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Research article

Numerical solution for the system of Lane-Emden type equations using cubic B-spline method arising in engineering

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Abstract: In this study, we develop a collocation method based on cubic B-spline functions for effectively solving the system of Lane-Emden type equations arising in physics, star structure, and astrophysics. To overcome the singularity behavior of the considered system at τ=0, we apply the L'Hôpital rule. Furthermore, we have carried out a convergence analysis of the proposed method and have demonstrated that it has a second-order convergence. To demonstrate the effectiveness, accuracy, simplicity, and practicality of the method, five test problems are solved numerically and the maximum absolute errors of the proposed method are compared with those of some existing methods.

Keywords: cubic spline; cubic B-spline; Lane-Emden equation; differential equations; initial value problem
Mathematics Subject Classification: 34K32, 34K34

1. Introduction

The study of nonlinear systems of singular initial value problems has recently attracted many mathematicians and physicists [1–12]. One of the systems in this category is the following Lane-Emden system of the form:

$$\frac{d^2\omega_1(\tau)}{d\tau^2}+\frac{\delta_1}{\tau}\frac{d\omega_1(\tau)}{d\tau}+\hbar_1(\omega_1(\tau),\omega_2(\tau))=\aleph_1(\tau), \tag{1}$$

$$\frac{d^2\omega_1(\tau)}{d\tau^2} + \frac{\delta_1}{\tau}\frac{d\omega_1(\tau)}{d\tau} + \hbar_1(\omega_1(\tau), \omega_2(\tau)) = \aleph_1(\tau),$$

subject to

$$\omega_1(0) = \varepsilon_1, \omega'_1(0) = 0,$$
$$\omega_2(0) = \vartheta_1, \omega'_2(0) = 0, \tag{2}$$

where $\aleph_1, \aleph_2$ are source functions, $\tau \in [0,1]$, $\delta_1, \delta_2, \varepsilon_1$, and ϑ_1 are real constants. The Lane-Emden equation, which was first investigated by astrophysicists Jonathan Homer Lane and Robert Emden, comes in various scientific applications in two kinds. For $\hbar\big(\omega(\tau)\big) = \omega^\gamma(\tau)$, (1) is called the Lane-Emden equation of index γ, or of the first kind, that is a fundamental equation in the theory of star structure [13–19]. It represents the temperature variant of a spherical gas cloud subject to the principles of thermodynamics and its molecules' mutual attraction, see [20–26] and references therein. In astrophysics, the Lane-Emden equation represents Poisson's equation for the gravitational potential of a spherically symmetric, polytropic fluid, and self-gravitating at hydrostatic equilibrium [27]. Moreover, an important area of application for this type of equation is the study of species' diffusive transit and chemical reactions inside porous catalyst particles [27]. However, for $\hbar\big(\omega(\tau)\big) = e^{\omega(\tau)}$, (1) is called the Lane-Emden equation of the second kind that describes the dimensionless density distribution in a sphere of isothermal gas [28]. The singular behavior that arises at $\tau = 0$ is the primary difficulty of the Lane-Emden equations.

Recently, modeling a variety of physical and chemical phenomena, including chemical reactions, population evolution, and pattern formation leads to the system of Lane-Emden equations [7]. Therefore, numerous approaches have been proposed for the solutions of scalar and system of Lane-Emden equations [1,2,10–18,42–47].

Spline methods employing piecewise polynomial functions have been demonstrated to be convenient methods for obtaining numerical solutions to many challenging models in science, engineering, and mathematics due to their simplicity of implementation and efficiency [29–34]. One of the well-known spline methods is the so-called B-spline (the "B" stands for basis) functions, which were first proposed by Schoenberg in 1946. The B-spline functions [35,36] have recently been a valuable tool in numerical computation, approximation theory, and image processing as they have various useful properties such as numerical stability of computations, local effects of coefficient changes, and built-in smoothness between neighboring polynomial pieces. The degrees of B-spline and the collocation points are the main factors that play a significant role in the execution of the technique and affect the outcomes to be achieved up to a required level of accuracy.

One of the most efficient and versatile techniques for obtaining approximate solutions is the cubic B-spline method (CBSM). The CBSM is a third-order piecewise polynomial constructed from a combination of recursive formulas referred to as the cubic B-spline basis. The derivation of the B-spline basis and the construction of the B-spline function are thoroughly discussed in [37,38]. In recent years, the CBSM has been successfully applied to various mathematical problems [39]. This demonstrates the effectiveness and usefulness of spline approaches through their numerous successful implementations. Therefore, this paper investigates the approximate solution of systems of the Lane-Emden equations using the CBSM.

This paper is organized as follows, in the next section, we present the basic preliminaries of the method. A short summary of cubic B-Spline method is presented in Section 3. We show the convergence analysis in Section 4. Finally, we present some numerical examples in Section 5.

2. Preliminaries

In this section, we introduce some basic facts regarding cubic B-spline approximation. Assume that the interval $\Gamma = [\alpha, \beta]$ can be divided into k subintervals $[\tau_i, \tau_{i+1}]$ via

$$\tau_i = \alpha + i\Lambda,\ i = 0, \cdots, k,$$

where

$$\Lambda = (\beta - \alpha)/k.$$

The linear space of the cubic spline over the given partition is

$$\mathrm{M}_3(I) = \{\mu(\tau) \in C^2(I): \mu(\tau)|_{I_i} \in P_3, i = 0, \dots, k-1\},$$

where $\mu(\tau)|_{I_i}$ indicates the restriction of $\mu(\tau)$ to I_i and P_3 indicates the set of cubic polynomials in one-variable. The dimension of linear space $\mathrm{M}_3(I)$ is $(k+3)$. Extend $\Gamma = [\alpha, \beta]$ to

$$\bar{\Gamma} = [\alpha - 3\Lambda, \beta + 3\Lambda]$$

with the equidistant knots

$$\tau_i = \alpha + i\Lambda, i = -3,\ \dots, k+3.$$

The cubic B-spline function

$$\mathrm{K}_i(\tau), i = -1, \dots, k+1,$$

is given by [40]

$$\mathrm{K}_i(\tau) = \begin{cases} \dfrac{(\tau-\tau_i)^3}{6\Lambda^3}, & \tau \in [\tau_i, \tau_{i+1}], \\ \dfrac{(\tau-\tau_i)^3}{6\Lambda^3} - 2\dfrac{(\tau-\tau_{i+1})^3}{3\Lambda^3}, & \tau \in [\tau_{i+1}, \tau_{i+2}], \\ \dfrac{(\tau_{i+4}-\tau)^3}{6\Lambda^3} - 2\dfrac{(\tau_{i+3}-\tau)^3}{3\Lambda^3}, & \tau \in [\tau_{i+2}, \tau_{i+3}], \\ \dfrac{(\tau_{i+4}-\tau)^3}{6\Lambda^3}, & \tau \in [\tau_{i+3}, \tau_{i+4}], \\ 0, & \text{else}. \end{cases}$$

The $\mathrm{K}_i(\tau), i = -1, \dots, k+1$, form basis splines of $\mathrm{M}_3(I)$. The values of $\mathrm{K}_i(\tau), \mathrm{K}_i'(\tau)$ and $\mathrm{K}_i''(\tau)$ at the knots are recorded in Table 1.

Table 1. The values of $\mathrm{K}_i(\tau), \mathrm{K}_i'(\tau)$ and $\mathrm{K}_i''(\tau)$ at the knots.

	τ_{i-1}	τ_i	τ_{i+1}	else
$\mathrm{K}_i(\tau)$	$\frac{1}{6}$	$\frac{4}{6}$	$\frac{1}{6}$	0
$\mathrm{K}_i'(\tau)$	$\frac{1}{2\Lambda}$	0	$-\frac{1}{2\Lambda}$	0
$\mathrm{K}_i''(\tau)$	$\frac{1}{\Lambda^2}$	$\frac{-2}{\Lambda^2}$	$\frac{1}{\Lambda^2}$	0

For a sufficiently smooth function $\rho(\tau)$, there is a unique cubic spline $\mu(\tau) \in \mathrm{M}_3(I)$ fulfilling the interpolation conditions

$$\mu(\tau_i) = \rho(\tau_i), i = 0, \ldots, k,$$

and

$$\mu'(\alpha) = \rho'(\alpha),$$

such that

$$\mu(\tau) = \sum_{i=-1}^{k+1} \lambda_i \mathrm{K}_i(\tau), \tag{3}$$

where $\lambda_i{}'$s are constants to be estimated.

Using (3), we get

$$\mu(\tau_j) = \sum_{i=-1}^{k+1} \lambda_i \, \mathrm{K}_i(\tau_j) = \frac{\lambda_{j-1} + 4\lambda_j + \lambda_{j+1}}{6}, \tag{4}$$

$$\mu'(\tau_j) = \sum_{i=-1}^{k+1} \lambda_i \, \mathrm{K}'_i(\tau_j) = \frac{\lambda_{j+1} - \lambda_{j-1}}{2\Lambda}, \tag{5}$$

$$\mu''(\tau_j) = \sum_{i=-1}^{k+1} \lambda_i \mathrm{K}''_i(\tau_j) = \frac{\lambda_{j-1} - 2\lambda_j + \lambda_{j+1}}{\Lambda^2}. \tag{6}$$

Equations (4)–(6) are the most important relations in deriving the CBSM.

3. Cubic B-spline method

In this section, we present the cubic B-spline method for (1) and (2). Let

$$\mu_1(\tau) = \sum_{i=-1}^{k+1} \lambda_i \mathrm{K}_i(\tau)$$

and

$$\mu_2(\tau) = \sum_{i=-1}^{k+1} \eta_i \mathrm{K}_i(\tau)$$

denote the approximate solutions of system (1) and (2) of $\omega_1(\tau)$ and $\omega_2(\tau)$, respectively. To overcome the singularity of (1) at $\tau = 0$, we apply the L'Hôpital's rule to the second term as τ approaches zero, to achieve

$$\begin{cases}(1+\delta_1)\dfrac{d^2\omega_1(\tau)}{d\tau^2}+\hbar_1(\omega_1(\tau),\omega_2(\tau))=\aleph_1(\tau),\\(1+\delta_2)\dfrac{d^2\omega_2(\tau)}{d\tau^2}+\hbar_2(\omega_1(\tau),\omega_2(\tau))=\aleph_2(\tau),\end{cases}\text{for } \tau=0,$$

$$\begin{cases}\dfrac{d^2\omega_1(\tau)}{d\tau^2}+\dfrac{\delta_1}{\tau}\dfrac{d\omega_1(\tau)}{d\tau}+\hbar_1(\omega_1(\tau),\omega_2(\tau))=\aleph_1(\tau),\\\dfrac{d^2\omega_2(\tau)}{d\tau^2}+\dfrac{\delta_2}{\tau}\dfrac{d\omega_2(\tau)}{d\tau}+\hbar_2(\omega_1(\tau),\omega_2(\tau))=\aleph_2(\tau),\end{cases}\text{for } \tau\neq 0. \tag{7}$$

By discretizing (7), we get

$$\begin{cases}(1+\delta_1)\dfrac{d^2\omega_1(\tau_0)}{d\tau^2}+\hbar_1(\omega_1(\tau_0),\omega_2(\tau_0))=\aleph_1(\tau_0),\\(1+\delta_2)\dfrac{d^2\omega_2(\tau_0)}{d\tau^2}+\hbar_2(\omega_1(\tau_0),\omega_2(\tau_0))=\aleph_2(\tau_0),\end{cases}$$

$$\begin{cases}\dfrac{d^2\omega_1(\tau_j)}{d\tau^2}+\dfrac{\delta_1}{\tau_j}\dfrac{d\omega_1(\tau_j)}{d\tau}+\hbar_1(\omega_1(\tau_j),\omega_2(\tau_j))=\aleph_1(\tau_j),\\\dfrac{d^2\omega_2(\tau_j)}{d\tau^2}+\dfrac{\delta_2}{\tau_j}\dfrac{d\omega_2(\tau_j)}{d\tau}+\hbar_2(\omega_1(\tau_j),\omega_2(\tau_j))=\aleph_2(\tau_j),\end{cases} \tag{8}$$

where $j=1,\cdots,k$. By using (2)–(5), (8) becomes

$$\begin{cases}(1+\delta_1)\left(\dfrac{\lambda_{-1}-2\lambda_0+\lambda_1}{\Lambda^2}\right)+\hbar_1(\varepsilon_1,\vartheta_1)=\aleph_1(\tau_0),\\(1+\delta_2)\left(\dfrac{\eta_{-1}-2\eta_0+\eta_1}{\Lambda^2}\right)+\hbar_2(\varepsilon_1,\vartheta_1)=\aleph_2(\tau_0),\end{cases}$$

$$\begin{cases}\left(\dfrac{\lambda_{i-1}-2\lambda_i+\lambda_{i+1}}{\Lambda^2}\right)+\dfrac{\delta_1}{\tau_j}\left(\dfrac{\lambda_{j+1}-\lambda_{j-1}}{2\Lambda}\right)+\hbar_1\left(\dfrac{\lambda_{i-1}+4\lambda_i+\lambda_{i+1}}{6},\ \dfrac{\eta_{i-1}+4\eta_i+\eta_{i+1}}{6}\right)\\=\aleph_1(\tau_j),\ j=1,\cdots,k,\\\left(\dfrac{\eta_{i-1}-2\eta_i+\eta_{i+1}}{\Lambda^2}\right)+\dfrac{\delta_2}{\tau_j}\left(\dfrac{\eta_{j+1}-\eta_{j-1}}{2\Lambda}\right)+\hbar_2\left(\dfrac{\lambda_{i-1}+4\lambda_i+\lambda_{i+1}}{6},\ \dfrac{\eta_{i-1}+4\eta_i+\eta_{i+1}}{6}\right)\\=\aleph_2(\tau_j),\ \ j=1,\cdots,k.\end{cases} \tag{9}$$

The initial conditions (2) as well provide the following four equations

$$\omega_1(0)=\varepsilon_1=\frac{\lambda_{-1}+4\lambda_0+\lambda_1}{6}, \tag{10}$$

$$\omega'_1(0)=0=\frac{\lambda_1-\lambda_{-1}}{2\Lambda}, \tag{11}$$

$$\omega_2(0)=\vartheta_1=\frac{\eta_{-1}+4\eta_0+\eta_1}{6}, \tag{12}$$

$$\omega''_2(0)=0=\frac{\eta_1-\eta_{-1}}{2\Lambda}. \tag{13}$$

Equations (9)–(13) give us $2(k+3)$ nonlinear equations with λ_i and η_i, $i=-1,\dots,k+1$ as unknowns. Upon solving this system, we determine the coefficients of

$$\mu_1(\tau)=\sum_{i=-1}^{k+1}\lambda_i \mathrm{K}_i(\tau)$$

and

$$\mu_2(\tau)=\sum_{i=-1}^{k+1}\eta_i \mathrm{K}_i(\tau).$$

4. Convergence analysis

In this section, we analyze the convergence for the proposed method. For this purpose, we assume $\omega_1(\tau),\omega_2(\tau)\in C^5[0,1]$. From (3)–(5), we get [39]

$$\Lambda\left[\mu_j'(\tau_{i-1})+4\mu_j'(\tau_i)+\mu_j'(\tau_{i+1})\right]=3\left[\omega_j(\tau_{i+1})-\omega_j(\tau_{i-1})\right], \tag{14}$$

$$\Lambda^2\mu_j''(\tau_i)=6\left[\mu_j(\tau_{i+1})-\mu_j(\tau_i)\right]-2\Lambda\left[2\mu_j'(\tau_i)+\mu_j'(\tau_{i+1})\right], \tag{15}$$

$j=1,2$. Using the shifting operator, $E\left(\mu_j(\tau_i)\right)=\mu_j(\tau_{i+1})$, (14) may be written as

$$\frac{\Lambda}{6}(E^{-1}+4+E)\mu_j'(\tau_i)=\frac{1}{2}(E-E^{-1})\omega_j(\tau_i), \tag{16}$$

$j=1,2$. Since $E=e^{\Lambda D}$ and $D\equiv d/d\tau$, one can get

$$\begin{aligned} e^{\Lambda D}+e^{-\Lambda D}&=2\sum_{k=0}^{\infty}\frac{(\Lambda D)^{2k}}{(2k)!},\\ e^{\Lambda D}-e^{-\Lambda D}&=2\sum_{k=0}^{\infty}\frac{(\Lambda D)^{2k+1}}{(2k+1)!}. \end{aligned} \tag{17}$$

Therefore, using (17), (16) can be expressed as

$$\left(1+\frac{1}{3}\sum_{k=1}^{\infty}\frac{(\Lambda D)^{2k}}{(2k)!}\right)\mu_j'(\tau_i)=\left(\sum_{k=0}^{\infty}\frac{(\Lambda D)^{2k+1}}{(2k+1)!}\right)\omega_j(\tau_i). \tag{18}$$

Simplifying (18) gives

$$\mu_j'(\tau_i) = \left(\sum_{k=0}^{\infty} \frac{(\Lambda D)^{2k+1}}{(2k+1)!}\right)\left(1+\frac{1}{3}\sum_{k=1}^{\infty}\frac{(\Lambda D)^{2k}}{(2k)!}\right)^{-1} \omega_j(\tau_i)$$
$$= (\mathrm{D}+\frac{\Lambda^2\mathrm{D}^3}{3!}+\frac{\Lambda^4\mathrm{D}^5}{5!}+\cdots)\left(1-\left(\frac{\Lambda^2\mathrm{D}^2}{6}+\frac{\Lambda^4\mathrm{D}^4}{72}+\cdots\right)\right. \tag{19}$$
$$\left.+\left(\frac{\Lambda^2 D^2}{6}+\frac{\Lambda^4 D^4}{72}+\cdots\right)^2+\cdots\right)\omega_j(\tau_i)$$
$$= D\left(1-\frac{\Lambda^4 D^4}{180}+\frac{\Lambda^6 D^6}{1512}-\cdots\right)\omega_j(\tau_i).$$

Therefore,

$$\mu_j'(\tau_i) = \omega_j'(\tau_i) - \frac{\Lambda^4}{180}\omega_j^{(5)}(\tau_i) + \cdots. \tag{20}$$

Similarly, (15) gives

$$\mu_j''(\tau_i) = \omega_j''(\tau_i) - \frac{1}{12}\Lambda^2\omega_j^{(4)}(\tau_i) + \frac{1}{360}\Lambda^4\omega_j^{(6)}(\tau_i) + O(\Lambda^6), \tag{21}$$

At this point, the error functions $e_j(\tau), j = 1,2,$ are stated as follows:

$$e_1(\tau_i) = \aleph_1(\tau_j) - \frac{d^2\omega_1(\tau_j)}{d\tau^2} - \frac{\delta_1}{\tau_j}\frac{d\omega_1(\tau_j)}{d\tau} - \hbar_1(\omega_1(\tau_j),\omega_2(\tau_j))$$
$$= \frac{d^2\mu_1(\tau_j)}{d\tau^2} + \frac{\delta_1}{\tau_j}\frac{d\mu_1(\tau_j)}{d\tau} + \hbar_1\left(\mu_1(\tau_j),\mu_2(\tau_j)\right)$$
$$- \frac{d^2\omega_1(\tau_j)}{d\tau^2} - \frac{\delta_1}{\tau_j}\frac{d\omega_1(\tau_j)}{d\tau} - \hbar_1(\omega_1(\tau_j),\omega_2(\tau_j))$$
$$= \left[\frac{d^2\mu_1(\tau_j)}{d\tau^2} - \frac{d^2\omega_1(\tau_j)}{d\tau^2}\right] + \frac{\delta_1}{\tau_j}\left[\frac{d\mu_1(\tau_j)}{d\tau} - \frac{d\omega_1(\tau_j)}{d\tau}\right], \tag{22}$$
$$e_2(\tau_i) = \aleph_2(\tau_j) - \frac{d^2\omega_2(\tau_j)}{d\tau^2} - \frac{\delta_2}{\tau_j}\frac{d\omega_2(\tau_j)}{d\tau} - \hbar_2(\omega_1(\tau_j),\omega_2(\tau_j))$$
$$= \frac{d^2\mu_2(\tau_j)}{d\tau^2} + \frac{\delta_2}{\tau_j}\frac{d\mu_2(\tau_j)}{d\tau} + \hbar_2\left(\mu_1(\tau_j),\mu_2(\tau_j)\right)$$
$$- \frac{d^2\omega_2(\tau_j)}{d\tau^2} - \frac{\delta_2}{\tau_j}\frac{d\omega_2(\tau_j)}{d\tau} - \hbar_2(\omega_1(\tau_j),\omega_2(\tau_j))$$
$$= \left[\frac{d^2\mu_2(\tau_j)}{d\tau^2} - \frac{d^2\omega_2(\tau_j)}{d\tau^2}\right] + \frac{\delta_2}{\tau_j}\left[\frac{d\mu_2(\tau_j)}{d\tau} - \frac{d\omega_2(\tau_j)}{d\tau}\right],$$

where $j = 1,\ldots,k$. Substitute (20) and (21) in (22) yields

$$\begin{aligned}\|e_1(\tau_i)\|_\infty &= O(\Lambda^2),\\ \|e_2(\tau_i)\|_\infty &= O(\Lambda^2).\end{aligned} \tag{23}$$

For i=0, we get

$$\begin{aligned}
e_1(\tau_0) &= \aleph_1(\tau_0) - (1+\delta_1)\frac{d^2\omega_1(\tau_0)}{d\tau^2} - \hbar_1\big(\omega_1(\tau_0),\omega_2(\tau_0)\big)\\
&= (1+\delta_1)\frac{d^2\mu_1(\tau_0)}{d\tau^2} + \hbar_1\big(\mu_1(\tau_0),\mu_2(\tau_0)\big)\\
&-(1+\delta_1)\frac{d^2\omega_1(\tau_0)}{d\tau^2} - \hbar_1(\omega_1(\tau_0),\omega_2(\tau_0)\\
&= (1+\delta_1)\left[\frac{d^2\mu_1(\tau_0)}{d\tau^2} - \frac{d^2\omega_1(\tau_0)}{d\tau^2}\right],\\
e_2(\tau_0) &= \aleph_2(\tau_0) - (1+\delta_2)\frac{d^2\omega_2(\tau_0)}{d\tau^2} - \hbar_2(\omega_1(\tau_0),\omega_2(\tau_0))\\
&= (1+\delta_2)\frac{d^2\omega_2(\tau_0)}{d\tau^2} + \hbar_2\big(\mu_1(\tau_0),\mu_2(\tau_0)\big)\\
&-(1+\delta_2)\frac{d^2\omega_2(\tau_0)}{d\tau^2} - \hbar_1(\omega_1(\tau_0),\omega_2(\tau_0)\\
&= (1+\delta_2)\left[\frac{d^2\omega_2(\tau_0)}{d\tau^2} - \frac{d^2\omega_2(\tau_0)}{d\tau^2}\right].
\end{aligned} \tag{24}$$

Using (21) in (24), we have

$$\begin{aligned}\|e_1(\tau_0)\|_\infty &= O(\Lambda^2),\\ \|e_2(\tau_0)\|_\infty &= O(\Lambda^2).\end{aligned} \tag{25}$$

Therefore, from (23) and (25), the truncation error for the considered system is $O(\Lambda^2)$.

5. Numerical results

In this section, we present the numerical solution to (1) and (2) using the cubic B-spline technique. Several problems are examined to prove the accuracy and efficiency of the proposed method using the absolute errors between the approximate solutions and the exact solutions ($|e_j(\tau)|$) for various *k*. All the results are calculated by using MATHEMATICA 12.

Problem 1. Consider the following system [24]

$$\begin{aligned}\frac{d^2\omega_1(\tau)}{d\tau^2} + \frac{3}{\tau}\frac{d\omega_1(\tau)}{d\tau} - 4(\omega_1(\tau)+\omega_2(\tau)) = 0,\\ \frac{d^2\omega_2(\tau)}{d\tau^2} + \frac{2}{\tau}\frac{d\omega_2(\tau)}{d\tau} + 3(\omega_1(\tau)+\omega_2(\tau)) = 0,\end{aligned} \tag{26}$$

subject to

$$\begin{aligned} \omega_1(0) = 1, \qquad \omega_1'(0) = 0, \\ \omega_2(0) = 1, \qquad \omega_2'(0) = 0, \end{aligned} \tag{27}$$

where the exact solutions are $\omega_1(\tau) = 1 + \tau^2$ and $\omega_2(\tau) = 1 - \tau^2$. In this example, we use $\Lambda = 0.1$. The numerical results and the exact solution at the grid points are listed in Table 2. We can conclude from Table 2 that the obtained numerical results are in excellent agreement with the exact solutions. We note that for this problem, our results are exact and the achieved errors are only due to round-off calculations. The CPU time for this problem, with $\Lambda = 0.1$, is 0.0156s.

Table 2. Numerical results for Problem 1.

τ	$\omega_1(\tau)$	$\mu_1(\tau)(\Lambda = 0.1)$	$\lvert e_1(\tau)\rvert$	$\omega_2(\tau)$	$\mu_2(\tau)(\Lambda = 0.1)$	$\lvert e_2(\tau)\rvert$
0.0	1	1	0	1	1	2.22045×10^{-16}
0.1	1.01	1.01	0	0.99	0.99	2.22045×10^{-16}
0.2	1.04	1.04	0	0.96	0.96	2.22045×10^{-16}
0.3	1.09	1.09	2.22045×10^{-16}	0.91	0.91	2.22045×10^{-16}
0.4	1.16	1.16	2.22045×10^{-16}	0.84	0.84	3.33067×10^{-16}
0.5	1.25	1.25	2.22045×10^{-16}	0.75	0.75	3.33067×10^{-16}
0.6	1.36	1.36	4.44089×10^{-16}	0.64	0.64	3.33067×10^{-16}
0.7	1.49	1.49	2.22045×10^{-16}	0.51	0.51	4.44089×10^{-16}
0.8	1.61	1.61	0	0.36	0.36	4.44089×10^{-16}
0.9	1.81	1.81	2.22045×10^{-16}	0.19	0.19	5.55112×10^{-16}
1.0	2	2	0	0	5.64363×10^{-16}	5.64363×10^{-16}

Problem 2. Consider the following system [27]

$$\begin{aligned} \omega_1''(\tau) + \frac{2}{\tau}\omega_1'(\tau) - (4\tau^2 + 6)\omega_1(\tau) + \omega_2(\tau) = \tau^4 - \tau^3, \\ \omega_2''(\tau) + \frac{8}{\tau}\omega_2'(\tau) + \omega_1(\tau) + \tau\omega_2(\tau) = e^{\tau^2} + \tau^5 - \tau^4 + 44\tau^2 - 30\tau, \end{aligned} \tag{28}$$

subject to

$$\begin{aligned} \omega_1(0) = 1, \omega_1'(0) = 0, \\ \omega_2(0) = 0, \omega_2'(0) = 0, \end{aligned} \tag{29}$$

where the exact solutions are

$$\omega_1(\tau) = e^{\tau^2}$$

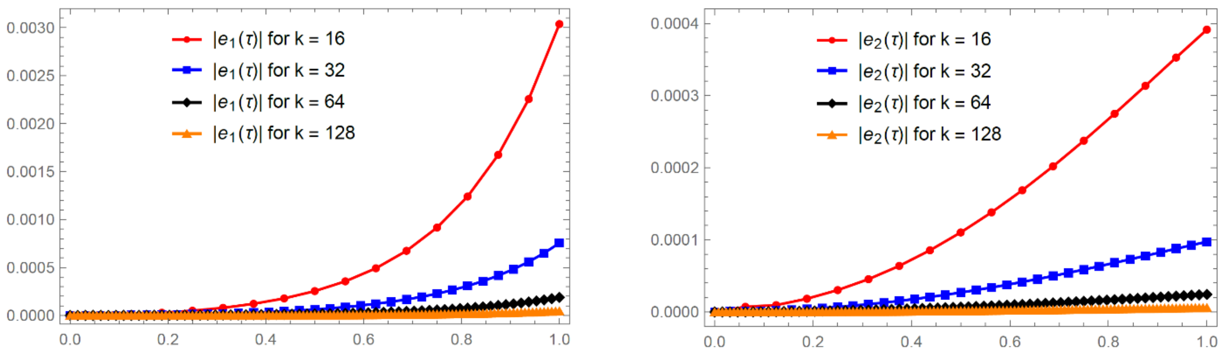
and

$$\omega_2(\tau) = \tau^4 - \tau^3.$$

The obtained numerical and exact solutions, with different values of Λ, are depicted in Figure 1. The absolute errors between exact and numerical results, with $\Lambda = 0.1$ and 0.01, are presented in Tables 3 and 4. In Table 5, we compare the maximum absolute error of CBSM with those of [27]. Our results seem to be better than those of [27]. The CPU time for this problem, with $\Lambda = 0.1$ and 0.01, are 0.0156s and 0.0625s respectively.

Table 3. Numerical result for approximate solution of $\omega_1(\tau)$ in Problem 2.

τ	$\omega_1(\tau)$	$\mu_1(\tau)(\Lambda = 0.1)$	$\lvert e_1(\tau)\rvert$	$\mu_1(\tau)(\Lambda = 0.01)$	$\lvert e_1(\tau)\rvert$
0.0	1	1	0	1	0
0.1	1.01005	1.01008	3.38838×10^{-5}	1.01005	1.71898×10^{-7}
0.2	1.04081	1.04091	9.51843×10^{-5}	1.04081	7.1747×10^{-7}
0.3	1.09417	1.09438	2.01533×10^{-4}	1.09417	1.75608×10^{-6}
0.4	1.17351	1.17389	3.80997×10^{-4}	1.17351	3.5098×10^{-6}
0.5	1.28403	1.2847	6.71612×10^{-4}	1.28403	6.3527×10^{-6}
0.6	1.43333	1.43446	1.13524×10^{-3}	1.43334	1.08922×10^{-5}
0.7	1.63232	1.63419	1.87147×10^{-3}	1.63233	1.81055×10^{-5}
0.8	1.89648	1.89952	3.04087×10^{-3}	1.89651	2.9567×10^{-5}
0.9	2.24791	2.25281	4.90399×10^{-3}	2.24796	4.78283×10^{-5}
1.0	2.71828	2.72617	7.88713×10^{-3}	2.71836	7.70601×10^{-5}



(a) Absolute errors for $\omega_1(\tau)$.

(b) Absolute errors for $\omega_2(\tau)$.

Figure 1. Absolute error functions for Problem 2.

Table 4. Numerical result for approximate solution of $\omega_2(\tau)$ in Problem 2.

τ	$\omega_2(\tau)$	$\mu_2(\tau)(\Lambda = 0.1)$	$\lvert e_2(\tau)\rvert$	$\mu_2(\tau)(\Lambda = 0.01)$	$\lvert e_2(\tau)\rvert$
0.0	0	5.99863×10^{-31}	5.99863×10^{-31}	0	0
0.1	−0.0009	−0.000853346	4.66538×10^{-5}	−0.000899887	1.13186×10^{-7}
0.2	−0.0064	−0.0063379	6.21025×10^{-5}	−0.00639955	4.45865×10^{-7}
0.3	-0.0189	−0.0187802	1.19801×10^{-4}	−0.01889900	9.98298×10^{-7}
0.4	-0.0384	−0.0382022	1.97763×10^{-4}	−0.0383982	1.76662×10^{-6}
0.5	-0.0625	−0.0622052	2.94843×10^{-4}	−0.0624973	2.74383×10^{-6}
0.6	-0.0864	−0.0859881	4.11936×10^{-4}	−0.0863961	3.91802×10^{-6}
0.7	-0.1029	−0.102353	5.46598×10^{-4}	−0.102895	5.2695×10^{-6}
0.8	-0.1024	−0.101704	6.95542×10^{-4}	−0.102393	6.76623×10^{-6}
0.9	-0.0729	−0.0720466	8.5344×10^{-4}	−0.0728916	8.35631×10^{-6}
1.0	0	0.00101164	1.01164×10^{-3}	9.95549×10^{-6}	9.95549×10^{-6}

Table 5. Comparison of maximum absolute error of Problem 2.

	$\mu_1(\tau)$	$\mu_2(\tau)$
CBSM($\Lambda = 0.1$)	7.88×10^{-3}	1.01×10^{-3}
CBSM($\Lambda = 0.01$)	7.70×10^{-5}	9.95×10^{-6}
[27] (N=5)	3.14×10^{-2}	4.19×10^{-5}
[27] (N=6)	6.11×10^{-4}	7.22×10^{-6}

Problem 3. Consider the following system [24,41]

$$\begin{aligned}&\omega_1''(\tau)+\frac{5}{\tau}\omega_1'(\tau)+8(e^{\omega_1(\tau)}+2e^{-\frac{\omega_2(\tau)}{2}})=0,\\&\omega_2''(\tau)+\frac{3}{\tau}\omega_2'(\tau)-8(e^{\frac{\omega_1(\tau)}{2}}+e^{-\omega_2(\tau)})=0,\end{aligned} \tag{30}$$

subject to

$$\begin{aligned}&\omega_1(0)=1-2\ln(2)\,,\omega_1'(0)=0,\\&\omega_2(0)=1+2\ln(2)\,,\omega_2'(0)=0,\end{aligned} \tag{31}$$

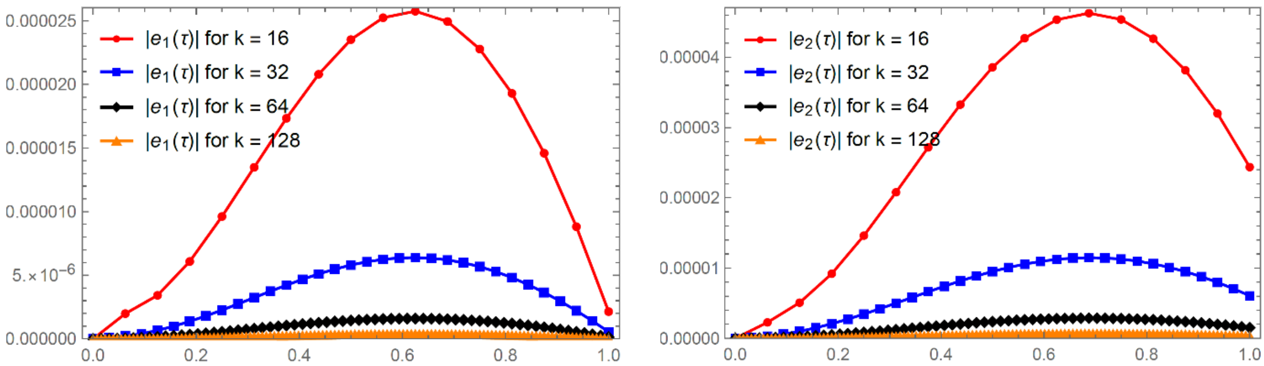
where the exact solutions are

$$\omega_1(\tau)=1-2\ln(\tau^2+2)$$

and

$$\omega_2(\tau)=1+2\ln(\tau^2+2).$$

We depicted our numerical and exact solutions with different values of Λ in Figure 2. The absolute errors between exact and numerical results are reported in Tables 6 and 7. Table 8 compares the maximum absolute error of CBSM with those of the method in [41]. It appears that our findings are superior to the outcomes presented in [41]. The CPU time for this problem, with $\Lambda = 0.1$ and 0.01, are 0.0156s and 0.0781s respectively.



(a) Absolute errors for $\omega_1(\tau)$.　　(b) Absolute errors for $\omega_2(\tau)$.

Figure 2. Absolute error functions for Problem 3.

Table 6. Numerical result for approximate solution of $\omega_1(\tau)$ in Problem 3.

τ	$\omega_1(\tau)$	$\mu_1(\tau)(\Lambda = 0.1)$	$\lvert e_1(\tau)\rvert$	$\mu_1(\tau)(\Lambda = 0.01)$	$\lvert e_1(\tau)\rvert$
0.0	-0.386294	-0.386294	2.22045×10^{-16}	-0.386294	-2.22045×10^{-16}
0.1	-0.396269	-0.396257	1.28893×10^{-5}	-0.396269	4.13534×10^{-8}
0.2	-0.4259	-0.425878	2.11827×10^{-5}	-0.425899	1.5319×10^{-7}
0.3	-0.474328	-0.474293	3.503×10^{-5}	-0.474328	3.07878×10^{-7}
0.4	-0.540216	-0.540166	5.02533×10^{-5}	-0.540216	4.67063×10^{-7}
0.5	-0.62186	-0.621799	6.18095×10^{-5}	-0.62186	5.91857×10^{-7}
0.6	-0.717323	-0.717256	6.68475×10^{-5}	-0.717323	6.51126×10^{-7}
0.7	-0.824565	-0.824502	6.3593×10^{-5}	-0.824565	6.26415×10^{-7}
0.8	-0.941558	-0.941506	5.16047×10^{-5}	-0.941557	5.13027×10^{-7}
0.9	-1.06637	-1.06634	3.15988×10^{-5}	-1.06637	3.18094×10^{-7}
1.0	-1.19722	-1.19722	5.10219×10^{-6}	-1.19722	5.69956×10^{-8}

Table 7. Numerical result for approximate solution of $\omega_2(\tau)$ in Problem 3.

τ	$\omega_2(\tau)$	$\mu_2(\tau)(\Lambda = 0.1)$	$\lvert e_2(\tau)\rvert$	$\mu_2(\tau)(\Lambda = 0.01)$	$\lvert e_2(\tau)\rvert$
0.0	2.38629	2.38629	2.22045×10^{-16}	2.38629	2.22045×10^{-16}
0.1	2.39627	2.39625	1.47619×10^{-5}	2.39627	6.20966×10^{-8}
0.2	2.4259	2.42587	3.17866×10^{-5}	2.42599	2.32511×10^{-7}
0.3	2.47433	2.47427	5.42035×10^{-5}	2.47433	4.75075×10^{-7}
0.4	2.54022	2.54014	7.94034×10^{-5}	2.54022	7.3881×10^{-7}
0.5	2.62186	2.62176	1.01507×10^{-4}	2.62186	9.70554×10^{-7}
0.6	2.71732	2.71721	1.16022×10^{-4}	2.71732	1.12546×10^{-6}
0.7	2.82457	2.82445	1.1998×10^{-4}	2.82456	1.17361×10^{-6}
0.8	2.94156	2.94145	1.121×10^{-4}	2.94156	1.10206×10^{-6}
0.9	3.06637	3.06628	9.26097×10^{-5}	3.06637	9.13128×10^{-7}
1.0	3.19722	3.19716	6.2868×10^{-5}	3.19722	6.20546×10^{-7}

Table 8. Comparison of maximum absolute error of Problem 3.

	$\mu_1(\tau)$	$\mu_2(\tau)$
CBSM($\Lambda = 0.1$)	6.68×10^{-5}	1.19×10^{-4}
CBSM($\Lambda = 0.01$)	6.51×10^{-7}	1.17×10^{-6}
[41] (*j*=3)	1.47×10^{-3}	1.64×10^{-3}
[41] (*j*=4)	3.67×10^{-4}	4.10×10^{-4}

Problem 4. Consider the following system of LEE [24,27,40,41]

$$\begin{aligned} &\omega_1''(\tau) + \frac{1}{\tau}\omega_1'(\tau) - \omega_2^3(\tau)(\omega_1^2 + 1) = 0, \\ &\omega_2''(\tau) + \frac{3}{\tau}\omega_2'(\tau) + \omega_2^5(\tau)(\omega_1^2 + 3) = 0, \end{aligned} \tag{32}$$

subject to

$$\begin{aligned} &\omega_1(0) = 1, \omega'_1(0) = 0, \\ &\omega_2(0) = 1, \omega'_2(0) = 0, \end{aligned} \tag{33}$$

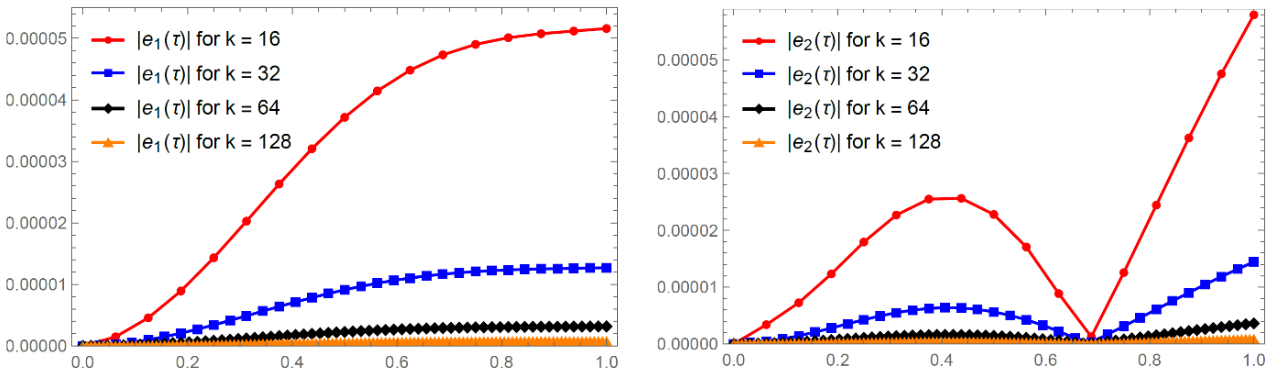
where the exact solutions are

$$\omega_1(\tau) = \sqrt{1 + \tau^2}$$

and

$$\omega_2(\tau) = \frac{1}{\sqrt{1 + \tau^2}}.$$

The achieved numerical results with $\Lambda = 0.1$ and 0.01 are exposed in Tables 9 and 10. Table 11 shows a comparison between the maximum absolute error of CBSM and that of the approaches discussed in [27,40,41]. Based on the results, it seems that our research outperforms the results reported in [27,40,41]. Absolute errors for different values of Λ are exposed in Figure 3. The CPU time for this problem, with $\Lambda = 0.1$ and 0.01, are 0.0156s and 0.0781s respectively.



(a) Absolute errors for $\omega_1(\tau)$.   (b) Absolute errors for $\omega_2(\tau)$.

Figure 3. Absolute error functions for Problem 4.

Table 9. Numerical result for approximate solution of $\omega_1(\tau)$ in Problem 4.

τ	$\omega_1(\tau)$	$\mu_1(\tau)(\Lambda = 0.1)$	$\lvert e_1(\tau)\rvert$	$\mu_1(\tau)(\Lambda = 0.01)$	$\lvert e_1(\tau)\rvert$
0.0	1	1	0	1	0
0.1	1.004988	1.004978	9.41481×10^{-6}	1.004987	6.21775×10^{-8}
0.2	1.019804	1.019776	2.80913×10^{-5}	1.019804	2.29506×10^{-7}
0.3	1.044031	1.043979	5.15984×10^{-5}	1.04403	4.61458×10^{-7}
0.4	1.077033	1.076957	7.60906×10^{-5}	1.077032	7.07716×10^{-7}
0.5	1.118034	1.117936	9.76652×10^{-5}	1.118033	9.25606×10^{-7}
0.6	1.16619	1.166076	1.13971×10^{-4}	1.166189	1.09013×10^{-6}
0.7	1.220656	1.220531	1.24491×10^{-4}	1.220654	1.19564×10^{-6}
0.8	1.280625	1.280495	1.30176×10^{-4}	1.280624	1.25163×10^{-6}
0.9	1.345362	1.34523	1.32854×10^{-4}	1.345361	1.27653×10^{-6}
1.0	1.414214	1.414079	1.34662×10^{-4}	1.414212	1.29187×10^{-6}

Table 10. Numerical result for approximate solution of $\omega_2(\tau)$ in Problem 4.

τ	$\omega_2(\tau)$	$\mu_2(\tau)(\Lambda = 0.1)$	$\lvert e_2(\tau)\rvert$	$\mu_2(\tau)(\Lambda = 0.01)$	$\lvert e_2(\tau)\rvert$
0.0	1	1	0	1	2.22045×10^{-16}
0.1	0.995037	0.995059	2.1554×10^{-5}	0.995037	9.03189×10^{-8}
0.2	0.980581	0.980623	4.20048×10^{-5}	0.980581	3.08008×10^{-7}
0.3	0.957826	0.957887	6.03609×10^{-5}	0.957827	5.34936×10^{-7}
0.4	0.928477	0.928545	6.87324×10^{-5}	0.928477	6.48155×10^{-7}
0.5	0.894427	0.894487	5.93414×10^{-5}	0.894428	5.76216×10^{-7}
0.6	0.857493	0.857525	3.17569×10^{-5}	0.857493	3.15637×10^{-7}
0.7	0.819232	0.819223	9.41889×10^{-6}	0.819232	8.61722×10^{-8}
0.8	0.780869	0.780811	5.74405×10^{-5}	0.780868	5.60283×10^{-7}
0.9	0.743294	0.743188	1.05817×10^{-4}	0.743293	1.04065×10^{-6}
1.0	0.707107	0.706957	1.4962×10^{-4}	0.707105	1.4772×10^{-6}

Table 11. Comparison of maximum absolute error of Problem 4.

	$\mu_1(\tau)$	$\mu_2(\tau)$
CBSM($\Lambda = 0.1$)	1.34×10^{-4}	1.49×10^{-4}
CBSM($\Lambda = 0.01$)	1.29×10^{-6}	1.47×10^{-6}
[27] (N=4)	6.44×10^{-4}	8.87×10^{-4}
[27] (N=5)	7.46×10^{-5}	6.08×10^{-5}
[40] (n=4)	4.86×10^{-4}	1.45×10^{-3}
[41] (j=3)	2.76×10^{-5}	1.31×10^{-4}
[41] (j=4)	6.87×10^{-6}	3.28×10^{-5}

Problem 5. Consider the following system of LEE [24,40]

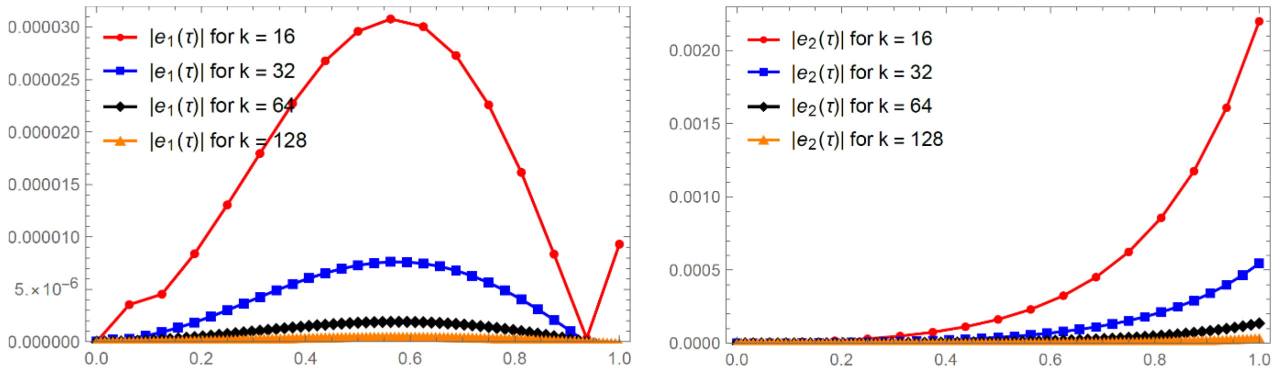
$$\begin{aligned} &\omega_1'' + \frac{8}{\tau}\omega_1'(\tau) + (18\omega_1(\tau) - 4\omega_1(\tau)\ln\omega_2(\tau)) = 0, \\ &\omega_2''(\tau) + \frac{4}{\tau}\omega_2'(\tau) + (4\omega_2(\tau)\ln\omega_1(\tau) - 10\omega_2(\tau)) = 0, \end{aligned} \tag{34}$$

subject to

$$\begin{aligned} &\omega_1(0) = 1, \omega_1'(0) = 0, \\ &\omega_2(0) = 1, \omega_2'(0) = 0, \end{aligned} \tag{35}$$

where the exact solutions are $\omega_1(\tau) = e^{-\tau^2}$ and $\omega_2(\tau) = e^{\tau^2}$.

Figure 4 represents the plot of our numerical and exact solutions for Problem 5 with different values of Λ. The absolute errors for $\Lambda = 0.1$ and 0.01 are presented in Tables 12 and 13. Table 14 displays a comparison between the maximum absolute error of CBSM and that of the approaches discussed in [40,41]. Our results indicate that they are better than the results reported in [40,41]. The CPU time for this problem, with $\Lambda = 0.1$ and 0.01, are 0.0156s and 0.0625s respectively. From all these tables and figures, we can observe that our numerical results are in good agreement with the exact ones. Moreover, it appears from our findings that the cubic B-spline method exhibits more accurate results than some existing methods.



(a) Absolute errors for $\omega_1(\tau)$.

(b) Absolute errors for $\omega_2(\tau)$.

Figure 4. Absolute error functions for Problem 5.

Table 12. Numerical result for approximate solution of $\omega_1(\tau)$ in Problem 5.

τ	$\omega_1(\tau)$	$\mu_1(\tau)(\Lambda = 0.1)$	$\lvert e_1(\tau)\rvert$	$\mu_1(\tau)(\Lambda = 0.01)$	$\lvert e_1(\tau)\rvert$
0.0	1	1	0	1	0
0.1	0.99005	0.990073	2.29673×10^{-5}	0.99005	5.53154×10^{-8}
0.2	0.960789	0.960817	$2.7658" \times 10^{-5}$	0.96079	2.03451×10^{-7}
0.3	0.913931	0.913979	4.76609×10^{-5}	0.913932	4.05169×10^{-7}
0.4	0.852144	0.85221	6.64108×10^{-5}	0.852144	6.04412×10^{-7}
0.5	0.778801	0.778879	7.81482×10^{-5}	0.778802	7.42993×10^{-7}
0.6	0.697676	0.697756	7.9594×10^{-5}	0.697677	7.74528×10^{-7}
0.7	0.612626	0.612695	6.83427×10^{-5}	0.612627	6.75274×10^{-7}
0.8	0.527292	0.527337	4.49264×10^{-5}	0.527293	4.49447×10^{-7}
0.9	0.444858	0.444871	1.24858×10^{-5}	0.444858	1.28099×10^{-7}
1.0	0.367879	0.367856	2.39084×10^{-5}	0.367879	2.37741×10^{-7}

Table 13. Numerical result for approximate solution of $\omega_2(\tau)$ in Problem 5.

τ	$\omega_2(\tau)$	$\mu_2(\tau)(\Lambda = 0.1)$	$\lvert e_2(\tau)\rvert$	$\mu_2(\tau)(\Lambda = 0.01)$	$\lvert e_2(\tau)\rvert$
0.0	1	1	0	1	0
0.1	1.01005	1.010078	2.82181×10^{-5}	1.01005	1.0345×10^{-7}
0.2	1.040811	1.040871	6.04278×10^{-5}	1.040811	4.35512×10^{-7}
0.3	1.094174	1.094299	1.24976×10^{-4}	1.094175	1.0806×10^{-6}
0.4	1.173511	1.173751	2.40286×10^{-4}	1.173513	2.19844×10^{-6}
0.5	1.284025	1.284458	4.32197×10^{-4}	1.284029	4.06313×10^{-6}
0.6	1.433329	1.434077	7.47197×10^{-4}	1.433337	7.12898×10^{-6}
0.7	1.632316	1.633578	1.26164×10^{-3}	1.632328	1.21425×10^{-5}
0.8	1.896481	1.898582	2.1011×10^{-3}	1.896501	1.21425×10^{-5}
0.9	2.247908	2.251381	3.47326×10^{-3}	2.247942	3.3724×10^{-5}
1.0	2.718282	2.724006	5.72394×10^{-3}	2.718338	5.56973×10^{-5}

Table 14. Comparison of maximum absolute error of Problem 5.

	$\mu_1(\tau)$	$\mu_2(\tau)$
CBSM($\Lambda = 0.1$)	7.95×10^{-5}	5.72×10^{-3}
CBSM($\Lambda = 0.01$)	7.74×10^{-7}	5.56×10^{-5}
[40] (n=4)	2.99×10^{-3}	9.00×10^{-3}
[41] (j=3)	4.54×10^{-4}	5.05×10^{-4}
[41] (j=4)	1.88×10^{-4}	1.26×10^{-4}

6. Conclusions

The system of Lane-Emden type equations describes a variety of phenomena in theoretical physics, star structure, and astrophysics. In this study, we introduce and examine the use of the cubic B-spline method for studying the solution of singular and nonlinear systems of Lane-Emden equations. To address the singularity that occurs at $\tau=0$, we use L'Hôpital's rule. We also evaluate the accuracy and validity of the proposed technique, demonstrating its success in solving the considered system. The presented test problems have shown the simplicity and applicability of the proposed method. We provide tabular and graphical representations to confirm its effectiveness, observing that our numerical solutions are in good agreement with the exact solutions. It is observed that our numerical solutions are in good agreement with the exact ones. Furthermore, we show that by decreasing the mesh size, the numerical results converge to the analytical solution, which confirms the convergence of the algorithm. It is noteworthy that the CPU time of the proposed method for each evaluated problem is under 1 second.

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Conflict of interest

The authors declare no conflicts of interest.

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