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Research article

Multiplicity results for some fourth-order elliptic equations with combined nonlinearities

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Abstract: In this paper, we regard with some fourth-order elliptic boundary value problems involving subcritical polynomial growth and subcritical (critical) exponential growth. Some new existence and multiplicity results are established by using variational methods combined Adams inequality.

Keywords: fourth-order elliptic Navier boundary value problem; the variational method; Adams inequality; subcritical and critical exponential growth
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1. Introduction

Consider with the following fourth-order elliptic Navier boundary problem

$$\begin{cases} \Delta^2 u + c\Delta u = \lambda a(x)|u|^{s-2}u + f(x,u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Delta^2 := \Delta(\Delta)$ denotes the biharmonic operator, $\Omega \subset \mathbb{R}^N (N \geq 4)$ is a smooth bounded domain, $c < \lambda_1$ (λ_1 is the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$) is a constant, $1 < s < 2, \lambda \geq 0$ is a parameter, $a \in L^\infty(\Omega), a(x) \geq 0, a(x) \not\equiv 0$, and $f \in C(\bar{\Omega} \times \mathbb{R}, \mathbb{R})$. It is well known that some of these fourth order elliptic problems appear in different areas of applied mathematics and physics. In the pioneer paper Lazer and Mckenna [13], they modeled nonlinear oscillations for suspensions bridges. It is worth mentioning that problem (1.1) can describe static deflection of an elastic plate in a fluid, see [21, 22]. The static form change of beam or the motion of rigid body can be described by the same problem. Equations of this type have received more and more attentions in recent years. For the case $\lambda = 0$, we refer the reader to [3, 7, 11, 14, 16, 17, 20, 23, 27, 29, 34–37] and the reference therein. In these papers, existence and multiplicity of solutions have been concerned under some assumptions on the nonlinearity f. Most of them considered the case $f(x, u) = b[(u + 1)^+ - 1]$ or f having asymptotically linear growth at infinity or f satisfying the famous Ambrosetti-Rabinowitz condition at infinity. Particularly, in the case $\lambda \neq 0$, that is, the combined nonlinearities for the fourth-order elliptic equations, Wei [33] obtained some existence and multiplicity by using the variational method. However, the author only considered the case that the nonlinearity f is asymptotically linear. When $\lambda = 1$, Pu et al. [26] did some similar work. There are some latest works for problem (1.1), for example [10, 18] and the reference therein. In this paper, we study problem (1.1) from two aspects. One is that we will obtain two multiplicity results when the nonlinearity f is superlinear at infinity and has the standard subcritical polynomial growth but not satisfy the Ambrosetti-Rabinowitz condition, the other is we can establish some existence results of multiple solutions when the nonlinearity f has the exponential growth but still not satisfy the Ambrosetti-Rabinowitz condition. In the first case, the standard methods for the verification of the compactness condition will fail, we will overcome it by using the functional analysis methods, i.e., Hahn-Banach Theorem combined the Resonance Theorem. In the last case, although the original version of the mountain pass theorem of Ambrosetti-Rabinowitz [1] is not directly applied for our purpose. Therefore, we will use a suitable version of mountain pass theorem and some new techniques to finish our goal.

When $N > 4$, there have been substantial lots of works (such as [3, 7, 11, 16, 17, 26, 34–37]) to study the existence of nontrivial solutions or the existence of sign-changing for problem (1.1). Furthermore, almost all of the works involve the nonlinear term $f(x, u)$ of a standard subcritical polynomial growth, say:

(SCP): There exist positive constants c_1 and $q \in (1, p^* - 1)$ such that

$$|f(x,t)| \leq c_1(1 + |t|^q) \text{ for all } t \in \mathbb{R} \text{ and } x \in \Omega,$$

where $p^* = \frac{2N}{N-4}$ expresses the critical Sobolev exponent. In this case, people can deal with problem (1.1) variationally in the Sobolev space $H^2(\Omega) \cap H_0^1(\Omega)$ owing to the some critical point theory, such as, the method of invariant sets of descent flow, mountain pass theorem and symmetric mountain pass theorem. It is worth while to note that since Ambrosetti and Rabinowitz presented the mountain pass theorem in their pioneer paper [1], critical point theory has become one of the main tools on looking for solutions to elliptic equation with variational structure. One of the important condition used in many works is the so-called Ambrosetti-Rabinowitz condition:

(AR) There exist $\theta > 2$ and $R > 0$ such that

$$0 < \theta F(x,t) \leq f(x,t)t, \text{ for } x \in \Omega \text{ and } |t| \geq R,$$

where $F(x,t) = \int_0^t f(x,s)ds$. A simple computation explains that there exist c_2, $c_3 > 0$ such that $F(x,t) \geq c_2|t|^\theta - c_3$ for all $(x,t) \in \bar{\Omega} \times \mathbb{R}$ and f is superlinear at infinity, i.e., $\lim_{t\to\infty} \frac{f(x,t)}{t} = +\infty$ uniformly in $x \in \Omega$. Thus problem (1.1) is called strict superquadratic if the nonlinearity f satisfies the (AR) condition. Notice that (AR) condition plays an important role in ensuring the boundedness of Palais-Smale sequences. However, there are many nonlinearities which are superlinear at infinity but do not satisfy above (AR) condition such as $f(x,t) = tln(1 + |t|^2) + |sint|t$.

In the recent years many authors tried to study problem (1.1) with $\lambda = 0$ and the standard Laplacian problem where (AR) is not assumed. Instead, they regard the general superquadratic condition:

(WSQC) The following limit holds

$$\lim_{|t|\to+\infty} \frac{F(x,t)}{t^2} = +\infty, \text{ uniformly for } x \in \Omega$$

with additional assumptions (see [3,5,7,11,12,15,17,19,24,26,31,37] and the references therein). In the most of them, there are some kind of monotonicity restrictions on the functions $F(x,t)$ or $\frac{f(x,t)}{t}$, or some convex property for the function $tf(x,t) - 2F(x,t)$.

In the case $N = 4$ and $c = 0$, motivated by the Adams inequality, there are a few works devoted to study the existence of nontrivial solutions for problem (1.1) when the nonlinearity f has the exponential growth, for example [15] and the references therein.

Now, we begin to state our main results: Let μ_1 be the first eigenvalue of $(\Delta^2 - c\Delta, H^2(\Omega) \cap H_0^1(\Omega))$ and suppose that $f(x,t)$ satisfies:

(H_1) $f(x,t)t \geq 0,\ \forall (x,t) \in \Omega \times \mathbb{R}$;

(H_2) $\lim\limits_{t \to 0} \frac{f(x,t)}{t} = f_0$ uniformly for a.e. $x \in \Omega$, where $f_0 \in [0, +\infty)$;

(H_3) $\lim\limits_{t \to \infty} \frac{F(x,t)}{t^2} = +\infty$ uniformly for a.e. $x \in \Omega$, where $F(x,t) = \int_0^t f(x,s)ds$.

In the case of $N > 4$, our results are stated as follows:

Theorem 1.1. *Assume that f has the standard subcritical polynomial growth on Ω (condition* (SCP)*) and satisfies (H_1)–(H_3). If $f_0 < \mu_1$ and $a(x) \geq a_0$ (a_0 is a positive constant), then there exists $\Lambda^* > 0$ such that for $\lambda \in (0, \Lambda^*)$, problem* (1.1) *has five solutions, two positive solutions, two negative solutions and one nontrivial solution.*

Theorem 1.2. *Assume that f has the standard subcritical polynomial growth on Ω (condition* (SCP)*) and satisfies (H_2) and (H_3). If $f(x,t)$ is odd in t.*

a) For every $\lambda \in R$, problem (1.1) *has a sequence of solutions $\{u_k\}$ such that $I_\lambda(u_k) \to \infty, k \to \infty$, definition of the functional I_λ will be seen in Section 2.*

b) If $f_0 < \mu_1$, for every $\lambda > 0$, problem (1.1) *has a sequence of solutions $\{u_k\}$ such that $I_\lambda(u_k) < 0$ and $I_\lambda(u_k) \to 0, k \to \infty$.*

Remark. Since our the nonlinear term $f(x,u)$ satisfies more weak condition (H_3) compared with the classical condition (AR), our Theorem 1.2 completely contains Theorem 3.20 in [32].

In case of $N = 4$, we have $p^* = +\infty$. So it's necessary to introduce the definition of the subcritical exponential growth and critical exponential growth in this case. By the improved Adams inequality (see [28] and Lemma 2.2 in Section 2) for the fourth-order derivative, namely,

$$\sup_{u \in H^2(\Omega) \cap H_0^1(\Omega), \|\Delta u\|_2 \leq 1} \int_\Omega e^{32\pi^2 u^2} dx \leq C|\Omega|.$$

So, we now define the subcritical exponential growth and critical exponential growth in this case as follows:

(SCE): f satisfies subcritical exponential growth on Ω, i.e., $\lim\limits_{t \to \infty} \frac{|f(x,t)|}{\exp(\alpha t^2)} = 0$ uniformly on $x \in \Omega$ for all $\alpha > 0$.

(CG): f satisfies critical exponential growth on Ω, i.e., there exists $\alpha_0 > 0$ such that

$$\lim_{t \to \infty} \frac{|f(x,t)|}{\exp(\alpha t^2)} = 0, \text{ uniformly on } x \in \Omega,\ \forall \alpha > \alpha_0,$$

and

$$\lim_{t \to \infty} \frac{|f(x,t)|}{\exp(\alpha t^2)} = +\infty, \text{ uniformly on } x \in \Omega,\ \forall \alpha < \alpha_0.$$

When $N = 4$ and f satisfies the subcritical exponential growth (SCE), our work is still to consider problem (1.1) where the nonlinearity f satisfies the (WSQC)-condition at infinity. As far as we know, this case is rarely studied by other people for problem (1.1) except for [24]. Hence, our results are new and our methods are technique since we successfully proved the compactness condition by using the Resonance Theorem combined Adams inequality and the truncated technique. In fact, the new idea derives from our work [25]. Our results are as follows:

Theorem 1.3. *Assume that f satisfies the subcritical exponential growth on Ω (condition (SCE)) and satisfies (H_1)–(H_3). If $f_0 < \mu_1$ and $a(x) \geq a_0$ (a_0 is a positive constant), then there exists $\Lambda^* > 0$ such that for $\lambda \in (0, \Lambda^*)$, problem (1.1) has five solutions, two positive solutions, two negative solutions and one nontrivial solution.*

Remark. Let $F(x,t) = t^2 e^{\sqrt{|t|}}, \forall (x,t) \in \Omega \times \mathbb{R}$. Then it satisfies that our conditions (H_1)–(H_3) but not satisfy the condition (AR). It's worth noting that we do not impose any monotonicity condition on $\frac{f(x,t)}{t}$ or some convex property on $tf(x,t) - 2F(x,t)$. Hence, our Theorem 1.3 completely extends some results contained in [15, 24] when $\lambda = 0$ in problem (1.1).

Theorem 1.4. *Assume that f satisfies the subcritical exponential growth on Ω (condition (SCE)) and satisfies (H_2) and (H_3). If $f_0 < \mu_1$ and $f(x,t)$ is odd in t.*

a) For $\lambda > 0$ small enough, problem (1.1) has a sequence of solutions $\{u_k\}$ such that $I_\lambda(u_k) \to \infty, k \to \infty$.

b) For every $\lambda > 0$, problem (1.1) has a sequence of solutions $\{u_k\}$ such that $I_\lambda(u_k) < 0$ and $I_\lambda(u_k) \to 0, k \to \infty$.

When $N = 4$ and f satisfies the critical exponential growth (CG), the study of problem (1.1) becomes more complicated than in the case of subcritical exponential growth. Similar to the case of the critical polynomial growth in $\mathbb{R}^N$ ($N \geq 3$) for the standard Laplacian studied by Brezis and Nirenberg in their pioneering work [4], our Euler-Lagrange functional does not satisfy the Palais-Smale condition at all level anymore. For the class standard Laplacian problem, the authors [8] used the extremal function sequences related to Moser-Trudinger inequality to complete the verification of compactness of Euler-Lagrange functional at some suitable level. Here, we still adapt the method of choosing the testing functions to study problem (1.1) without (AR) condition. Our result is as follows:

Theorem 1.5. *Assume that f has the critical exponential growth on Ω (condition (CG)) and satisfies (H_1)–(H_3). Furthermore, assume that*

(H_4) $\lim_{t\to\infty} f(x,t)\exp(-\alpha_0 t^2)t \geq \beta > \frac{64}{\alpha_0 r_0^4}$, uniformly in (x,t), where r_0 is the inner radius of Ω, i.e., $r_0 :=$ radius of the largest open ball $\subset \Omega$. and

(H_5) f is in the class (H_0), i.e., for any $\{u_n\}$ in $H^2(\Omega) \cap H_0^1(\Omega)$, if $u_n \rightharpoonup 0$ in $H^2(\Omega) \cap H_0^1(\Omega)$ and $f(x,u_n) \to 0$ in $L^1(\Omega)$, then $F(x,u_n) \to 0$ in $L^1(\Omega)$ (up to a subsequence).

If $f_0 < \mu_1$, then there exists $\Lambda^* > 0$ such that for $\lambda \in (0, \Lambda^*)$, problem (1.1) has at least four nontrivial solutions.

Remark. For standard biharmonic problems with Dirichlet boundary condition, Lam and Lu [15] have recently established the existence of nontrivial nonnegative solutions when the nonlinearity f has the critical exponential growth of order $\exp(\alpha u^2)$ but without satisfying the Ambrosetti- Rabinowitz condition. However, for problem (1.1) with Navier boundary condition involving critical exponential growth and the concave term, there are few works to study it. Hence our result is new and interesting.

The paper is organized as follows. In Section 2, we present some necessary preliminary knowledge and some important lemmas. In Section 3, we give the proofs for our main results. In Section 4, we give a conclusion.

2. Preliminaries and some lemmas

We let λ_k $(k = 1, 2, \cdots)$ denote the eigenvalue of $-\Delta$ in $H_0^1(\Omega)$, then $0 < \mu_1 < \mu_2 < \cdots < \mu_k < \cdots$ be the eigenvalues of $(\Delta^2 - c\Delta, H^2(\Omega) \cap H_0^1(\Omega))$ and $\varphi_k(x)$ be the eigenfunction corresponding to μ_k. Let X_k denote the eigenspace associated to μ_k. In fact, $\mu_k = \lambda_k(\lambda_k - c)$. Throughout this paper, we denote by $\|\cdot\|_p$ the $L^p(\Omega)$ norm , $c < \lambda_1$ in $\Delta^2 - c\Delta$ and the norm of u in $H^2(\Omega) \cap H_0^1(\Omega)$ will be defined by the

$$\|u\| := \left(\int_\Omega (|\Delta u|^2 - c|\nabla u|^2) dx \right)^{\frac{1}{2}}.$$

We also denote $E = H^2(\Omega) \cap H_0^1(\Omega)$.

Definition 2.1. Let $(\mathbb{E}, \|\cdot\|_{\mathbb{E}})$ be a real Banach space with its dual space $(\mathbb{E}^*, \|\cdot\|_{\mathbb{E}^*})$ and $I \in C^1(\mathbb{E}, \mathbb{R})$. For $c^* \in \mathbb{R}$, we say that I satisfies the $(\mathrm{PS})_{c^*}$ condition if for any sequence $\{x_n\} \subset \mathbb{E}$ with

$$I(x_n) \to c^*, I'(x_n) \to 0 \text{ in } \mathbb{E}^*,$$

there is a subsequence $\{x_{n_k}\}$ such that $\{x_{n_k}\}$ converges strongly in $\mathbb{E}$. Also, we say that I satisfy the $(\mathrm{C})_{c^*}$ condition if for any sequence $\{x_n\} \subset \mathbb{E}$ with

$$I(x_n) \to c^*, \ \|I'(x_n)\|_{\mathbb{E}^*}(1 + \|x_n\|_{\mathbb{E}}) \to 0,$$

there exists subsequence $\{x_{n_k}\}$ such that $\{x_{n_k}\}$ converges strongly in $\mathbb{E}$.

Definition 2.2. We say that $u \in E$ is the solution of problem (1.1) if the identity

$$\int_\Omega (\Delta u \Delta \varphi - c \nabla u \nabla \varphi) dx = \lambda \int_\Omega a(x)|u|^{s-2} u \varphi dx + \int_\Omega f(x, u) \varphi dx$$

holds for any $\varphi \in E$.

It is obvious that the solutions of problem (1.1) are corresponding with the critical points of the following C^1 functional:

$$I_\lambda(u) = \frac{1}{2}\|u\|^2 - \frac{\lambda}{s} \int_\Omega a(x)|u|^s dx - \int_\Omega F(x, u) dx, \quad u \in E.$$

Let $u^+ = \max\{u, 0\}, u^- = \min\{u, 0\}$.

Now, we concern the following problem

$$\begin{cases} \Delta^2 u + c\Delta u = \lambda a(x)|u^+|^{s-2} u^+ + f^+(x, u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where

$$f^+(x,t) = \begin{cases} f(x,t) & t \geq 0, \\ 0, & t < 0. \end{cases}$$

Define the corresponding functional $I_\lambda^+ : E \to \mathbb{R}$ as follows:

$$I_\lambda^+(u) = \frac{1}{2}\|u\|^2 - \frac{\lambda}{s}\int_\Omega a(x)|u^+|^s dx - \int_\Omega F^+(x,u)dx,$$

where $F^+(x,u) = \int_0^u f^+(x,s)ds$. Obviously, the condition (SCP) or (SCE) ((CG)) ensures that $I_\lambda^+ \in C^1(E,\mathbb{R})$. Let u be a critical point of I_λ^+, which means that u is a weak solution of problem (2.1). Furthermore, since the weak maximum principle (see [34]), it implies that $u \geq 0$ in Ω. Thus u is also a solution of problem (1.1) and $I_\lambda^+(u) = I_\lambda(u)$.

Similarly, we define

$$f^-(x,t) = \begin{cases} f(x,t) & t \leq 0, \\ 0, & t > 0, \end{cases}$$

and

$$I_\lambda^-(u) = \frac{1}{2}\|u\|^2 - \frac{\lambda}{s}\int_\Omega a(x)|u^-|^s dx - \int_\Omega F^-(x,u)dx,$$

where $F^-(x,u) = \int_0^u f^-(x,s)ds$. Similarly, we also have $I_\lambda^- \in C^1(E,\mathbb{R})$ and if v is a critical point of I_λ^- then it is a solution of problem (1.1) and $I_\lambda^-(v) = I_\lambda(v)$.

Prosition 2.1. *([6, 30]). Let $\mathbb{E}$ be a real Banach space and suppose that $I \in C^1(\mathbb{E},\mathbb{R})$ satisfies the condition*

$$\max\{I(0), I(u_1)\} \leq \alpha < \beta \leq \inf_{\|u\|=\rho} I(u),$$

for some $\alpha < \beta, \rho > 0$ and $u_1 \in \mathbb{E}$ with $\|u_1\| > \rho$. Let $c^* \geq \beta$ be characterized by

$$c^* = \inf_{\gamma\in\Gamma}\max_{0\leq t\leq 1} I(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1],\mathbb{E}), \gamma(0) = 0, \gamma(1) = u_1\}$ is the set of continuous paths joining 0 and u_1. Then, there exists a sequence $\{u_n\} \subset \mathbb{E}$ such that

$$I(u_n) \to c^* \geq \beta \text{ and } (1 + \|u_n\|)\|I'(u_n)\|_{\mathbb{E}^*} \to 0 \text{ as } n \to \infty.$$

Lemma 2.1. *([28]). Let $\Omega \subset \mathbb{R}^4$ be a bounded domain. Then there exists a constant $C > 0$ such that*

$$\sup_{u\in E,\|\Delta u\|_2\leq 1}\int_\Omega e^{32\pi^2u^2}dx < C|\Omega|,$$

and this inequality is sharp.

Next, we introduce the following a revision of Adams inequality:

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^4$ be a bounded domain. Then there exists a constant $C^* > 0$ such that*

$$\sup_{u\in E,\|u\|\leq 1}\int_\Omega e^{32\pi^2u^2}dx < C^*|\Omega|,$$

and this inequality is also sharp.

Proof. We will give a summarize proof in two different cases. In the case of $c \leq 0$ in the definition of $\|.\|$, if $\|u\| \leq 1$, we can deduce that $\|\Delta u\|_2 \leq 1$ and by using Lemma 2.1 combined with the Proposition 6.1 in [28], the conclusion holds.

In the case of $0 < c < \lambda_1$ in the definition of $\|.\|$, from Lemma 2.1, the proof and remark of Theorem 1 in [2] and the proof of Proposition 6.1 in [28], we still can establish this revised Adams inequality.

Lemma 2.3. *Assume* (H_1) *and* (H_3) *hold. If* f *has the standard subcritical polynomial growth on* Ω *(condition* (SCP)*), then* I_λ^+ (I_λ^-) *satisfies* $(\mathrm{C})_{c^*}$.

Proof. We only prove the case of I_λ^+. The arguments for the case of I_λ^- are similar. Let $\{u_n\} \subset E$ be a $(\mathrm{C})_{c^*}$ sequence such that

$$I_\lambda^+(u_n) = \frac{1}{2}\|u_n\|^2 - \frac{\lambda}{s}\int_\Omega a(x)|u_n^+|^s dx - \int_\Omega F^+(x, u_n)dx = c^* + \circ(1), \tag{2.2}$$

$$(1 + \|u_n\|)\|I_\lambda^{+'}(u_n)\|_{E^*} \to 0 \text{ as } n \to \infty. \tag{2.3}$$

Obviously, (2.3) implies that

$$\langle I_\lambda^{+'}(u_n), \varphi\rangle = \langle u_n, \varphi\rangle - \lambda\int_\Omega a(x)|u_n^+|^{s-2}u_n^+\varphi dx - \int_\Omega f^+(x, u_n(x))\varphi dx = \circ(1). \tag{2.4}$$

Step 1. We claim that $\{u_n\}$ is bounded in E. In fact, assume that

$$\|u_n\| \to \infty, \quad \text{as } n \to \infty.$$

Define

$$v_n = \frac{u_n}{\|u_n\|}.$$

Then, $\|v_n\| = 1$, $\forall n \in \mathbf{N}$ and then, it is possible to extract a subsequence (denoted also by $\{v_n\}$) converges weakly to v in E, converges strongly in $L^p(\Omega)(1 \leq p < p^*)$ and converges v a.e. $x \in \Omega$.

Dividing both sides of (2.2) by $\|u_n\|^2$, we get

$$\int_\Omega \frac{F^+(x, u_n)}{\|u_n\|^2}dx \to \frac{1}{2}. \tag{2.5}$$

Set

$$\Omega_+ = \{x \in \Omega : v(x) > 0\}.$$

By (H_3) , we imply that

$$\frac{F^+(x, u_n)}{u_n^2}v_n^2 \to \infty, \quad x \in \Omega_+. \tag{2.6}$$

If $|\Omega_+|$ is positive, since Fatou's lemma, we get

$$\lim_{n\to\infty}\int_\Omega \frac{F^+(x, u_n)}{\|u_n\|^2}dx \geq \lim_{n\to\infty}\int_{\Omega_+}\frac{F^+(x, u_n)}{u_n^2}v_n^2 dx = +\infty,$$

which contradicts with (2.5). Thus, we have $v \leq 0$. In fact, we have $v = 0$. Indeed, again using (2.3), we get

$$(1 + \|u_n\|)|\langle I_\lambda^{+'}(u_n), v\rangle| \leq \circ(1)\|v\|.$$

Thus, we have

$$\begin{aligned}\int_\Omega(\Delta u_n\Delta v-c\nabla u_n\nabla v)dx &\le \int_\Omega(\Delta u_n\Delta v-c\nabla u\nabla v)dx-\lambda\int_\Omega a(x)|u_n^+|^{s-2}u_n^+vdx\\ &- \int_\Omega f^+(x,u_n)vdx\le\frac{\circ(1)\|v\|}{1+\|u_n\|},\end{aligned}$$

by noticing that since $v\le 0$, $f^+(x,u_n)v\le 0$ a.e. $x\in\Omega$, thus $-\int_\Omega f^+(x,u_n)vdx\ge 0$. So we get

$$\int_\Omega(\Delta v_n\Delta v-c\nabla v_n\nabla v)dx\to 0.$$

On the other hand, from $v_n\rightharpoonup v$ in E, we have

$$\int_\Omega(\Delta v_n\Delta v-c\nabla v_n\nabla v)dx\to\|v\|^2$$

which implies $v=0$.

Dividing both sides of (2.4) by $\|u_n\|$, for any $\varphi\in E$, then there exists a positive constant $M(\varphi)$ such that

$$\left|\int_\Omega\frac{f^+(x,u_n)}{\|u_n\|}\varphi dx\right|\le M(\varphi),\ \forall n\in\mathbf{N}. \tag{2.7}$$

Set

$$\mathbf{f}_n(\varphi)=\int_\Omega\frac{f^+(x,u_n)}{\|u_n\|}\varphi dx,\ \varphi\in E.$$

Thus, by (SCP), we know that $\{\mathbf{f}_n\}$ is a family bounded linear functionals defined on E. Combing (2.7) with the famous Resonance Theorem, we get that $\{|\mathbf{f}_n|\}$ is bounded, where $|\mathbf{f}_n|$ denotes the norm of $\mathbf{f}_n$. It means that

$$|\mathbf{f}_n|\le C_*. \tag{2.8}$$

Since $E\subset L^{\frac{p^*}{p^*-q}}(\Omega)$, using the Hahn-Banach Theorem, there exists a continuous functional $\hat{\mathbf{f}}_n$ defined on $L^{\frac{p^*}{p^*-q}}(\Omega)$ such that $\hat{\mathbf{f}}_n$ is an extension of $\mathbf{f}_n$, and

$$\hat{\mathbf{f}}_n(\varphi)=\mathbf{f}_n(\varphi),\ \varphi\in E, \tag{2.9}$$

$$\|\hat{\mathbf{f}}_n\|_{\frac{p^*}{q}}=|\mathbf{f}_n|, \tag{2.10}$$

where $\|\hat{\mathbf{f}}_n\|_{\frac{p^*}{q}}$ denotes the norm of $\hat{\mathbf{f}}_n(\varphi)$ in $L^{\frac{p^*}{q}}(\Omega)$ which is defined on $L^{\frac{p^*}{p^*-q}}(\Omega)$.

On the other hand, from the definition of the linear functional on $L^{\frac{p^*}{p^*-q}}(\Omega)$, we know that there exists a function $S_n(x)\in L^{\frac{p^*}{q}}(\Omega)$ such that

$$\hat{\mathbf{f}}_n(\varphi)=\int_\Omega S_n(x)\varphi(x)dx,\ \varphi\in L^{\frac{p^*}{p^*-q}}(\Omega). \tag{2.11}$$

So, from (2.9) and (2.11), we obtain

$$\int_\Omega S_n(x)\varphi(x)dx=\int_\Omega\frac{f^+(x,u_n)}{\|u_n\|}\varphi dx,\ \varphi\in E,$$

which implies that

$$\int_\Omega \left(S_n(x) - \frac{f^+(x,u_n)}{\|u_n\|}\right)\varphi dx = 0,\ \varphi \in E.$$

According to the basic lemma of variational, we can deduce that

$$S_n(x) = \frac{f^+(x,u_n)}{\|u_n\|}\ \text{ a.e. } x \in \Omega.$$

Thus, by (2.8) and (2.10), we have

$$\|\hat{\mathbf{f}}_n\|_{\frac{p^*}{q}} = \|S_n\|_{\frac{p^*}{q}} = |\mathbf{f}_n| < C_*. \tag{2.12}$$

Now, taking $\varphi = v_n - v$ in (2.4), we get

$$\langle A(v_n), v_n - v\rangle - \lambda \int_\Omega a(x)|u_n^+|^{s-2}u_n^+ v_n dx - \int_\Omega \frac{f^+(x,u_n)}{\|u_n\|} v_n dx \to 0, \tag{2.13}$$

where $A : E \to E^*$ defined by

$$\langle A(u), \varphi\rangle = \int_\Omega \Delta u \Delta\varphi dx - c\int_\Omega \nabla u \nabla\varphi dx,\ u, \varphi \in E.$$

By the Hölder inequality and (2.12), we obtain

$$\int_\Omega \frac{f^+(x,u_n)}{\|u_n\|} v_n dx \to 0.$$

Then from (2.13), we can conclude that

$$v_n \to v\ \text{ in } E.$$

This leads to a contradiction since $\|v_n\| = 1$ and $v = 0$. Thus, $\{u_n\}$ is bounded in E.

Step 2. We show that $\{u_n\}$ has a convergence subsequence. Without loss of generality, we can suppose that

$$\begin{aligned} &u_n \rightharpoonup u \ \text{ in } \ E, \\ &u_n \to u \text{ in } L^\gamma(\Omega),\ \forall 1 \le \gamma < p^*, \\ &u_n(x) \to u(x) \text{ a.e. } x \in \Omega. \end{aligned}$$

Now, it follows from f satisfies the condition (SCP) that there exist two positive constants $c_4, c_5 > 0$ such that

$$f^+(x,t) \le c_4 + c_5|t|^q,\ \forall (x,t) \in \Omega \times \mathbb{R},$$

then

$$\begin{aligned} &\left|\int_\Omega f^+(x,u_n)(u_n - u)dx\right| \\ &\le c_4 \int_\Omega |u_n - u|dx + c_5 \int_\Omega |u_n - u||u_n|^q dx \end{aligned}$$

$$\leq c_4 \int_\Omega |u_n - u| dx + c_5 \left(\int_\Omega (|u_n|^q)^{\frac{p^*}{q}} \, dx \right)^{\frac{q}{p^*}} \left(\int_\Omega |u_n - u|^{\frac{p^*}{p^*-q}} dx \right)^{\frac{p^*-q}{p^*}}.$$

Similarly, since $u_n \rightharpoonup u$ in E, $\int_\Omega |u_n - u| dx \to 0$ and $\int_\Omega |u_n - u|^{\frac{p^*}{p^*-q}} dx \to 0$.

Thus, from (2.4) and the formula above, we obtain

$$\langle A(u_n), u_n - u \rangle \to 0, \text{ as } n \to \infty.$$

So, we get $\|u_n\| \to \|u\|$. Thus we have $u_n \to u$ in E which implies that I_λ^+ satisfies $(\mathrm{C})_{c^*}$.

Lemma 2.4. *Let $\varphi_1 > 0$ be a μ_1-eigenfunction with $\|\varphi_1\| = 1$ and assume that (H_1)–(H_3) and* (SCP) *hold. If $f_0 < \mu_1$, then:*

(i) *For $\lambda > 0$ small enough, there exist $\rho, \alpha > 0$ such that $I_\lambda^\pm(u) \geq \alpha$ for all $u \in E$ with $\|u\| = \rho$,*

(ii) *$I_\lambda^\pm(t\varphi_1) \to -\infty$ as $t \to +\infty$.*

Proof. Since (SCP) and (H_1)–(H_3), for any $\varepsilon > 0$, there exist $A = A(\varepsilon)$, M large enough and $B = B(\varepsilon)$ such that for all $(x, s) \in \Omega \times \mathbb{R}$,

$$F^\pm(x, s) \leq \frac{1}{2}(f_0 + \epsilon)s^2 + A|s|^q, \tag{2.14}$$

$$F^\pm(x, s) \geq \frac{M}{2} s^2 - B. \tag{2.15}$$

Choose $\varepsilon > 0$ such that $(f_0 + \varepsilon) < \mu_1$. By (2.14), the Poincaré inequality and the Sobolev embedding, we obtain

$$\begin{aligned} I_\lambda^\pm(u) &\geq \frac{1}{2}\|u\|^2 - \frac{\lambda\|a\|_\infty}{s} \int_\Omega |u|^s dx - \int_\Omega F^\pm(x, u) dx \\ &\geq \frac{1}{2}\|u\|^2 - \frac{\lambda\|a\|_\infty}{s} \int_\Omega |u|^s dx - \frac{f_0 + \varepsilon}{2}\|u\|_2^2 - A \int_\Omega |u|^q dx \\ &\geq \frac{1}{2}\left(1 - \frac{f_0 + \varepsilon}{\mu_1}\right)\|u\|^2 - \lambda K\|u\|^s - C^{**}\|u\|^q \\ &\geq \|u\|^2 \left(\frac{1}{2}\left(1 - \frac{f_0 + \varepsilon}{\mu_1}\right) - \lambda K\|u\|^{s-2} - C^{**}\|u\|^{q-2} \right), \end{aligned}$$

where K, C^{**} are constant.

Write

$$h(t) = \lambda K t^{s-2} + C^{**} t^{q-2}.$$

We can prove that there exists t^* such that

$$h(t^*) < \frac{1}{2}\left(1 - \frac{f_0 + \varepsilon}{\mu_1}\right).$$

In fact, letting $h'(t) = 0$, we get

$$t^* = \left(\frac{\lambda K(2 - s)}{C^{**}(q - 2)} \right)^{\frac{1}{q-s}}.$$

According to the knowledge of mathematical analysis, $h(t)$ has a minimum at $t = t^*$. Denote

$$\vartheta = \frac{K(2-s)}{C^{**}(q-2)},\ \hat{\vartheta} = \frac{s-2}{q-s},\ \bar{\vartheta} = \frac{q-2}{q-s},\ \nu = \frac{1}{2}\left(1 - \frac{f_0+\varepsilon}{\mu_1}\right).$$

Taking t^* in $h(t)$, we get

$$h(t^*) < \nu,\ 0 < \lambda < \Lambda^*,$$

where $\Lambda^* = (\frac{\nu}{K\vartheta^{\hat{\vartheta}} + C^{**}\vartheta^{\bar{\vartheta}}})^{\frac{1}{\bar{\vartheta}}}$. So, part (i) holds if we take $\rho = t^*$.

On the other hand, from (2.15), we get

$$I_\lambda^+(t\varphi_1) \le \frac{1}{2}\left(1 - \frac{M}{\mu_1}\right)t^2 - t^s\frac{\lambda}{s}\int_\Omega a(x)|\varphi_1|^s dx + B|\Omega| \to -\infty \text{ as } t \to +\infty.$$

Similarly, we have

$$I_\lambda^-(t(-\varphi_1)) \to -\infty,\ \text{as } t \to +\infty.$$

Thus part (ii) holds.

Lemma 2.5. *Let $\varphi_1 > 0$ be a μ_1-eigenfunction with $\|\varphi_1\| = 1$ and assume that (H_1)–(H_3) and* (SCE)*(or* (CG)*) hold. If $f_0 < \mu_1$, then:*

(i) *For $\lambda > 0$ small enough, there exist $\rho, \alpha > 0$ such that $I_\lambda^\pm(u) \ge \alpha$ for all $u \in E$ with $\|u\| = \rho$,*
(ii) *$I_\lambda^\pm(t\varphi_1) \to -\infty$ as $t \to +\infty$.*

Proof. From (SCE) (or (CG)) and (H_1)-(H_3), for any $\varepsilon > 0$, there exist $A_1 = A_1(\varepsilon)$, M_1 large enough, $B_1 = B_1(\varepsilon)$, $\kappa_1 > 0$ and $q_1 > 2$ such that for all $(x, s) \in \Omega \times \mathbb{R}$,

$$F^\pm(x,s) \le \frac{1}{2}(f_0+\epsilon)s^2 + A_1\exp(\kappa_1 s^2)|s|^{q_1}, \tag{2.16}$$

$$F^\pm(x,s) \ge \frac{M_1}{2}s^2 - B_1. \tag{2.17}$$

Choose $\varepsilon > 0$ such that $(f_0 + \varepsilon) < \mu_1$. By (2.16), the Hölder inequality and the Adams inequality (see Lemma 2.2), we obtain

$$\begin{aligned}
I_\lambda^\pm(u) &\ge \frac{1}{2}\|u\|^2 - \frac{\lambda\|a\|_\infty}{s}\int_\Omega |u|^s dx - \int_\Omega F^\pm(x,u)dx \\
&\ge \frac{1}{2}\|u\|^2 - \frac{\lambda\|a\|_\infty}{s}\int_\Omega |u|^s dx - \frac{f_0+\varepsilon}{2}\|u\|_2^2 - A_1\int_\Omega \exp(\kappa_1 u^2)|u|^{q_1}dx \\
&\ge \frac{1}{2}\left(1 - \frac{f_0+\varepsilon}{\mu_1}\right)\|u\|^2 - \lambda K\|u\|^s - A_1\left(\int_\Omega \exp(\kappa_1 r_1\|u\|^2(\frac{|u|}{\|u\|})^2\right)dx)^{\frac{1}{r_1}}\left(\int_\Omega |u|^{r_1' q}dx\right)^{\frac{1}{r_1'}} \\
&\ge \frac{1}{2}\left(1 - \frac{f_0+\varepsilon}{\mu_1}\right)\|u\|^2 - \lambda K\|u\|^s - \hat{C}^{**}\|u\|^{q_1},
\end{aligned}$$

where $r_1 > 1$ sufficiently close to 1, $\|u\| \le \sigma$ and $\kappa_1 r_1\sigma^2 < 32\pi^2$. Remained proof is completely similar to the proof of part (i) of Lemma 2.4, we omit it here. So, part (i) holds if we take $\|u\| = \rho > 0$ small enough.

On the other hand, from (2.17), we get

$$I_\lambda^+(t\varphi_1) \le \frac{1}{2}\left(1-\frac{M_1}{\mu_1}\right)t^2 - t^s\frac{\lambda}{s}\int_\Omega a(x)|\varphi_1|^s dx + B_1|\Omega| \to -\infty \text{ as } t \to +\infty.$$

Similarly, we have

$$I_\lambda^-(t(-\varphi_1)) \to -\infty, \text{ as } t \to +\infty.$$

Thus part (ii) holds.

Lemma 2.6. *Assume* (H_1) *and* (H_3) *hold. If* f *has the subcritical exponential growth on* Ω *(condition* (SCE)*), then* I_λ^+ *(* I_λ^-*) satisfies* $(\mathrm{C})_{c^*}$.

Proof. We only prove the case of I_λ^+. The arguments for the case of I_λ^- are similar. Let $\{u_n\} \subset E$ be a $(\mathrm{C})_{c^*}$ sequence such that the formulas (2.2)–(2.4) in Lemma 2.3 hold.

Now, according to the previous section of Step 1 of the proof of Lemma 2.3, we also obtain that the formula (2.7) holds. Set

$$\mathbf{f}_n(\varphi) = \int_\Omega \frac{f^+(x,u_n)}{\|u_n\|}\varphi dx, \ \varphi \in E.$$

Then from for any $u \in E$, $e^{\alpha u^2} \in L^1(\Omega)$ for all $\alpha > 0$, we can draw a conclusion that $\{\mathbf{f}_n\}$ is a family bounded linear functionals defined on E. Using (2.7) and the famous Resonance Theorem, we know that $\{|\mathbf{f}_n|\}$ is bounded, where $|\mathbf{f}_n|$ denotes the norm of $\mathbf{f}_n$. It means that the formula (2.8) (see the proof of Lemma 2.3) holds.

Since $E \subset L^{q_0}(\Omega)$ for some $q_0 > 1$, using the Hahn-Banach Theorem, there exists a continuous functional $\hat{\mathbf{f}}_n$ defined on $L^{q_0}(\Omega)$ such that $\hat{\mathbf{f}}_n$ is an extension of $\mathbf{f}_n$, and

$$\hat{\mathbf{f}}_n(\varphi) = \mathbf{f}_n(\varphi), \ \varphi \in E, \tag{2.18}$$

$$\|\hat{\mathbf{f}}_n\|_{q_0^*} = |\mathbf{f}_n|, \tag{2.19}$$

where $\|\hat{\mathbf{f}}_n\|_{q_0^*}$ is the norm of $\hat{\mathbf{f}}_n(\varphi)$ in $L^{q_0^*}(\Omega)$ which is defined on $L^{q_0}(\Omega)$ and q_0^* is the dual number of q_0.

By the definition of the linear functional on $L^{q_0}(\Omega)$, we know that there is a function $S_n(x) \in L^{q_0^*}(\Omega)$ such that

$$\hat{\mathbf{f}}_n(\varphi) = \int_\Omega S_n(x)\varphi(x)dx, \ \varphi \in L^{q_0}(\Omega). \tag{2.20}$$

Similarly to the last section of the Step 1 of the proof of Lemma 2.3, we can prove that $(\mathrm{C})_{c^*}$ sequence $\{u_n\}$ is bounded in E. Next, we show that $\{u_n\}$ has a convergence subsequence. Without loss of generality, assume that

$$\begin{aligned}
&\|u_n\| \le \beta^*,\\
&u_n \rightharpoonup u \ \text{ in } \ E,\\
&u_n \to u \text{ in } L^\gamma(\Omega), \ \forall\gamma \ge 1,\\
&u_n(x) \to u(x) \text{ a.e. } x \in \Omega.
\end{aligned}$$

Since f has the subcritical exponential growth (SCE) on Ω, we can find a constant $C_{\beta^*} > 0$ such that

$$|f^+(x,t)| \le C_{\beta^*}\exp\left(\frac{32\pi^2}{k(\beta^*)^2}t^2\right), \ \forall (x,t) \in \Omega \times \mathbb{R}.$$

Thus, from the revised Adams inequality (see Lemma 2.2),

$$\begin{aligned}
&\left|\int_\Omega f^+(x,u_n)(u_n-u)dx\right| \\
&\le C_{\beta^*}\left(\int_\Omega \exp\left(\frac{32\pi^2}{(\beta^*)^2}u_n^2\right)dx\right)^{\frac{1}{k}}|u_n-u|_{k'} \\
&\le C_{**}|u_n-u|_{k'}\to 0,
\end{aligned}$$

where $k>1$ and k' is the dual number of k. Similar to the last proof of Lemma 2.3, we have $u_n\to u$ in E which means that I_λ^+ satisfies $(\mathrm{C})_{c^*}$.

Lemma 2.7. *Assume* (H_3) *holds. If* f *has the standard subcritical polynomial growth on* Ω *(condition* (SCP)*), then* I_λ *satisfies* $(\mathrm{PS})_{c^*}$.

Proof. Let $\{u_n\}\subset E$ be a $(\mathrm{PS})_{c^*}$ sequence such that

$$\frac{\|u_n\|^2}{2}-\frac{\lambda}{s}\int_\Omega a(x)|u_n|^s dx-\int_\Omega F(x,u_n)dx\to c^*, \tag{2.21}$$

$$\int_\Omega \Delta u_n\Delta\varphi dx-c\int_\Omega \nabla u_n\nabla\varphi dx-\lambda\int_\Omega a(x)|u_n|^{s-2}u_n\varphi dx-\int_\Omega f(x,u_n)\varphi dx=\circ(1)\|\varphi\|,\ \varphi\in E. \tag{2.22}$$

Step 1. To prove that $\{u_n\}$ has a convergence subsequence, we first need to prove that it is a bounded sequence. To do this, argue by contradiction assuming that for a subsequence, which is still denoted by $\{u_n\}$, we have

$$\|u_n\|\to\infty.$$

Without loss of generality, assume that $\|u_n\|\ge 1$ for all $n\in\mathbf{N}$ and let

$$v_n=\frac{u_n}{\|u_n\|}.$$

Clearly, $\|v_n\|=1$, $\forall n\in\mathbf{N}$ and then, it is possible to extract a subsequence (denoted also by $\{v_n\}$) converges weakly to v in E, converges strongly in $L^p(\Omega)(1\le p<p^*)$ and converges v a.e. $x\in\Omega$.

Dividing both sides of (2.21) by $\|u_n\|^2$, we obtain

$$\int_\Omega \frac{F(x,u_n)}{\|u_n\|^2}dx\to\frac{1}{2}. \tag{2.23}$$

Set

$$\Omega_0=\{x\in\Omega: v(x)\ne 0\}.$$

By (H_3) , we get that

$$\frac{F(x,u_n)}{u_n^2}v_n^2\to\infty,\quad x\in\Omega_0. \tag{2.24}$$

If $|\Omega_0|$ is positive, from Fatou's lemma, we obtain

$$\lim_{n\to\infty}\int_\Omega \frac{F(x,u_n)}{\|u_n\|^2}dx\ge\lim_{n\to\infty}\int_{\Omega_0}\frac{F(x,u_n)}{u_n^2}v_n^2dx=+\infty,$$

which contradicts with (2.23).

Dividing both sides of (2.22) by $\|u_n\|$, for any $\varphi \in E$, then there exists a positive constant $M(\varphi)$ such that

$$\left|\int_\Omega \frac{f(x,u_n)}{\|u_n\|}\varphi dx\right| \le M(\varphi),\ \forall n \in \mathbf{N}. \tag{2.25}$$

Set

$$\mathbf{f}_n(\varphi) = \int_\Omega \frac{f(x,u_n)}{\|u_n\|}\varphi dx,\ \varphi \in E.$$

Thus, by (SCP), we know that $\{\mathbf{f}_n\}$ is a family bounded linear functionals defined on E. By (2.25) and the famous Resonance Theorem, we get that $\{|\mathbf{f}_n|\}$ is bounded, where $|\mathbf{f}_n|$ denotes the norm of $\mathbf{f}_n$. It means that

$$|\mathbf{f}_n| \le \tilde{C}_*. \tag{2.26}$$

Since $E \subset L^{\frac{p^*}{p^*-q}}(\Omega)$, using the Hahn-Banach Theorem, there exists a continuous functional $\hat{\mathbf{f}}_n$ defined on $L^{\frac{p^*}{p^*-q}}(\Omega)$ such that $\hat{\mathbf{f}}_n$ is an extension of $\mathbf{f}_n$, and

$$\hat{\mathbf{f}}_n(\varphi) = \mathbf{f}_n(\varphi),\ \varphi \in E, \tag{2.27}$$

$$\|\hat{\mathbf{f}}_n\|_{\frac{p^*}{q}} = |\mathbf{f}_n|, \tag{2.28}$$

where $\|\hat{\mathbf{f}}_n\|_{\frac{p^*}{q}}$ denotes the norm of $\hat{\mathbf{f}}_n(\varphi)$ in $L^{\frac{p^*}{q}}(\Omega)$ which is defined on $L^{\frac{p^*}{p^*-q}}(\Omega)$.

Remained proof is completely similar to the last proof of Lemma 2.3, we omit it here.

Lemma 2.8. *Assume (H_3) holds. If f has the subcritical exponential growth on Ω (condition* (SCE)*), then I_λ satisfies* $(\mathrm{PS})_{c^*}$.

Proof. Combining the previous section of the proof of Lemma 2.7 with slightly modifying the last section of the proof of Lemma 2.6, we can prove it. So we omit it here.

To prove the next Lemma, we firstly introduce a sequence of nonnegative functions as follows. Let $\Phi(t) \in C^\infty[0,1]$ such that

$$\Phi(0) = \Phi'(0) = 0,$$
$$\Phi(1) = \Phi'(1) = 0.$$

We let

$$H(t) = \begin{cases} \frac{1}{n}\Phi(nt), & \text{if } t \le \frac{1}{n}, \\ t, & \text{if } \frac{1}{n} < t < 1 - \frac{1}{n}, \\ 1 - \frac{1}{n}\Phi(n(1-t)), & \text{if } 1 - \frac{1}{n} \le t \le 1, \\ 1, & \text{if } 1 \le t, \end{cases}$$

and $\psi_n(r) = H((lnn)^{-1} ln\frac{1}{r})$. Notice that $\psi_n(x) \in E$, B the unit ball in $\mathbb{R}^N$, $\psi_n(x) = 1$ for $|x| \le \frac{1}{n}$ and , as it was proved in [2],

$$\|\Delta\psi_n\|_2 = 2\sqrt{2}\pi(lnn)^{-\frac{1}{2}}A_n = \|\psi_n\| + \circ(1),\ \text{as } n \to \infty.$$

where $0 \le \lim\limits_{n\to\infty} A_n \le 1$. Thus, we take $x_0 \in \Omega$ and $r_0 > 0$ such that $B(x_0, r) \subset \Omega$, denote

$$\Psi_n(x) = \begin{cases} \frac{\psi_n(|x-x_0|)}{\|\psi_n\|}, & \text{if } x \in B(x_0, r_0), \\ 0, & \text{if } x \in \Omega \backslash B(x_0, r_0). \end{cases}$$

Lemma 2.9. *Assume* (H_1) *and* (H_4) *hold. If* f *has the critical exponential growth on* Ω *(condition* (CG)*), then there exists n such that*

$$\max\{I^{\pm}_{\lambda}(\pm t\Psi_n) : t \geq 0\} < \frac{16\pi^2}{\alpha_0}.$$

Proof. We only prove the case of I^{+}_{λ}. The arguments for the case of I^{-}_{λ} are similar. Assume by contradiction that this is not the case. So, for all n, this maximum is larger or equal to $\frac{16\pi^2}{\alpha_0}$. Let $t_n > 0$ be such that

$$\mathcal{I}^{+}_{\lambda}(t_n\Psi_n) \geq \frac{16\pi^2}{\alpha_0}. \tag{2.29}$$

From (H_1) and (2.29), we conclude that

$$t_n^2 \geq \frac{32\pi^2}{\alpha_0}. \tag{2.30}$$

Also at $t = t_n$, we have

$$t_n - t_n^{s-1}\lambda \int_{\Omega} a(x)|\Psi_n|^s dx - \int_{\Omega} f(x, t_n\Psi_n)\Psi_n dx = 0,$$

which implies that

$$t_n^2 \geq t_n^s\lambda \int_{\Omega} a(x)|\Psi_n|^s dx + \int_{B(x_0,r_0)} f(x, t_n\Psi_n)t_n\Psi_n dx. \tag{2.31}$$

Since (H_4), for given $\epsilon > 0$ there exists $R_\epsilon > 0$ such that

$$tf(x,t) \geq (\beta - \epsilon)\exp\left(\alpha_0 t^2\right),\ t \geq R_\epsilon.$$

So by (2.31), we deduce that, for large n

$$t_n^2 \geq t_n^s\lambda \int_{\Omega} a(x)|\Psi_n|^s dx + (\beta - \epsilon)\frac{\pi^2}{2} r_0^4 \exp\left[\left((\frac{t_n}{A_n})^2\frac{\alpha_0}{32\pi^2} - 1\right)4lnn\right]. \tag{2.32}$$

By (2.30), the inequality above is true if, and only if

$$\lim_{n\to\infty} A_n = 1 \ \text{ and } \ t_n \to \left(\frac{32\pi^2}{\alpha_0}\right)^{\frac{1}{2}}. \tag{2.33}$$

Set

$$A_n^* = \{x \in B(x_0, r_0) : t_n\Psi_n(x) \geq R_\epsilon\},\quad B_n = B(x_0, r_0) \setminus A_n^*,$$

and break the integral in (2.31) into a sum of integrals over A_n^* and B_n. By simple computation, we have

$$\left[\frac{32\pi^2}{\alpha_0}\right] \geq (\beta - \epsilon)\lim_{n\to\infty}\int_{B(x_0,r_0)} \exp\left[\alpha_0 t_n^2|\Psi_n(x)|^2\right]dx - (\beta - \epsilon)r_0^4\frac{\pi^2}{2}. \tag{2.34}$$

The last integral in (2.34), denote I_n is evaluated as follows:

$$I_n \geq (\beta - \epsilon)r_0^4\pi^2.$$

Thus, finally from (2.34) we get

$$\left[\frac{32\pi^2}{\alpha_0}\right] \geq (\beta - \epsilon)r_0^4\frac{\pi^2}{2},$$

which means $\beta \leq \frac{64}{\alpha_0 r_0^4}$. This results in a contradiction with (H_4).

To conclude this section we state the Fountain Theorem of Bartsch [32].

Define

$$Y_k = \oplus_{j=1}^{k} X_j, \;\; Z_k = \overline{\oplus_{j\geq k} X_j}. \tag{2.35}$$

Lemma 2.10. *(Dual Fountain Theorem). Assume that $I_\lambda \in C^1(\mathbb{E}, \mathbb{R})$ satisfies the* $(\mathrm{PS})_c^*$ *condition (see [32]), $I_\lambda(-u) = I_\lambda(u)$. If for almost every $k \in \mathbf{N}$, there exist $\rho_k > r_k > 0$ such that*

(i) $a_k := \inf\limits_{u\in Z_k, \|u\|=\rho_k} I_\lambda(u) \geq 0$,

(ii) $b_k := \max\limits_{u\in Y_k, \|u\|=r_k} I_\lambda(u) < 0$,

(iii) $b_k = \inf\limits_{u\in Z_k, \|u\|=\rho_k} I_\lambda(u) \to 0$, *as* $k \to \infty$,

then I_λ has a sequence of negative critical values converging 0*.*

3. Proofs of the main results

Proof of Theorem 1.1. For $I_\lambda^{\pm}$, we first demonstrate that the existence of local minimum $v_\pm$ with $I_\lambda^{\pm}(v_\pm) < 0$. We only prove the case of I_λ^+. The arguments for the case of I_λ^- are similar.

For ρ determined in Lemma 2.4, we write

$$\bar{B}(\rho) = \{u \in E, \; \|u\| \leq \rho\}, \;\; \partial B(\rho) = \{u \in E, \; \|u\| = \rho\}.$$

Then $\bar{B}(\rho)$ is a complete metric space with the distance

$$\mathrm{dist}(u, v) = \|u - v\|, \quad \forall u, v \in \bar{B}(\rho).$$

From Lemma 2.4, we have for $0 < \lambda < \Lambda^*$,

$$I_\lambda^+(u)|_{\partial B(\rho)} \geq \alpha > 0.$$

Furthermore, we know that $I_\lambda^+ \in C^1(\bar{B}(\rho), \mathbb{R})$, hence I_λ^+ is lower semi-continuous and bounded from below on $\bar{B}(\rho)$. Set

$$c_1^* = \inf\{I_\lambda^+(u), u \in \bar{B}(\rho)\}.$$

Taking $\tilde{\phi} \in C_0^\infty(\Omega)$ with $\tilde{\phi} > 0$, and for $t > 0$, we get

$$\begin{aligned} I_\lambda^+(t\tilde{\phi}) &= \frac{t^2}{2}\|\tilde{\phi}\|^2 - \frac{\lambda t^s}{s}\int_\Omega a(x)|\tilde{\phi}|^s dx - \int_\Omega F^+(x, t\tilde{\phi})dx \\ &\leq \frac{t^2}{2}\|\tilde{\phi}\|^2 - \frac{\lambda t^s}{s}\int_\Omega a(x)|\tilde{\phi}|^s dx \\ &< 0, \end{aligned}$$

for all $t > 0$ small enough. Hence, $c_1^* < 0$.

Since Ekeland's variational principle and Lemma 2.4, for any $m > 1$, there exists u_m with $\|u_m\| < \rho$ such that

$$I_\lambda^+(u_m) \to c_1^*, \quad I_\lambda^{+'}(u_m) \to 0.$$

Hence, there exists a subsequence still denoted by $\{u_m\}$ such that

$$u_m \to v_+, \quad I_\lambda^{+'}(v_+) = 0.$$

Thus v_+ is a weak solution of problem (1.1) and $I_\lambda^+(v_+) < 0$. In addition, from the maximum principle, we know $v_+ > 0$. By a similar way, we obtain a negative solution v_- with $I_\lambda^-(v_-) < 0$.

On the other hand, from Lemmas 2.3 and 2.4, the functional I_λ^+ has a mountain pass-type critical point u_+ with $I_\lambda^+(u_+) > 0$. Again using the maximum principle, we have $u_+ > 0$. Hence, u_+ is a positive weak solution of problem (1.1). Similarly, we also obtain a negative mountain pass-type critical point u_- for the functional I_λ^-. Thus, we have proved that problem (1.1) has four different nontrivial solutions. Next, our method to obtain the fifth solution follows the idea developed in [33] for problem (1.1). We can assume that v_+ and v_- are isolated local minima of I_λ. Let us denote by b_λ the mountain pass critical level of I_λ with base points v_+, v_- :

$$b_\lambda = \inf_{\gamma\in\Gamma}\max_{0\le t\le 1} I_\lambda(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1], E), \gamma(0) = v_+, \gamma(1) = v_-\}$. We will show that $b_\lambda < 0$ if λ is small enough. To this end, we regard

$$I_\lambda(tv_\pm) = \frac{t^2}{2}\|v_\pm\|^2 - \frac{\lambda t^s}{s}\int_\Omega a(x)|v_\pm|^s dx - \int_\Omega F(x, tv_\pm)dx.$$

We claim that there exists $\delta > 0$ such that

$$I_\lambda(tv_\pm) < 0,\ \forall t \in (0,1),\ \forall \lambda \in (0,\delta). \tag{3.1}$$

If not, we have $t_0 \in (0,1)$ such that $I_\lambda(t_0 v_\pm) \ge 0$ for λ small enough. Similarly, we also have $I_\lambda(tv_\pm) < 0$ for $t > 0$ small enough. Let $\rho_0 = t_0\|v_\pm\|$ and $\check{c}_*^\pm = \inf\{I_\lambda^\pm(u), u \in \bar{B}(\rho_0)\}$. Since previous arguments, we obtain a solution $v_\pm^*$ such that $I_\lambda(v_\pm^*) < 0$, a contradiction. Hence, (3.1) holds.

Now, let us consider the 2-dimensional plane Π_2 containing the straightlines tv_- and tv_+, and take $v \in \Pi_2$ with $\|v\| = \epsilon$. Note that for such v one has $\|v\|_s = c_s\epsilon$. Then we get

$$I_\lambda(v) \le \frac{\epsilon^2}{2} - \frac{\lambda}{s}c_s^s h_0 \epsilon^s.$$

Thus, for small ϵ,

$$I_\lambda(v) < 0. \tag{3.2}$$

Consider the path $\bar{\gamma}$ obtained gluing together the segments $\{tv_- : \epsilon\|v_-\|^{-1} \le t \le 1\}, \{tv_+ : \epsilon\|v_+\|^{-1} \le t \le 1\}$ and the arc $\{v \in \Pi_2 : \|v\| = \epsilon\}$. by (3.1)and (3.2), we get

$$b_\lambda \le \max_{v\in\bar{\gamma}} I_\lambda(v) < 0,$$

which verifies the claim. Since the (PS) condition holds because of Lemma 2.3 , the level$\{I_\lambda(v) = b_\lambda\}$ carries a critical point v_3 of I_λ, and v_3 is different from $v_\pm$.

Proof of Theorem 1.2. We first use the symmetric mountain pass theorem to prove the case of *a*). It follows from our assumptions that the functional I_λ is even. Since the condition (SCP), we know that (I_1') of Theorem 9.12 in [30] holds. Furthermore, by condition (H_3), we easily verify that (I_2') of Theorem 9.12 also holds. Hence, by Lemma 2.7, our theorem is proved.

Next we use the dual fountain theorem (Lemma 2.10) to prove the case of *b*). Since Lemma 2.7, we know that the functional I_λ satisfies $(\mathrm{PS})_c^*$ condition. Next, we just need to prove the conditions (i)-(iii) of Lemma 2.10.

First, we verify (i) of Lemma 2.10. Define

$$\beta_k := \sup_{u \in Z_k, \|u\|=1} \|u\|_s.$$

From the conditions (SCP) and (H_2), we get, for $u \in Z_k, \|u\| \leq R$,

$$\begin{aligned} I_\lambda(u) &\geq \frac{\|u\|^2}{2} - \lambda\beta_k^s \frac{\|u\|^s}{s} - \frac{f_0 + \epsilon}{2}\|u\|_2^2 - c_6\|u\|^q \\ &\geq \frac{1}{4}(1 - \frac{f_0 + \epsilon}{\mu_1})\|u\|^2 - \lambda\beta_k^s \frac{\|u\|^s}{s}. \end{aligned} \tag{3.3}$$

Here, R is a positive constant and $\epsilon > 0$ small enough. We take $\rho_k = (4\mu_1\lambda\beta_k^s/[(\mu_1 - f_0 - \epsilon)s])^{\frac{1}{2-s}}$. Since $\beta_k \to 0, k \to \infty$, it follows that $\rho_k \to 0, k \to \infty$. There exists k_0 such that $\rho_k \leq R$ when $k \geq k_0$. Thus, for $k \geq k_0, u \in Z_k$ and $\|u\| = \rho_k$, we have $I_\lambda(u) \geq 0$ and (i) holds. The verification of (ii) and (iii) is standard, we omit it here.

Proof of Theorem 1.3. According to our assumptions, similar to previous section of the proof of Theorem 1.1, we obtain that the existence of local minimum $v_\pm$ with $I_\lambda^\pm(v_\pm) < 0$. In addition, by Lemmas 2.5 and 2.6, for $I_\lambda^\pm$, we obtain two mountain pass type critical points u_+ and u_- with positive energy. Similar to the last section of the proof of Theorem 1.1, we can also get another solution u_3, which is different from $v_\pm$ and $u_\pm$. Thus, this proof is completed.

Proof of Theorem 1.4. We first use the symmetric mountain pass theorem to prove the case of a). It follows from our assumptions that the functional I_λ is even. Since the condition (SCE), we know that (I_1') of Theorem 9.12 in [30] holds. In fact, similar to the proof of (i) of Lemma 2.5, we can conclude it. Furthermore, by condition (H_3), we easily verify that (I_2') of Theorem 9.12 also holds. Hence, by Lemma 2.8, our theorem is proved.

Next we use the dual fountain theorem (Lemma 2.10) to prove the case of b). Since Lemma 2.8, we know that the functional I_λ satisfies $(\mathrm{PS})_c^*$ condition. Next, we just need to prove the conditions (i)-(iii) of Lemma 2.10.

First, we verify (i) of Lemma 2.10. Define

$$\beta_k := \sup_{u \in Z_k, \|u\|=1} \|u\|_s.$$

From the conditions (SCE), (H_2) and Lemma 2.2, we get, for $u \in Z_k, \|u\| \leq R$,

$$\begin{aligned} I_\lambda(u) &\geq \frac{\|u\|^2}{2} - \lambda\beta_k^s \frac{\|u\|^s}{s} - \frac{f_0 + \epsilon}{2}\|u\|_2^2 - c_7\|u\|^q \\ &\geq \frac{1}{4}(1 - \frac{f_0 + \epsilon}{\mu_1})\|u\|^2 - \lambda\beta_k^s \frac{\|u\|^s}{s}. \end{aligned} \tag{3.4}$$

Here, R is a positive constant small enough and $\epsilon > 0$ small enough. We take $\rho_k = (4\mu_1\lambda\beta_k^s/[(\mu_1 - f_0 - \epsilon)s])^{\frac{1}{2-s}}$. Since $\beta_k \to 0, k \to \infty$, it follows that $\rho_k \to 0, k \to \infty$. There exists k_0 such that $\rho_k \leq R$ when $k \geq k_0$. Thus, for $k \geq k_0, u \in Z_k$ and $\|u\| = \rho_k$, we have $I_\lambda(u) \geq 0$ and (i) holds. The verification of (ii) and (iii) is standard, we omit it here.

Proof of Theorem 1.5. According to our assumptions, similar to previous section of the proof of Theorem 1.1, we obtain that the existence of local minimum $v_\pm$ with $I_\lambda^\pm(v_\pm) < 0$. Now, we show that I_λ^+

has a positive mountain pass type critical point. Since Lemmas 2.5 and 2.9, then there exists a $(C)_{c_M}$ sequence $\{u_n\}$ at the level $0 < c_M \leq \frac{16\pi^2}{\alpha_0}$. Similar to previous section of the proof of Lemma 2.6, we can prove that $(C)_{c_M}$ sequence $\{u_n\}$ is bounded in E. Without loss of generality, we can suppose that

$$u_n \rightharpoonup u_+ \ \text{ in } E.$$

Following the proof of Lemma 4 in [9], we can imply that u_+ is weak of problem (1.1). So the theorem is proved if u_+ is not trivial. However, we can get this due to our technical assumption (H_5). Indeed, assume $u_+ = 0$, similarly as in [9], we obtain $f^+(x, u_n) \to 0$ in $L^1(\Omega)$. Since (H_5), $F^+(x, u_n) \to 0$ in $L^1(\Omega)$ and we get

$$\lim_{n\to\infty} \|u_n\|^2 = 2c_M < \frac{32\pi^2}{\alpha_0},$$

and again following the proof in [9], we get a contradiction.

We claim that v_+ and u_+ are distinct. Since the previous proof, we know that there exist sequence $\{u_n\}$ and $\{v_n\}$ in E such that

$$u_n \to v_+,\ I^+_\lambda(u_n) \to c^+_* < 0,\ \langle I^{+'}_\lambda(u_n), u_n\rangle \to 0, \tag{3.5}$$

and

$$v_n \rightharpoonup u_+,\ I^+_\lambda(v_n) \to c_M > 0,\ \langle I^{+'}_\lambda(v_n), v_n\rangle \to 0. \tag{3.6}$$

Now, argue by contradiction that $v_+ = u_+$. Since we also have $v_n \rightharpoonup v_+$ in E, up to subsequence, $\lim\limits_{n\to\infty} \|v_n\| \geq \|v_+\| > 0$. Setting

$$w_n = \frac{v_n}{\|v_n\|},\ \ w_0 = \frac{v_+}{\lim_{n\to\infty}\|v_n\|},$$

we know that $\|w_n\| = 1$ and $w_n \rightharpoonup w_0$ in E.

Now, we consider two possibilities:

$$\text{(i) } \|w_0\| = 1, \quad \text{(ii) } \|w_0\| < 1.$$

If (i) happens, we have $v_n \to v_+$ in E, so that $I^+_\lambda(v_n) \to I^+_\lambda(v_+) = c^+_*$. This is a contradiction with (3.5) and (3.6).

Now, suppose that (ii) happens. We claim that there exists $\delta > 0$ such that

$$h\alpha_0\|v_n\|^2 \leq \frac{32\pi^2}{1 - \|w_0\|^2} - \delta \tag{3.7}$$

for n large enough. In fact, by the proof of v_+ and Lemma 2.9, we get

$$0 < c_M < c^+_* + \frac{16\pi^2}{\alpha_0}. \tag{3.8}$$

Thus, we can choose $h > 1$ sufficiently close to 1 and $\delta > 0$ such that

$$h\alpha_0\|v_n\|^2 \leq \frac{16\pi^2}{c_M - I^+_\lambda(v_+)}\|v_n\|^2 - \delta.$$

Since $v_n \rightharpoonup v_+$, by condition (H_5), up to a subsequence, we conclude that

$$\frac{1}{2}\|v_n\|^2 = c_M + \frac{\lambda}{s}\int_\Omega a(x)v^s_+ dx + \int_\Omega F^+(x, v_+)dx + \circ(1). \tag{3.9}$$

Thus, for n sufficiently large we get

$$h\alpha_0\|v_n\|^2 \leq 32\pi^2 \frac{c_M + \frac{\lambda}{s}\int_\Omega a(x)v_+^s dx + \int_\Omega F^+(x, v_+)dx + \circ(1)}{c_M - I_\lambda^+(v_+)} - \delta. \tag{3.10}$$

Thus, from (3.9) and the definition of w_0, (3.10) implies (3.7) for n large enough.

Now, taking $\tilde{h} = (h + \epsilon)\alpha_0\|v_n\|^2$, it follows from (3.7) and a revised Adams inequality (see [28]), we have

$$\int_\Omega \exp((h + \epsilon)\alpha_0\|v_n\|^2|w_n|^2 dx \leq C \tag{3.11}$$

for $\epsilon > 0$ small enough. Thus, from our assumptions and the Hölder inequality we get $v_n \to v_+$ and this is absurd.

Similarly, we can find a negative mountain pass type critical point u_- which is different that v_-. Thus, the proof is completed.

4. Conclusions

In this research, we mainly studied the existence and multiplicity of nontrivial solutions for the fourth-order elliptic Navier boundary problems with exponential growth. Our method is based on the variational methods, Resonance Theorem together with a revised Adams inequality.

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Conflict of interest

There is no conflict of interest.

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