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Research article

Multiple positive solutions for system of mixed Hadamard fractional boundary value problems with (p_1, p_2)-Laplacian operator

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Abstract: In this paper, we investigate the existence of positive solutions of a system of Riemann-Liouville Hadamard differential equations with p-Laplacian operators under various combinations of superlinearity and sublinearity. We apply the Guo-Krasnosel'skii fixed point theorem for the proof of the existence results.

Keywords: Hadamard fractional system; positive solutions; boundary value problems; p-Laplacian; fixed point theorem
Mathematics Subject Classification: 34A08, 34B15, 34B18, 34B27

1. Introduction

In recent decades, interest in the study of fractional differential equations has increased due to the intensive development of the theory of fractional calculus itself and its applications in various fields of science and mathematics due to its high accuracy and applicability [7, 8, 17, 30, 31, 36]. Compared with integer differential equation, fractional differential can better describe some physical phenomena, so scientists from various fields pay great attention to them. For more details on some results about fractional differential equations, we refer readers to [3, 5, 18–21, 25].

A p-Laplacian differential equation was first introduced by Leibenson [22] when he studied turbulent flow in a porous medium. By converting this fundamental mechanical problem into the existence of solutions of the following p-Laplacian differential equation: $\phi_p(u'(t))' = f(t, u(t)), \quad t \in (0, 1),$ where $\phi_p(s) = |s|^{p-2}s$ $(p > 1)$ is the p-Laplacian operator. Its inverse function is denoted by $\phi_q(s)$ with $\phi_q(s) = |s|^{q-2}s$ and p, q satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Some scholars found that the fractional-order differential models are better than the integer-order differential models for problems in various areas of science such as physics, water management, electrical grids and many others [13, 23, 33, 37, 47, 48]. Consequently, the research of fractional differential equations with p-Laplacian operator BVP has already become a focus in recent years, and has developed very rapidly [1, 14, 24, 26, 32, 38–42, 45, 46, 50, 52]. However, we should point out that most of the work on this subject in recent years has been based on fractional differential equations of the Riemann-Liouville and Caputo.

In 1892, Hadamard [12] introduced another type of fractional derivations, i.e., Hadamard type fractional differential equations, which differ from the previous ones in that the kernel of the integral and derivative contains a logarithmic function with arbitrary exponent. Details and properties of Hadamard fractional derivative and integral can be found in [2, 4, 6, 9, 11, 15, 16, 27–29, 34, 35, 43, 44, 49, 51, 53–55].

Motivated by the above works, the existence of several positive solutions for the following system of nonlinear fractional differential equations with p-Laplacian is investigated. We consider the nonlinear Hadamard FDE with (p_1, p_2)-Laplacian operator

$$\begin{cases} \mathfrak{D}_{1^+}^{m_1}(\phi_{p_1}(\mathfrak{D}_{1^+}^{n_1}\upsilon(\tau))) + \mathfrak{f}_1(\tau, \upsilon(\tau), \omega(\tau)) = 0, \ \tau \in (1, e), \\ \mathfrak{D}_{1^+}^{m_2}(\phi_{p_2}(\mathfrak{D}_{1^+}^{n_2}\omega(\tau))) + \mathfrak{f}_2(\tau, \upsilon(\tau), \omega(\tau)) = 0, \ \tau \in (1, e), \end{cases} \tag{1.1}$$

with coupled boundary conditions

$$\begin{cases} \upsilon(1) = \upsilon'(1) = \upsilon''(1) = 0, \ \ \mathfrak{D}_{1^+}^{n_1}\upsilon(1) = 0, \ \ \lambda_1\mathfrak{D}_{1^+}^{\gamma_1}\upsilon(e) = \mu_1\mathfrak{D}_{1^+}^{\gamma_2}\omega(\eta), \\ \omega(1) = \omega'(1) = \omega''(1) = 0, \ \ \mathfrak{D}_{1^+}^{n_2}\omega(1) = 0, \ \ \lambda_2\mathfrak{D}_{1^+}^{\delta_1}\omega(e) = \mu_2\mathfrak{D}_{1^+}^{\delta_2}\upsilon(\xi), \end{cases} \tag{1.2}$$

where $n_i, m_i, \gamma_i, \delta_i \in R$, $n_i \in (3, 4], m_i \in (0, 1], i = 1, 2$, $\gamma_1, \delta_1 \in [1, 2], \gamma_2 \in [1, \delta_1], \delta_2 \in [0, \gamma_1]$, $\eta, \xi \in (1, e)$, $\mathfrak{D}_{1^+}^{\chi}$ denotes the Hadamard fractional order χ (for $\chi = m_j, n_j, \gamma_j, \delta_j$), λ_j, μ_j, $j = 1, 2$ are real positive constants, $p_1, p_2 > 1$, $\phi_{p_k}(s) = |s|^{p_k-2}s$, $\phi_{p_k}^{-1} = \phi_{q_k}$, $\frac{1}{p_k} + \frac{1}{q_k} = 1, k = 1, 2$ and $\mathfrak{f}_1, \mathfrak{f}_2 \in C([1, e] \times [0, \infty) \times [0, \infty), [0, \infty))$.

The paper is organized as follows. In section 2, we give some properties of Green's functions that will be needed later. We also state the Guo-Krasnosel'skii fixed point theorem for cone preserving operators and prove an important lemma used in the proofs of our main results. In Section 3, we establish several results for the problems (1.1) and (1.2).

2. Preliminaries

In this section, we will come up with some definitions and lemmas that will be worn in the proof of used by our main results.

Definition 2.1. *[31] The Hadamard fractional derivative of order $\alpha > 0$ of a function $f : [1, \infty) \to R$ is given by*

$$D_{1^+}^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\Big(t\frac{d}{dt}\Big)^n \int_1^t \Big(\log\frac{t}{s}\Big)^{n-\alpha-1} f(s)\frac{ds}{s}, \quad n-1 < \alpha < n,$$

where $n = [\alpha] + 1$, $[\alpha]$ represent the integer part of the real number α and $\log(\cdot) = \log_e(\cdot)$.

Definition 2.2. *[31] The Hadamard fractional integral of order $\alpha > 0$ is given by*

$$I^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_1^t \Big(\log\frac{t}{s}\Big)^{\alpha-1} f(s)\frac{ds}{s}, \ \ \alpha > 0,$$

provided that integral exists.

In this section, we build the Green's functions and their bounds for the corresponding Hadamard FDE.

We consider the Hadamard FDE

$$\begin{cases} \mathfrak{D}_{1^+}^{n_1}\upsilon(\tau)+x(\tau)=0, \quad 1<\tau<e, \\ \mathfrak{D}_{1^+}^{n_2}\omega(\tau)+y(\tau)=0, \quad 1<\tau<e, \\ \upsilon(1)=\upsilon'(1)=\upsilon''(1)=0, \quad \lambda_1\mathfrak{D}_{1^+}^{\gamma_1}\upsilon(e)=\mu_1\mathfrak{D}_{1^+}^{\gamma_2}\omega(\eta), \\ \omega(1)=\omega'(1)=\omega''(1)=0, \quad \lambda_2\mathfrak{D}_{1^+}^{\delta_1}\omega(e)=\mu_2\mathfrak{D}_{1^+}^{\delta_2}\upsilon(\xi), \end{cases} \tag{2.1}$$

where $x, y \in C[1, e]$. We introduce the following number

$$\Delta = \frac{\lambda_1\lambda_2\Gamma(n_1)\Gamma(n_2)}{\Gamma(n_1-\gamma_1)\Gamma(n_2-\delta_1)} - \frac{\mu_1\mu_2\Gamma(n_1)\Gamma(n_2)}{\Gamma(n_1-\delta_2)\Gamma(n_2-\gamma_2)}(\log\xi)^{n_1-\delta_2-1}(\log\eta)^{n_2-\gamma_2-1}.$$

Lemma 2.1. *If $\Delta \neq 0$, then the problem* (2.1) *has a unique solution which is given by*

$$\begin{cases} \upsilon(\tau)=\int_1^e \sigma_1(\tau,\varsigma)x(\varsigma)\frac{d\varsigma}{\varsigma}+\int_1^e \sigma_2(\tau,\varsigma)y(\varsigma)\frac{d\varsigma}{\varsigma}, \\ \omega(\tau)=\int_1^e \sigma_3(\tau,\varsigma)y(\varsigma)\frac{d\varsigma}{\varsigma}+\int_1^e \sigma_4(\tau,\varsigma)x(\varsigma)\frac{d\varsigma}{\varsigma}, \end{cases} \tag{2.2}$$

where

$$\begin{aligned} \sigma_1(\tau,\varsigma) &= \zeta_1(\tau,\varsigma)+\frac{(\log\tau)^{n_1-1}\mu_1\mu_2\Gamma(n_2)(\log\eta)^{n_2-\gamma_2-1}}{\Delta\Gamma(n_2-\gamma_2)}\zeta_2(\xi,\varsigma), \\ \sigma_2(\tau,\varsigma) &= \frac{(\log\tau)^{n_1-1}\mu_1\lambda_2\Gamma(n_2)}{\Delta\Gamma(n_2-\delta_1)}\zeta_3(\eta,\varsigma), \\ \sigma_3(\tau,\varsigma) &= \zeta_4(\tau,\varsigma)+\frac{(\log\tau)^{n_2-1}\lambda_1\lambda_2\Gamma(n_1)(\log\xi)^{n_1-\gamma_2-1}}{\Delta\Gamma(n_1-\gamma_2)}\zeta_3(\eta,\varsigma), \\ \sigma_4(\tau,\varsigma) &= \frac{(\log\tau)^{n_2-1}\mu_2\lambda_1\Gamma(n_1)}{\Delta\Gamma(n_1-\gamma_1)}\zeta_2(\xi,\varsigma), \quad \forall\, \tau,\varsigma\in[1,e], \end{aligned} \tag{2.3}$$

in which

$$\begin{aligned} \zeta_1(\tau,\varsigma) &= \frac{1}{\Gamma(n_1)}\begin{cases} (\log\tau)^{n_1-1}(1-\log\varsigma)^{n_1-\gamma_1-1}-\left(\log\frac{\tau}{\varsigma}\right)^{n_1-1}, & 1\le\varsigma\le\tau\le e, \\ (\log\tau)^{n_1-1}(1-\log\varsigma)^{n_1-\gamma_1-1}, & 1\le\tau\le\varsigma\le e, \end{cases} \\ \zeta_2(\tau,\varsigma) &= \frac{1}{\Gamma(n_1-\delta_2)}\begin{cases} (\log\tau)^{n_1-\delta_2-1}(1-\log\varsigma)^{n_1-\gamma_1-1}-\left(\log\frac{\tau}{\varsigma}\right)^{n_1-\delta_2-1}, & 1\le\varsigma\le\tau\le e, \\ (\log\tau)^{n_1-\delta_2-1}(1-\log\varsigma)^{n_1-\gamma_1-1}, & 1\le\tau\le\varsigma\le e, \end{cases} \\ \zeta_3(\tau,\varsigma) &= \frac{1}{\Gamma(n_2-\gamma_2)}\begin{cases} (\log\tau)^{n_2-\gamma_2-1}(1-\log\varsigma)^{n_2-\delta_1-1}-\left(\log\frac{\tau}{\varsigma}\right)^{n_2-\gamma_2-1}, & 1\le\varsigma\le\tau\le e, \\ (\log\tau)^{\alpha_2-\gamma_2-1}(1-\log\varsigma)^{n_2-\delta_1-1}, & 1\le\tau\le\varsigma\le e, \end{cases} \\ \zeta_4(\tau,\varsigma) &= \frac{1}{\Gamma(n_2)}\begin{cases} (\log\tau)^{n_2-1}(1-\log\varsigma)^{n_2-\delta_1-1}-\left(\log\frac{\tau}{\varsigma}\right)^{n_2-1}, & 1\le\varsigma\le\tau\le e, \\ (\log\tau)^{n_2-1}(1-\log\varsigma)^{n_2-\delta_1-1}, & 1\le\tau\le\varsigma\le e. \end{cases} \end{aligned} \tag{2.4}$$

Proof. As stated in [17], the Hadamard FDE's solution in (2.1) can be expressed as

$$\upsilon(\tau)=c_1(\log\tau)^{n_1-1}+c_2(\log\tau)^{n_1-2}+c_3(\log\tau)^{n_1-3}+c_4(\log\tau)^{n_1-4}-\frac{1}{\Gamma(n_1)}\int_1^t\left(\log\frac{\tau}{\varsigma}\right)^{n_1-1}x(\varsigma)\frac{d\varsigma}{\varsigma},$$

$$\omega(\tau)=d_1(\log\tau)^{n_2-1}+d_2(\log\tau)^{n_2-2}+d_3(\log\tau)^{n_2-3}+d_4(\log\tau)^{n_2-4}-\frac{1}{\Gamma(n_2)}\int_1^\tau\left(\log\frac{\tau}{\varsigma}\right)^{n_2-1}y(\varsigma)\frac{d\varsigma}{\varsigma},$$

for some $c_j, d_j \in R$; $j = 1,2,3,4$. From the boundary condition $\upsilon(1) = \upsilon'(1) = \upsilon''(1) = 0, \omega(1) = \omega'(1) = \omega''(1) = 0$, we have $c_j = d_j = 0$; $j = 2,3,4$. Hence

$$\begin{aligned} \upsilon(\tau) &= c_1(\log\tau)^{n_1-1} - \frac{1}{\Gamma(n_1)}\int_1^\tau \Big(\log\frac{\tau}{\varsigma}\Big)^{n_1-1} x(\varsigma)\frac{d\varsigma}{\varsigma}, \\ \omega(\tau) &= d_1(\log\tau)^{n_2-1} - \frac{1}{\Gamma(n_2)}\int_1^\tau \Big(\log\frac{\tau}{\varsigma}\Big)^{n_2-1} y(\varsigma)\frac{d\varsigma}{\varsigma}, \end{aligned}$$

and we have

$$\begin{aligned} \mathfrak{D}_{1^+}^{\gamma_1}\upsilon(\tau) &= c_1\frac{\Gamma(n_1)}{\Gamma(n_1-\gamma_1)}(\log\tau)^{n_1-\gamma_1-1} - \frac{1}{\Gamma(n_1-\gamma_1)}\int_1^\tau \Big(\log\frac{\tau}{\varsigma}\Big)^{n_1-\gamma_1-1} x(\varsigma)\frac{d\varsigma}{\varsigma}, \\ \mathfrak{D}_{1^+}^{\delta_1}\omega(\tau) &= d_1\frac{\Gamma(n_2)}{\Gamma(n_2-\delta_1)}(\log\tau)^{n_2-\delta_1-1} - \frac{1}{\Gamma(n_2-\delta_1)}\int_1^\tau \Big(\log\frac{\tau}{\varsigma}\Big)^{n_2-\delta_1-1} y(\varsigma)\frac{d\varsigma}{\varsigma}, \\ \mathfrak{D}_{1^+}^{\gamma_2}\omega(\tau) &= d_1\frac{\Gamma(n_2)}{\Gamma(n_2-\gamma_2)}(\log\tau)^{n_2-\gamma_2-1} - \frac{1}{\Gamma(n_2-\gamma_2)}\int_1^\tau \Big(\log\frac{\tau}{\varsigma}\Big)^{n_2-\gamma_2-1} y(\varsigma)\frac{d\varsigma}{\varsigma}, \\ \mathfrak{D}_{1^+}^{\delta_2}\upsilon(\tau) &= c_1\frac{\Gamma(n_1)}{\Gamma(n_1-\delta_2)}(\log\tau)^{n_1-\delta_2-1} - \frac{1}{\Gamma(n_1-\delta_2)}\int_1^\tau \Big(\log\frac{\tau}{\varsigma}\Big)^{n_1-\delta_2-1} x(\varsigma)\frac{d\varsigma}{\varsigma}, \end{aligned}$$

and from the boundary conditions $\lambda_1\mathfrak{D}_{1^+}^{\gamma_1}\upsilon(e) = \mu_1\, \mathfrak{D}_{1^+}^{\gamma_2}\omega(\eta),\ \lambda_2\, \mathfrak{D}_{1^+}^{\delta_1}\omega(e) = \mu_2\mathfrak{D}_{1^+}^{\delta_2}\upsilon(\xi)$ we have

$$\begin{aligned} & c_1\frac{\lambda_1\Gamma(n_1)}{\Gamma(n_1-\gamma_1)} + d_1\frac{-\mu_1\Gamma(n_2)(\log\eta)^{n_2-\gamma_2-1}}{\Gamma(n_2-\gamma_2)} \\ &\quad = \lambda_1\int_1^e \frac{(1-\log\varsigma)^{n_1-\gamma_1-1}}{\Gamma(n_1-\gamma_1)}x(\varsigma)\frac{d\varsigma}{\varsigma} - \mu_1\int_1^\eta \frac{(\log\frac{\eta}{\varsigma})^{n_2-\gamma_2-1}}{\Gamma(n_2-\gamma_2)}y(\varsigma)\frac{d\varsigma}{\varsigma}, \end{aligned}$$

and

$$\begin{aligned} & c_1\frac{-\mu_2\Gamma(n_1)(\log\xi)^{n_1-\delta_2-1}}{\Gamma(n_1-\delta_2)} + d_1\frac{\lambda_2\Gamma(n_2)}{\Gamma(n_2-\delta_1)} \\ &\quad = \lambda_2\int_1^e \frac{(1-\log\varsigma)^{n_2-\delta_1-1}}{\Gamma(n_2-\delta_1)}y(\varsigma)\frac{d\varsigma}{\varsigma} - \mu_2\int_1^\xi \frac{(\log\frac{\xi}{\varsigma})^{n_1-\delta_2-1}}{\Gamma(n_1-\delta_2)}x(\varsigma)\frac{d\varsigma}{\varsigma}. \end{aligned}$$

Solving for c_1 and d_1, we have

$$\begin{aligned} c_1 = \frac{1}{\Delta}\bigg[& \frac{\lambda_1\lambda_2\Gamma(n_2)}{\Gamma(n_1-\gamma_1)\Gamma(n_2-\delta_1)}\int_1^e (1-\log\varsigma)^{n_1-\gamma_1-1}x(\varsigma)\frac{d\varsigma}{\varsigma} \\ & - \frac{\mu_1\lambda_2\Gamma(n_2)}{\Gamma(n_2-\delta_1)\Gamma(n_2-\gamma_2)}\int_1^\eta \Big(\log\frac{\eta}{\varsigma}\Big)^{n_2-\gamma_2-1}y(\varsigma)\frac{d\varsigma}{\varsigma} \\ & + \frac{\mu_1\lambda_2\Gamma(n_2)(\log\eta)^{n_2-\gamma_2-1}}{\Gamma(n_2-\gamma_2)\Gamma(n_2-\delta_1)}\int_1^e (1-\log\varsigma)^{n_2-\delta_1-1}y(\varsigma)\frac{d\varsigma}{\varsigma} \\ & - \frac{\mu_1\mu_2\Gamma(n_2)(\log\eta)^{n_2-\gamma_2-1}}{\Gamma(n_1-\delta_2)\Gamma(n_2-\gamma_2)}\int_1^\xi \Big(\log\frac{\xi}{\varsigma}\Big)^{n_1-\delta_2-1}x(\varsigma)\frac{d\varsigma}{\varsigma}\bigg], \end{aligned}$$

and

$$
\begin{aligned}
d_1 = \frac{1}{\Delta}\Big[&\frac{\lambda_1\lambda_2\Gamma(n_1)}{\Gamma(n_1-\gamma_1)\Gamma(n_2-\delta_1)}\int_1^e (1-\log\varsigma)^{n_2-\delta_1-1}y(\varsigma)\frac{d\varsigma}{\varsigma}\\
&-\frac{\lambda_1\mu_2\Gamma(n_1)}{\Gamma(n_1-\gamma_1)\Gamma(n_1-\delta_2)}\int_1^{\xi}\Big(\log\frac{\xi}{\varsigma}\Big)^{n_1-\delta_2-1}x(\varsigma)\frac{d\varsigma}{\varsigma}\\
&+\frac{\lambda_1\mu_2\Gamma(n_1)(\log\xi)^{n_1-\delta_2-1}}{\Gamma(n_1-\delta_2)\Gamma(n_1-\gamma_1)}\int_1^e (1-\log\varsigma)^{n_1-\gamma_1-1}x(\varsigma)\frac{d\varsigma}{\varsigma}\\
&-\frac{\mu_1\mu_2\Gamma(n_1)(\log\xi)^{n_1-\delta_2-1}}{\Gamma(n_1-\delta_2)\Gamma(n_2-\gamma_2)}\int_1^{\eta}\Big(\log\frac{\eta}{\varsigma}\Big)^{n_2-\gamma_2-1}y(\varsigma)\frac{d\varsigma}{\varsigma}\Big].
\end{aligned}
$$

Accordingly, we have

$$
\begin{aligned}
\upsilon(\tau) = &-\frac{1}{\Gamma(n_1)}\int_1^{\tau}\Big(\log\frac{\tau}{\varsigma}\Big)^{n_1-1}x(\varsigma)\frac{d\varsigma}{\varsigma}+\frac{(\log\tau)^{n_1-1}}{\Delta}\Big[\frac{\lambda_1\lambda_2\Gamma(n_2)}{\Gamma(n_1-\gamma_1)\Gamma(n_2-\delta_1)}\times\\
&\times\int_1^e (1-\log\varsigma)^{n_1-\gamma_1-1}x(\varsigma)\frac{d\varsigma}{\varsigma}-\frac{\mu_1\lambda_2\Gamma(n_2)}{\Gamma(n_2-\delta_1)\Gamma(n_2-\gamma_2)}\int_1^{\eta}\Big(\log\frac{\eta}{\varsigma}\Big)^{n_2-\gamma_2-1}y(\varsigma)\frac{d\varsigma}{\varsigma}\\
&+\frac{\mu_1\lambda_2\Gamma(n_2)(\log\eta)^{n_2-\gamma_2-1}}{\Gamma(n_2-\gamma_2)\Gamma(n_2-\delta_1)}\int_1^e (1-\log\varsigma)^{n_2-\delta_1-1}y(\varsigma)\frac{d\varsigma}{\varsigma}\\
&-\frac{\mu_1\mu_2\Gamma(n_2)(\log\eta)^{n_2-\gamma_2-1}}{\Gamma(n_1-\delta_2)\Gamma(n_2-\gamma_2)}\int_1^{\xi}\Big(\log\frac{\xi}{\varsigma}\Big)^{n_1-\delta_2-1}x(\varsigma)\frac{d\varsigma}{\varsigma}\Big]\\
&=\frac{1}{\Gamma(n_1)}\Big\{\int_1^{\tau}\Big[(\log\tau)^{n_1-1}(1-\log\tau)^{n_1-\gamma_1-1}-\big(\log\frac{\tau}{\varsigma}\big)^{n_1-1}\Big]x(\varsigma)\frac{d\varsigma}{\varsigma}\\
&\quad+\int_{\tau}^e (\log\tau)^{n_1-1}(1-\log\varsigma)^{n_1-\gamma_1-1}x(\varsigma)\frac{d\varsigma}{\varsigma}\Big\}-\frac{1}{\Gamma(n_1)}\int_1^e(\log\tau)^{n_1-1}(1-\log\varsigma)^{n_1-\gamma_1-1}x(\varsigma)\frac{d\varsigma}{\varsigma}\\
&\quad+\frac{\lambda_1\lambda_2(\log\tau)^{n_1-1}\Gamma(n_2)}{\Delta\Gamma(n_1-\gamma_1)\Gamma(n_2-\delta_1)}\int_1^e(1-\log\varsigma)^{n_1-\gamma_1-1}x(\varsigma)\frac{d\varsigma}{\varsigma}\\
&-\frac{\mu_1\lambda_2(\log\tau)^{n_1-1}\Gamma(n_2)}{\Delta\Gamma(n_2-\delta_1)\Gamma(n_2-\gamma_2)}\int_1^{\eta}\Big(\log\frac{\eta}{\varsigma}\Big)^{n_2-\gamma_2-1}y(\varsigma)\frac{d\varsigma}{\varsigma}\\
&+\frac{\mu_1\lambda_2(\log\tau)^{n_1-1}\Gamma(n_2)(\log\eta)^{n_2-\gamma_2-1}}{\Delta\Gamma(n_2-\gamma_2)\Gamma(n_2-\delta_1)}\int_1^e(1-\log\varsigma)^{n_2-\delta_1-1}y(\varsigma)\frac{d\varsigma}{\varsigma}\\
&-\frac{\mu_1\mu_2(\log\tau)^{n_1-1}\Gamma(n_2)(\log\eta)^{n_2-\gamma_2-1}}{\Delta\Gamma(n_1-\delta_2)\Gamma(n_2-\gamma_2)}\int_1^{\xi}\Big(\log\frac{\xi}{\varsigma}\Big)^{n_1-\delta_2-1}x(\varsigma)\frac{d\varsigma}{\varsigma}\\
&=\frac{1}{\Gamma(n_1)}\Big\{\int_1^{\tau}\Big[(\log\tau)^{n_1-1}(1-\log\varsigma)^{n_1-\gamma_1-1}-\big(\log\frac{\tau}{\varsigma}\big)^{n_1-1}\Big]x(\varsigma)\frac{d\varsigma}{\varsigma}\\
&\quad+\int_{\tau}^e(\log\tau)^{n_1-1}(1-\log\varsigma)^{n_1-\gamma_1-1}x(\varsigma)\frac{d\varsigma}{\varsigma}\Big\}\\
&\quad-\frac{\lambda_1\lambda_2\Gamma(n_2)}{\Delta\Gamma(n_1-\gamma_1)\Gamma(n_2-\delta_1)}\int_1^e(\log\tau)^{n_1-1}(1-\log\varsigma)^{n_1-\gamma_1-1}x(\varsigma)\frac{d\varsigma}{\varsigma}\\
&\quad+\frac{\mu_1\mu_2\Gamma(n_2)(\log\eta)^{n_2-\gamma_2-1}(\log\xi)^{n_1-\delta_2-1}}{\Delta\Gamma(n_1-\delta_2)\Gamma(n_2-\gamma_2)}\int_1^e(\log\tau)^{n_1-1}(1-\log\varsigma)^{n_1-\gamma_1-1}x(\varsigma)\frac{d\varsigma}{\varsigma}\\
&\quad+\frac{\lambda_1\lambda_2(\log\tau)^{n_1-1}\Gamma(n_2)}{\Delta\Gamma(n_1-\gamma_1)\Gamma(n_2-\delta_1)}\int_1^e(1-\log\varsigma)^{n_1-\gamma_1-1}x(\varsigma)\frac{d\varsigma}{\varsigma}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{\mu_1\lambda_2(\log\tau)^{n_1-1}\Gamma(n_2)}{\Delta\Gamma(n_2-\delta_1)\Gamma(n_2-\gamma_2)}\int_1^\eta\Big(\log\frac{\eta}{\varsigma}\Big)^{n_2-\gamma_2-1}y(\varsigma)\frac{d\varsigma}{\varsigma}\\
&+ \frac{\mu_1\lambda_2(\log\tau)^{n_1-1}\Gamma(n_2)(\log\eta)^{n_2-\gamma_2-1}}{\Delta\Gamma(n_2-\gamma_2)\Gamma(n_2-\delta_1)}\int_1^e(1-\log\varsigma)^{n_2-\delta_1-1}y(\varsigma)\frac{d\varsigma}{\varsigma}\\
&- \frac{\mu_1\mu_2(\log\tau)^{n_1-1}\Gamma(n_2)(\log\eta)^{n_2-\gamma_2-1}}{\Delta\Gamma(n_1-\delta_2)\Gamma(n_2-\gamma_2)}\int_1^\xi\Big(\log\frac{\xi}{\varsigma}\Big)^{n_1-\delta_2-1}x(\varsigma)\frac{d\varsigma}{\varsigma}
\end{aligned}
$$

$$
\begin{aligned}
\upsilon(\tau) =& \frac{1}{\Gamma(n_1)}\Big\{\int_1^\tau\Big[(\log\tau)^{n_1-1}(1-\log\varsigma)^{n_1-\gamma_1-1}-\big(\log\frac{\tau}{\varsigma}\big)^{n_1-1}\Big]x(\varsigma)\frac{d\varsigma}{\varsigma}\\
&\quad+\int_\tau^e(\log\tau)^{n_1-1}(1-\log\varsigma)^{n_1-\gamma_1-1}x(\varsigma)\frac{d\varsigma}{\varsigma}\Big\}\\
&+\frac{\mu_1\mu_2\Gamma(n_2)\log\tau)^{n_1-1}(\log\eta)^{n_2-\gamma_2-1}}{\Delta\Gamma(n_1-\delta_2)\Gamma(n_2-\gamma_2)}\Big[\int_1^e(\log\xi)^{n_1-\delta_2-1}(1-\log\varsigma)^{n_1-\gamma_1-1}x(\varsigma)\frac{d\varsigma}{\varsigma}\\
&-\int_1^\xi\Big(\log\frac{\xi}{\varsigma}\Big)^{n_1-\delta_2-1}x(\varsigma)\frac{d\varsigma}{\varsigma}\Big]+\frac{\mu_1\lambda_2\Gamma(n_2)\log\tau)^{n_1-1}}{\Delta\Gamma(n_2-\gamma_2)\Gamma(n_2-\delta_1)}\times\\
&\times\Big[\int_1^e(\log\eta)^{n_2-\gamma_2-1}(1-\log\varsigma)^{n_2-\delta_1-1}y(\varsigma)\frac{d\varsigma}{\varsigma}-\int_1^\eta\Big(\log\frac{\eta}{\varsigma}\Big)^{n_2-\gamma_2-1}y(\varsigma)\frac{d\varsigma}{\varsigma}\Big]\\
=&\frac{1}{\Gamma(n_1)}\Big\{\int_1^\tau\Big[(\log\tau)^{n_1-1}(1-\log\varsigma)^{n_1-\gamma_1-1}-\big(\log\frac{\tau}{\varsigma}\big)^{n_1-1}\Big]x(\varsigma)\frac{d\varsigma}{\varsigma}\\
&+\int_\tau^e(\log\tau)^{n_1-1}(1-\log\varsigma)^{n_1-\gamma_1-1}x(\varsigma)\frac{d\varsigma}{\varsigma}\Big\}+\frac{(\log\tau)^{n_1-1}\mu_1\mu_2\Gamma(n_2)(\log\eta)^{n_2-\gamma_2-1}}{\Delta\Gamma(n_1-\delta_2)\Gamma(n_2-\gamma_2)}\\
&\times\Big\{\Big[\int_1^\xi(\log\xi)^{n_1-\delta_2-1}(1-\log\varsigma)^{n_1-\gamma_1-1}-\Big(\log\frac{\xi}{\varsigma}\Big)^{n_1-\delta_2-1}\Big]x(\varsigma)\frac{d\varsigma}{\varsigma}\\
&+\int_\xi^e(\log\xi)^{n_1-\delta_2-1}(1-\log\varsigma)^{n_1-\gamma_1-1}x(\varsigma)\frac{d\varsigma}{\varsigma}\Big\}+\frac{(\log\tau)^{n_1-1}\mu_1\lambda_2\Gamma(n_2)}{\Delta\Gamma(n_2-\gamma_2)\Gamma(n_2-\delta_1)}\\
&\times\Big\{\Big[\int_1^\eta(\log\eta)^{n_2-\gamma_2-1}(1-\log\varsigma)^{n_2-\delta_1-1}-\Big(\log\frac{\eta}{\varsigma}\Big)^{n_2-\gamma_2-1}\Big]y(\varsigma)\frac{d\varsigma}{\varsigma}\\
&\quad+\int_\eta^e(\log\eta)^{n_2-\gamma_2-1}(1-\log\varsigma)^{n_2-\delta_1-1}y(\varsigma)\frac{d\varsigma}{\varsigma}\Big\}\\
=&\int_1^e\zeta_1(\tau,\varsigma)x(\varsigma)\frac{d\varsigma}{\varsigma}+\frac{(\log\tau)^{n_1-1}\mu_1\mu_2\Gamma(n_2)(\log\eta)^{n_2-\gamma_2-1}}{\Delta\Gamma(n_2-\gamma_2)}\int_1^e\zeta_2(\xi,\varsigma)x(\varsigma)\frac{d\varsigma}{\varsigma}\\
&\quad+\frac{(\log\tau)^{n_1-1}\mu_1\lambda_2\Gamma(n_2)}{\Delta\Gamma(n_2-\delta_1)}\int_1^e\zeta_3(\eta,\varsigma)y(\varsigma)\frac{d\varsigma}{\varsigma}\\
\upsilon(\tau) =& \int_1^e\sigma_1(\tau,\varsigma)x(\varsigma)\frac{d\varsigma}{\varsigma}+\int_1^e\sigma_2(\tau,\varsigma)y(\varsigma)\frac{d\varsigma}{\varsigma}.
\end{aligned}
$$

In a similar approach, we conclude that

$$
\begin{aligned}
\omega(\tau) =& \int_1^e\zeta_4(\tau,\varsigma)y(\varsigma)\frac{d\varsigma}{\varsigma}+\frac{(\log\tau)^{n_2-1}\lambda_1\lambda_2\Gamma(n_1)(\log\xi)^{n_1-\delta_1-1}}{\Delta\Gamma(n_1-\delta_2)}\int_1^e\zeta_3(\eta,\varsigma)y(\varsigma)\frac{d\varsigma}{\varsigma}\\
&+\frac{(\log\tau)^{n_2-1}\mu_2\lambda_1\Gamma(n_1)}{\Delta\Gamma(n_1-\gamma_1)}\int_1^e\zeta_2(\xi,\varsigma)x(\varsigma)\frac{d\varsigma}{\varsigma}=\int_1^e\sigma_3(\tau,\varsigma)y(\varsigma)\frac{d\varsigma}{\varsigma}+\int_1^e\sigma_4(\tau,\varsigma)x(\varsigma)\frac{d\varsigma}{\varsigma}.
\end{aligned}
$$

Thus, we get (2.2)

Lemma 2.2. *Let* $3 < n_i \le 4, 0 < m_i \le 1$ *for* $i = 1,2$ *and* $h, k \in C[1,e]$. *Formerly the unique solution of*

$$\begin{cases} \mathfrak{D}_{1^+}^{m_1}(\phi_{p_1}(\mathfrak{D}_{1^+}^{n_1}\upsilon(\tau))) + h(\tau) = 0,\ \tau \in (1,e), \\ \mathfrak{D}_{1^+}^{m_2}(\phi_{p_2}(\mathfrak{D}_{1^+}^{n_2}\omega(\tau))) + k(\tau) = 0,\ \tau \in (1,e), \\ \upsilon(1) = \upsilon'(1) = \upsilon''(1) = 0,\ \ \mathfrak{D}_{1^+}^{\alpha_1}\upsilon(1) = 0,\ \ \lambda_1\ \mathfrak{D}_{1^+}^{\gamma_1}\upsilon(e) = \mu_1\ \mathfrak{D}_{1^+}^{\gamma_2}\omega(\eta), \\ \omega(1) = \omega'(1) = \omega''(1) = 0,\ \ \mathfrak{D}_{1^+}^{\alpha_2}\omega(1) = 0,\ \ \lambda_2\ \mathfrak{D}_{1^+}^{\delta_1}\omega(e) = \mu_2\ \mathfrak{D}_{1^+}^{\delta_2}\upsilon(\xi), \end{cases} \tag{2.5}$$

is

$$\begin{aligned} \upsilon(\tau) = & \int_1^e \sigma_1(\tau,\varsigma)\phi_{q_1}\Big(\frac{1}{\Gamma(m_1)}\int_1^{\varsigma}\big(\log\frac{\varsigma}{\ell}\big)^{m_1-1}h(\ell)\frac{d\ell}{\ell}\Big)\frac{d\varsigma}{\varsigma} \\ & + \int_1^e \sigma_2(\tau,\varsigma)\phi_{q_2}\Big(\frac{1}{\Gamma(m_2)}\int_1^{\varsigma}\big(\log\frac{\varsigma}{\ell}\big)^{m_2-1}k(\ell)\frac{d\ell}{\ell}\Big)\frac{d\varsigma}{\varsigma},\ \ \tau \in (1,e), \\ \omega(\tau) = & \int_1^e \sigma_3(\tau,\varsigma)\phi_{q_2}\Big(\frac{1}{\Gamma(m_2)}\int_1^{\varsigma}\big(\log\frac{\varsigma}{\ell}\big)^{m_2-1}k(\ell)\frac{d\ell}{\ell}\Big)\frac{d\varsigma}{\varsigma} \\ & + \int_1^e \sigma_4(\tau,\varsigma)\phi_{q_1}\Big(\frac{1}{\Gamma(m_1)}\int_1^{\varsigma}\big(\log\frac{\varsigma}{\ell}\big)^{m_1-1}h(\ell)\frac{d\ell}{\ell}\Big)\frac{d\varsigma}{\varsigma},\ \ \tau \in (1,e). \end{aligned} \tag{2.6}$$

Proof. In fact, let $\phi = \mathfrak{D}_{1^+}^{n_1}\upsilon,\ \omega = \phi_{p_1}(\phi)$ and $\psi = \mathfrak{D}_{1^+}^{n_2}\omega,\ \upsilon = \phi_{p_2}(\psi)$. Formerly, the solution of the IVP

$$\begin{cases} \mathfrak{D}_{1^+}^{m_1}\omega(\tau) + h(\tau) = 0,\ \ \tau \in (1,e), \\ \mathfrak{D}_{1^+}^{m_2}\upsilon(\tau) + k(\tau) = 0,\ \ \tau \in (1,e), \\ \omega(1) = 0,\ \vartheta(1) = 0. \end{cases} \tag{2.7}$$

By a similar justification to Lemma 2.1, we have $0 < m_i \le 1, i = 1,2$. An similar integral equation for (2.7) is given by

$$\begin{aligned} \omega(\tau) &= c_1(\log\tau)^{m_1-1} - I_{1^+}^{m_1}h(\tau),\ \ \tau \in (1,e), \\ \upsilon(\tau) &= d_1(\log\tau)^{m_2-1} - I_{1^+}^{m_2}k(\tau),\ \ \tau \in (1,e). \end{aligned}$$

From the relation $\omega(1) = \upsilon(1) = 0$, we get $c_1 = 0, d_1 = 0$ and consequently

$$\omega(\tau) = -I_{1^+}^{m_1}h(\tau),\ \ \upsilon(\tau) = -I_{1^+}^{m_2}k(\tau),\ \ \tau \in (1,e). \tag{2.8}$$

Noting that $\mathfrak{D}_{1^+}^{n_1}\upsilon = \phi,\ \phi = \phi_{p_1}^{-1}(\omega)$ and $\mathfrak{D}_{1^+}^{n_2}\omega = \psi,\ \psi = \phi_{p_2}^{-1}(\upsilon)$ we have from (2.8) that the solution of (2.7) satisfies

$$\begin{cases} \mathfrak{D}_{1^+}^{n_1}\upsilon = \phi_{p_1}^{-1}(-I_{1^+}^{m_1}h(\tau)),\ \tau \in (1,e), \\ \mathfrak{D}_{1^+}^{n_2}\omega = \phi_{p_2}^{-1}(-I_{1^+}^{m_2}k(\tau)),\ \tau \in (1,e), \\ \upsilon(1) = \upsilon'(1) = \upsilon''(1) = 0,\ \ \lambda_1\ \mathfrak{D}_{1^+}^{\gamma_1}\upsilon(e) = \mu_1\ \mathfrak{D}_{1^+}^{\gamma_2}\omega(\eta), \\ \omega(1) = \omega'(1) = \omega''(1) = 0,\ \ \lambda_2\ \mathfrak{D}_{1^+}^{\delta_1}\omega(e) = \mu_2\ \mathfrak{D}_{1^+}^{\delta_2}\upsilon(\xi). \end{cases} \tag{2.9}$$

By Lemma 2.1, the solution of (2.9) can be put down as

$$\begin{aligned} \upsilon(\tau) &= -\int_1^e \sigma_1(\tau,\varsigma)\phi_{p_1}^{-1}(-I_{1^+}^{m_1}h(\varsigma))\frac{d\varsigma}{\varsigma} - \int_1^e \sigma_2(\tau,\varsigma)\phi_{p_2}^{-1}(-I_{1^+}^{m_2}k(\varsigma))\frac{d\varsigma}{\varsigma},\ \ \tau \in (1,e), \\ \omega(\tau) &= -\int_1^e \sigma_3(\tau,\varsigma)\phi_{p_2}^{-1}(-I_{1^+}^{m_2}k(\varsigma))\frac{d\varsigma}{\varsigma} - \int_1^e \sigma_4(\tau,\varsigma)\phi_{p_1}^{-1}(-I_{1^+}^{m_1}h(\varsigma))\frac{d\varsigma}{\varsigma},\ \ \tau \in (1,e), \end{aligned}$$

since $h(\varsigma) \geq 0, k(\varsigma) \geq 0, \varsigma \in [1, e]$, we have

$$\phi_{p_1}^{-1}(-I_{1^+}^{m_1} h(\varsigma)) = -\phi_{q_1}(I_{1^+}^{m_1} h(\varsigma)) \text{ and } \phi_{p_2}^{-1}(-I_{1^+}^{m_2} k(\varsigma)) = -\phi_{q_2}(I_{1^+}^{m_2} k(\varsigma)), \varsigma \in [1, e],$$

which implies that the solution of equation (2.7) is

$$\begin{cases} \upsilon(\tau) &= \int_1^e \sigma_1(\tau,\varsigma)\phi_{q_1}\Big(\frac{1}{\Gamma(m_1)}\int_1^\varsigma \big(\log\frac{\varsigma}{\ell}\big)^{m_1-1} h(\ell)\frac{d\ell}{\ell}\Big)\frac{d\varsigma}{\varsigma} \\ &\quad + \int_1^e \sigma_2(\tau,\varsigma)\phi_{q_2}\Big(\frac{1}{\Gamma(m_2)}\int_1^\varsigma \big(\log\frac{\varsigma}{\ell}\big)^{m_2-1} k(\ell)\frac{d\ell}{\ell}\Big)\frac{d\varsigma}{\varsigma}, \quad \tau \in (1, e), \\ \omega(\tau) &= \int_1^e \sigma_3(\tau,\varsigma)\phi_{q_2}\Big(\frac{1}{\Gamma(m_2)}\int_1^\varsigma \big(\log\frac{\varsigma}{\ell}\big)^{m_2-1} k(\ell)\frac{d\ell}{\ell}\Big)\frac{d\varsigma}{\varsigma} \\ &\quad + \int_1^e \sigma_4(\tau,\varsigma)\phi_{q_1}\Big(\frac{1}{\Gamma(m_1)}\int_1^\varsigma \big(\log\frac{\varsigma}{\ell}\big)^{m_1-1} h(\ell)\frac{d\ell}{\ell}\Big)\frac{d\varsigma}{\varsigma}, \quad \tau \in (1, e). \end{cases}$$

Lemma 2.3. *If* $\Delta > 0$*, then the Green's functions* $\sigma_i(\tau, \varsigma), i = 1, 2, 3, 4$*, defined respectively by* (2.3) *have the successive properties:*

(A1): $\sigma_i(\tau, \varsigma) \geq 0$, for all $\tau, \varsigma \in [1, e]$,
(A2): $\sigma_i(\tau, \varsigma) \leq \sigma_i(e, \varsigma)$, for all $(\tau, \varsigma) \in [1, e] \times [1, e]$,
(A3): $\sigma_i(\tau, \varsigma) \geq \aleph\, \sigma_i(e, \varsigma)$, for all $(\tau, \varsigma) \in I \times (1, e)$, *where* $I = [e^{1/4}, e^{3/4}]$, $\aleph = \min\{(\frac{1}{4})^{n_1-1}, (\frac{1}{4})^{n_2-1}\}$.

Proof. The Green's function $\sigma_i(t, \varsigma), i = 1, 2, 3, 4$ is given in (2.3).
$(A1)$: (i) For $\tau \leq \varsigma$, we have $\zeta_1(\tau, \varsigma) \geq 0$. Let $\varsigma \leq \tau$, we get

$$\begin{aligned} \zeta_1(\tau,\varsigma) &= \frac{1}{\Gamma(n_1)}\Big[(\log\tau)^{n_1-1}(1-\log\varsigma)^{n_1-\gamma_1-1} - \Big(\log\frac{\tau}{\varsigma}\Big)^{n_1-1}\Big] \\ &= \frac{1}{\Gamma(n_1)}\Big[(\log\tau)^{n_1-1}(1-\log\varsigma)^{n_1-\gamma_1-1} - \Big(1-\frac{\log\varsigma}{\log\tau}\Big)^{n_1-1}(\log\tau)^{n_1-1}\Big] \\ &\geq \frac{(\log\tau)^{n_1-1}}{\Gamma(n_1)}[(1-\log\varsigma)^{n_1-\gamma_1-1} - (1-\log\varsigma)^{n_1-1}] \\ &= \frac{(\log\tau)^{n_1-1}}{\Gamma(n_1)}[(1-\log\varsigma)^{-\gamma_1} - 1](1-\log\varsigma)^{n_1-1} \\ &= \frac{(\log\tau)^{n_1-1}}{\Gamma(n_1)}\Big[\Big(1+\gamma_1(\log\varsigma) + \frac{\gamma_1(\gamma_1+1)}{2}(\log\varsigma)^2 + \cdots\Big) - 1\Big](1-\log\varsigma)^{n_1-1} \\ &= \frac{(\log\tau)^{n_1-1}}{\Gamma(n_1)}\Big[\gamma_1(\log\varsigma) + O(\log\varsigma)^2\Big](1-\log\varsigma)^{n_1-1} \geq 0. \end{aligned}$$

(ii) In fact, if $\tau \leq \varsigma$, obviously $\zeta_2(\tau, \varsigma) \geq 0$ holds. If $\varsigma \leq \tau$, we have

$$\begin{aligned} \zeta_2(\tau,\varsigma) &= \frac{1}{\Gamma(n_1-\delta_2)}\Big[(\log\tau)^{n_1-\delta_2-1}(1-\log\varsigma)^{n_1-\gamma_1-1} - \Big(\log\frac{\tau}{\varsigma}\Big)^{n_1-\delta_2-1}\Big] \\ &\geq \frac{(\log\tau)^{n_1-\delta_2-1}}{\Gamma(n_1-\delta_2)}\Big[(1-\log\varsigma)^{n_1-\gamma_1-1} - (1-\log\varsigma)^{n_1-\delta_2-1}\Big] \\ &= \frac{(\log\tau)^{n_1-\delta_2-1}}{\Gamma(n_1-\delta_2)}\Big[(1-\log\varsigma)^{-(\gamma_1-\delta_2)} - 1\Big](1-\log\varsigma)^{n_1-\delta_2-1} \\ &= \frac{(\log\tau)^{n_1-\delta_2-1}}{\Gamma(n_1-\delta_2)}\Big[(\gamma_1-\delta_2)(\log\varsigma) + O(\log\varsigma)^2\Big](1-\log\varsigma)^{n_1-\delta_2-1} \geq 0. \end{aligned}$$

(iii) Let $\tau \le \varsigma$, we have $\zeta_3(\tau,\varsigma) \ge 0$. For $\varsigma \le \tau$, we get

$$\begin{aligned}
\zeta_3(\tau,\varsigma) &= \frac{1}{\Gamma(n_2-\gamma_2)}\Big[(\log\tau)^{n_2-\gamma_2-1}(1-\log\varsigma)^{n_2-\delta_1-1} - \Big(\log\frac{\tau}{\varsigma}\Big)^{n_2-\gamma_2-1}\Big] \\
&\ge \frac{(\log\tau)^{n_2-\gamma_2-1}}{\Gamma(n_2-\gamma_2)}\Big[(1-\log\varsigma)^{n_2-\delta_1-1} - (1-\log\varsigma)^{n_2-\gamma_2-1}\Big] \\
&= \frac{(\log\tau)^{n_2-\gamma_2-1}}{\Gamma(n_1-\gamma_2)}\Big[(1-\log\varsigma)^{-(\delta_1-\gamma_2)} - 1\Big](1-\log\varsigma)^{n_2-\gamma_2-1} \\
&= \frac{(\log\tau)^{n_2-\gamma_2-1}}{\Gamma(n_1-\gamma_2)}\Big[(\delta_1-\gamma_2)(\log\varsigma) + O(\log\varsigma)^2\Big](1-\log\varsigma)^{n_2-\gamma_2-1} \ge 0.
\end{aligned}$$

(iv) For $\tau \le \varsigma$, we have $\zeta_4(\tau,\varsigma) \ge 0$. Let $\varsigma \le \tau$, we have

$$\begin{aligned}
\zeta_4(\tau,\varsigma) &= \frac{1}{\Gamma(n_2)}\Big[(\log\tau)^{n_2-1}(1-\log\varsigma)^{n_2-\delta_1-1} - \Big(\log\frac{\tau}{\varsigma}\Big)^{n_2-1}\Big] \\
&\ge \frac{(\log\tau)^{n_2-1}}{\Gamma(n_2)}[(1-\log\varsigma)^{n_2-\delta_1-1} - (1-\log\varsigma)^{n_2-1}] \\
&= \frac{(\log\tau)^{n_2-1}}{\Gamma(n_2)}[(1-\log\varsigma)^{-\delta_1} - 1](1-\log\varsigma)^{n_2-1} \\
&= \frac{(\log\tau)^{n_2-1}}{\Gamma(n_2)}\Big[(1+\delta_1(\log\varsigma) + \frac{\delta_1(\delta_1+1)}{2}(\log\varsigma)^2 + \cdots) - 1](1-\log\varsigma)^{n_2-1} \\
&= \frac{(\log\tau)^{n_2-1}}{\Gamma(n_2)}\Big[\delta_1(\log\varsigma) + O(\log\varsigma)^2](1-\log\varsigma)^{n_2-1} \ge 0,
\end{aligned}$$

which implies that $\sigma_i(\tau,\varsigma) \ge 0$, for all $\tau,\varsigma \in [1,e], i = 1,2,3,4$.

$(A2)$: (i) Let $\tau \le \varsigma$, we have

$$\frac{d}{d\tau}\zeta_1(\tau,\varsigma) = \frac{1}{\Gamma(n_1)}[(n_1-1)(\log\tau)^{n_1-2}(1-\log\varsigma)^{n_1-\gamma_1-1}] \ge 0.$$

For $\varsigma \le \tau$, we get

$$\begin{aligned}
\frac{d}{d\tau}\zeta_1(\tau,\varsigma) &= \frac{1}{\Gamma(n_1)}\Big[(n_1-1)(\log\tau)^{n_1-2}(1-\log\varsigma)^{n_1-\gamma_1-1} - (n_1-1)\Big(\log\frac{\tau}{\varsigma}\Big)^{n_1-2}\Big] \\
&\ge \frac{(n_1-1)(\log\tau)^{n_1-2}}{\Gamma(n_1)}\Big[(1-\log\varsigma)^{n_1-\gamma_1-1} - (1-\log\varsigma)^{n_1-2}\Big] \\
&= \frac{(n_1-1)(\log\tau)^{n_1-2}}{\Gamma(n_1)}\Big[1-(1-\log\varsigma)^{\gamma_1-1}\Big](1-\log\varsigma)^{n_1-\gamma_1-1} \\
&= \frac{(n_1-1)(\log\tau)^{n_1-2}}{\Gamma(n_1)}\Big[(\gamma_1-1)(\log\varsigma) + O(\log\varsigma)^2\Big](1-\log\varsigma)^{n_1-\gamma_1-1} \ge 0.
\end{aligned}$$

Thus, for all $(\tau,\varsigma) \in [1,e]\times[1,e]$, we get

$$\frac{d}{d\tau}\sigma_1(\tau,\varsigma) = \frac{d}{d\tau}\zeta_1(\tau,\varsigma) + \frac{(\alpha_1-1)(\log\tau)^{n_1-2}\mu_1\mu_2\Gamma(n_2)(\log\xi)^{n_2-\gamma_1-1}}{\Delta\Gamma(n_2-\gamma_1)}\zeta_2(\xi,\varsigma) \ge 0.$$

(ii) Let $\tau \leq \varsigma$, we have

$$\frac{d}{d\tau}\zeta_4(\tau,\varsigma) = \frac{1}{\Gamma(n_2)}\big[(n_2-1)(\log\tau)^{n_2-2}(1-\log\varsigma)^{n_2-\gamma_2-1}\big] \geq 0.$$

For $\varsigma \leq \tau$, we get

$$\begin{aligned}
\frac{d}{d\tau}\zeta_4(\tau,\varsigma) &= \frac{1}{\Gamma(n_2)}\Big[(n_2-1)(\log\tau)^{n_2-2}(1-\log\varsigma)^{n_2-\gamma_2-1} - (n_2-1)\Big(\log\frac{\tau}{\varsigma}\Big)^{n_2-2}\Big] \\
&= \frac{(n_2-1)}{\Gamma n_2}\Big[(\log\tau)^{n_2-2}(1-\log\varsigma)^{n_2-\gamma_2-1} - \Big(1-\frac{\log\varsigma}{\log\tau}\Big)^{n_2-2}(\log t)^{n_2-2}\Big] \\
&\geq \frac{(n_2-1)(\log\tau)^{n_2-2}}{\Gamma(n_2)}\Big[(1-\log\varsigma)^{n_2-\gamma_2-1} - (1-\log\varsigma)^{n_2-2}\Big] \\
&= \frac{(n_2-1)(\log\tau)^{n_2-2}}{\Gamma(n_2)}\Big[1-(1-\log\varsigma)^{\gamma_2-1}\Big](1-\log\varsigma)^{n_2-\gamma_2-1} \\
&= \frac{(n_2-1)(\log\tau)^{n_2-2}}{\Gamma(n_2)}\Big[(\gamma_2-1)(\log\varsigma) + O(\log\varsigma)^2\Big](1-\log\varsigma)^{n_2-\gamma_2-1} \geq 0.
\end{aligned}$$

Thus, for all $(\tau,\varsigma) \in [1,e]\times[1,e]$, we get

$$\frac{d}{d\tau}\sigma_3(\tau,\varsigma) = \frac{d}{d\tau}\zeta_4(\tau,\varsigma) + \frac{(n_1-1)(\log\tau)^{n_2-2}\lambda_1\lambda_2\Gamma(n_1)(\log\eta)^{n_1-\gamma_2-1}}{\Delta\Gamma(n_1-\gamma_2)}\zeta_3(\xi,\varsigma) \geq 0.$$

Similarly, we get $\frac{d}{d\tau}\sigma_2(\tau,\varsigma) \geq 0$ and $\frac{d}{d\tau}\sigma_4(\tau,\varsigma) \geq 0$, that implies $\sigma_i(\tau,\varsigma), i = 1,2,3,4$, are the monotone nondecreasing functions, so

$$\sigma_i(\tau,\varsigma) \leq \sigma_i(e,\varsigma), \text{ for all } (\tau,\varsigma) \in [1,e]\times[1,e], i = 1,2,3,4.$$

($A3$) : (i) For $\tau \leq \varsigma$, we have

$$\zeta_1(\tau,\varsigma) = \frac{1}{\Gamma(n_1)}\Big[(\log\tau)^{n_1-1}(1-\log\varsigma)^{n_1-\gamma_1-1} = (\log\tau)^{n_1-1}\zeta_1(e,\varsigma), \quad \text{for } \varsigma \in (1,e).$$

For $\varsigma \leq \tau$, we get

$$\begin{aligned}
\zeta_1(\tau,\varsigma) &= \frac{1}{\Gamma(n_1)}\Big[(\log\tau)^{n_1-1}(1-\log\varsigma)^{n_1-\gamma_1-1} - \Big(\log\frac{\tau}{\varsigma}\Big)^{n_1-1}\Big] \\
&\geq \frac{(\log\tau)^{n_1-1}}{\Gamma(n_1)}[(1-\log\varsigma)^{n_1-\gamma_1-1} - (1-\log\varsigma)^{n_1-1}] \\
&= (\log\tau)^{n_1-1}\zeta_1(e,\varsigma), \quad \text{for } \varsigma \in (1,e).
\end{aligned}$$

Thus,

$$\zeta_1(\tau,\varsigma) \geq (\log\tau)^{n_1-1}\zeta_1(e,\varsigma), \quad \text{for all } (\tau,\varsigma) \in [1,e]\times(1,e).$$

Then

$$\begin{aligned}
\sigma_1(\tau,\varsigma) &= \zeta_1(\tau,\varsigma) + \frac{(\log\tau)^{n_1-1}\mu_1\mu_2\Gamma(n_2)(\log\eta)^{n_2-\gamma_2-1}}{\Delta\Gamma(n_2-\gamma_2)}\zeta_2(\xi,\varsigma) \\
&\geq (\log\tau)^{n_1-1}\zeta_1(e,\varsigma) + \frac{(\log\tau)^{n_1-1}\mu_1\mu_2\Gamma(n_2)(\log\eta)^{n_2-\gamma_2-1}}{\Delta\Gamma(n_2-\gamma_2)}\zeta_2(\xi,\varsigma) \\
&= (\log\tau)^{n_1-1}\sigma_1(e,\varsigma) \geq \Big(\frac{1}{4}\Big)^{n_1-1}\sigma_1(e,\varsigma), \quad \text{for all } (\tau,\varsigma) \in I\times(1,e).
\end{aligned}$$

(ii) $\sigma_2(\tau,\varsigma)=\frac{(\log\tau)^{n_1-1}\mu_1\lambda_2\Gamma(n_2)}{\Delta\Gamma(n_2-\delta_1)}\zeta_3(\eta,\varsigma)\geq\left(\frac{1}{4}\right)^{n_1-1}\sigma_2(e,\varsigma),\ \text{ for all } (\tau,\varsigma)\in I\times(1,e).$
(iii) For $\tau\leq\varsigma$, we have

$$\zeta_4(\tau,\varsigma)=\frac{1}{\Gamma(n_2)}\Big[(\log\tau)^{n_2-1}(1-\log\varsigma)^{n_2-\delta_1-1}\Big]=\big(\log\tau\big)^{n_2-1}\zeta_4(e,\varsigma),\ \text{ for } \varsigma\in(1,e).$$

For $\varsigma\leq\tau$, we get

$$\begin{aligned}\zeta_4(\tau,\varsigma)&=\frac{1}{\Gamma(n_2)}\Big[(\log\tau)^{n_2-1}(1-\log\varsigma)^{n_2-\delta_1-1}-\Big(\log\frac{\tau}{\varsigma}\Big)^{n_2-1}\Big]\\&\geq\frac{(\log\tau)^{n_2-1}}{\Gamma(n_2)}[(1-\log\varsigma)^{n_2-\delta_1-1}-(1-\log\varsigma)^{n_2-1}]\\&=(\log\tau)^{n_2-1}\zeta_4(e,\varsigma),\ \text{ for } \varsigma\in(1,e).\end{aligned}$$

Thus,

$$\zeta_4(\tau,\varsigma)\geq\big(\log\tau\big)^{n_2-1}\zeta_4(e,\varsigma),\ \text{ for all } (\tau,\varsigma)\in[1,e]\times(1,e).$$

Then

$$\begin{aligned}\sigma_3(\tau,\varsigma)&=\zeta_4(\tau,\varsigma)+\frac{(\log\tau)^{n_2-1}\lambda_1\lambda_2\Gamma(n_1)(\log\xi)^{n_1-\gamma_2-1}}{\Delta\Gamma(n_1-\gamma_2)}\zeta_3(\eta,\varsigma)\\&\geq(\log\tau)^{n_2-1}\zeta_4(e,\varsigma)+\frac{(\log\tau)^{n_2-1}\lambda_1\lambda_2\Gamma(n_1)(\log\xi)^{n_1-\gamma_2-1}}{\Delta\Gamma(n_1-\gamma_2)}\zeta_3(\eta,\varsigma)\\&=\big(\log\tau\big)^{n_2-1}\sigma_3(e,\varsigma)\geq\Big(\frac{1}{4}\Big)^{n_2-1}\sigma_3(e,\varsigma),\ \text{ for all } (\tau,\varsigma)\in I\times(1,e).\end{aligned}$$

(iv) $\sigma_4(\tau,\varsigma)=\frac{(\log\tau)^{n_2-1}\mu_2\lambda_1\Gamma(n_1)}{\Delta\Gamma(n_1-\gamma_1)}\zeta_2(\xi,\varsigma)\geq\left(\frac{1}{4}\right)^{n_2-1}\sigma_4(e,\varsigma),\ \text{ for all } (\tau,\varsigma)\in I\times(1,e).$
Therefore, we have $\sigma_i(\tau,\varsigma)\geq\aleph\sigma_i(e,\varsigma)$ for all $(\tau,\varsigma)\in I\times(1,e)$, $i=1,2,3,4$, where $I=[e^{1/4},e^{3/4}]$, $\aleph=\min\{(\frac{1}{4})^{n_1-1},(\frac{1}{4})^{n_2-1}\}$.

We consider the Banach space $\mathcal{X}=C[1,e]$ with the norm $\|\cdot\|$ and the Banach space $\mathcal{Y}=\mathcal{X}\times\mathcal{X}$ with the norm $\|(\upsilon,\omega)\|=\max\{\|\upsilon\|,\|\omega\|\}$; $\|\upsilon\|=\max\limits_{t\in[1,e]}|\upsilon(t)|$; $\|\omega\|=\max\limits_{t\in[1,e]}|\omega(t)|$. We define the cone

$$\mathcal{P}=\{(\upsilon,\omega)\in\mathcal{Y};\ \upsilon(t)\geq0,\ \omega(t)\geq0,\forall\ t\in[1,e],\min_{t\in I}\{\upsilon(t)+\omega(t)\}\geq\aleph\|(\upsilon,\omega)\|\},$$

where $I=[e^{1/4},e^{3/4}]$, $\aleph=\min\{(\frac{1}{4})^{n_1-1},(\frac{1}{4})^{n_2-1}\}$.

Consider the coupled system of integral equations

$$\begin{cases}\upsilon(\tau)&=\int_1^e\sigma_1(\tau,\varsigma)\phi_{q_1}\Big(\frac{1}{\Gamma(m_1)}\int_1^\varsigma\big(\log\frac{\varsigma}{\ell}\big)^{m_1-1}\mathfrak{f}_1(\ell,\upsilon(\ell),\omega(\ell))\frac{d\ell}{\ell}\Big)\frac{d\varsigma}{\varsigma}\\&\quad+\int_1^e\sigma_2(\tau,\varsigma)\phi_{q_2}\Big(\frac{1}{\Gamma(m_2)}\int_1^\varsigma\big(\log\frac{\varsigma}{\ell}\big)^{m_2-1}\mathfrak{f}_2(\ell,\upsilon(\ell),\omega(\ell))\frac{d\ell}{\ell}\Big)\frac{d\varsigma}{\varsigma},\ \ \tau\in(1,e),\\\omega(\tau)&=\int_1^e\sigma_3(\tau,\varsigma)\phi_{q_2}\Big(\frac{1}{\Gamma(m_2)}\int_1^\varsigma\big(\log\frac{\varsigma}{\ell}\big)^{m_2-1}\mathfrak{f}_2(\ell,\upsilon(\ell),\omega(\ell))\frac{d\ell}{\ell}\Big)\frac{d\varsigma}{\varsigma}\\&\quad+\int_1^e\sigma_4(\tau,\varsigma)\phi_{q_1}\Big(\frac{1}{\Gamma(m_1)}\int_1^\varsigma\big(\log\frac{\varsigma}{\ell}\big)^{m_1-1}\mathfrak{f}_1(\ell,\upsilon(\ell),\omega(\ell))\frac{d\ell}{\ell}\Big)\frac{d\varsigma}{\varsigma},\ \ \tau\in(1,e).\end{cases}\tag{2.10}$$

By Lemma 2.2, $(\upsilon,\omega)\in\mathcal{Y}$ is a solution of boundary value problems (1.1) and (1.2) if and only if it is a solution of the system of integral Eq (2.10).

Next, define the operators $\Upsilon_1,\Upsilon_2:\mathcal{Y}\to\mathcal{X}$ and $\Upsilon:\mathcal{Y}\to\mathcal{Y}$ by

$$\Upsilon(\upsilon,\omega)=(\Upsilon_1(\upsilon,\omega),\Upsilon_2(\upsilon,\omega)),\tag{2.11}$$

where

$$\Upsilon_1(\upsilon,\omega)=\int_1^e \sigma_1(\tau,\varsigma)\phi_{q_1}\Big(\frac{1}{\Gamma(m_1)}\int_1^\varsigma \Big(\log\frac{\varsigma}{\ell}\Big)^{m_1-1}\mathfrak{f}_1(\ell,\upsilon(\ell),\omega(\ell))\frac{d\ell}{\ell}\Big)\frac{d\varsigma}{\varsigma}$$
$$+\int_1^e \sigma_2(\tau,\varsigma)\phi_{q_2}\Big(\frac{1}{\Gamma(m_2)}\int_1^\varsigma \Big(\log\frac{\varsigma}{\ell}\Big)^{m_2-1}\mathfrak{f}_2(\ell,\upsilon(\ell),\omega(\ell))\frac{d\ell}{\ell}\Big)\frac{d\varsigma}{\varsigma},\quad \tau\in(1,e),$$
$$\Upsilon_2(\upsilon,\omega)=\int_1^e \sigma_3(\tau,\varsigma)\phi_{q_2}\Big(\frac{1}{\Gamma(m_2)}\int_1^\varsigma \Big(\log\frac{\varsigma}{\ell}\Big)^{m_2-1}\mathfrak{f}_2(\ell,\upsilon(\ell),\omega(\ell))\frac{d\ell}{\ell}\Big)\frac{d\varsigma}{\varsigma}$$
$$+\int_1^e \sigma_4(\tau,\varsigma)\phi_{q_1}\Big(\frac{1}{\Gamma(m_1)}\int_1^\varsigma \Big(\log\frac{\varsigma}{\ell}\Big)^{m_1-1}\mathfrak{f}_1(\ell,\upsilon(\ell),\omega(\ell))\frac{d\ell}{\ell}\Big)\frac{d\varsigma}{\varsigma},\quad \tau\in(1,e).$$

Then, $(\upsilon,\omega)\in\mathcal{Y}$ is a solution of boundary value problems (1.1) and (1.2) if and only if it is a fixed point of the operator Υ.

Lemma 2.4. *The operator* Υ *defined by* (2.11)*, then* $\Upsilon:\mathcal{P}\to\mathcal{P}$ *is a completely continuous.*

Proof. By using standard arguments, we can easily show that, the operator Υ is completely continuous and we need only to prove $\Upsilon(\mathcal{P})\subset\mathcal{P}$. Let $(\upsilon,\omega)\in\mathcal{P}$ be an arbitrary element. Then by Lemma 2.3, we have

$$\|\Upsilon_1(\upsilon,\omega)\|\leq\int_1^e \sigma_1(e,\varsigma)\phi_{q_1}\Big(\frac{1}{\Gamma(m_1)}\int_1^\varsigma \Big(\log\frac{\varsigma}{\ell}\Big)^{m_1-1}\mathfrak{f}_1(\ell,\upsilon(\ell),\omega(\ell))\frac{d\ell}{\ell}\Big)\frac{d\varsigma}{\varsigma}$$
$$+\int_1^e \sigma_2(e,\varsigma)\phi_{q_2}\Big(\frac{1}{\Gamma(m_2)}\int_1^\varsigma \Big(\log\frac{\varsigma}{\ell}\Big)^{m_2-1}\mathfrak{f}_2(\ell,\upsilon(\ell),\omega(\ell))\frac{d\ell}{\ell}\Big)\frac{d\varsigma}{\varsigma},$$
$$\|\Upsilon_2(\upsilon,\omega)\|\leq\int_1^e \sigma_3(e,\varsigma)\phi_{q_2}\Big(\frac{1}{\Gamma(m_2)}\int_1^\varsigma \Big(\log\frac{\varsigma}{\ell}\Big)^{m_2-1}\mathfrak{f}_2(\ell,\upsilon(\ell),\omega(\ell))\frac{d\ell}{\ell}\Big)\frac{d\varsigma}{\varsigma}$$
$$+\int_1^e \sigma_4(e,\varsigma)\phi_{q_1}\Big(\frac{1}{\Gamma(m_1)}\int_1^\varsigma \Big(\log\frac{\varsigma}{\ell}\Big)^{m_1-1}\mathfrak{f}_1(\ell,\upsilon(\ell),\omega(\ell))\frac{d\ell}{\ell}\Big)\frac{d\varsigma}{\varsigma},$$

and

$$\min_{\tau\in I}\Upsilon_1(\upsilon,\omega)=\min_{\tau\in I}\Big[\int_1^e \sigma_1(\tau,\varsigma)\phi_{q_1}\Big(\frac{1}{\Gamma(m_1)}\int_1^\varsigma \Big(\log\frac{\varsigma}{\ell}\Big)^{m_1-1}\mathfrak{f}_1(\ell,\upsilon(\ell),\omega(\ell))\frac{d\ell}{\ell}\Big)\frac{d\varsigma}{\varsigma}$$
$$+\int_1^e \sigma_2(\tau,\varsigma)\phi_{q_2}\Big(\frac{1}{\Gamma(m_2)}\int_1^\varsigma \Big(\log\frac{\varsigma}{\ell}\Big)^{m_2-1}\mathfrak{f}_2(\ell,\upsilon(\ell),\omega(\ell))\frac{d\ell}{\ell}\Big)\frac{d\varsigma}{\varsigma}\Big]$$
$$\geq\Big(\frac{1}{4}\Big)^{n_1-1}\Big[\int_1^e \sigma_1(e,\varsigma)\phi_{q_1}\Big(\frac{1}{\Gamma(m_1)}\int_1^\varsigma \Big(\log\frac{\varsigma}{\ell}\Big)^{m_1-1}\mathfrak{f}_1(\ell,\upsilon(\ell),\omega(\ell))\frac{d\ell}{\ell}\Big)\frac{d\varsigma}{\varsigma}$$
$$+\int_1^e \sigma_2(e,\varsigma)\phi_{q_2}\Big(\frac{1}{\Gamma(m_2)}\int_1^\varsigma \Big(\log\frac{\varsigma}{\ell}\Big)^{m_2-1}\mathfrak{f}_2(\ell,\upsilon(\ell),\omega(\ell))\frac{d\ell}{\ell}\Big)\frac{d\varsigma}{\varsigma}\Big]$$
$$\geq\aleph\|\Upsilon_1(\upsilon,\omega)\|.$$

Similarly, $\min_{\tau\in I}\Upsilon_2(\upsilon,\omega)(\tau)\geq\aleph\|\Upsilon_2(\upsilon,\omega)\|$. Therefore

$$\min_{\tau\in I}\{\Upsilon_1(\upsilon,\omega)(\tau)+\Upsilon_2(\upsilon,\omega)(\tau)\}\geq\aleph\|\Upsilon_1(\upsilon,\omega)\|+\aleph\|\Upsilon_2(\upsilon,\omega)\|=\aleph\|(\Upsilon_1(\upsilon,\omega),\Upsilon_2(\upsilon,\omega))\|=\aleph\|\Upsilon(\upsilon,\omega)\|.$$

Hence, we get $\Upsilon(\mathcal{P})\subset\mathcal{P}$.

Next, we prove that Υ is a completely continuous operator. For this, let $\mho\subset\mathcal{P}$ be any bounded set. Then there exist M and N such that

$$\mathfrak{f}_1(\tau,\upsilon(\tau),\omega(\tau))\leq \mathrm{M},\quad \mathfrak{f}_2(\tau,\upsilon(\tau),\omega(\tau))\leq \mathrm{N}.$$

Then, for any $(\upsilon, \omega) \in \mho$, it follows from Lemma 2.3, we have

$$\begin{aligned}
\Upsilon_1(\upsilon,\omega) &\le \int_1^e \sigma_1(e,\varsigma)\phi_{q_1}\Big(\frac{1}{\Gamma(m_1)}\int_1^{\varsigma}\big(\log\frac{\varsigma}{\ell}\big)^{m_1-1}\mathfrak{f}_1(\ell,\upsilon(\ell),\omega(\ell))\frac{d\ell}{\ell}\Big)\frac{d\varsigma}{\varsigma}\\
&\quad+\int_1^e \sigma_2(e,\varsigma)\phi_{q_2}\Big(\frac{1}{\Gamma(m_2)}\int_1^{\varsigma}\big(\log\frac{\varsigma}{\ell}\big)^{m_2-1}\mathfrak{f}_2(\ell,\upsilon(\ell),\omega(\ell))\frac{d\ell}{\ell}\Big)\frac{d\varsigma}{\varsigma}\\
&\le \int_1^e \sigma_1(e,\varsigma)\phi_{q_1}\Big(\frac{\mathrm{M}}{\Gamma(m_1)}\int_1^{\varsigma}\big(\log\frac{\varsigma}{\ell}\big)^{m_1-1}\frac{d\ell}{\ell}\Big)\frac{d\varsigma}{\varsigma}\\
&\quad+\int_1^e \sigma_2(e,\varsigma)\phi_{q_2}\Big(\frac{\mathrm{N}}{\Gamma(m_2)}\int_1^{\varsigma}\big(\log\frac{\varsigma}{\ell}\big)^{m_2-1}\frac{d\ell}{\ell}\Big)\frac{d\varsigma}{\varsigma}\\
&\le \Big(\frac{\mathrm{M}}{\Gamma(m_1+1)}\Big)^{q_1-1}\int_1^e \sigma_1(e,\varsigma)(\log\varsigma)^{m_1(q_1-1)}\frac{d\varsigma}{\varsigma}\\
&\quad+\Big(\frac{\mathrm{N}}{\Gamma(m_2+1)}\Big)^{q_2-1}\int_1^e \sigma_2(e,\varsigma)(\log\varsigma)^{m_2(q_2-1)}\frac{d\varsigma}{\varsigma}\\
&< \infty.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\Upsilon_2(\upsilon,\omega) &\le \Big(\frac{\mathrm{N}}{\Gamma(m_2+1)}\Big)^{q_2-1}\int_1^e \sigma_3(e,\varsigma)(\log\varsigma)^{m_2(q_2-1)}\frac{d\varsigma}{\varsigma}\\
&\quad+\Big(\frac{\mathrm{M}}{\Gamma(m_1+1)}\Big)^{q_1-1}\int_1^e \sigma_4(e,\varsigma)(\log\varsigma)^{m_1(q_1-1)}\frac{d\varsigma}{\varsigma}\\
&< \infty.
\end{aligned}$$

So, the operator Υ is uniformly bounded.

Next, we show that Υ is equicontinuous. For this, set $\mathrm{L}_1 = \max_{\tau\in I}\mathfrak{f}_1|(\tau,\upsilon(\tau),\omega(\tau))|$, and $\mathrm{L}_2 = \max_{\tau\in I}\mathfrak{f}_2|(\tau,\upsilon(\tau),\omega(\tau))|$. Choose $\tau_1,\tau_2 \in I$ such that $\tau_1 < \tau_2$. Therefore, for $(\upsilon,\omega) \in \mathcal{P}$, we have

$$\begin{aligned}
&|\Upsilon_1(\upsilon(\tau_2),\omega(\tau_2)) - \Upsilon_1(\upsilon(\tau_1),\omega(\tau_1))|\\
&\le \int_1^e |\sigma_1(\tau_2,\varsigma)-\sigma_1(\tau_1,\varsigma)|\Big|\phi_{q_1}\Big(\frac{1}{\Gamma(m_1)}\int_1^{\varsigma}\big(\log\frac{\varsigma}{\ell}\big)^{m_1-1}\mathfrak{f}_1(\ell,\upsilon(\ell),\omega(\ell))\frac{d\ell}{\ell}\Big)\frac{1}{\varsigma}\Big|d\varsigma\\
&\quad+\int_1^e |\sigma_2(\tau_1,\varsigma)-\sigma_2(\tau_2,\varsigma)|\Big|\phi_{q_2}\Big(\frac{1}{\Gamma(m_2)}\int_1^{\varsigma}\big(\log\frac{\varsigma}{\ell}\big)^{m_2-1}\mathfrak{f}_2(\ell,\upsilon(\ell),\omega(\ell))\frac{d\ell}{\ell}\Big)\frac{1}{\varsigma}\Big|d\varsigma\\
&\le \int_1^e |\sigma_1(\tau_2,\varsigma)-\sigma_1(\tau_1,\varsigma)|\Big|\phi_{q_1}\Big(\frac{\mathrm{L}_1}{\Gamma(m_1)}\int_1^{\varsigma}\big(\log\frac{\varsigma}{\ell}\big)^{m_1-1}\frac{d\ell}{\ell}\Big)\frac{1}{\varsigma}\Big|d\varsigma\\
&\quad+\int_1^e |\sigma_2(\tau_1,\varsigma)-\sigma_2(\tau_2,\varsigma)|\Big|\phi_{q_2}\Big(\frac{\mathrm{L}_2}{\Gamma(m_2)}\int_1^{\varsigma}\big(\log\frac{\varsigma}{\ell}\big)^{m_2-1}\frac{d\ell}{\ell}\Big)\frac{1}{\varsigma}\Big|d\varsigma\\
&\le \left(\frac{\mathrm{L}_1}{\Gamma(m_1+1)}\right)^{q_1-1}\int_1^e |\sigma_1(\tau_2,\varsigma)-\sigma_1(\tau_1,\varsigma)|\frac{|\log\varsigma^{m_1(q_1-1)}|}{\varsigma}d\varsigma\\
&\quad+\left(\frac{\mathrm{L}_2}{\Gamma(m_2+1)}\right)^{q_2-1}\int_1^e |\sigma_2(\tau_2,\varsigma)-\sigma_2(\tau_1,\varsigma)|\frac{|\log\varsigma^{m_2(q_2-1)}|}{\varsigma}d\varsigma\\
&\le \left(\frac{\mathrm{L}_1}{\Gamma(m_1+1)}\right)^{q_1-1}\int_1^e \frac{1}{\varsigma}|\sigma_1(\tau_2,\varsigma)-\sigma_1(\tau_1,\varsigma)|d\varsigma
\end{aligned}$$

$$+\left(\frac{\mathrm{L}_2}{\Gamma(m_2+1)}\right)^{q_2-1}\int_1^e \frac{1}{\varsigma}|\sigma_2(\tau_2,\varsigma)-\sigma_2(\tau_1,\varsigma)|d\varsigma. \tag{2.12}$$

Next, setting

$$\begin{aligned}
\zeta_{10}(\tau,\varsigma) &= \frac{1}{\Gamma(n_1)}\Big[(\log\tau)^{n_1-1}(1-\log\varsigma)^{n_1-\gamma_1-1}-(\log\frac{\tau}{\varsigma})^{n_1-1}\Big],\\
\zeta_{11}(\tau,\varsigma) &= \frac{1}{\Gamma(n_1)}\Big[(\log\tau)^{n_1-1}(1-\log\varsigma)^{n_1-\gamma_1-1}\Big],\\
\zeta_{20}(\tau,\varsigma) &= \frac{1}{\Gamma(n_1-\delta_2)}\Big[(\log\tau)^{n_1-\delta_2-1}(1-\log\varsigma)^{n_1-\gamma_1-1}-(\log\frac{\tau}{\varsigma})^{n_1-\delta_2-1}\Big],\\
\zeta_{21}(\tau,\varsigma) &= \frac{1}{\Gamma(n_1-\delta_2)}\Big[(\log\tau)^{n_1-\delta_2-1}(1-\log\varsigma)^{n_1-\gamma_1-1}\Big],\\
ß &= \frac{\mu_1\mu_2\Gamma(n_2)(\log\eta)^{n_2-\gamma_2-1}}{\Delta\Gamma(n_2-\gamma_2)}.
\end{aligned}$$

Then from (2.4), we have

$$\zeta_1(\tau,\varsigma)=\begin{cases}\zeta_{10}(\tau,\varsigma), & 1\le\varsigma\le\tau\le e,\\ \zeta_{11}(\tau,\varsigma), & 1\le\tau\le\varsigma\le e,\end{cases}$$

$$\zeta_2(\tau,\varsigma)=\begin{cases}\zeta_{20}(\tau,\varsigma), & 1\le\varsigma\le\tau\le e,\\ \zeta_{21}(\tau,\varsigma), & 1\le\tau\le\varsigma\le e.\end{cases}$$

From (2.3), we obtain

$$\sigma_1(\tau,\varsigma)=\zeta_1(\tau,\varsigma)+ß(\log\tau)^{n_1-1}\zeta_2(\xi,\varsigma).$$

Now, let

$$\begin{aligned}
\sigma_{10}(\tau,\varsigma) &= \zeta_{10}(\tau,\varsigma)+ß(\log\tau)^{n_1-1}\zeta_{20}(\xi,\varsigma),\\
\sigma_{11}(\tau,\varsigma) &= \zeta_{11}(\tau,\varsigma)+ß(\log\tau)^{n_1-1}\zeta_{21}(\xi,\varsigma).
\end{aligned}$$

Now, Consider

$$\begin{aligned}
\int_1^e \frac{1}{\varsigma}|\sigma_1(\tau_2,\varsigma)-\sigma_1(\tau_1,\varsigma)|d\varsigma &= \int_1^{\tau_1}\frac{1}{\varsigma}|\sigma_{10}(\tau_2,\varsigma)-\sigma_{10}(\tau_1,\varsigma)|d\varsigma+\int_{\tau_1}^{\tau_2}\frac{1}{\varsigma}|\sigma_{10}(\tau_2,\varsigma)-\sigma_{11}(\tau_2,\varsigma)|d\varsigma\\
&\quad+\int_{\tau_2}^e\frac{1}{\varsigma}|\sigma_{11}(\tau_2,\varsigma)-\sigma_{11}(\tau_1,\varsigma)|d\varsigma,
\end{aligned} \tag{2.13}$$

where

$$\begin{aligned}
\int_1^{\tau_1}\frac{1}{\varsigma}|\sigma_{10}(\tau_2,\varsigma)-\sigma_{10}(\tau_1,\varsigma)|d\varsigma &\le \int_1^e \frac{|(1-\log\varsigma)^{n_1-\gamma_1-1}|}{\Gamma(n_1)\varsigma}|(\log\tau_2)^{n_1-1}-(\log\tau_1)^{n_1-1}|d\varsigma\\
&+\int_1^{\tau_1}\frac{1}{\Gamma(n_1)}\frac{1}{\varsigma}\Big|(\log\frac{\tau_2}{\varsigma})^{n_1-1}-(\log\frac{\tau_1}{\varsigma})^{n_1-1}\Big|d\varsigma\\
&+\frac{ß}{\Gamma(n_1-\delta_2)}\int_1^e\frac{1}{\varsigma}|(1-\log\varsigma)^{n_1-\gamma_1-1}||(\log\tau_2)^{2n_1-\delta_2-1}-(\log\tau_1)^{2n_1-\delta_2-1}|d\varsigma\\
&+\frac{ß}{\Gamma(n_1-\delta_2)}\int_1^e\frac{1}{\varsigma}|(\log\tau_2)^{n_1-1}(\log\frac{\tau_2}{\varsigma})^{n_1-\delta_2-1}-(\log\tau_1)^{n_1-1}(\log\frac{\tau_1}{\varsigma})^{n_1-\delta_2-1}|d\varsigma
\end{aligned}$$

$$\leq \frac{1}{(n_1-\gamma_1)\Gamma(n_1)}|(\log\tau_2)^{n_1-1}-(\log\tau_1)^{n_1-1}|+\frac{1}{\Gamma(n_1+1)}|(\log\tau_2)^{n_1}-(\log\tau_1)^{n_1}|$$
$$+\frac{ß}{(n_1-\gamma_1)\Gamma(n_1-\delta_2)}|(\log\tau_2)^{2n_1-\delta_2-1}-(\log\tau_1)^{2n_1-\delta_2-1}|$$
$$+\frac{ß}{(n_1-\delta_2)\Gamma(n_1-\delta_2)}|(\log\tau_2)|^{n_1-1}\Big[|\log\tau_2-\log\tau_1|^{n_1-\delta_2}+|(\log\tau_1)^{n_1-\delta_2}-(\log\tau_2)^{n_1-\delta_2}|\Big]$$
$$\to 0 \quad \text{as} \quad \tau_1\to\tau_2.$$

Similarly, we can prove that

$$\int_{\tau_1}^{\tau_2}\frac{1}{\varsigma}|\sigma_{10}(\tau_2,\varsigma)-\sigma_{11}(\tau_2,\varsigma)|d\varsigma\to 0 \quad \text{as} \quad \tau_1\to\tau_2,$$

and

$$\int_{\tau_2}^{e}\frac{1}{\varsigma}|\sigma_{11}(\tau_2,\varsigma)-\sigma_{11}(\tau_1,\varsigma)|d\varsigma\to 0 \quad \text{as} \quad \tau_1\to\tau_2.$$

From (2.13),

$$\int_{1}^{e}\frac{1}{\varsigma}|\sigma_{1}(\tau_2,\varsigma)-\sigma_{1}(\tau_1,\varsigma)|d\varsigma\to 0 \quad \text{as} \quad \tau_1\to\tau_2. \tag{2.14}$$

Similar to the above arguements, we can prove that

$$\int_{1}^{e}\frac{1}{\varsigma}|\sigma_{2}(\tau_2,\varsigma)-\sigma_{2}(\tau_1,\varsigma)|d\varsigma\to 0 \quad \text{as} \quad \tau_1\to\tau_2. \tag{2.15}$$

From (2.14), (2.15) and (2.12), we obtain

$$|\Upsilon_1(\upsilon(\tau_2),\omega(\tau_2))-\Upsilon_1(\upsilon(\tau_1),\omega(\tau_1))|\to 0 \quad \text{as} \quad \tau_1\to\tau_2.$$

Repeating arguements similar to those above, it can be proved that

$$|\Upsilon_2(\upsilon(\tau_2),\omega(\tau_2))-\Upsilon_2(\upsilon(\tau_1),\omega(\tau_1))|\to 0 \quad \text{as} \quad \tau_1\to\tau_2.$$

We conclude that $\Upsilon(\mathcal{P})$ is equicontinuous. Hence, by Arzela-Ascoli theorem, $\Upsilon:\mathcal{P}\to\mathcal{P}$ is completely continuous.

Theorem 2.1. *[Krasnosel'skii [10]] Let X be a Banach space, $K\subseteq X$ be a cone, and suppose that Ω_1,Ω_2 are open subsets of X with $0\in\Omega_1$ and $\overline{\Omega}_1\subset\Omega_2$. Suppose further that $T: K\cap(\overline{\Omega}_2\backslash\Omega_1)\to K$ is completely continuous operator such that either*
(i) $\|Tu\|\leq\|u\|,\ u\in K\cap\partial\Omega_1$ and $\|Tu\|\geq\|u\|,\ u\in K\cap\partial\Omega_2$, or
(ii) $\|Tu\|\geq\|u\|,\ u\in K\cap\partial\Omega_1$ and $\|Tu\|\leq\|u\|,\ u\in K\cap\partial\Omega_2$
holds. Then T has a fixed point in $K\cap(\overline{\Omega}_2\backslash\Omega_1)$.

3. Main results

In this section we investigate the existence of multiple positive solutions of problems (1.1) and (1.2) under some assumptions on the functions $\mathfrak{f}_i; i = 1, 2, 3, 4$. For notational convenience,

$$\mathfrak{f}_{10} = \lim_{\upsilon\to 0}\inf \inf_{\tau\in I\subset(1,e)} \frac{\mathfrak{f}_1(\tau,\upsilon,\omega)}{\phi_{p_1}(\upsilon)},\quad \mathfrak{f}_1^0 = \lim_{\upsilon\to 0}\sup \sup_{\tau\in[1,e]} \frac{\mathfrak{f}_1(\tau,\upsilon,\omega)}{\phi_{p_1}(\upsilon)},$$
$$\mathfrak{f}_{20} = \lim_{\omega\to 0}\inf \inf_{\tau\in I\subset(1,e)} \frac{\mathfrak{f}_2(\tau,\upsilon,\omega)}{\phi_{p_2}(\omega)},\quad \mathfrak{f}_2^0 = \lim_{\omega\to 0}\sup \sup_{\tau\in[1,e]} \frac{\mathfrak{f}_2(\tau,\upsilon,\omega)}{\phi_{p_2}(\omega)},$$
$$\mathfrak{f}_{1\infty} = \lim_{\upsilon\to \infty}\inf \inf_{\tau\in I\subset(1,e)} \frac{\mathfrak{f}_1(\tau,\upsilon,\omega)}{\phi_{p_1}(\upsilon)},\quad \mathfrak{f}_1^\infty = \lim_{\upsilon\to \infty}\sup \sup_{\tau\in[1,e]} \frac{\mathfrak{f}_1(\tau,\upsilon,\omega)}{\phi_{p_1}(\upsilon)},$$
$$\mathfrak{f}_{2\infty} = \lim_{\omega\to \infty}\inf \inf_{\tau\in I\subset(1,e)} \frac{\mathfrak{f}_2(\tau,\upsilon,\omega)}{\phi_{p_2}(\omega)},\quad \mathfrak{f}_2^\infty = \lim_{\upsilon\to \infty}\sup \sup_{\tau\in[1,e]} \frac{\mathfrak{f}_2(\tau,\upsilon,\omega)}{\phi_{p_2}(\omega)}.$$

For convenience of the reader, we denote

$$\sigma_i^\star = \frac{1}{2}\Big[\Big(\frac{1}{\Gamma(m_i+1)}\Big)^{q_i-1}\int_1^e \sigma_i(e,\varsigma)(\log\varsigma)^{m_i(q_i-1)}\frac{d\varsigma}{\varsigma}\Big]^{-1},\ \ i = 1, 2,$$
$$\rho_i^\star = \frac{1}{2}\Big[\Big(\frac{1}{\Gamma(m_i+1)}\Big)^{q_i-1}\int_{e^{1/4}}^{e^{3/4}} \aleph\sigma_i(e,\varsigma)\big(\log\varsigma - 1/4\big)^{m_i(q_i-1)}\frac{d\varsigma}{\varsigma}\Big]^{-1},\ \ i = 1, 2.$$

From now we will use the following assumptions:

(C1) $\mathfrak{f}_{i0} \in (\phi_{p_i}(\frac{\rho_i^\star}{\aleph}), \infty]$, $\mathfrak{f}_{i\infty} \in (\phi_{p_i}(\frac{\rho_i^\star}{\aleph}), \infty]$.
(C2) $\mathfrak{f}_i^0 \in [0, \phi_{p_i}(\sigma_i^\star))$, $\mathfrak{f}_i^\infty \in [0, \phi_{p_i}(\sigma_i^\star))$.
(C3) There exist constants $d_i \in (0, \sigma_i^\star)$ and $\lambda_1 > 0$ such that

$$\mathfrak{f}_i(\tau,\upsilon,\omega) \le \phi_{p_i}(d_i\lambda_1),\ \ \tau\in[1,e], 0\le \upsilon,\omega\le\lambda_1.$$

(C4) There exist constants $d_i^\star \in (\rho_i^\star, \infty)$ and $\lambda_2 > 0, [e^{1/4}, e^{3/4}] \subset (1, e)$ such that

$$\mathfrak{f}_i(\tau,\upsilon,\omega) \ge \phi_{p_i}(d_i^\star\lambda_2),\ \ \tau\in[e^{1/4}, e^{3/4}], \aleph\lambda_2\le \upsilon,\omega\le\lambda_2.$$

.

Theorem 3.1. *Assume that* $(A2), (A3), (C1)$ *and* $(C3)$ *hold, then problems* (1.1) *and* (1.2) *has at least two positive solutions* (υ_1, ω_1) *and* (υ_2, ω_2) *such that* $0 < \|(\upsilon_1, \omega_1)\| < \lambda_1 < \|(\upsilon_2, \omega_2)\|$.

Proof. Firstly, by condition (C_3), there exist constants $d_i \in (0, \sigma_i^\star)$ and $\lambda_1 > 0$ such that

$$\mathfrak{f}_i(\tau,\upsilon,\omega) \le \phi_{p_i}(d_i\lambda_1),\ \ \tau\in[1,e],\ 0\le \upsilon,\omega\le\lambda_1,$$

Set $\Omega_{\lambda_1} = \{(\upsilon,\omega) \in \mathcal{P} : \|(\upsilon,\omega)\| < \lambda_1\}$ for any $(\upsilon,\omega) \in \partial\Omega_{\lambda_1}$, then

$$\begin{aligned}\Upsilon_1(\upsilon,\omega) &= \int_1^e \sigma_1(\tau,\varsigma)\phi_{q_1}\Big(\frac{1}{\Gamma(m_1)}\int_1^\varsigma\big(\log\frac{\varsigma}{\ell}\big)^{m_1-1}\mathfrak{f}_1(\ell,\upsilon(\ell),\omega(\ell))\frac{d\ell}{\ell}\Big)\frac{d\varsigma}{\varsigma}\\ &\quad+\int_1^e \sigma_2(\tau,\varsigma)\phi_{q_2}\Big(\frac{1}{\Gamma(m_2)}\int_1^\varsigma\big(\log\frac{\varsigma}{\ell}\big)^{m_2-1}\mathfrak{f}_2(\ell,\upsilon(\ell),\omega(\ell))\frac{d\ell}{\ell}\Big)\frac{d\varsigma}{\varsigma}\\ &\le \int_1^e \sigma_1(e,\varsigma)\phi_{q_1}\Big(\frac{1}{\Gamma(m_1)}\int_1^\varsigma\big(\log\frac{\varsigma}{\ell}\big)^{m_1-1}\phi_{p_1}(d_1\lambda_1)\frac{d\ell}{\ell}\Big)\frac{d\varsigma}{\varsigma}\\ &\quad+\int_1^e \sigma_2(\tau,\varsigma)\phi_{q_2}\Big(\frac{1}{\Gamma(m_2)}\int_1^\varsigma\big(\log\frac{\varsigma}{\ell}\big)^{m_2-1}\phi_{p_2}(d_2\lambda_1)\frac{d\ell}{\ell}\Big)\frac{d\varsigma}{\varsigma}\end{aligned}$$

$$\leq \lambda_1\Big[\sigma_1^\star\Big(\frac{1}{\Gamma(m_1+1)}\Big)^{q_1-1}\int_1^e \sigma_1(e,\varsigma)(\log\varsigma)^{m_1(q_1-1)}\frac{d\varsigma}{\varsigma}$$
$$+\sigma_2^\star\Big(\frac{1}{\Gamma(m_2+1)}\Big)^{q_2-1}\int_1^e \sigma_2(e,\varsigma)(\log\varsigma)^{m_2(q_2-1)}\frac{d\varsigma}{\varsigma}\Big]$$
$$= \lambda_1 = \|(\upsilon,\omega)\|.$$

So $\|\Upsilon_1(\upsilon,\omega)\| \leq \|(\upsilon,\omega)\|, (\upsilon,\omega)\in\partial\Omega_{\lambda_1}$. In a similar manner, we may take $\|\Upsilon_2(\upsilon,\omega)\| \leq \lambda_1 = \|(\upsilon,\omega)\|, (\upsilon,\omega)\in\partial\Omega_{\lambda_1}$. Consequently

$$\|\Upsilon(\upsilon,\omega)\| = \max\{\|\Upsilon_1(\upsilon,\omega)\|, \|\Upsilon_2(\upsilon,\omega)\|\} \leq \lambda_1 = \|(\upsilon,\omega)\|, \quad (\upsilon,\omega)\in\partial\Omega_{\lambda_1}. \tag{3.1}$$

Secondly, with the first relation of condition $(C1)$, $\mathfrak{f}_{i0} \in (\phi_{p_i}(\frac{\rho_i^\star}{\aleph}),\infty]$ there exists a real number $\lambda \in (0,\lambda_1)$ such that

$$\mathfrak{f}_1(\tau,\upsilon,\omega) \geq \phi_{p_1}(\upsilon)\phi_{p_1}\Big(\frac{\rho_1^\star}{\aleph}\Big),\ \tau\in I,\ \ 0<\upsilon\leq\lambda,\ \omega\geq 0,$$
$$\mathfrak{f}_2(\tau,\upsilon,\omega) \geq \phi_{p_2}(\omega)\phi_{p_2}\Big(\frac{\rho_2^\star}{\aleph}\Big),\ \tau\in I,\ \ 0<\omega\leq\lambda,\ \upsilon\geq 0.$$

Set $\Omega_{\lambda_1} = \{(\upsilon,\omega)\in\mathcal{P} : \|(\upsilon,\omega)\| < \lambda_1\}$. For any $(\upsilon,\omega)\in\partial\Omega_{\lambda_1}$, we have $\lambda_1 = \|(\upsilon,\omega)\| \geq \min_{\tau\in I}(\upsilon(\tau)+\omega(\tau)) \geq \aleph\|(\upsilon,\omega)\| = \aleph\lambda_1$ then;

$$\Upsilon_1(\upsilon,\omega) = \int_1^e \sigma_1(\tau,\varsigma)\phi_{q_1}\Big(\frac{1}{\Gamma(m_1)}\int_1^\varsigma\big(\log\frac{\varsigma}{\ell}\big)^{m_1-1}\mathfrak{f}_1(\ell,\upsilon(\ell),\omega(\ell))\frac{d\ell}{\ell}\Big)\frac{d\varsigma}{\varsigma}$$
$$+\int_1^e \sigma_2(\tau,\varsigma)\phi_{q_2}\Big(\frac{1}{\Gamma(m_2)}\int_1^\varsigma\big(\log\frac{\varsigma}{\ell}\big)^{m_2-1}\mathfrak{f}_2(\ell,\upsilon(\ell),\omega(\ell))\frac{d\ell}{\ell}\Big)\frac{d\varsigma}{\varsigma}$$
$$\geq \int_{e^{1/4}}^{e^{3/4}} \aleph\sigma_1(e,\varsigma)\phi_{q_1}\Big(\frac{1}{\Gamma(m_1)}\int_{e^{1/4}}^\varsigma\big(\log\frac{\varsigma}{\ell}\big)^{m_1-1}\phi_{p_1}(\upsilon)\phi_{p_1}\Big(\frac{\rho_1^\star}{\aleph}\Big)\frac{d\ell}{\ell}\Big)\frac{d\varsigma}{\varsigma}$$
$$+\int_{e^{1/4}}^{e^{3/4}} \aleph\sigma_2(e,\varsigma)\phi_{q_2}\Big(\frac{1}{\Gamma(m_2)}\int_{e^{1/4}}^\varsigma\big(\log\frac{\varsigma}{\ell}\big)^{m_2-1}\phi_{p_2}(\omega)\phi_{p_2}\Big(\frac{\rho_2^\star}{\aleph}\Big)\frac{d\ell}{\ell}\Big)\frac{d\varsigma}{\varsigma}$$
$$\geq \int_{e^{1/4}}^{e^{3/4}} \aleph\sigma_1(e,\varsigma)\phi_{q_1}\Big(\frac{1}{\Gamma(m_1)}\int_{e^{1/4}}^\varsigma\big(\log\frac{\varsigma}{\ell}\big)^{m_1-1}\phi_{p_1}(\aleph\lambda_1)\phi_{p_1}\Big(\frac{\rho_1^\star}{\aleph}\Big)\frac{d\ell}{\ell}\Big)\frac{d\varsigma}{\varsigma}$$
$$+\int_{e^{1/4}}^{e^{3/4}} \aleph\sigma_2(e,\varsigma)\phi_{q_2}\Big(\frac{1}{\Gamma(m_2)}\int_{e^{1/4}}^\varsigma\big(\log\frac{\varsigma}{\ell}\big)^{m_2-1}\phi_{p_2}(\aleph\lambda_1)\phi_{p_2}\Big(\frac{\rho_2^\star}{\aleph}\Big)\frac{d\ell}{\ell}\Big)\frac{d\varsigma}{\varsigma}$$
$$\geq \lambda_1\Big[\rho_1^\star\Big(\frac{1}{\Gamma(m_1+1)}\Big)^{q_1-1}\int_{e^{1/4}}^{e^{3/4}} \aleph\sigma_1(e,\varsigma)(\log\varsigma-1/4)^{m_1(q_1-1)}\frac{d\varsigma}{\varsigma}$$
$$+\rho_2^\star\Big(\frac{1}{\Gamma(m_2+1)}\Big)^{q_2-1}\int_{e^{1/4}}^{e^{3/4}} \aleph\sigma_2(e,\varsigma)(\log\varsigma-1/4)^{m_2(q_2-1)}\frac{d\varsigma}{\varsigma}\Big]$$
$$= \lambda_1 = \|(\upsilon,\omega)\|.$$

Therefore, we obtain

$$\|\Upsilon(\upsilon,\omega)\| = \max\{\|\Upsilon_1(\upsilon,\omega)\|, \|\Upsilon_2(\upsilon,\omega)\|\} \geq \lambda_1 = \|(\upsilon,\omega)\|, \text{ for any } (\upsilon,\omega)\in\partial\Omega_{\lambda_1} \tag{3.2}$$

Hence thirdly, with the second relation of condition ($C1$), $\mathfrak{f}_{i\infty} \in (\phi_{p_i}(\frac{\rho_i^\star}{\aleph}), \infty]$, there exist real numbers $R^\star,\ R^{\star\star}$, such that

$$\mathfrak{f}_1(\tau, \upsilon, \omega) \geq \phi_{p_1}(\upsilon)\phi_{p_1}(\frac{\rho_1^\star}{\aleph}), \text{ for all } \tau \in I,\ \ \upsilon \geq R^\star,\ \omega \geq 0,$$
$$\mathfrak{f}_2(\tau, \upsilon, \omega) \geq \phi_{p_2}(\omega)\phi_{p_2}(\frac{\rho_2^\star}{\aleph}), \text{ for all } \tau \in I,\ \ \omega \geq R^{\star\star},\ \upsilon \geq 0.$$

Choose $R_2 = \max\{2\lambda_1, \frac{R^\star}{\aleph}, \frac{R^{\star\star}}{\aleph}\}$. Set $\Omega_{R_2} = \{(\upsilon, \omega) \in \mathcal{P} : \|(\upsilon, \omega)\| < R_2\}$. For any $(\upsilon, \omega) \in \partial\Omega_{R_2}$, we have

$$R_2 = \|(\upsilon, \omega)\| \geq \upsilon(\tau) \geq \aleph\|(\upsilon, \omega)\| \geq \aleph R_2 \geq R_2^\star,\ \ 1 \leq \tau \leq e,$$
$$R_2 = \|(\upsilon, \omega)\| \geq \omega(\tau) \geq \aleph\|(\upsilon, \omega)\| \geq \aleph R_2 \geq R_2^{\star\star},\ \ 1 \leq \tau \leq e.$$

Thus, for any $(\upsilon, \omega) \in \partial\Omega_{R_2}$, we have

$$\mathfrak{f}_1(\tau, \upsilon, \omega) \geq \phi_{p_1}(\upsilon(t))\phi_{p_1}\Big(\frac{\rho_1^\star}{\aleph}\Big) \geq \phi_{p_1}(\aleph R_2)\phi_{p_1}\Big(\frac{\rho_1^\star}{\aleph}\Big), \text{ for all } \tau \in I,$$
$$\mathfrak{f}_2(\tau, \upsilon, \omega) \geq \phi_{p_2}(\omega(\tau))\phi_{p_2}\Big(\frac{\rho_2^\star}{\aleph}\Big) \geq \phi_{p_2}(\aleph R_2)\phi_{p_2}\Big(\frac{\rho_2^\star}{\aleph}\Big), \text{ for all } \tau \in I.$$

$$\begin{aligned}
\Upsilon_1(\upsilon, \omega) = & \int_1^e \sigma_1(\tau, \varsigma)\phi_{q_1}\Big(\frac{1}{\Gamma(m_1)}\int_1^\varsigma \big(\log\frac{\varsigma}{\ell}\big)^{m_1-1}\mathfrak{f}_1(\ell, \upsilon(\ell), \omega(\ell))\frac{d\ell}{\ell}\Big)\frac{d\varsigma}{\varsigma} \\
& + \int_1^e \sigma_2(\tau, \varsigma)\phi_{q_2}\Big(\frac{1}{\Gamma(m_2)}\int_1^\varsigma \big(\log\frac{\varsigma}{\ell}\big)^{m_2-1}\mathfrak{f}_2(\ell, \upsilon(\ell), \omega(\ell))\frac{d\ell}{\ell}\Big)\frac{d\varsigma}{\varsigma} \\
\geq & \int_{e^{1/4}}^{e^{3/4}} \aleph\sigma_1(e, \varsigma)\phi_{q_1}\Big(\frac{1}{\Gamma(m_1)}\int_{e^{1/4}}^\varsigma \big(\log\frac{\varsigma}{\ell}\big)^{m_1-1}\phi_{p_1}(\upsilon)\phi_{p_1}\Big(\frac{\rho_1^\star}{\aleph}\Big)\frac{d\ell}{\ell}\Big)\frac{d\varsigma}{\varsigma} \\
& + \int_{e^{1/4}}^{e^{3/4}} \aleph\sigma_2(e, \varsigma)\phi_{q_2}\Big(\frac{1}{\Gamma(m_2)}\int_{e^{1/4}}^\varsigma \big(\log\frac{\varsigma}{\ell}\big)^{m_2-1}\phi_{p_2}(\omega)\phi_{p_2}\Big(\frac{\rho_2^\star}{\aleph}\Big)\frac{d\ell}{\ell}\Big)\frac{d\varsigma}{\varsigma} \\
\geq & \int_{e^{1/4}}^{e^{3/4}} \aleph\sigma_1(e, \varsigma)\phi_{q_1}\Big(\frac{1}{\Gamma(m_1)}\int_{e^{1/4}}^\varsigma \big(\log\frac{\varsigma}{\ell}\big)^{m_1-1}\phi_{p_1}(\aleph R_2)\phi_{p_1}\Big(\frac{\rho_1^\star}{\gamma}\Big)\frac{d\ell}{\ell}\Big)\frac{d\varsigma}{\varsigma} \\
& + \int_{e^{1/4}}^{e^{3/4}} \aleph\sigma_2(e, \varsigma)\phi_{q_2}\Big(\frac{1}{\Gamma(m_2)}\int_{e^{1/4}}^\varsigma \big(\log\frac{\varsigma}{\ell}\big)^{m_2-1}\phi_{p_2}(\aleph R_2)\phi_{p_2}\Big(\frac{\rho_2^\star}{\aleph}\Big)\frac{d\ell}{\ell}\Big)\frac{d\varsigma}{\varsigma} \\
\geq & R_2\Big[\rho_1^\star\Big(\frac{1}{\Gamma(m_1+1)}\Big)^{q_1-1}\int_{e^{1/4}}^{e^{3/4}} \aleph\sigma_1(e, \varsigma)(\log\varsigma - 1/4)^{m_1(q_1-1)}\frac{d\varsigma}{\varsigma} \\
& + \rho_2^\star\Big(\frac{1}{\Gamma(m_2+1)}\Big)^{q_2-1}\int_{e^{1/4}}^{e^{3/4}} \aleph\sigma_2(e, \varsigma)(\log\varsigma - 1/4)^{m_2(q_2-1)}\frac{d\varsigma}{\varsigma}\Big] \\
= & R_2 = \|(\upsilon, \omega)\|.
\end{aligned}$$

Hence, we obtain

$$\|\Upsilon(\upsilon, \omega)\| = \max\{\|\Upsilon_1(\upsilon, \omega)\|\|\Upsilon_2(\upsilon, \omega)\|\} \geq R_2 = \|(\upsilon, \omega)\|, \text{ for any } (\upsilon, \omega) \in \partial\Omega_{R_2}. \tag{3.3}$$

Therefore, by (3.1)–(3.3) and Theorem 2.1, Υ has a fixed point $(\upsilon_1, \omega_1) \in (\overline{\Omega}_{R_2} \setminus \Omega_\lambda)$ and a fixed point $(\upsilon_2, \omega_2) \in (\overline{\Omega}_\lambda \setminus \Omega_{R_2})$. That is to say $(\upsilon_1, \omega_1); (\upsilon_2, \omega_2)$ are both positive solutions of problems (1.1) and (1.2) such that $0 < \|(\upsilon_1, \omega_1)\| < \lambda_1 < \|(\upsilon_2, \omega_2)\|$.

Theorem 3.2. *Assume that* (A2), (A3), (C2) *and* (C4) *hold, then problems* (1.1) *and* (1.2) *has at least two solutions* (υ_1, ω_1) *and* (υ_2, ω_2) *satisfying* $0 < \|(\upsilon_1, \omega_1)\| < \lambda_2 < \|(\upsilon_2, \omega_2)\|$.

Proof. Firstly, by condition (C_4), there exist constants $d_i^\star \in (\rho_i^\star, \infty)$ and $\lambda_2 > 0$ such that

$$\mathfrak{f}_i(\tau, \upsilon, \omega) \geq \phi_{p_1}(d_i^\star \lambda_2), \text{ for all } \tau \in I, \ \ \aleph\lambda_2 \leq \upsilon, \omega \leq \lambda_2.$$

Set $\Omega_{\lambda_2} = \{(\upsilon, \omega) \in \mathcal{P} : \|(\upsilon, \omega)\| < \lambda_2\}$. For any $(\upsilon, \omega) \in \partial\Omega_{\lambda_2}$, we have

$$\lambda_2 = \|(\upsilon, \omega)\| \geq \min_{\tau\in I}(\upsilon(\tau) + \omega(\tau)) \geq \aleph\|(\upsilon, \omega)\| = \aleph\lambda_2,$$

then

$$\begin{aligned}
\Upsilon_1(\upsilon, \omega) &= \int_1^e \sigma_1(\tau, \varsigma)\phi_{q_1}\Big(\frac{1}{\Gamma(m_1)}\int_1^\varsigma \big(\log\frac{\varsigma}{\ell}\big)^{m_1-1}\mathfrak{f}_1(\ell, \upsilon(\ell), \omega(\ell))\frac{d\ell}{\ell}\Big)\frac{d\varsigma}{\varsigma}\\
&\quad + \int_1^e \sigma_2(\tau, \varsigma)\phi_{q_2}\Big(\frac{1}{\Gamma(m_2)}\int_1^\varsigma \big(\log\frac{\varsigma}{\ell}\big)^{m_2-1}\mathfrak{f}_2(\ell, \upsilon(\ell), \omega(\ell))\frac{d\ell}{\ell}\Big)\frac{d\varsigma}{\varsigma}\\
&\geq \int_{e^{1/4}}^{e^{3/4}} \aleph\sigma_1(e, \varsigma)\phi_{q_1}\Big(\frac{1}{\Gamma(m_1)}\int_{e^{1/4}}^\varsigma \big(\log\frac{\varsigma}{\ell}\big)^{m_1-1}\phi_{p_1}(d_1^\star\lambda_2)\frac{d\ell}{\ell}\Big)\frac{d\varsigma}{\varsigma}\\
&\quad + \int_{e^{1/4}}^{e^{3/4}} \aleph\sigma_2(e, \varsigma)\phi_{q_2}\Big(\frac{1}{\Gamma(m_2)}\int_{e^{1/4}}^\varsigma \big(\log\frac{\varsigma}{\ell}\big)^{m_2-1}\phi_{p_2}(d_2^\star\lambda_2)\frac{d\ell}{\ell}\Big)\frac{d\varsigma}{\varsigma}\\
&\geq \int_{e^{1/4}}^{e^{3/4}} \aleph\sigma_1(e, \varsigma)\phi_{q_1}\Big(\frac{1}{\Gamma(m_1)}\int_{e^{1/4}}^\varsigma \big(\log\frac{\varsigma}{\ell}\big)^{m_1-1}\phi_{p_1}(d_1^\star\lambda_2)\frac{d\ell}{\ell}\Big)\frac{d\varsigma}{\varsigma}\\
&\quad + \int_{e^{1/4}}^{e^{3/4}} \aleph\sigma_2(e, \varsigma)\phi_{q_2}\Big(\frac{1}{\Gamma(m_2)}\int_{e^{1/4}}^\varsigma \big(\log\frac{\varsigma}{\ell}\big)^{m_2-1}\phi_{p_2}(d_2^\star\lambda_2)\frac{d\ell}{\ell}\Big)\frac{d\varsigma}{\varsigma}\\
&\geq \lambda_2\Big[\rho_1^\star\Big(\frac{1}{\Gamma(m_1+1)}\Big)^{q_1-1}\int_{e^{1/4}}^{e^{3/4}} \aleph\sigma_1(e, \varsigma)(\log\varsigma - 1/4)^{m_1(q_1-1)}\frac{d\varsigma}{\varsigma}\\
&\quad + \rho_2^\star\Big(\frac{1}{\Gamma(m_2+1)}\Big)^{q_2-1}\int_{e^{1/4}}^{e^{3/4}} \aleph\sigma_2(e, \varsigma)(\log\varsigma - 1/4)^{m_2(q_2-1)}\frac{d\varsigma}{\varsigma}\Big]\\
&= \lambda_2 = \|(\upsilon, \omega)\|.
\end{aligned}$$

Hence

$$\|\Upsilon(\upsilon, \omega)\| = \max\{\|\Upsilon_1(\upsilon, \omega)\|, \|\Upsilon_2(\upsilon, \omega)\|\} \geq \lambda_2 = \|(\upsilon, \omega)\|, \text{ for any } (\upsilon, \omega) \in \partial\Omega_{\lambda_2} \tag{3.4}$$

Secondly, with the first relation of condition (C_2), $\mathfrak{f}_i^0 \in [0, \phi_{p_i}(\sigma_i^\star))$; there exist a real number $r_2 \in (0, \lambda_2)$ such that

$$\begin{aligned}
\mathfrak{f}_1(\tau, \upsilon, \omega) \leq \phi_{p_1}(\upsilon\sigma_1^\star) \leq \phi_{p_1}(r_2\sigma_1^\star), 1 \leq \tau \leq e, \ 0 \leq \upsilon \leq r_2, \ \omega \geq 0,\\
\mathfrak{f}_2(\tau, \upsilon, \omega) \leq \phi_{p_2}(\omega\sigma_2^\star) \leq \phi_{p_2}(r_2\sigma_2^\star), 1 \leq \tau \leq e, \ 0 \leq \omega \leq r_2, \ \upsilon \geq 0.
\end{aligned}$$

Set $\Omega_{r_2} = \{(\upsilon, \omega) \in \mathcal{P} : \|(\upsilon, \omega)\| < r_2\}$ for any $(\upsilon, \omega) \in \partial\Omega_{r_2}$, then

$$\begin{aligned}
\Upsilon_1(\upsilon, \omega) &= \int_1^e \sigma_1(\tau, \varsigma)\phi_{q_1}\Big(\frac{1}{\Gamma(m_1)}\int_1^\varsigma \big(\log\frac{\varsigma}{\ell}\big)^{m_1-1}\mathfrak{f}_1(\ell, \upsilon(\ell), \omega(\ell))\frac{d\ell}{\ell}\Big)\frac{d\varsigma}{\varsigma}\\
&\quad + \int_1^e \sigma_2(\tau, \varsigma)\phi_{q_2}\Big(\frac{1}{\Gamma(m_2)}\int_1^\varsigma \big(\log\frac{\varsigma}{\ell}\big)^{m_2-1}\mathfrak{f}_2(\ell, \upsilon(\ell), \omega(\ell))\frac{d\ell}{\ell}\Big)\frac{d\varsigma}{\varsigma}\\
&\leq \int_1^e \sigma_1(e, \varsigma)\phi_{q_1}\Big(\frac{1}{\Gamma(m_1)}\int_1^\varsigma \big(\log\frac{\varsigma}{\ell}\big)^{m_1-1}\phi_{p_1}(r_2\sigma_1^\star)\frac{d\ell}{\ell}\Big)\frac{d\varsigma}{\varsigma}\\
&\quad + \int_1^e \sigma_2(\tau, \varsigma)\phi_{q_2}\Big(\frac{1}{\Gamma(m_2)}\int_1^\varsigma \big(\log\frac{\varsigma}{\ell}\big)^{m_2-1}\phi_{p_2}(r_2\sigma_2^\star)\frac{d\ell}{\ell}\Big)\frac{d\varsigma}{\varsigma}
\end{aligned}$$

$$
\begin{aligned}
&\le r_2\Big[\sigma_1^\star\Big(\frac{1}{\Gamma(m_1+1)}\Big)^{q_1-1}\int_1^e \sigma_1(e,\varsigma)(\log\varsigma)^{m_1(q_1-1)}\frac{d\varsigma}{\varsigma}\\
&\quad+\sigma_2^\star\Big(\frac{1}{\Gamma(m_2+1)}\Big)^{q_2-1}\int_1^e \sigma_2(e,\varsigma)(\log\varsigma)^{m_2(q_2-1)}\frac{d\varsigma}{\varsigma}\Big]\\
&= r_2 = \|(\upsilon,\omega)\|.
\end{aligned}
$$

So $\|\Upsilon_1(\upsilon,\omega)\|\le\|(\upsilon,\omega)\|, (\upsilon,\omega)\in\partial\Omega_{r_2}$. In a similar manner, we deduce $\|\Upsilon_2(\upsilon,\omega)\|\le\|(\upsilon,\omega)\|, (\upsilon,\omega)\in\partial\Omega_{r_2}$. Hence

$$
\|\Upsilon(\upsilon,\omega)\| = \max\{\|\Upsilon_1(\upsilon,\omega)\|,\|\Upsilon_2(\upsilon,\omega)\|\}\le r_2 = \|(\upsilon,\omega)\|,\ \ (\upsilon,\omega)\in\partial\Omega_{r_2}. \tag{3.5}
$$

Thirdly, with the second relation of condition $(C2)$ $\mathfrak{f}_i^\infty\in[0,\phi_{p_i}(\sigma_1^\star)$, there exists a positive number $R^\star$ such that

$$
\begin{aligned}
&\mathfrak{f}_1(\tau,\upsilon,\omega)\le\phi_{p_1}(\upsilon\sigma_1^\star),\ \ 0\le\tau\le e,\ \upsilon\ge R^\star,\ \omega\ge 0,\\
&\mathfrak{f}_2(\tau,\upsilon,\omega)\le\phi_{p_2}(\omega\sigma_2^\star),\ \ 0\le\tau\le e,\ \omega\ge R^\star,\ \upsilon\ge 0.
\end{aligned}
$$

We now consider two situations

Case (i) The functions $\mathfrak{f}_i$ is bounded on $[0,\infty)$, then we choose a positive number $k>0$ such that

$$
\mathfrak{f}_i(\tau,\upsilon,\omega)\le\phi_{p_i}(k\sigma_i^\star);\ 1\le\tau\le e,\upsilon,\omega\ge 0,\ i=1,2.
$$

Let $R_2=\max\{2\lambda_2,k\}$ and $\Omega_{R_2}=\{(\upsilon,\omega)\in\mathcal{P}:\|(\upsilon,\omega)\|<R_2\}$.

For any $(\upsilon,\omega)\in\partial\Omega_{R_2}$, we have

$$
\begin{aligned}
\Upsilon_1(\upsilon,\omega) &= \int_1^e\sigma_1(\tau,\varsigma)\phi_{q_1}\Big(\frac{1}{\Gamma(m_1)}\int_1^\varsigma\Big(\log\frac{\varsigma}{\ell}\Big)^{m_1-1}\mathfrak{f}_1(\ell,\upsilon(\ell),\omega(\ell))\frac{d\ell}{\ell}\Big)\frac{d\varsigma}{\varsigma}\\
&\quad+\int_1^e\sigma_2(\tau,\varsigma)\phi_{q_2}\Big(\frac{1}{\Gamma(m_2)}\int_1^\varsigma\Big(\log\frac{\varsigma}{\ell}\Big)^{m_2-1}\mathfrak{f}_2(\ell,\upsilon(\ell),\omega(\ell))\frac{d\ell}{\ell}\Big)\frac{d\varsigma}{\varsigma}\\
&\le\int_1^e\sigma_1(e,\varsigma)\phi_{q_1}\Big(\frac{1}{\Gamma(m_1)}\int_1^\varsigma\Big(\log\frac{\varsigma}{\ell}\Big)^{m_1-1}\phi_{p_1}(k\sigma_1^\star)\frac{d\ell}{\ell}\Big)\frac{d\varsigma}{\varsigma}\\
&\quad+\int_1^e\sigma_2(\tau,\varsigma)\phi_{q_2}\Big(\frac{1}{\Gamma(m_2)}\int_1^\varsigma\Big(\log\frac{\varsigma}{\ell}\Big)^{m_2-1}\phi_{p_2}(k\sigma_2^\star)\frac{d\ell}{\ell}\Big)\frac{d\varsigma}{\varsigma}\\
&\le R_2\Big[\sigma_1^\star\Big(\frac{1}{\Gamma(m_1+1)}\Big)^{q_1-1}\int_1^e \sigma_1(e,\varsigma)(\log\varsigma)^{m_1(q_1-1)}\frac{d\varsigma}{\varsigma}\\
&\quad+\sigma_2^\star\Big(\frac{1}{\Gamma(m_2+1)}\Big)^{q_2-1}\int_1^e \sigma_2(e,\varsigma)(\log\varsigma)^{m_2(q_2-1)}\frac{d\varsigma}{\varsigma}\Big]\\
&= R_2 = \|(\upsilon,\omega)\|.
\end{aligned}
$$

So $\|\Upsilon_1(\upsilon,\omega)\|\le\|(\upsilon,\omega)\|, (\upsilon,\omega)\in\partial\Omega_{R_2}$. In a similar manner, we deduce $\|\Upsilon_2(\upsilon,\omega)\|\le\|(\upsilon,\omega)\|, (\upsilon,\omega)\in\partial\Omega_{R_2}$. Hence

$$
\|\Upsilon(\upsilon,\omega)\| = \max\{\|\Upsilon_1(\upsilon,\omega)\|,\|\Upsilon_2(\upsilon,\omega)\|\}\le R_2 = \|(\upsilon,\omega)\|,\ \ (\upsilon,\omega)\in\partial\Omega_{R_2}. \tag{3.6}
$$

Case (ii) The functions $\mathfrak{f}_1$ and $\mathfrak{f}_2$ are unbounded. We can choose a positive number $R_2=\max\{2\lambda_2,R^\star\}$, such that

$$
\mathfrak{f}_i(\tau,\upsilon,\omega)\le\mathfrak{f}_i(\tau,R_2,R_2);\ \ 1\le\tau\le e,\ 0\le\upsilon,\omega\le R_2,\ i=1,2.
$$

Set $\Omega_{R_2} = \{(\upsilon,\omega)\in\mathcal{P} : \|(\upsilon,\omega)\| < R_2\}$. For any $(\upsilon,\omega)\in\partial\Omega_{R_2}$, we have

$$\begin{aligned}
\Upsilon_1(\upsilon,\omega) &= \int_1^e \sigma_1(\tau,\varsigma)\phi_{q_1}\Big(\frac{1}{\Gamma(m_1)}\int_1^{\varsigma}\big(\log\frac{\varsigma}{\ell}\big)^{m_1-1}\mathfrak{f}_1(\ell,\upsilon(\ell),\omega(\ell))\frac{d\ell}{\ell}\Big)\frac{d\varsigma}{\varsigma}\\
&\quad+\int_1^e \sigma_2(\tau,\varsigma)\phi_{q_2}\Big(\frac{1}{\Gamma(m_2)}\int_1^{\varsigma}\big(\log\frac{\varsigma}{\ell}\big)^{m_2-1}\mathfrak{f}_2(\ell,\upsilon(\ell),\omega(\ell))\frac{d\ell}{\ell}\Big)\frac{d\varsigma}{\varsigma}\\
&\le \int_1^e \sigma_1(\tau,\varsigma)\phi_{q_1}\Big(\frac{1}{\Gamma(m_1)}\int_1^{\varsigma}\big(\log\frac{\varsigma}{\ell}\big)^{m_1-1}\mathfrak{f}_1(\ell,R_2,R_2)\frac{d\ell}{\ell}\Big)\frac{d\varsigma}{\varsigma}\\
&\quad+\int_1^e \sigma_2(\tau,\varsigma)\phi_{q_2}\Big(\frac{1}{\Gamma(m_2)}\int_1^{\varsigma}\big(\log\frac{\varsigma}{\ell}\big)^{m_2-1}\mathfrak{f}_2(\ell,R_2,R_2)\frac{d\ell}{\ell}\Big)\frac{d\varsigma}{\varsigma}\\
&\le \int_1^e \sigma_1(e,\varsigma)\phi_{q_1}\Big(\frac{1}{\Gamma(m_1)}\int_1^{\varsigma}\big(\log\frac{\varsigma}{\ell}\big)^{m_1-1}\phi_{p_1}(R_2\sigma_1^{\star})\frac{d\ell}{\ell}\Big)\frac{d\varsigma}{\varsigma}\\
&\quad+\int_1^e \sigma_2(\tau,\varsigma)\phi_{q_2}\Big(\frac{1}{\Gamma(m_2)}\int_1^{\varsigma}\big(\log\frac{\varsigma}{\ell}\big)^{m_2-1}\phi_{p_2}(R_2\sigma_2^{\star})\frac{d\ell}{\ell}\Big)\frac{d\varsigma}{\varsigma}\\
&\le R_2\Big[\sigma_1^{\star}\Big(\frac{1}{\Gamma(m_1+1)}\Big)^{q_1-1}\int_1^e \sigma_1(e,\varsigma)(\log\varsigma)^{m_1(q_1-1)}\frac{d\varsigma}{\varsigma}\\
&\quad+\sigma_2^{\star}\Big(\frac{1}{\Gamma(m_2+1)}\Big)^{q_2-1}\int_1^e \sigma_2(e,\varsigma)(\log\varsigma)^{m_2(q_2-1)}\frac{d\varsigma}{\varsigma}\Big]\\
&= R_2 = \|(\upsilon,\omega)\|,
\end{aligned}$$

and $\|\Upsilon_1(\upsilon,\omega)\| \le \|(\upsilon,\omega)\|, (\upsilon,\omega)\in\partial\Omega_{R_2}$. Similarly, we have $\|\Upsilon_2(\upsilon,\omega)\| \le \|(\upsilon,\omega)\|, (\upsilon,\omega)\in\partial\Omega_{R_2}$.

$$\|\Upsilon(\upsilon,\omega)\| = \max\{\|\Upsilon_1(\upsilon,\omega)\|, \|\Upsilon_2(\upsilon,\omega)\|\} \le R_2 = \|(\upsilon,\omega)\|,\ \ (\upsilon,\omega)\in\partial\Omega_{R_2}. \tag{3.7}$$

Therefore, by (3.4)–(3.7) and Theorem 2.1, Υ has a fixed point $(\upsilon_1,\omega_1)\in(\Omega_{\lambda_2}\setminus\overline{\Omega}_{r_2})$ and a fixed point $(\upsilon_2,\omega_2)\in(\Omega_{R_2}\setminus\overline{\Omega}_{\lambda_2})$. That is to say $(\upsilon_1,\omega_1),(\upsilon_2,\omega_2)$ are both positive solutions of problems (1.1) and (1.2) such that $0<\|(\upsilon_1,\omega_1)\|<\lambda_2<\|(\upsilon_2,\omega_2)\|$.

Conclusions

In this paper, we obtained several sufficient conditions for the existence and multiplicity of positive solutions for a coupled system of nonlinear Hadamard fractional boundary value problems with (p_1, p_2)-Laplacian operator by using Guo-Krasnosel'skii fixed point theorem. Our results will be a useful contribution to the existing literature on the topic of Hadamard fractional order differential equations.

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Conflict of interest

The authors declare that they have no competing interests.

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