we introduce a new variable which shows the past history of Eq (1.1), that is,

$$\eta^t(x,s) = \eta(x,t,s) = \int_0^s u(x,t-r)dr, \quad s \geq 0, \tag{2.1}$$

and

$$\eta_t^t(x,s) = u(x,t) - \eta_s^t(x,s), \quad s \geq 0, \tag{2.2}$$

where $\eta_t = \frac{\partial}{\partial t}\eta$, $\eta_s = \frac{\partial}{\partial s}\eta$.

Therefore, according to (1.6), (2.1) and (2.2), the problem (1.1) can be transformed into the following system:

$$\begin{cases} u_t - \varepsilon(t)\triangle u_t - \triangle u - \int_0^\infty \mu(s)\triangle\eta^t(s)ds + f(u) = g, \\ \eta_t^t = -\eta_s^t + u, \end{cases} \tag{2.3}$$

with the corresponding initial conditions

$$\begin{cases} u(x,t) = 0, \quad x \in \partial\Omega, \quad t \geq \tau, \\ \eta^t(x,s) = 0, \quad (x,s) \in \partial\Omega \times \mathbb{R}^+, \ t \geq \tau, \\ u(x,\tau) = u_\tau(x), \quad x \in \Omega, \ \tau \in \mathbb{R}, \\ \eta(x,\tau,s) = \eta^\tau(x,s) = \int_0^s u_\tau(x,\tau-r)dr, \quad (x,s) \in \Omega \times \mathbb{R}^+, \ \tau \in \mathbb{R}. \end{cases} \tag{2.4}$$

First, we give some spaces and the corresponding norms used in the paper. Usually, let $\|\cdot\|_{L^p(\Omega)}$ be the norm of $L^p(\Omega)$ $(p \geq 1)$. In particularly, let $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ be the scalar product and norm of $\mathrm{H} = L^2(\Omega)$, respectively. The Laplacian $A = -\triangle$ with Dirichlet boundary conditions is a positive operator on H with domain $H^2(\Omega) \cap H_0^1(\Omega)$. Then, we consider the family of Hilbert spaces $\mathrm{H}_s = \mathrm{D}(A^{s/2})$, $\forall s \in \mathbb{R}$, with the standard inner products and norms, respectively,

$$\langle\cdot,\cdot\rangle_s = \langle\cdot,\cdot\rangle_{D(A^{s/2})} = \langle A^{s/2}\cdot, A^{s/2}\cdot\rangle, \quad \|\cdot\|_s = \|A^{s/2}\cdot\|.$$

Especially, $\mathrm{H}_{-1} = H^{-1}(\Omega)$, $\mathrm{H}_0 = \mathrm{H}$, $\mathrm{H}_1 = H_0^1(\Omega)$, $\mathrm{H}_2 = H^2(\Omega) \cap H_0^1(\Omega)$.

Therefore, we define the space $\mathcal{H}_t$ with the time-dependent norm

$$\|u\|^2_{\mathcal{H}_t} = \|u\|^2 + \varepsilon(t)\|u\|_1^2.$$

According to the definition of memory kernel, we introduce the Hilbert (history) space

$$\mathcal{M}^1 = L^2_\mu(\mathbb{R}^+; \mathrm{H}_1) = \{\eta^t : \mathbb{R}^+ \to \mathrm{H}_1 : \int_0^\infty \mu(s)\|\eta^t(s)\|_1 ds < \infty\}$$

with the corresponding inner product and norm

$$\langle \eta^t, \xi^t \rangle_{\mu,1} = \langle \eta^t, \xi^t \rangle_{\mathcal{M}^1} = \int_0^\infty \mu(s)\langle \eta^t(s), \xi^t(s)\rangle_1 ds,$$

$$\|\eta^t\|^2_{\mu,1} = \|\eta^t\|^2_{\mathcal{M}^1} = \int_0^\infty \mu(s)\|\eta^t(s)\|_1^2 ds.$$

Now, combining the above spaces, we have the time-dependent space

$$\mathscr{U}_t = \mathcal{H}_t \times \mathcal{M}^1$$

endowed with the norm

$$\|z\|^2_{\mathscr{U}_t} = \|u\|^2 + \varepsilon(t)\|u\|_1^2 + \|\eta^t\|^2_{\mu,1}.$$

Note that the dual space of X is denoted as X^*. As a convenience, we choose C as a positive constant depending on the subscript that may be different from line to line or in the same line throughout the paper.

Second, we recall some notations, some abstract results and standard conclusions in order to obtain compactness; see [11,13,20,25,27]. For every $t \in \mathbb{R}$, let X_t be a family of normed spaces, and we introduce the $R-$ball of X_t:

$$\mathbb{B}_{X_t}(R) = \{u \in X_t : \|u\|^2_{X_t} \le R\}.$$

Definition 2.1. [11] *Let $\{X_t\}_{t\in\mathbb{R}}$ be a family of normed spaces. A process is a two-parameter family of mappings $\{U(t,\tau) : X_\tau \to X_t,\ t \ge \tau \in \mathbb{R}\}$ with properties*

(i) $U(\tau,\tau) = Id$ is the identity on X_τ, $\tau \in \mathbb{R}$;

(ii) $U(t,s)U(s,\tau) = U(t,\tau)$, $\forall\ t \ge s \ge \tau$.

Definition 2.2. [11] *A family $\mathfrak{D} = \{D_t\}_{t\in\mathbb{R}}$ of bounded sets $D_t \subset X_t$ is called uniformly bounded if there exists a constant $R > 0$ such that $D_t \subset \mathbb{B}_{X_t}(R)$, $\forall\ t \in \mathbb{R}$.*

Definition 2.3. [11] *A time-dependent absorbing set for the process $\{U(t,\tau)\}_{t\ge\tau}$ is a uniformly bounded family $\mathfrak{B} = \{B_t\}_{t\in\mathbb{R}}$ with the following property: For every $R > 0$ there exists a t_0 such that*

$$U(t,\tau)\mathbb{B}_{X_\tau}(R) \subset B_t, \ \textit{for all}\ \tau \le t - t_0.$$

Definition 2.4. [11] *The time-dependent global attractor for $\{U(t,\tau)\}_{t\ge\tau}$ is the smallest family $\mathfrak{A} = \{A_t\}_{t\in\mathbb{R}}$ such that*

(i) *each A_t is compact in X_t;*

(ii) *$\mathfrak{A}$ is pullback attracting, i.e., it is uniformly bounded, and the limit*

$$\lim_{\tau\to-\infty} dist_{X_t}(U(t,\tau)D_\tau, A_t) = 0$$

holds for every uniformly bounded family $\mathfrak{D} = \{D_t\}_{t\in\mathbb{R}}$ *and every fixed* $t \in \mathbb{R}$.

Definition 2.5. [11] *We say* $\mathfrak{A} = \{A_t\}_{t\in\mathbb{R}}$ *is invariant if*

$$U(t,\tau)A_\tau = A_t,\ \forall\ t\ \geq\ \tau.$$

Definition 2.6. [27] *We say that a process* $\{U(t,\tau)\}_{t\geq\tau}$ *in a family of normed spaces* $\{X_t\}_{t\in\mathbb{R}}$ *is pullback asymptotically compact if and only if for any fixed* $t \in \mathbb{R}$, *bounded sequence* $\{x_n\}_{n=1}^\infty \subset X_{\tau_n}$ *and any* $\{\tau_n\}_{n=1}^\infty \subset \mathbb{R}^{-t}$ *with* $\tau_n \to -\infty$ *as* $n \to \infty$, *the sequence* $\{U(t,\tau_n)x_n\}_{n=1}^\infty$ *has a convergent subsequence, where* $\mathbb{R}^{-t} = \{\tau : \tau \in \mathbb{R}, \tau \leq t\}$.

Definition 2.7. [27] *Let* $\{X_t\}_{t\in\mathbb{R}}$ *be a family of Banach spaces and* $\mathfrak{C} = \{C_t\}_{t\in\mathbb{R}}$ *be a family of uniformly bounded subsets of* $\{X_t\}_{t\in\mathbb{R}}$. *We call a function* $\psi_\tau^t(\cdot,\cdot)$ *defined on* $X_t \times X_t$ *a contractive function on* $C_\tau \times C_\tau$ *if for any fixed* $t \in \mathbb{R}$ *and any sequence* $\{x_n\}_{n=1}^\infty \subset C_\tau$, *there is a subsequence* $\{x_{n_k}\}_{n=1}^\infty \subset \{x_n\}_{n=1}^\infty$ *such that*

$$\lim_{k\to\infty}\lim_{l\to\infty}\psi_\tau^t(x_{n_k}, x_{n_l}) = 0.$$

Theorem 2.8. [27] *Let* $\{U(t,\tau)\}_{t\geq\tau}$ *be a process* $\{X_t\}_{t\in\mathbb{R}}$ *and have a pullback absorbing family* $\mathfrak{B} = \{B_t\}_{t\in\mathbb{R}}$. *Moreover, assume that for any* $\epsilon > 0$ *there exist* $T(\epsilon) \leq t$, $\psi_T^t \in \mathfrak{C}(B_T)$ *such that*

$$\|U(t,T)x - U(t,T)y\|_{X_t} \leq \epsilon + \psi_T^t(x,y),\ \forall x,y \in B_T,$$

for any fixed $t \in \mathbb{R}$. *Then,* $\{U(t,\tau)\}_{t\geq\tau}$ *is pullback asymptotically compact.*

Theorem 2.9. [27] *Let* $\{U(t,\tau)\}_{t\geq\tau}$ *be a process in a family of Banach spaces* $\{X_t\}_{t\in\mathbb{R}}$. *Then,* $U(\cdot,\cdot)$ *has a time-dependent global attractor* $\mathcal{A} = \{A_t\}_{t\in\mathbb{R}}$ *satisfying* $A_t = \bigcap_{s\leq t}\overline{\bigcup_{\tau\leq s} U(t,\tau)B_\tau}$ *if and only if*

(i) $\{U(t,\tau)\}_{t\geq\tau}$ *has a pullback absorbing family* $\mathfrak{B} = \{B_t\}_{t\in\mathbb{R}}$;

(ii) $\{U(t,\tau)\}_{t\geq\tau}$ *is pullback asymptotically compact.*

Lemma 2.10. [13,20] *Assume that the memory function* κ *satisfies (1.6)–(1.8), and then for any* $T > \tau$, $\eta^t \in C([\tau,T], L^2_\mu(\mathbb{R}^+;\mathrm{H}_1))$ *such that*

$$\begin{aligned}-\langle\eta_s^t,\eta^t\rangle_{\mu,1} &= -\frac{1}{2}\int_0^\infty \mu(s)\frac{d}{ds}\|\nabla\eta^t(s)\|^2 ds\\ &= \left[-\frac{1}{2}\mu(s)\|\nabla\eta^t(s)\|^2\right]_0^\infty + \frac{1}{2}\int_0^\infty \mu'(s)\|\nabla\eta^t(s)\|^2 ds\\ &\leq 0.\end{aligned} \tag{2.5}$$

Lemma 2.11. [25] *(Aubin-Lions Lemma) Assume that* X, B *and* Y *are three Banach spaces with* $X \hookrightarrow\hookrightarrow B$ *and* $B \hookrightarrow Y$. *Let* f_n *be bounded in* $L^p([0,T],B)$ $(1 \leq p < \infty)$. *Suppose* f_n *satisfies*

(i) f_n *is bounded in* $L^p([0,T],X)$;

(ii) $\frac{\partial f_n}{\partial t}$ *is bounded in* $L^p([0,T],Y)$.

Then, f_n *is relatively compact in* $L^p([0,T],B)$.

3. Well-posedness

Next, we give the definition of a weak solution and prove well-posedness of the weak solution for the problem (2.3)–(2.4) by using the Faedo-Galerkin method from [14, 38].

Definition 3.1. *The function $z = (u, \eta^t) = (u(x,t), \eta^t(x,s))$ defined in $\Omega \times [\tau, T]$ is said to be a weak solution for the problem (2.3)–(2.4) with the initial data $z_\tau \in \mathbb{B}_{\mathscr{U}_\tau}(R_0) \subset \mathscr{U}_\tau$, $-\infty < \tau < T < +\infty$ if z satisfies*

(i) $z \in C([\tau, T], \mathscr{U}_t)$, $(x,t) \in \Omega \times [\tau, T]$;

(ii) for any $\theta = (v, \xi^t) \in \mathcal{H}_t \times L^2_\mu(\mathbb{R}^+; \mathrm{H}_1)$, the equality

$$\langle u_t, v\rangle + \varepsilon(t)\langle \nabla u_t, \nabla v\rangle + \langle \nabla u, \nabla v\rangle + \langle \eta^t, v\rangle_{\mu,1} + \langle f(u), v\rangle = \langle g, v\rangle$$

and

$$\langle \eta^t_t, \xi^t\rangle_{\mu,1} = -\langle \eta^t_s, \xi^t\rangle_{\mu,1} + \langle u, \xi^t\rangle_{\mu,1}$$

hold for a.e. $[\tau, T]$.

Theorem 3.2. *Assume that (1.2)–(1.8) hold and $g \in H^{-1}(\Omega)$, and then for any initial data $z_\tau = (u_\tau, \eta^\tau) \in \mathbb{B}_{\mathscr{U}_\tau}(R_0) \subset \mathscr{U}_\tau$ and any $\tau \in \mathbb{R}$, there exists a unique solution z for the problem (2.3)–(2.4) such that $z = (u, \eta^t) \in C([\tau, T], \mathscr{U}_t)$ for any fixed $T > \tau$. Furthermore, the solution depends on the initial data continuously in $\mathscr{U}_t$.*

Proof. Assume that ω_k is the eigenfunction of $A = -\triangle$ with Dirichlet boundary value in H_1, and then $\{\omega_k\}_{k=1}^\infty$ is a standard orthogonal basis of H and is also an orthogonal basis in H_1. The corresponding eigenvalues are denoted by $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_j \le \cdots$, $\lambda_j \to \infty$ with $A\omega_k = \lambda_k \omega_k$, $\forall k \in \mathbb{N}$. Our proof will be finished through the following four steps.

⋆ **Faedo-Galerkin scheme.**

Given an integer m, we denote by P_m the projection on the subspace $span\{\omega_1, \cdots, \omega_m\}$ in $H^1_0(\Omega)$ and Q_m the projection on the subspace $span\{e_1, \cdots, e_m\} \subset L^2_\mu(\mathbb{R}^+, \mathrm{H}_1)$ in $L^2_\mu(\mathbb{R}^+, \mathrm{H}_1)$. For every fixed m, we look for function $u^m(t) = P_m u = \Sigma_{k=1}^m a_m^k(t)\omega_k$ and $\eta^{t,m}(s) = Q_m \eta^t = \Sigma_{k=1}^m b_m^k(t) e_k(s)$ satisfying the following system:

$$\begin{cases} \langle u_t^m, \omega_k\rangle + \langle \varepsilon(t) A u_t^m, \omega_k\rangle + \langle A u^m, \omega_k\rangle + \langle \eta^{t,m}, \omega_k\rangle_{\mu,1} = \langle g, \omega_k\rangle - \langle f(u^m), \omega_k\rangle, \\ \langle \eta_t^{t,m}, e_k\rangle_{\mu,1} = -\langle \eta_s^{t,m}, e_k\rangle_{\mu,1} + \langle u^m, e_k\rangle_{\mu,1}, \\ z_\tau^m = \langle P_m u_\tau, Q_m \eta^\tau\rangle. \end{cases} \tag{3.1}$$

Applying the divergence theorem to the term $\langle \int_0^\infty \Delta \eta^{t,m} ds, \omega_k\rangle$, we obtain a system of ordinary differential equations in the variables $a_m^k(t)$ and $b_m^k(t)$ of the form

$$\begin{cases} \frac{d}{dt} a_m^j + \lambda_j \varepsilon(t) \frac{d}{dt} a_m^j + \lambda_j a_m^j + \Sigma_{k=1}^m b_m^k \langle e_k, \omega_j\rangle_{\mu,1} = \langle g, \omega_j\rangle - \langle f(u^m), \omega_j\rangle, \\ \frac{d}{dt} b_m^j = \Sigma_{k=1}^m a_m^k \langle \omega_k, e_j\rangle_{\mu,1} - \Sigma_{k=1}^m b_m^k \langle e_k', e_j\rangle_{\mu,1}, \end{cases}$$

with initial conditions

$$a_m^j(\tau) = \langle u_\tau, \omega_j\rangle, \;\; b_m^j(\tau) = \langle \eta^\tau, e_j\rangle_{\mu,1}, \;\; j,k = 0, 2 \cdots, m,$$

which satisfy

$$\Sigma_{k=1}^m a_m^k(\tau)\omega_j \to u_\tau \;\text{ in } \mathcal{H}_t,$$

$$\Sigma_{k=1}^{m} b_m^k(\tau) e_j \to \eta^\tau \ \text{ in } \mathcal{M}^1.$$

Thereby, there exists a continuous solution of the problem (2.3)–(2.4) on an interval $[\tau, T]$ by the standard existence theory for ordinary differential equations. Then, we will prove the convergence of $z^m(t) = (u^m, \eta^{t,m})$.

⋆ **Energy estimates.**

Multiplying the first and the second equation of (3.1) by a_m^k and b_m^k respectively and summing from 1 to m about k, we have

$$\begin{aligned} &\frac{d}{dt}(\|u^m\|^2 + \varepsilon(t)\|u^m\|_1^2 + \|\eta^{t,m}\|_{\mu,1}^2) + (2 - \varepsilon'(t))\|u^m\|_1^2 \\ =& -2\langle \eta^{t,m}, \eta_s^{t,m}\rangle_{\mu,1} - 2\langle f(u^m), u^m\rangle + 2\langle g, u^m\rangle. \end{aligned} \tag{3.2}$$

It follows from (1.5) and Poincaré's inequality that there exist $c > 0$ and $\nu > 0$ such that

$$\langle f(u^m), u^m\rangle \geq -\frac{1}{2}(1-\nu)\|u^m\|_1^2 - c. \tag{3.3}$$

According to (2.5), Hölder's inequality and Young's inequality, we have

$$\langle g, u^m\rangle \leq \frac{1}{\nu}\|g\|_{-1}^2 + \frac{\nu}{4}\|u^m\|_1^2, \tag{3.4}$$

$$\langle \eta^{t,m}, \eta_s^{t,m}\rangle_{\mu,1} \geq 0. \tag{3.5}$$

By (3.2)–(3.5) and the decreasing property of $\varepsilon(t)$, we get

$$\frac{d}{dt}(\|u^m\|^2 + \varepsilon(t)\|u^m\|_1^2 + \|\eta^{t,m}\|_{\mu,1}^2) + \frac{\nu}{2}\|u^m\|_1^2 \leq \frac{2}{\nu}\|g\|_{-1}^2 + 2c. \tag{3.6}$$

Integrating from τ to t at the sides of (3.6), we obtain

$$\|z^m\|_{\mathscr{U}_t}^2 + \frac{\nu}{2}\int_\tau^t \|u^m(s)\|_1^2 ds \leq R, \tag{3.7}$$

where

$$R = \|z_\tau^m\|_{\mathscr{U}_\tau}^2 + (t-\tau)(\frac{2}{\nu}\|g\|_{-1}^2 + 2c).$$

Therefore, we deduce from (3.7) that for any fixed $T > t$,

$$\{u^m\}_m^\infty \text{ is bounded in } L^\infty([\tau, T], \mathcal{H}_t)) \cap L^2([\tau, T], \mathrm{H}_1), \tag{3.8}$$

$$\{\eta^{t,m}\}_m^\infty \text{ is bounded in } L^\infty([\tau, T], L_\mu^2(\mathbb{R}^+, \mathrm{H}_1)). \tag{3.9}$$

Combining (1.2), (3.7) and the embedding inequality (c_1 is embedding constant), we arrive at

$$\begin{aligned} \int_\tau^T \int_\Omega |f(u^m)|^{\frac{2N}{N+2}} dxdt &\leq C_{N,c_1}\int_\tau^T \|u^m(s)\|_1^{\frac{2N}{N-2}} ds + C_{N,(T-\tau),|\Omega|} \\ &\leq C_{N,c_1,\varepsilon(T),N} R^{\frac{N}{N-2}}(T-\tau) + C_{N,(T-\tau),|\Omega|}. \end{aligned} \tag{3.10}$$

So, we infer from (3.10) that

$$\{f(u^m)\}_{m=1}^{\infty} \text{ is bounded in } L^{\frac{2N}{N+2}}([\tau,T],L^{\frac{2N}{N+2}}(\Omega)). \tag{3.11}$$

Next, we verify the uniform estimate for u_t^m. Multiplying the first equation of (3.1) by $\partial_t a_m^k$ and summing from 1 to m yields

$$\frac{d}{dt}E(t) = -\langle \eta^{t,m}, u_t^m\rangle_{\mu,1} - \|u_t^m\|^2 - \varepsilon(t)\|u_t^m\|_1^2, \tag{3.12}$$

where

$$E(t) = \frac{1}{2}\|u^m\|_1^2 + \langle F(u^m),1\rangle - \langle g,u^m\rangle.$$

Applying (1.4), (1.5) and embedding inequality, we have

$$\langle F(u^m),1\rangle \geq -\frac{1}{2}(1-\nu)\|u^m\|_1^2 - c, \tag{3.13}$$

$$\begin{aligned}|\langle F(u^m),1\rangle| &\leq C(\|u^m\|^2 + \|u^m\|_{L^{\frac{2N}{N-2}}(\Omega)}^{\frac{2N}{N-2}})\\ &\leq C(\|u^m\|^2 + c_1\|u^m\|_1^{\frac{2N}{N-2}}).\end{aligned} \tag{3.14}$$

Thereby, due to (3.4), (3.13) and (3.14),

$$E(t) \geq \frac{\nu}{4}\|u^m\|_1^2 - \frac{1}{\nu}\|g\|_{-1}^2 - c, \tag{3.15}$$

$$E(t) \leq (\frac{1}{2}+\frac{\nu}{4})\|u^m\|_1^2 + C\|u^m\|^2 + Cc_1\|u^m\|_1^{\frac{2N}{N-2}} + \frac{1}{\nu}\|g\|_{-1}^2. \tag{3.16}$$

Moreover,

$$|\langle \eta^{t,m}, u_t^m\rangle_{\mu,1}| \leq \frac{\kappa(0)}{2\varepsilon(t)}\|\eta^{t,m}\|_{\mu,1}^2 + \frac{\varepsilon(t)}{2}\|u_t^m\|_1^2. \tag{3.17}$$

Hence, it follows from (3.7), (3.12) and (3.17) that

$$\frac{d}{dt}E(t) + \frac{1}{2}\|u_t^m\|^2 + \frac{\varepsilon(t)}{2}\|u_t^m\|_1^2 \leq \frac{R\kappa(0)}{2\varepsilon(t)} \leq \frac{R\kappa(0)}{2\varepsilon(T)} \tag{3.18}$$

for $t \in [\tau,T]$. Integrating from s to t at the sides of (3.18) and combining with (3.7) yield

$$\begin{aligned}&E(t) + \frac{1}{2}\int_s^t (\|u_t^m(r)\|^2 + \varepsilon(r)\|u_t^m(r)\|_1^2)dr\\ \leq& E(s) + \frac{R\kappa(0)}{2\varepsilon(T)}(t-s)\\ \leq& \frac{2+\nu}{4\varepsilon(T)}R + CR + \frac{Cc_1}{\varepsilon(T)^{\frac{N}{N-2}}}R^{\frac{N}{N-2}} + \frac{1}{\nu}\|g\|_{-1}^2 + \frac{R\kappa(0)}{2\varepsilon(T)}(t-s)\end{aligned} \tag{3.19}$$

for any $s \in (\tau,T]$. By (3.15) and (3.19), we arrive at

$$\frac{\nu}{4}\|u^m\|_1^2 + \frac{1}{2}\int_\tau^T (\|u_t^m(r)\|^2 + \varepsilon(r)\|u_t^m(r)\|_1^2)dr \leq \rho_1 \tag{3.20}$$

for fixed $T > t$ and $s \to \tau$, where

$$\rho_1 = (\frac{2+\nu}{4\varepsilon(T)} + C)R + \frac{Cc_1}{\varepsilon(T)^{\frac{N}{N-2}}} R^{\frac{N}{N-2}} + + \frac{R\kappa(0)}{2\varepsilon(T)}(T-\tau) + \frac{2}{\nu}\|g\|_{-1}^2 + c.$$

So, (3.20) implies

$$\{u_t^m\}_{m=1}^{\infty} \text{ is bounded in } L^2([\tau, T], \mathcal{H}_t). \tag{3.21}$$

⋆ **Existence of solutions.**

Step 1. Combining (3.8), (3.9) and (3.21), we find that there are $u \in L^\infty([\tau, T], \mathcal{H}_t) \cap L^2([\tau, T], \mathrm{H}_1)$, $\eta^t \in L^\infty([\tau, T], L^2_\mu(\mathbb{R}^+, \mathrm{H}_1))$, $\chi \in L^{\frac{2N}{N+2}}([\tau, T], L^{\frac{2N}{N+2}}(\Omega))$, $u_t \in L^2([\tau, T], \mathcal{H}_t)$ and a subsequence of $\{u^m\}_{m=1}^{\infty}$ (still denoted as $\{u^m\}_{m=1}^{\infty}$) such that

$$u^m \to u \text{ weak-star in } L^\infty([\tau, T], \mathcal{H}_t)), \tag{3.22}$$

$$u^m \to u \text{ weakly in } L^2([\tau, T], \mathrm{H}_1), \tag{3.23}$$

$$\eta^{t,m} \to \eta^t \text{ weakly in } L^\infty([\tau, T], \mathcal{M}^1), \tag{3.24}$$

$$f(u^m) \to \chi \text{ weakly in } L^{\frac{2N}{N+2}}([\tau, T], L^{\frac{2N}{N+2}}(\Omega)), \tag{3.25}$$

$$u_t^m \to u_t \text{ weakly in } L^2([\tau, T], \mathcal{H}_t). \tag{3.26}$$

Applying (3.7), (3.20) and Lemma 2.11, we can know that there exists a subsequence of $\{u^m\}_{m=1}^{\infty}$ (still denoted as $\{u^m\}_{m=1}^{\infty}$) such that

$$u^m \to u \ \text{ in } L^2([\tau, T], L^2(\Omega)),$$

which shows

$$u^m \to u, \ \text{ a.e. in } \Omega \times [\tau, T]. \tag{3.27}$$

From (3.27) and the continuity of f, we get

$$f(u^m) \to f(u), \quad \text{a.e. in } \Omega \times [\tau, T],$$

which combines with Lebesgue term by term integral theorem and the uniqueness of the limit, and we get $\chi = f(u)$.

Next, we have

$$\begin{aligned} &u_t^m - u_t^n - \varepsilon(t)\triangle(u_t^m - u_t^n) - \triangle(u^m - u^n) \\ &= \int_0^\infty \mu(s)\triangle(\eta^{t,m}(s) - \eta^{t,n}(s))ds - (f(u^m) - f(u^n)). \end{aligned} \tag{3.28}$$

Multiplying Eq (3.28) by $u^m - u^n$ and integrating on Ω, we get

$$\begin{aligned} &\frac{d}{dt}\left(\|u^m - u^n\|^2 + \varepsilon(t)\|u^m - u^n\|_1^2 + \|\eta^{t,m} - \eta^{t,n}\|_{\mu,1}^2\right) + (2 - \varepsilon'(t))\|u^m - u^n\|_1^2 \\ &= -2\langle \eta^{t,m} - \eta^{t,n}, \eta_s^{t,m} - \eta_s^{t,n}\rangle - \langle f(u^m) - f(u^n), 2(u^m - u^n)\rangle. \end{aligned} \tag{3.29}$$

It follows from (1.4), (2.5), (3.20) and the monotonicity of $\varepsilon(t)$ that

$$\begin{aligned}&\frac{d}{dt}\left(\|u^m-u^n\|^2+\varepsilon(t)\|u^m-u^n\|_1^2+\|\eta^{t,m}-\eta^{t,n}\|_{\mu,1}^2\right)\\ \le& C(1+\|u^m\|_1^{\frac{4}{N-2}}+\|u^n\|_1^{\frac{4}{N-2}})\|u^m-u^n\|_1^2\\ \le& C_{\rho_1,\nu,N}\frac{L+1}{\varepsilon(T)}\left(\|u^m-u^n\|^2+\varepsilon(t)\|u^m-u^n\|_1^2+\|\eta^{t,m}-\eta^{t,n}\|_{\mu,1}^2\right)\end{aligned} \tag{3.30}$$

for any $t\in[\tau,T]$. Using Gronwall's lemma yields

$$\|z^m-z^n\|_{\mathscr{U}_t}^2\le e^{C_{\rho_1,\nu,N,\varepsilon(T),L}(t-\tau)}\|z^m(\tau)-z^n(\tau)\|_{\mathscr{U}_\tau}^2, \tag{3.31}$$

which implies

$$\{z^m\}_{m=1}^{\infty} \text{ is a Cauchy sequence in } C([\tau,T],\mathscr{U}_t).$$

From the uniqueness of the limit, we know that

$$z^m\to z \text{ uniformly in } C([\tau,T],\mathscr{U}_t),\quad \text{for all } T>\tau.$$

Therefore, we have the following conclusion:

$$z\in C([\tau,T],\mathscr{U}_t). \tag{3.32}$$

Thus, when $m\to\infty$, $z^m(\tau)\to z_\tau$ in $\mathscr{U}_t$.

Step 2. Choose a test function $\theta(t)=(v,\xi^t)=(\Sigma_{k=1}^{\bar N}a_m^k(t)\omega_k,\Sigma_{k=1}^{\bar N}b_m^k(t)e_k)\in C([\tau,T],\mathscr{U}_t)$ for fixed $\bar N$. For $m\ge\bar N$, multiplying the first and the second equation of (3.1) by a_m^k and b_m^k respectively, summing from 1 to $\bar N$ and integrating from τ to T, we get

$$\begin{cases}\int_\tau^T[\langle u_t^m,v\rangle+\varepsilon(t)\langle\nabla u_t^m,\nabla v\rangle]dt+\int_\tau^T\langle\nabla u^m,\nabla v\rangle dt+\int_\tau^T\langle\eta^{t,m},v\rangle_{\mu,1}dt\\ \qquad+\int_\tau^T\langle f(u^m),v\rangle dt=\int_\tau^T\langle g,v\rangle dt,\\ \int_\tau^T\langle\eta_t^{t,m},\xi^t\rangle_{\mu,1}dt=-\int_\tau^T\langle\eta_s^{t,m},\xi^t\rangle_{\mu,1}dt+\int_\tau^T\langle u^m,\xi^t\rangle_{\mu,1}dt.\end{cases} \tag{3.33}$$

By (3.22)–(3.26), (3.33), we have

$$\begin{cases}\int_\tau^T[\langle u_t,v\rangle+\varepsilon(t)\langle\nabla u_t,\nabla v\rangle]dt+\int_\tau^T\langle\nabla u,\nabla v\rangle dt+\int_\tau^T\langle\eta^{t},v\rangle_{\mu,1}dt\\ \qquad+\int_\tau^T\langle f(u),v\rangle dt=\int_\tau^T\langle g,v\rangle dt,\\ \int_\tau^T\langle\eta_t^{t},\xi^t\rangle_{\mu,1}dt=-\int_\tau^T\langle\eta_s^{t},\xi^t\rangle_{\mu,1}dt+\int_\tau^T\langle u,\xi^t\rangle_{\mu,1}dt.\end{cases} \tag{3.34}$$

Owing to the arbitrariness of T, for *a.e.* $[\tau,T]$,

$$\langle u_t,v\rangle+\varepsilon(t)\langle\nabla u_t,\nabla v\rangle+\langle\nabla u,\nabla v\rangle+\langle\eta^t,v\rangle_{\mu,1}+\langle f(u),v\rangle=\langle g,v\rangle, \tag{3.35}$$

$$\langle\eta_t^t,\xi^t\rangle_{\mu,1}=-\langle\eta_s^t,\xi^t\rangle_{\mu,1}+\langle u,\xi^t\rangle_{\mu,1}. \tag{3.36}$$

Step 3. We now verify $z(\tau) = z_\tau$. Indeed, it is obvious that $z(\tau)$ is meaningful according to $z \in C([\tau, T], \mathscr{U}_t)$. Choose the function $\theta(t) \in C^1([\tau, T], \mathscr{U}_t)$ with $(v(T), \xi^T) = (0, 0)$, and then

$$\begin{cases} -\int_\tau^T [\langle u, v_t\rangle + \varepsilon(t)\langle\nabla u_t, \nabla v\rangle]dt + \int_\tau^T \langle\nabla u, \nabla v\rangle dt + \int_\tau^T \langle\eta^t, v\rangle_{\mu,1} dt \\ \qquad + \int_\tau^T \langle f(u), v\rangle dt = \int_\tau^T \langle g, v\rangle dt + \langle u(\tau), v(\tau)\rangle, \\ -\int_\tau^T \langle\eta^t, \xi_t^t\rangle_{\mu,1} dt = -\int_\tau^T \langle\eta_s^t, \xi^t\rangle_{\mu,1} dt + \int_\tau^T \langle u, \xi^t\rangle_{\mu,1} dt + \langle\eta(\tau), \xi(\tau)\rangle_{\mu,1}, \end{cases} \tag{3.37}$$

by (3.34). From (3.33), we can have

$$\begin{cases} -\int_\tau^T [\langle u^m, v_t\rangle - \varepsilon(t)\langle\nabla u_t^m, \nabla v\rangle]dt + \int_\tau^T \langle\nabla u^m, \nabla v\rangle dt + \int_\tau^T \langle\eta^{t,m}, v\rangle_{\mu,1} dt \\ \qquad + \int_\tau^T \langle f(u^m), v\rangle dt = \int_\tau^T \langle g, v\rangle dt + \langle\eta^\tau, \xi(\tau)\rangle, \\ -\int_\tau^T \langle\eta^{t,m}, \xi_t^t\rangle_{\mu,1} dt = -\int_\tau^T \langle\eta_s^{t,m}, \xi^t\rangle_{\mu,1} dt + \int_\tau^T \langle u^m, \xi^t\rangle_{\mu,1} dt + \langle\eta^m(\tau), \xi(\tau)\rangle. \end{cases} \tag{3.38}$$

Because of $z_m(\tau) \to z_\tau$ $(m \to \infty)$, it follows from (3.38) that

$$\begin{cases} -\int_\tau^T [\langle u, v_t\rangle + \varepsilon(t)\langle\nabla u_t, \nabla v\rangle]dt + \int_\tau^T \langle\nabla u, \nabla v\rangle dt + \int_\tau^T \langle\eta^t, v\rangle_{\mu,1} dt \\ \qquad + \int_\tau^T \langle f(u), v\rangle dt = \int_\tau^T \langle g, v\rangle dt + \langle u_\tau, v(\tau)\rangle, \\ -\int_\tau^T \langle\eta^t, \xi_t^t\rangle_{\mu,1} dt = -\int_\tau^T \langle\eta_s^t, \xi^t\rangle_{\mu,1} dt + \int_\tau^T \langle u, \xi^t\rangle_{\mu,1} dt + \langle\eta^\tau, \xi^\tau\rangle_{\mu,1}. \end{cases} \tag{3.39}$$

Combining (3.37), (3.39) and the arbitrariness of $\theta(\tau) = (v(\tau), \xi(\tau))$ yields

$$z(\tau) = z_\tau. \tag{3.40}$$

Hence, the existence of the weak solution is obtained by (3.32), (3.35), (3.36) and (3.40).

⋆ **Uniqueness and continuity of solutions.**

Assume that $z^i = (u^i, \eta^{t,i})$ $(i = 1, 2)$ are two solutions of the problem (2.3)–(2.4) with the initial data $z_\tau^i = (u_\tau^i, \eta^{\tau,i})$, respectively. For convenience, define $\bar{u} = u^1 - u^2$, $\bar{\eta}^t = \eta^{t,1} - \eta^{t,2}$, and then $\bar{z}(t) = z^1(t) - z^2(t) = (\bar{u}, \bar{\eta}^t)$ satisfies the following equation:

$$\begin{cases} \bar{u}_t - \varepsilon(t)\triangle\bar{u}_t - \Delta\bar{u} - \int_0^\infty \mu(s)\triangle\bar{\eta}^t(s)ds + f(u^1) - f(u^2) = 0, \\ \bar{\eta}_t^t = -\bar{\eta}_s^t + \bar{u}, \end{cases}$$

with initial data

$$\bar{z}(x, \tau) = \bar{z}_\tau = z_\tau^1 - z_\tau^2.$$

Repeating the proof of (3.31) yields

$$\|\bar{z}(t)\|_{\mathscr{U}_t}^2 \le e^{C_{\rho_1,\nu,N,\varepsilon(T),L}(t-\tau)}\|\bar{z}_\tau\|_{\mathscr{U}_\tau}^2. \tag{3.41}$$

Thereby, (3.41) implies the uniqueness of the solution as well as the property of continuous dependence of the solution on initial data. □

According to Theorem 3.2, we can define a continuous process $\{U(t, \tau)\}_{t \ge \tau}$ generated by the solution of the problem (2.3)–(2.4), where the mapping

$$U(t, \tau) : \mathscr{U}_\tau \to \mathscr{U}_t, \quad t \ge \tau \in \mathbb{R},$$

and $U(t, \tau)z_\tau = z(t)$, $z_\tau \in \mathscr{U}_\tau$.

4. Time-dependent global attractor

At first, we consider the existence of time-dependent absorbing sets for the solution process $\{U(t,\tau)\}_{t\geq\tau}$ in $\mathscr{U}_t$.

Lemma 4.1. *Assume that (1.2), (1.3), (1.5)–(1.8) hold, $g \in H^{-1}(\Omega)$, and $z_\tau = (u_\tau, \eta^\tau) \in \mathbb{B}_{\mathscr{U}_\tau}(R_0) \subset \mathscr{U}_\tau$, and then there exists $R_1 > 0$ such that $\mathfrak{B} = \{B_t\}_{t\in\mathbb{R}} = \{\mathbb{B}_{\mathscr{U}_t}(R_1)\}_{t\in\mathbb{R}}$ is a time-dependent absorbing set in $\mathscr{U}_t$ for the process $\{U(t,\tau)\}_{t\geq\tau}$ corresponding to the problem (2.3)–(2.4).*

Proof. Multiplying the first equation of (2.3) by u and repeating the estimate of Theorem 3.2, we conclude that

$$\frac{d}{dt}E_1(t) + (1-\varepsilon'(t))\|u\|_1^2 + \frac{\nu}{2}\|u\|_1^2 \leq \frac{2}{\nu}\|g\|_{-1}^2 + 2c, \tag{4.1}$$

where

$$E_1(t) = \|u\|^2 + \varepsilon(t)\|u\|_1^2 + \|\eta^t\|_{\mu,1}^2.$$

According to (1.2), (1.3), (4.1) and Poincaré's inequality, we get

$$\frac{d}{dt}E_1(t) + \frac{\varepsilon(t)}{L}\|u\|_1^2 + \frac{\nu}{4}\|u\|_1^2 + \frac{\nu\lambda_1}{4}\|u\|^2 \leq \frac{2}{\nu}\|g\|_{-1}^2 + 2c. \tag{4.2}$$

To reconstruct $E_1(t)$, we introduce a new function

$$\Psi_1(t) = \int_0^\infty \kappa(s)\|\eta^t(s)\|_1^2 ds \tag{4.3}$$

by using the idea of [15]. Due to (1.7), we have

$$\Psi_1(t) \leq \Theta\|\eta^t\|_{\mu,1}^2 \leq \Theta E_1(t). \tag{4.4}$$

In addition, taking the derivative with respect to t at the sides of (4.3) and combining (1.6) and (1.7), we find that

$$\begin{aligned}\frac{d}{dt}\Psi_1(t) &= -\|\eta^t\|_{\mu,1}^2 + 2\int_0^\infty \kappa(s)\langle\nabla\eta^t(s), \nabla u(s)\rangle ds \\ &\leq -\frac{1}{2}\|\eta^t\|_{\mu,1}^2 + 2\Theta^2\kappa(0)\|u\|_1^2.\end{aligned} \tag{4.5}$$

Therefore, for fixed $\nu > 0$, we define the function

$$\Phi_1(t) = E_1(t) + \frac{\nu}{8\Theta^2\kappa(0)}\Psi_1(t). \tag{4.6}$$

Combining (4.2), (4.5) and (4.6), we have

$$\frac{d}{dt}\Phi_1(t) + \frac{\varepsilon(t)}{L}\|u\|_1^2 + \frac{\nu\lambda_1}{4}\|u\|^2 + \frac{\nu}{16\Theta^2\kappa(0)}\|\eta^t\|_{\mu,1}^2 \leq \frac{2}{\nu}\|g\|_{-1}^2 + 2c.$$

Choose $\sigma_1 = \min\{\frac{1}{2L}, \frac{\nu\lambda_1}{8L}, \frac{\nu}{32\Theta^2\kappa(0)}\} > 0$, and then

$$\frac{d}{dt}\Phi_1(t) + 2\sigma_1 E_1(t) \leq \frac{2}{\nu}\|g\|_{-1}^2 + 2c. \tag{4.7}$$

For small enough ν, we have

$$E_1(t) \leq \Phi_1(t) \leq 2E_1(t). \tag{4.8}$$

It follows from (4.7) and (4.8) that

$$\frac{d}{dt}\Phi_1(t) + \sigma_1\Phi_1(t) \leq \frac{2}{\nu}\|g\|_{-1}^2 + 2c. \tag{4.9}$$

By Gronwall's lemma, we see that

$$\Phi_1(t) \leq e^{-\sigma_1(t-\tau)}\Phi_1(\tau) + \frac{1}{\sigma_1}(\frac{2}{\nu}\|g\|_{-1}^2 + 2c). \tag{4.10}$$

Hence, we conclude from (4.8) and (4.10) that

$$E_1(t) \leq 2e^{-\sigma_1(t-\tau)}E_1(\tau) + \frac{1}{\sigma_1}(\frac{2}{\nu}\|g\|_{-1}^2 + 2c),$$

which shows

$$\|u\|^2 + \varepsilon(t)\|u\|_1^2 + \|\eta^t\|_{\mu,1}^2 \leq R_1$$

for any $t \geq t^* = \tau + \frac{1}{\sigma_1}\ln\frac{4E_1(\tau)}{R_1}$, where

$$R_1 = \frac{2}{\sigma_1}(\frac{2}{\nu}\|g\|_{-1}^2 + 2c).$$

So, $B_t = \{z = (u, \eta^t) \in \mathscr{U}_t : \|z(t)\|_{\mathscr{U}_t}^2 \leq R_1\}$ is a time-dependent absorbing set in $\mathscr{U}_t$ for the solution process $\{U(t,\tau)\}_{t\geq\tau}$. The proof is finished. □

We next verify the pullback asymptotical compactness for the process $\{U(t,\tau)\}_{t\geq\tau}$ corresponding to the problem (2.3)–(2.4).

Theorem 4.2. *Assume that (1.2), (1.3) and (1.5)–(1.8) hold, and then the process $\{U(t,\tau)\}_{t\geq\tau}$ of the problem (2.3)–(2.4) is pullback asymptotic compact in $\mathscr{U}_t$.*

Proof. Assume that $z^n = (u^n, \eta^{t,n})$, $z^m = (u^m, \eta^{t,m})$ are two solutions of the problem (2.3)–(2.4) with initial data z_τ^n, $z_\tau^m \in \mathbb{B}_{\mathscr{U}_\tau}(R_0)$, respectively. Without loss of generality, we assume $\tau \leq T_1 < t$ for every fixed T_1. As a convenience, let $w(t) = u^n(t) - u^m(t)$, $\zeta^t = \eta^{t,n} - \eta^{t,m}$, and then $(w(t), \zeta^t)$ satisfies the following equation:

$$\begin{cases} w_t - \varepsilon(t)\triangle w_t - \triangle w - \int_0^\infty \mu(s)\triangle\zeta^t(s)ds + f(u^n) - f(u^m) = 0, \\ \zeta_t^t = -\zeta_s^t + w, \quad t \geq \tau, \end{cases} \tag{4.11}$$

with

$$w(x, T_1) = w_{T_1} = u_{T_1}^n - u_{T_1}^m, \quad \zeta^{T_1} = \eta^{T_1,n} - \eta^{T_1,m}.$$

Multiplying the first Eq (4.11) by w and integrating in Ω, we can get

$$\frac{d}{dt}E_2(t) + (1 - \varepsilon'(t))\|w\|_1^2 + \|w\|_1^2 \leq -2\langle f(u^n) - f(u^m), w\rangle, \tag{4.12}$$

where

$$E_2(t) = \|w\|^2 + \varepsilon(t)\|w\|_1^2 + \|\zeta^t\|_{\mu,1}^2.$$

By (1.2), (1.3) and Poincaré's inequality, we have

$$\frac{d}{dt}E_2(t) + \frac{\varepsilon(t)}{L}\|w\|_1^2 + \frac{1}{2}\|w\|_1^2 + \frac{\lambda_1}{2}\|w\|^2 \leq -2\langle f(u^n) - f(u^m), w\rangle. \tag{4.13}$$

Set

$$\Psi_2(t) = \int_0^\infty \kappa(s)\|\zeta^t(s)\|_1^2 ds,$$

$$\Phi_2(t) = E_2(t) + \frac{\nu}{4\Theta^2\kappa(0)}\Psi_2(t).$$

Then, applying similar arguments as used in the proof of Theorem 4.1, we get

$$\frac{d}{dt}\Phi_2(t) + 2\sigma_2 E_2(t) \leq -2\langle f(u^n) - f(u^m), w\rangle, \tag{4.14}$$

$$E_2(t) \leq \Phi_2(t) \leq 2E_2(t), \tag{4.15}$$

where $0 < \sigma_2 = \min\{\frac{1}{2L}, \frac{\lambda_1}{4}, \frac{\nu}{16\Theta^2\kappa(0)}\}$ and ν is small enough. Combining (4.14) and (4.15), we find that

$$\frac{d}{dt}\Phi_2(t) + \sigma_2\Phi_2(t) \leq -2\langle f(u^n) - f(u^m), w\rangle. \tag{4.16}$$

Integrating from s to t at both sides of (4.16), we arrive

$$\Phi_2(t) \leq \Phi_2(s) - 2\int_s^t \langle f(u^n(r)) - f(u^m(r)), w(r)\rangle dr. \tag{4.17}$$

At the same time, integrating from T_1 to t at both sides of (4.16), we have

$$\int_{T_1}^t \Phi_2(r)dr \leq \frac{1}{\sigma_2}\Phi_2(T_1) - \frac{2}{\sigma_2}\int_{T_1}^t \langle f(u^n(r)) - f(u^m(r)), w(r)\rangle dr. \tag{4.18}$$

Then, integrating over $[T_1, t]$ about variable s at both sides of (4.17), we obtain that

$$(t - T_1)\Phi_2(t) \leq \int_{T_1}^t \Phi_2(r)dr - 2\int_{T_1}^t\int_s^t \langle f(u^n(r)) - f(u^m(r)), w(r)\rangle drds. \tag{4.19}$$

Due to (4.18) and (4.19), we get

$$\begin{aligned}\Phi_2(t) \leq& \frac{\Phi_2(T_1)}{\sigma_2(t - T_1)} - \frac{2}{\sigma_2(t - T_1)}\int_{T_1}^t \langle f(u^n(r)) - f(u^m(r)), w(r)\rangle dr \\ &- \frac{2}{t - T_1}\int_{T_1}^t\int_s^t \langle f(u^n(r)) - f(u^m(r)), w(r)\rangle drds.\end{aligned} \tag{4.20}$$

By (4.15) and (4.20), we can see

$$E_2(t) \leq \frac{2E_2(T_1)}{\sigma_2(t - T_1)} + \psi_{T_1}^t(u_{T_1}^m, u_{T_1}^m),$$

where

$$\psi^t_{T_1}(u^m_{T_1}, u^m_{T_1}) = -\frac{2}{\sigma_2(t-T_1)}\int_{T_1}^t \langle f(u^n(r)) - f(u^m(r)), w(r)\rangle dr$$
$$-\frac{2}{t-T_1}\int_{T_1}^t\int_s^t \langle f(u^n(r)) - f(u^m(r)), w(r)\rangle drds.$$

For some fixed t, let $t > T_1 \geq \tau$ such that, with $t - T_1$ large enough, we conclude that $\frac{2E_2(T_1)}{\sigma_2(t-T_1)} \leq \epsilon$ for any $\epsilon > 0$. Next, we will prove $\psi^t_{T_1} \in \mathfrak{C}(B_{T_1})$ for each fixed T_1. Indeed, assume that $z^k = (u^k, \eta^{t,k})$ is a solution of the problem (2.3)–(2.4) with initial data $z^k_\tau \in \mathbb{B}_{\mathscr{U}_\tau}(R_0)$. Then, $u^k_t \in L^2([T_1, t], \mathcal{H}_t)$, and $u^k \in L^2([T_1, t], H^1_0(\Omega))$ by using the same arguments of Theorem 3.2. Hence, it follows from Lemma 2.11 that there is a convergent subsequence of u^k (denoted as u^{k_i}) such that

$$\lim_{i\to\infty}\lim_{j\to\infty}\int_{T_1}^t \|u^{k_i}(r) - u^{k_j}(r)\|^2 dr = 0.$$

This shows

$$u^{k_i} \to u^{k_j}, \ \ a.e.\ (x,t) \in \Omega \times [T_1, t]. \tag{4.21}$$

In view of (3.25), (4.21) and the continuity of f,

$$f(u^{k_i}) \to f(u^{k_j}), \ \ a.e.\ (x,t) \in \Omega \times [T_1, t]. \tag{4.22}$$

Hence, by (4.21) and (4.22), we have

$$\lim_{i\to\infty}\lim_{j\to\infty}\int_{T_1}^t \langle f(u^{k_i}(r)) - f(u^{k_j}(r)), (u^{k_i} - u^{k_j})\rangle dr = 0. \tag{4.23}$$

For some fixed t, $\int_s^t \langle f(u^{k_i}(r)) - f(u^{k_j}(r)), (u^{k_i} - u^{k_j})\rangle dr$ ($s \in [T_1, t]$) is bounded. Using the Lebesgue dominated convergence theorem yields

$$\lim_{i\to\infty}\lim_{j\to\infty}\int_{T_1}^t\int_s^t \langle f(u^{k_i}(r)) - f(u^{k_j}(r)), (u^{k_i} - u^{k_j})\rangle drds = 0. \tag{4.24}$$

According to (4.23) and (4.24), we conclude that $\psi^t_{T_1} \in \mathfrak{C}(B_{T_1})$. Consequently,

$$\|U(t, T_1)|u^n_{T_1} - U(t, T_1)u^m_{T_1}\| \leq \epsilon + \psi^t_{T_1}(u^n_{T_1}, u^m_{T_1}).$$

Thereby, it follows from Theorem 2.8 that the process $\{U(t,\tau)\}_{t\geq\tau}$ is pullback asymptotic compact in $\mathscr{U}_t$. □

Theorem 4.3. *The process $\{U(t,\tau)\}_{t\geq\tau}$ generated by the problem (2.3)–(2.4) has an invariant time-dependent global attractor $\mathfrak{A} = \{A_t\}_{t\in\mathbb{R}}$ in $\mathscr{U}_t$.*

Proof. Combining Lemma 4.1 and Theorem 4.2, we get easily the existence of the invariant time-dependent global attractor $\mathfrak{A} = \{A_t\}_{t\in\mathbb{R}}$ for the problem (2.3)–(2.4). □

5. Conclusions and discussion

This research examines the nonclassical diffusion equation with memory and time-dependent coefficient. Using the contractive function method and some delicate estimates, we obtained the existence and uniqueness of the time-dependent global attractor for the problem (1.1). It is well known that attractors, as a powerful tool for studying dynamical systems, can well characterize long-term behaviors. The time-dependent global attractor obtained can describe the asymptotic behavior of Eq (1.1). This study is helpful to more accurately observe the sensitivity of the physical model to the disturbance, the external force and the internal friction, and it provides a better theoretical basis for the study of solid mechanics, heat conduction and relaxation of high viscosity liquids and non-Newtonian fluids.

At present, the study of nonlinear development equations has become one of the important topics in the intersection of dynamic systems, differential equations and nonlinear analysis, while it is a hot topic to explore the dynamic behavior of dissipative partial differential equations with time-dependent coefficients and their related problems in the field of infinite dimensional dynamic systems in recent years. Although some theoretical results and applications on time-dependent space have been obtained, the time-dependent global attractor alone is not a good way to describe the dynamic behavior of the system. Moreover, it is expected that our results will help to describe the observability of attractors in numerical simulation more clearly. Naturally, there are two interesting problems: Can we establish the existence and stability theory of the time-dependent exponential attractor? How does one study the long-term behavior for the nonclassical diffusion equation model by combining with numerical simulation methods, because the nonclassical diffusion equation model we studied is particular and complex?

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Conflict of interest

The authors declare no conflict of interest.

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