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Research article

Sustainable practices to reduce environmental impact of industry using interaction aggregation operators under interval-valued Pythagorean fuzzy hypersoft set

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Abstract: Optimization techniques can be used to find the optimal combination of inputs and parameters and help identify the most efficient solution. Aggregation operators (AOs) play a prominent role in discernment between two circulations of prospect and pull out anxieties from that insight. The most fundamental objective of this research is to extend the interaction AOs to the interval-valued Pythagorean fuzzy hypersoft set (IVPFHSS), the comprehensive system of the interval-valued Pythagorean fuzzy soft set (IVPFSS). The IVPFHSS adroitly contracts with defective and ambagious facts compared to the prevalent Pythagorean fuzzy soft set and interval-valued intuitionistic fuzzy hypersoft set (IVIFHSS). It is the dominant technique for enlarging imprecise information in decision-making (DM). The most important intention of this exploration is to intend interactional operational laws for IVPFHSNs. We extend the AOs to interaction AOs under IVPFHSS setting such as interval-valued Pythagorean fuzzy hypersoft interactive weighted average (IVPFHSIWA) and interval-valued Pythagorean fuzzy hypersoft interactive weighted geometric (IVPFHSIWG) operators. Also, we study the significant properties of the proposed operators, such as Idempotency, Boundedness, and Homogeneity. Still, the prevalent multi-criteria group decision-making (MCGDM) approaches consistently carry irreconcilable consequences. Meanwhile, our proposed MCGDM model is deliberate to accommodate these shortcomings. By utilizing a developed mathematical model and optimization technique, Industry 5.0 can achieve digital green innovation, enabling the development of sustainable processes that significantly decrease environmental impact. The impacts show that the intentional model is more operative and consistent in conducting inaccurate data based on IVPFHSS.

Keywords: hypersoft set; interval-valued Pythagorean fuzzy hypersoft set; IVPFHSIWA operator; IVPFHSIWG operator; MCGDM
Mathematics Subject Classification: 03E72, 68T35, 90B50

1. Introduction

In recent years, there has been a growing interest in using fuzzy mathematical models and optimization techniques in digital green innovation for Industry 5.0. This is due to the increasing need for sustainable industry practices and the need to optimize processes to minimize environmental impact. The environmental impact of industry refers to the negative effects of industrial activities on the environment, including air and water pollution, deforestation, greenhouse gas emissions, and biodiversity loss. Industrial activities can have both direct and indirect impacts on the environment. Fuzzy mathematical models are a type of mathematical model that can handle uncertain or vague data. This is particularly useful in digital green innovation, where data may be incomplete or imprecise. Fuzzy logic can model complex relationships between inputs and outputs and help identify optimal solutions without a clear answer. Optimization techniques can be used to bargain the best possible solution to a problem. In the context of digital green innovation, this might involve optimizing processes to reduce energy consumption or minimize waste. Optimization techniques can be used to find the optimal combination of inputs and parameters and help identify the most efficient solution. Fuzzy mathematical models and optimization techniques offer a variety of applications in digital green innovation for Industry 5.0. One significant application is the optimization of energy-efficient buildings. Using these techniques, engineers and architects can develop more energy-efficient designs that reduce energy consumption and minimize environmental impact. In addition, these techniques can be used to identify the optimal combination of renewable energy sources for a particular location. Renewable energy sources such as solar, wind, and hydropower can provide clean and sustainable energy, but the optimal combination for a particular location depends on climate, geography, and available resources. Fuzzy mathematical models and optimization techniques can be used to identify the most effective and efficient combination of renewable energy sources for a specific location, reducing greenhouse gas emissions and minimizing environmental impact. Furthermore, optimization techniques can be used to improve supply chain efficiency and reduce the environmental impact of transportation and logistics. By optimizing transportation routes and modes, companies can reduce emissions and fuel consumption, resulting in a more sustainable supply chain. Optimization techniques can also identify the optimal time to maintain equipment and vehicles, reducing downtime and improving efficiency.

Overall, fuzzy mathematical models and optimization techniques provide numerous opportunities for digital green innovation in Industry 5.0. By utilizing these techniques, businesses and industries can develop more sustainable processes, technologies, and supply chains with a reduced environmental impact, helping to create a more sustainable future. In addition to fuzzy mathematical models and optimization techniques, another area of interest for digital green innovation in Industry 5.0 is Multiple Criteria Group Decision Making (MCGDM) mathematical models and optimization. MCGDM is a decision-making approach that is particularly useful when multiple criteria or objectives need to be considered. In digital green innovation, this might involve balancing economic, environmental, and social factors in decision-making processes. MCGDM can help identify optimal solutions that take into account multiple factors. Optimization techniques can be used to identify the best possible solution to a given problem. In the context of MCGDM, this might involve identifying the optimal trade-off between different criteria or objectives. Optimization techniques can help identify the most efficient solution that satisfies all criteria. MCGDM mathematical models and optimization techniques can be used to develop more sustainable practices in Industry 5.0. For example, these techniques can optimize supply chains to minimize the environmental impact or develop green technologies that balance economic and environmental factors. Most verdicts are reserved when the objectives and boundaries are ordinarily unstipulated or indistinct in realistic surroundings. The theory of fuzzy sets (FS), originated by Zadeh [1], provides a means of dealing with equivocal and tentative data in the DM process. Xiao [2] introduced a cost-aware, fault-tolerant and reliable strategy for operator development on fuzzy complex event processing systems based on the TOPSIS technique. Turksen [3] proposed an interval-valued FS (IVFS) with basic operations. Existing FS and IVFS cannot provide evidence about a substitute non-membership degree (NMD). However, the current FS and IVFS cannot handle situations where experts consider an NMD in the DM process. Atanassov [4] introduced the intuitionistic fuzzy set (IFS) to overawed this inadequacy. Wang and Liu [5] proposed several operations for IFS. Xiao [6] presented the distance measure for IFS and utilized it in pattern classification problems. Atanassov [7] extended IFS to an interval-valued IFS (IVIFS). Xiao [8] introduced the novel evidential fuzzy multi-criteria decision-making (MCDM) method, is proposed by integrating Dempster-Shafer theory with belief entropy. Despite these advancements, the existing IFS cannot handle inconsistent and conflicting data, as it assumes a linear relationship between membership degree (MD) and NMD. When the MD and NMD values of a team of experts exceed 1 (e.g., MD = 0.6 and NMD = 0.7), the current IFS cannot effectively handle the situation as $0.6 + 0.7 \geq 1$.

Over the past few decades, various AOs have been settled and widely studied in several extensions of FS and non-classical decision theory circles. These AOs' well-established structures have originated their approach into frequent application fields such as economics, biology, education, knowledge-based systems, and robotics. The utility of AOs in a field depends on how well the mathematical properties of AOs correspond to the fusion procedure of elementary facts. Some AOs have mathematical properties that can be deduced as behavioral parameters, i.e., their values stimulus the operator's behavior. Previous research on AOs and their applications has concentrated on the domain exemplification competency of the power of DM. In addition, AOs appearances are imperative in the performance of factors, which is essential for specialists, particularly when these interactive constraints are deputations of domain-specific facts that are openly or statistically tough to an extent. In this case, the behavioral parameters of these AOs become more than just another mathematical feature. Yager [9] introduced the Pythagorean fuzzy set (PFS) to address the limitations of existing fuzzy set theories, which could not handle inconsistent and uncertain data. The PFS corrects these errors by revising the basic condition $\kappa + \delta \leq 1$ to $\kappa^2 + \delta^2 \leq 1$. Khan et al. [10] introduced the dissimilarity measure and refined the VIKOR method for PFS. Rahman et al. [11] extended the PFS theory by proposing the Einstein-weighted geometric AOs, which were then used to develop a multi-attribute group decision-making (MAGDM) technique. Huang et al. [12] protracted the MULTIMOORA technique for PFS based on distance measure and score function. Zhang and Xu [13] further expanded on the operational rules of PFS and applied the TOPSIS to address MCDM problems. Lin et al. [14] developed the directional correlation coefficient measures for PFS and used their presented measures in medical diagnosis and cluster analysis. Wei and Lu [15] introduced the power aggregation operators for PFS and deliberated their essential properties, demonstrating their use in a decision-making system for multi-attribute decision-making (MADM). Akram et al. [16] clarify the application of the ELECTRE II technique for group decision-making in a complex Pythagorean fuzzy context. Xiao and Ding [17] established the divergence measure between PFS by compelling the advantage of the Jensen–Shannon divergence. They also developed a competent DM algorithm to resolve medical diagnoses. Wang and Li [18] investigated the interactions between Pythagorean fuzzy numbers (PFNs) and power Bonferroni mean operators. Lin et al. [19] presented the partitioned Bonferroni mean AOs for linguistic PFS with their essential properties. Khan et al. [20] developed the Archimedean AOs for T-spherical fuzzy sets and established a MADM model to determine DM complications. Akram et al. [21] developed the complex Pythagorean fuzzy N-soft VIKOR technique that can direct an excessive contract of linguistic inaccuracy and imprecision intrinsic in human valuations. Zhang [22] proposed a unique DM method based on similarity measures to address MCGDM obstacles in PFS scenarios. Riaz and Farid [23] protracted the hybrid AOs for picture fuzzy sets and established a DM technique to resolve MCDM problems. Lin et al. [24] proposed the partitioned Heronian mean AOs for picture fuzzy sets and developed a MADM model to resolve DM issues. Finally, Peng and Yang [25] extended the theory to include interval-valued Pythagorean fuzzy sets (IVPFS) and proposed a DM system based on their proposed method. Lin et al. [26] evaluated the Internet of Things platforms as an MCDM problem since it comprises numerous concerns. A novel incorporated MCDM technique is put forward for handling this problem. Rahman et al. [27] expanded weighted geometric AOs to interval-valued Pythagorean fuzzy sets (IVPFS) and developed a DM method based on these operators. Lin et al. [28] developed the weighted AOs for linguistic q-rung orthopair fuzzy sets and established a MADM technique based on their developed operators to solve DM issues. While these methods have a broad range of applications, there are some limitations to using them in parametric chemistry due to their inadequacy in dealing with uncertainties and vagueness. To address these issues, Molodtsov [29] introduced the concept of soft sets (SS) and their properties for handling chaos and ambiguity. Building on this idea, Maji et al. [30] proposed basic operations for SS and later established fuzzy soft sets by combining fuzzy and soft sets theories [31]. They also extended the concept to intuitionistic fuzzy soft sets (IFSS) [32], for which Arora and Garg [33] developed AOs and a DM technology based on these operators. Das [34] established a group DM technique for fuzzy parameterized intuitionistic multi-fuzzy N-soft set. Additionally, Jiang et al. [35] presented interval-valued IFSS (IVIFSS). They discussed its essential properties, while Zulqarnain et al. [36] proposed a TOPSIS technique based on the correlation coefficient (CC) for IVIFSS to solve MADM problems. Peng et al. [37] extended the concept to PFSS by incorporating both PFS and SS, and Zulqarnain et al. [38] developed AOs for IVPFSS and presented a MAGDM approach for solving real-world problems.

The theory of hypersoft sets (HSS), proposed by Smarandache [39], encompasses compound sub-parameters in the parametric function f, which represents the cartesian product with n features. HSS is considered the most suitable model for handling multiple sub-attributes of the parameters in association with SS and other established concepts. There are various HSS approaches with corresponding DM methods. The possibility IFHSS was introduced by Rahman et al. [40], and they established DM methods using similarity measures. Rahman et al. [41] demonstrated a DM methodology for neutrosophic HSS. Saeed et al. [42] utilized neutrosophic hypersoft mapping to diagnose the brain tumor. Zulqarnain et al. [43] protracted the AOs for IFHSS and extended the DM methodologies consuming their settled AOs. Zulqarnain et al. [44] extended IFHSS to PFHSS with fundamental operations. Zulqarnain et al. [45] raised the AOs in the IVPFHSS setting and demonstrated an MCDM technique to resolve DM complications. The selection and assessment of suppliers is a crucial article of proficient movement. Variations in current management policies use suppliers' evaluations dignified from various perspectives, comprising environmentally friendly and public desires. Consequently, this delinquent is called sustainable supplier selection (SSS) in the literature and is a reference delinquent for MCGDM. At the same time, multiple authorizations [46–49] designate the necessity for an MCGDM method for worker selection, concentrating on proper lexical deliberations for environmental facts and expert expectations.

1.1. Motivation

The IVPFHSS is a composite structure that combines the properties of both IVPFS and HSS. IVIFSS and IVPFSS are widely used systematic tools for handling known and uncertain information. The use of AOs in DM is crucial, as it enables the combination of information from multiple sources for a comprehensive evaluation. However, the current AOs for IVPFHSS are inadequate in dealing with imprecise and uncertain data in DM development. Additionally, the model proposes that the NMD in interval form is an independent NMD (MD). Therefore, these models are not effective in providing clear preferences for alternatives. To address this issue, incorporating IVPFHSNs over interaction AOs is an intriguing topic. We propose the interaction AOs for IVPFHSS, such as the IVPFHSIWA and IVPFHSIWG operators. These AOs can be compared to prevalent fusion extensions of FS. The models discussed above suggest that the overall MD (NMD) is determined by its compatible NMD (MD) interval values. Therefore, the outcome of the core replica is adverse, and the partiality of the alternative cannot be appropriately constituted. So, incorporating these interval-valued Pythagorean fuzzy hypersoft numbers (IVPFHSNs) with their interactions is an encouraging topic. The AOs defined in [45] are not abundant to check the data on better concepts and have a spongy ability to acquire precise outcomes. Such as U = $\{\mathrm{u}_1, \mathrm{u}_2\}$ be a set of two experts with weights ω_i = $(0.6,\ 0.4)^T$ and $\mathrm{e}_1, \mathrm{e}_2$ be the selected factors with their compatible sub-parameters, such as $\mathrm{e}_1 = \{\mathrm{e}_{11}, \mathrm{e}_{12}\}$ and $\mathrm{e}_2 = \{\mathrm{e}_{21}\}$. Where $\mathfrak{L}'$ be a 2-tuple cartesian product of the considered factors, can be indicated as $\mathfrak{L}' = \mathrm{e}_1 \times \mathrm{e}_2 = \{\mathrm{e}_{11}, \mathrm{e}_{12}\} \times \{\mathrm{e}_{21}\} = \{(\mathrm{e}_{11}, \mathrm{e}_{21}), (\mathrm{e}_{12}, \mathrm{e}_{21})\} = \{\check{d}_1, \check{d}_2\}$ with weights ν_j = $(0.4, .0.6)^T$. Let $\mathfrak{I}$ be an alternate, then preferences of experts can be précised as $\mathfrak{I}$ = $\begin{bmatrix} ([0.7, 0.8], [0.0, 0.0]) & ([0.2, 0.6], [0.3, 0.5]) \\ ([0.3, 0.6], [0.5, 0.7]) & ([0.5, 0.7], [0.1, 0.6]) \end{bmatrix}$ longsighted the sub-parameters of the planned aspects in terms of IVPFHSNs. Then, we conquered the collected value by the IVPFHSWA [45] operator is $\langle [0.8333, 0.9487], [0.0, 0.0] \rangle$. Similarly, we engaged the IVPFHSWG [45] operator and achieved a collected value $\langle [0.3584, 0.6505], [0.0, 0.0] \rangle$. This shows that there is no effect on the collective consequence $\delta_{\check{d}_k}$. As $\delta_{\check{d}_k}$ = $\delta_{\check{d}_{11}}$ = $[0.0, 0.0]$, $\delta_{\check{d}_{12}}$ = $[0.5, 0.7]$, $\delta_{\check{d}_{21}}$ = $[0.3, 0.5]$, and $\delta_{\check{d}_{22}}$ = $[0.1, 0.6]$, which is arbitrary. An amended consolidating approach appeals to investigators to crack baffling and unsatisfactory details. Consequently, the significances of these AOs are unreliable, and no extra information for substitutes is specified. Hence, integrating these IVPFHSNs over AOs is a stimulating theme. The methodologies taken in [45] are unsatisfactory in scrutinizing the facts with a reflection on established theory and clear implications. Then, we originate the composed value using the IVPFHSWA and IVPFHSWG operators unable to deliver the proper evaluation considering the interaction. So, we claim that the developed interaction AOs for IVPFHSS is an enhanced classification technique that fascinates detectives to crack incomprehensible and inadequate specifics.

1.2. Significant contribution

We implement a strategy for picking sustainable suppliers using IVPFHSS information to address these inadequacies. An enriched consolidating system fascinates researchers to defect inexplicable and scarce data to address these inadequacies. IVPFHSS is energetic in DM interpreting the consideration concerns by accumulating abundant foundations into a particular value. The existing AOs for IVPFHSS cannot manage the situation while the information of any sub-attribute is given in the form of intervals. It is a novel amalgam configuration to handle ambiguous complications through the DM process. Therefore, to instigate the present investigation of IVPFHSS, we will discuss extant interaction AOs founded on asymmetrical facts. The fundamental purposes of the extant exploration are specified as follows:

(1) To capitalize on the advantages of incorporating multiple sub-attributes of parameters in the DM coordination, we expanded the interaction AOs of IVPFHSS.
(2) IVPFHSS interaction AOs are known as elegant dominant AOs. In some cases, basic AOs function labeling does not respond to accurate determination of the DM procedure. For this, the prevailing AOs need to be modified. We will propose interactional operational laws for interval-valued Pythagorean fuzzy hypersoft numbers (IVPFHSNs) to reveal these barriers.
(3) Based on the developed interactional operational laws, the IVPFHSIWA and IVPFHSIWG operators have been introduced with their desirable properties.
(4) A new algorithm based on the planned operators has been established to demonstrate the MCGDM problems.
(5) Supplier selection is a deferential aspect of thermal power plant equipment as it appreciates the actual surroundings for all features. Supplier selection is a strenuous but substantial phase in proficient development. The constructor's proficiency, efficiency, and eccentricity will suffer due to the absence of a supplier.
(6) A comparative study of the developed MCGDM model and prevalent methodologies is delivered to reflect the efficacy and supremacy of our protracted model.

The organizational structure of this article assumes the following: The second part deals with some elementary notions that support the configuration of our development of the advanced study. Section 3 proposes some new algebraic operations for IVPFHSS considering the interaction. Also, IVPFHSIWA and IVPFHSIWG operators will be introduced with their basic characteristics in the same section. Section 4 presents the MCGDM method based on the proposed interaction AOs. A numerical example is discussed in the same section to verify the practicality of established supplier selection techniques in thermal equipment plants. In addition, a brief comparative analysis is performed to confirm the potential of the method developed in Section 5.

2. Preliminaries

This section comprises some fundamental definitions that will organize the subsequent work.

Definition 2.1. [29] A soft set over U is a pair (Ω, A), where A is a non-empty set of attributes, and Ω is a mapping from A to the power set of U, denoted by $P(U)$.
In other words,

$$(\Omega, A) = \{\Omega(t) \in \mathcal{P}(U): t \in A, \Omega(t) = \emptyset \; if \; t \notin A\}.$$

Definition 2.2. [25] Let A be any subset of U, and let $[a, b]$ be an interval in the set of real numbers. An interval-valued Pythagorean fuzzy set A over U is defined as a mapping:

$$\Omega: A \to [0,1].$$

That assigns an MD and NMD to each element t in A, such that the MD and NMD of an element t in A is an interval $[\kappa_A^l(t), \kappa_A^u(t)]$ and $[\delta_A^l(t), \delta_A^u(t)]$ w.r.t. U, where $\kappa_A^l(t)$ and $\kappa_A^u(t)$ are the lower and upper bounds of the MD interval and $\delta_A^l(t)$ and $\delta_A^u(t)$ are the lower and upper bounds of the NMD interval, respectively. Also, satisfied the $0 \leq (\kappa_A^u(t))^2 + (\delta_A^u(t))^2 \leq 1$.

Definition 2.3. [38] The pair (Ω, N) is called an IVPFSS over U and is defined as follows:

$$\Omega: \mathbb{N} \to \wp K^U.$$

Here, $\wp K^U$ denotes the collection of interval-valued Pythagorean fuzzy subsets of the universe of discourse U. Also, it can be represented as:

$$(\Omega, \mathbb{N}) = \{x, ([\kappa_A^l(t), \kappa_A^u(t)], [\delta_A^l(t), \delta_A^u(t)]) | t \in A\}.$$

Where, $MD = [\kappa_A^l(t), \kappa_A^u(t)]$, $NMD = [\delta_A^l(t), \delta_A^u(t)]$, $\kappa_A^l(t), \kappa_A^u(t), \delta_A^l(t), \delta_A^u(t) \in [0,1]$ and fulfilled the consequent state $0 \leq (\kappa_A^u(t))^2 + (\delta_A^u(t))^2 \leq 1$ and $A \subset \mathbb{N}$.

The studies mentioned above cannot deal with the situation when any expert considers the sub-attribute of any deliberated parameter. Smarandache [39] proposed the hypersoft set to handle such complications.

Definition 2.4. [39] Let U be a universal set and $\mathcal{P}(\mathrm{U})$ be a power set over U, and $\mathrm{t} = \{\mathrm{t}_1, \mathrm{t}_2, \mathrm{t}_3, \ldots, \mathrm{t}_n\}$,$(\mathrm{n} \geq 1)$ and $\mathrm{T_i}$ designated the set of parameters and their corresponding multi-sub-parameters, such as $\mathrm{T_i} \cap \mathrm{T_j} = \varphi$, where $i \neq j$ for each $n \geq 1$ and $i, j \in \{1,2,3 \ldots n\}$. Suppose $\mathrm{T_1} \times \mathrm{T_2} \times \mathrm{T_3} \times \ldots \times \mathrm{T_n} = \ddot{\mathrm{A}} = \{\mathrm{d}_{1h} \times \mathrm{d}_{2k} \times \cdots \times \mathrm{d}_{nl}\}$ be an assortment of multi sub-attributes, where $1 \leq \mathrm{h} \leq \alpha$, $1 \leq \mathrm{k} \leq \beta$, and $1 \leq \mathrm{l} \leq \gamma$, and $\alpha, \beta, \gamma \in \mathrm{N}$. Then the pair $(\mathcal{F}, T_1 \times T_2 \times T_3 \times \ldots \times T_n) = (\Omega, \ddot{\mathrm{A}})$ is known as HSS and is defined as follows:

$$\Omega: T_1 \times T_2 \times T_3 \times \ldots \times T_n = \ddot{\mathrm{A}} \to \mathcal{P}(\mathrm{U}).$$

Also, it is defined as

$$(\Omega, \ddot{\mathrm{A}}) = \{\check{\mathrm{d}}, \Omega_{\ddot{\mathrm{A}}}(\check{\mathrm{d}}) : \check{\mathrm{d}} \in \ddot{\mathrm{A}},\ \Omega_{\ddot{\mathrm{A}}}(\check{\mathrm{d}}) \in \mathcal{P}(\mathrm{U})\}.$$

Definition 2.5. [45] Let U be a universal set and $\mathcal{P}(\mathrm{U})$ be a power set over U, and $\mathrm{t} = \{\mathrm{t}_1, \mathrm{t}_2, \mathrm{t}_3, \ldots, \mathrm{t}_n\}$,$(\mathrm{n} \geq 1)$ and $\mathrm{T_i}$ denoted the set of attributes and their compatible multi-sub-attributes, such as $\mathrm{T_i} \cap \mathrm{T_j} = \varphi$, where $i \neq j$ for each $n \geq 1$ and $i, j \in \{1,2,3 \ldots n\}$. Suppose $T_1 \times T_2 \times T_3 \times \cdots \times T_n = \ddot{A} = \{d_{1h} \times d_{2k} \times \cdots \times d_{nl}\}$ be an assortment of multi sub-parameters, where $1 \leq h \leq \alpha$, $1 \leq k \leq \beta$, and $1 \leq l \leq \gamma$, and $\alpha, \beta, \gamma \in \mathrm{N}$. Then the pair $(\mathcal{F}, T_1 \times T_2 \times T_3 \times \ldots \times T_n) = (\Omega, \ddot{\mathrm{A}})$ is called IVPFHSS, and its mapping can be defined as:

$$\Omega: T_1 \times T_2 \times T_3 \times \ldots \times T_n = \ddot{\mathrm{A}} \to IVPFHS^U.$$

Also, it is defined as $(\Omega, \ddot{A}) = \{(\check{d}, \Omega_{\ddot{A}}(\check{d})) : \check{d} \in \ddot{A},\ \Omega_{\ddot{A}}(\check{d}) \in IVPFHS^U \in [0,1]\}$, where $\Omega_{\ddot{A}}(\check{d}) = \{\langle \zeta, \kappa_{\Omega(\check{d})}(\zeta), \delta_{\Omega(\check{d})}(\zeta) \rangle : \zeta \in U\}$, and $\kappa_{\Omega(\check{d})}(\zeta) = [\kappa_{\Omega(\check{d})}^l(\zeta), \kappa_{\Omega(\check{d})}^u(\zeta)]$, $\delta_{\Omega(\check{d})}(\zeta) = [\delta_{\Omega(\check{d})}^l(\zeta), \delta_{\Omega(\check{d})}^u(\zeta)]$, $\kappa_{\Omega(\check{d})}(\zeta)$ be the MD interval and $\delta_{\Omega(\check{d})}(\zeta)$ be the NMD interval, such as $\kappa_{\Omega(\check{d})}^l(\zeta), \kappa_{\Omega(\check{d})}^u(\zeta), \delta_{\Omega(\check{d})}^l(\zeta), \delta_{\Omega(\check{d})}^u(\zeta) \in [0,1]$, and $0 \leq (\kappa_{\Omega(\check{d})}^u(\zeta))^2 + (\delta_{\Omega(\check{d})}^u(\zeta))^2 \leq 1$. It can be written as $\mathcal{F} = ([\kappa_{\Omega(\check{d})}^l(\zeta), \kappa_{\Omega(\check{d})}^u(\zeta)], [\delta_{\Omega(\check{d})}^l(\zeta), \delta_{\Omega(\check{d})}^u(\zeta)])$.

The score and accuracy functions used to calculate the alternative rank for IVPFHSS can be defined as follows:
If $\mathcal{F} = \left(\left[\kappa^l_{\Omega(\breve{d})}(\zeta), \kappa^u_{\Omega(\breve{d})}(\zeta)\right], \left[\delta^l_{\Omega(\breve{d})}(\zeta), \delta^u_{\Omega(\breve{d})}(\zeta)\right]\right)$ be an IVPFHSN. Then

$$S(\mathcal{F}) = \frac{\left(\kappa^l_{\Omega(\breve{d})}(\zeta)\right)^2 + \left(\kappa^u_{\Omega(\breve{d})}(\zeta)\right)^2 - \left(\delta^l_{\Omega(\breve{d})}(\zeta)\right)^2 - \left(\delta^u_{\Omega(\breve{d})}(\zeta)\right)^2}{2}. \tag{2.1}$$

And

$$A(\mathcal{F}) = \frac{\left(\kappa^l_{\Omega(\breve{d})}(\zeta)\right)^2 + \left(\kappa^u_{\Omega(\breve{d})}(\zeta)\right)^2 + \left(\delta^l_{\Omega(\breve{d})}(\zeta)\right)^2 + \left(\delta^u_{\Omega(\breve{d})}(\zeta)\right)^2}{2}. \tag{2.2}$$

Definition 2.6. [45] Let $\mathcal{M}_{\breve{d}_k} = \left(\left[\kappa^l_{\breve{d}_k}, \kappa^u_{\breve{d}_k}\right], \left[\delta^l_{\breve{d}_k}, \delta^u_{\breve{d}_k}\right]\right)$, $\mathcal{M}_{\breve{d}_{11}} = \left(\left[\kappa^l_{\breve{d}_{11}}, \kappa^u_{\breve{d}_{11}}\right], \left[\delta^l_{\breve{d}_{11}}, \delta^u_{\breve{d}_{11}}\right]\right)$, and $\mathcal{M}_{\breve{d}_{12}} = \left(\left[\kappa^l_{\breve{d}_{12}}, \kappa^u_{\breve{d}_{12}}\right], \left[\delta^l_{\breve{d}_{12}}, \delta^u_{\breve{d}_{12}}\right]\right)$ be three IVPFHSNs and $\beta > 0$, we have

(1) $\mathcal{M}_{\breve{d}_{11}} \oplus \mathcal{M}_{\breve{d}_{12}} = \left(\left[\sqrt{{\kappa^l_{\breve{d}_{11}}}^2 + {\kappa^l_{\breve{d}_{12}}}^2 - {\kappa^l_{\breve{d}_{11}}}^2 {\kappa^l_{\breve{d}_{12}}}^2}, \sqrt{{\kappa^u_{\breve{d}_{11}}}^2 + {\kappa^u_{\breve{d}_{12}}}^2 - {\kappa^u_{\breve{d}_{11}}}^2 {\kappa^u_{\breve{d}_{12}}}^2}\right], \left[\delta^l_{\breve{d}_{11}}\delta^l_{\breve{d}_{12}}, \delta^u_{\breve{d}_{11}}\delta^u_{\breve{d}_{12}}\right]\right)$

(2) $\mathcal{M}_{\breve{d}_{11}} \otimes \mathcal{M}_{\breve{d}_{12}} = \left(\left[\kappa^l_{\breve{d}_{11}}\kappa^l_{\breve{d}_{12}}, \kappa^u_{\breve{d}_{11}}\kappa^u_{\breve{d}_{12}}\right], \left[\sqrt{{\delta^l_{\breve{d}_{11}}}^2 + {\delta^l_{\breve{d}_{12}}}^2 - {\delta^l_{\breve{d}_{11}}}^2 {\delta^l_{\breve{d}_{12}}}^2}, \sqrt{{\delta^u_{\breve{d}_{11}}}^2 + {\delta^u_{\breve{d}_{12}}}^2 - {\delta^u_{\breve{d}_{11}}}^2 {\delta^u_{\breve{d}_{12}}}^2}\right]\right)$

(3) $\beta\mathcal{M}_{\breve{d}_k} = \left(\left[\sqrt{1 - \left(1 - {\kappa^l_{\breve{d}_k}}^2\right)^\beta}, \sqrt{1 - \left(1 - {\kappa^u_{\breve{d}_k}}^2\right)^\beta}\right], \left[{\delta^l_{\breve{d}_k}}^\beta, {\delta^u_{\breve{d}_k}}^\beta\right]\right) = \left(\sqrt{1 - \left(1 - \left[\kappa^l_{\breve{d}_k}, \kappa^u_{\breve{d}_k}\right]^2\right)^\beta}, \left[{\delta^l_{\breve{d}_k}}^\beta, {\delta^u_{\breve{d}_k}}^\beta\right]\right)$

(4) ${\mathcal{M}_{\breve{d}_k}}^\beta = \left(\left[{\kappa^l_{\breve{d}_k}}^\beta, {\kappa^u_{\breve{d}_k}}^\beta\right], \left[\sqrt{1 - \left(1 - {\delta^l_{\breve{d}_k}}^2\right)^\beta}, \sqrt{1 - \left(1 - {\delta^u_{\breve{d}_k}}^2\right)^\beta}\right]\right) = \left(\left[{\kappa^l_{\breve{d}_k}}^\beta, {\kappa^u_{\breve{d}_k}}^\beta\right], \sqrt{1 - \left(1 - \left[\delta^l_{\breve{d}_k}, \delta^u_{\breve{d}_k}\right]^2\right)^\beta}\right)$.

Zulqarnain et al. [45] proposed AOs for an array of IVPFHSNs denoted as $\mathcal{M}_{e_{ij}}$, where ω_i and ν_j are weights assigned to professionals and attributes, respectively, subject to certain conditions: $\omega_i > 0$ and $\sum_{i=1}^{n} \omega_i = 1$; $\nu_j > 0$ and $\sum_{j=1}^{m} \nu_j = 1$.

$$\mathrm{IVPFHSWA}\left(\mathcal{M}_{\breve{d}_{11}}, \mathcal{M}_{\breve{d}_{12}}, \ldots\ldots\ldots, \mathcal{M}_{\breve{d}_{nm}}\right) = \left(\sqrt{1 - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - \left[\kappa^l_{\breve{d}_{ij}}, \kappa^u_{\breve{d}_{ij}}\right]^2\right)^{\omega_i}\right)^{\nu_j}}, \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\left[\delta^l_{\breve{d}_{ij}}, \delta^u_{\breve{d}_{ij}}\right]\right)^{\omega_i}\right)^{\nu_j}\right) \tag{2.3}$$

$$\mathrm{IVPFHSWG}\left(\mathcal{M}_{\breve{d}_{11}}, \mathcal{M}_{\breve{d}_{12}}, \ldots\ldots\ldots, \mathcal{M}_{\breve{d}_{nm}}\right) = \left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\left[\kappa^l_{\breve{d}_{ij}}, \kappa^u_{\breve{d}_{ij}}\right]\right)^{\omega_i}\right)^{\nu_j}, \sqrt{1 - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - \left[\delta^l_{\breve{d}_{ij}}, \delta^u_{\breve{d}_{ij}}\right]^2\right)^{\omega_i}\right)^{\nu_j}}\right) \tag{2.4}$$

Upon analyzing the IVPFHSWA and IVPFHSWG operators, it becomes apparent that they produce undesirable outcomes in certain cases. To report these concerns, we propose the interaction AOs for IVPFHSNs.

3. Interaction aggregation operators for interval-valued Pythagorean fuzzy hypersoft sets

This section will present interaction operational laws for IVPFHSNs and propose the IVPFHSIWA and IVPFHSIWG operators based on these laws.

Definition 3.1. Let $\mathcal{M}_{\breve{d}_k} = \left(\left[\kappa^l_{\breve{d}_k}, \kappa^u_{\breve{d}_k}\right], \left[\delta^l_{\breve{d}_k}, \delta^u_{\breve{d}_k}\right]\right)$, $\mathcal{M}_{\breve{d}_{11}} = \left(\left[\kappa^l_{\breve{d}_{11}}, \kappa^u_{\breve{d}_{11}}\right], \left[\delta^l_{\breve{d}_{11}}, \delta^u_{\breve{d}_{11}}\right]\right)$, and $\mathcal{M}_{\breve{d}_{12}} = \left(\left[\kappa^l_{\breve{d}_{12}}, \kappa^u_{\breve{d}_{12}}\right], \left[\delta^l_{\breve{d}_{12}}, \delta^u_{\breve{d}_{12}}\right]\right)$ be three IVPFHSNs and $\beta > 0$, and by algebraic norms, interactional operational laws for IVPFHSNs can be defined as:

(1) $\mathcal{M}_{\breve{d}_{11}} \oplus \mathcal{M}_{\breve{d}_{12}} =$

$$\left(\begin{array}{c}\left[\sqrt{{\kappa^l_{\breve{d}_{11}}}^2 + {\kappa^l_{\breve{d}_{12}}}^2 - {\kappa^l_{\breve{d}_{11}}}^2{\kappa^l_{\breve{d}_{12}}}^2}, \sqrt{{\kappa^u_{\breve{d}_{11}}}^2 + {\kappa^u_{\breve{d}_{12}}}^2 - {\kappa^u_{\breve{d}_{11}}}^2{\kappa^u_{\breve{d}_{12}}}^2}\right], \\ \left[\sqrt{{\delta^l_{\breve{d}_{11}}}^2 + {\delta^l_{\breve{d}_{12}}}^2 - {\delta^l_{\breve{d}_{11}}}^2{\delta^l_{\breve{d}_{12}}}^2 - {\kappa^l_{\breve{d}_{11}}}^2{\delta^l_{\breve{d}_{12}}}^2 - {\delta^l_{\breve{d}_{11}}}^2{\kappa^l_{\breve{d}_{12}}}^2}, \sqrt{{\delta^u_{\breve{d}_{11}}}^2 + {\delta^u_{\breve{d}_{12}}}^2 - {\delta^u_{\breve{d}_{11}}}^2{\delta^u_{\breve{d}_{12}}}^2 - {\kappa^u_{\breve{d}_{11}}}^2{\delta^u_{\breve{d}_{12}}}^2 - {\delta^u_{\breve{d}_{11}}}^2{\kappa^u_{\breve{d}_{12}}}^2}\right]\end{array}\right)$$

(2) $\mathcal{M}_{\breve{d}_{11}} \otimes \mathcal{M}_{\breve{d}_{12}} =$

$$\left(\begin{array}{c}\left[\sqrt{{\kappa^l_{\breve{d}_{11}}}^2 + {\kappa^l_{\breve{d}_{12}}}^2 - {\kappa^l_{\breve{d}_{11}}}^2{\kappa^l_{\breve{d}_{12}}}^2 - {\kappa^l_{\breve{d}_{11}}}^2{\delta^l_{\breve{d}_{12}}}^2 - {\delta^l_{\breve{d}_{11}}}^2{\kappa^l_{\breve{d}_{12}}}^2}, \sqrt{{\kappa^u_{\breve{d}_{11}}}^2 + {\kappa^u_{\breve{d}_{12}}}^2 - {\kappa^u_{\breve{d}_{11}}}^2{\kappa^u_{\breve{d}_{12}}}^2 - {\kappa^u_{\breve{d}_{11}}}^2{\delta^u_{\breve{d}_{12}}}^2 - {\delta^u_{\breve{d}_{11}}}^2{\kappa^u_{\breve{d}_{12}}}^2}\right], \\ \left[\sqrt{{\delta^l_{\breve{d}_{11}}}^2 + {\delta^l_{\breve{d}_{12}}}^2 - {\delta^l_{\breve{d}_{11}}}^2{\delta^l_{\breve{d}_{12}}}^2}, \sqrt{{\delta^u_{\breve{d}_{11}}}^2 + {\delta^u_{\breve{d}_{12}}}^2 - {\delta^u_{\breve{d}_{11}}}^2{\delta^u_{\breve{d}_{12}}}^2}\right]\end{array}\right)$$

(3) $$\beta\mathcal{M}_{\breve{d}_k} = \left(\sqrt{1 - \left(1 - \left[\kappa^l_{\breve{d}_k}, \kappa^u_{\breve{d}_k}\right]^2\right)^{\beta}}, \sqrt{\left(1 - \left[\kappa^l_{\breve{d}_k}, \kappa^u_{\breve{d}_k}\right]^2\right)^{\beta} - \left(1 - \left(\left[\kappa^l_{\breve{d}_k}, \kappa^u_{\breve{d}_k}\right]^2 + \left[\delta^l_{\breve{d}_k}, \delta^u_{\breve{d}_k}\right]^2\right)\right)^{\beta}}\right)$$

(4) $${\mathcal{M}_{\breve{d}_k}}^{\beta} = \left(\sqrt{\left(1 - \left[\delta^l_{\breve{d}_k}, \delta^u_{\breve{d}_k}\right]^2\right)^{\beta} - \left(1 - \left[\left(\left[\kappa^l_{\breve{d}_k}, \kappa^u_{\breve{d}_k}\right]^2 + \left[\delta^l_{\breve{d}_k}, \delta^u_{\breve{d}_k}\right]^2\right)\right]\right)^{\beta}}, \sqrt{1 - \left(1 - \left[\delta^l_{\breve{d}_k}, \delta^u_{\breve{d}_k}\right]^2\right)^{\beta}}\right).$$

We will present the average interactional aggregation operator with some important results and properties for IVPFHSS using the above-presented interactional operational laws for IVPFHSNs in the following.

Definition 3.2. Let $\mathcal{M}_{\breve{d}_k} = \left(\left[\kappa^l_{\breve{d}_k}, \kappa^u_{\breve{d}_k}\right], \left[\delta^l_{\breve{d}_k}, \delta^u_{\breve{d}_k}\right]\right)$ be a collection of IVPFHSNs, and ω_i and ν_j be the weights for specialists and multi sub-attributes, disparately, with certain circumstances $\omega_i > 0$, $\sum_{i=1}^{n} \omega_i = 1$; $\nu_j > 0$, $\sum_{j=1}^{m} \nu_j = 1$. Then, the IVPFHSIWA operator is defined as IVPFHSIWA: $\Psi^n \longrightarrow \Psi$

$$\text{IVPFHSIWA}(\mathcal{M}_{\breve{d}_{11}}, \mathcal{M}_{\breve{d}_{12}}, \dots \dots \dots, \mathcal{M}_{\breve{d}_{nm}}) = \oplus_{j=1}^{m} \nu_j \left(\oplus_{i=1}^{n} \omega_i \mathcal{M}_{\breve{d}_{ij}}\right).$$

Theorem 3.1. Let $\mathcal{M}_{\breve{d}_{ij}} = \left(\left[\kappa^l_{\breve{d}_{ij}}, \kappa^u_{\breve{d}_{ij}}\right], \left[\delta^l_{\breve{d}_{ij}}, \delta^u_{\breve{d}_{ij}}\right]\right)$ be a collection of IVPFHSNs, and the aggregated value is also an IVPFHSN, such as

$$\text{IVPFHSIWA}(\mathcal{M}_{\breve{d}_{11}}, \mathcal{M}_{\breve{d}_{12}}, \dots \dots \dots, \mathcal{M}_{\breve{d}_{nm}}) =$$

$$\left(\sqrt{1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\left[\kappa_{\breve{d}_{ij}}^{l},\kappa_{\breve{d}_{ij}}^{u}\right]^{2}\right)^{\omega_i}\right)^{\nu_j}},\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\left[\kappa_{\breve{d}_{ij}}^{l},\kappa_{\breve{d}_{ij}}^{u}\right]^{2}\right)^{\omega_i}\right)^{\nu_j}-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\left(\left[\kappa_{\breve{d}_{ij}}^{l},\kappa_{\breve{d}_{ij}}^{u}\right]^{2}+\left[\delta_{\breve{d}_{ij}}^{l},\delta_{\breve{d}_{ij}}^{u}\right]^{2}\right)\right)^{\omega_i}\right)^{\nu_j}}\right).$$

ω_i and ν_j be the weights for experts and parameters disparately, with assumed conditions $\omega_i > 0,\ \sum_{i=1}^{n}\omega_i = 1;\ \nu_j > 0,\ \sum_{j=1}^{m}\nu_j = 1$.

Proof. We shall prove the IVPFHSIWA operator by employing the principle of mathematical induction:
For $n = 1$, we get $\omega_1 = 1$. Then, we have

$$\text{IVPFHSIWA}\left(\mathcal{M}_{\breve{d}_{11}},\mathcal{M}_{\breve{d}_{12}},\ldots\ldots\ldots,\mathcal{M}_{\breve{d}_{1m}}\right)=\oplus_{j=1}^{m}\nu_j\mathcal{M}_{\breve{d}_{1j}}=$$

$$\left(\sqrt{1-\prod_{j=1}^{m}\left(1-\left[\kappa_{\breve{d}_{1j}}^{l},\kappa_{\breve{d}_{1j}}^{u}\right]^{2}\right)^{\nu_j}},\sqrt{\prod_{j=1}^{m}\left(1-\left[\kappa_{\breve{d}_{1j}}^{l},\kappa_{\breve{d}_{1j}}^{u}\right]^{2}\right)^{\nu_j}-\prod_{j=1}^{m}\left(1-\left(\left[\kappa_{\breve{d}_{1j}}^{l},\kappa_{\breve{d}_{1j}}^{u}\right]^{2}+\left[\delta_{\breve{d}_{1j}}^{l},\delta_{\breve{d}_{1j}}^{u}\right]^{2}\right)\right)^{\nu_j}}\right)=$$

$$\left(\sqrt{1-\prod_{j=1}^{m}\left(\prod_{i=1}^{1}\left(1-\left[\kappa_{\breve{d}_{ij}}^{l},\kappa_{\breve{d}_{ij}}^{u}\right]^{2}\right)^{\omega_i}\right)^{\nu_j}},\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{1}\left(1-\left[\kappa_{\breve{d}_{ij}}^{l},\kappa_{\breve{d}_{ij}}^{u}\right]^{2}\right)^{\omega_i}\right)^{\nu_j}-\prod_{j=1}^{m}\left(\prod_{i=1}^{1}\left(1-\left(\left[\kappa_{\breve{d}_{ij}}^{l},\kappa_{\breve{d}_{ij}}^{u}\right]^{2}+\left[\delta_{\breve{d}_{ij}}^{l},\delta_{\breve{d}_{ij}}^{u}\right]^{2}\right)\right)^{\omega_i}\right)^{\nu_j}}\right).$$

For $m = 1$, we get $\nu_1 = 1$. Then, we have

$$\text{IVPFHSIWA}\left(\mathcal{M}_{\breve{d}_{11}},\mathcal{M}_{\breve{d}_{21}},\ldots\ldots\ldots,\mathcal{M}_{\breve{d}_{n1}}\right)=\oplus_{i=1}^{n}\omega_i\mathcal{M}_{\breve{d}_{i1}}$$

$$=\left(\sqrt{1-\prod_{i=1}^{n}\left(1-\left[\kappa_{\breve{d}_{i1}}^{l},\kappa_{\breve{d}_{i1}}^{u}\right]^{2}\right)^{\omega_i}},\sqrt{\prod_{i=1}^{n}\left(1-\left[\kappa_{\breve{d}_{i1}}^{l},\kappa_{\breve{d}_{i1}}^{u}\right]^{2}\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\left(\left[\kappa_{\breve{d}_{i1}}^{l},\kappa_{\breve{d}_{i1}}^{u}\right]^{2}+\left[\delta_{\breve{d}_{i1}}^{l},\delta_{\breve{d}_{i1}}^{u}\right]^{2}\right)\right)^{\omega_i}}\right)$$

$$=\left(\sqrt{1-\prod_{j=1}^{1}\left(\prod_{i=1}^{n}\left(1-\left[\kappa_{\breve{d}_{ij}}^{l},\kappa_{\breve{d}_{ij}}^{u}\right]^{2}\right)^{\omega_i}\right)^{\nu_j}},\sqrt{\prod_{j=1}^{1}\left(\prod_{i=1}^{n}\left(1-\left[\kappa_{\breve{d}_{ij}}^{l},\kappa_{\breve{d}_{ij}}^{u}\right]^{2}\right)^{\omega_i}\right)^{\nu_j}-\prod_{j=1}^{1}\left(\prod_{i=1}^{n}\left(1-\left(\left[\kappa_{\breve{d}_{ij}}^{l},\kappa_{\breve{d}_{ij}}^{u}\right]^{2}+\left[\delta_{\breve{d}_{ij}}^{l},\delta_{\breve{d}_{ij}}^{u}\right]^{2}\right)\right)^{\omega_i}\right)^{\nu_j}}\right).$$

This demonstrates that the overhead statement holds for $n = 1$ and $m = 1$. Suppose it also holds for $m = \alpha_1 + 1, n = \alpha_2$ and $m = \alpha_1, n = \alpha_2 + 1$, such as

$$
= \left(\frac{\bigoplus_{j=1}^{\alpha_1+1} \nu_j \left(\bigoplus_{i=1}^{\alpha_2} \omega_i \mathcal{M}_{\breve{d}_{ij}} \right)}{} \sqrt{1 - \prod_{j=1}^{\alpha_1+1} \left(\prod_{i=1}^{\alpha_2} \left(1 - \left[\kappa_{\breve{d}_{ij}}^{l}, \kappa_{\breve{d}_{ij}}^{u} \right]^2 \right)^{\omega_i} \right)^{\nu_j}}, \sqrt{\prod_{j=1}^{\alpha_1+1} \left(\prod_{i=1}^{\alpha_2} \left(1 - \left[\kappa_{\breve{d}_{ij}}^{l}, \kappa_{\breve{d}_{ij}}^{u} \right]^2 \right)^{\omega_i} \right)^{\nu_j} - \prod_{j=1}^{\alpha_1+1} \left(\prod_{i=1}^{\alpha_2} \left(1 - \left(\left[\kappa_{\breve{d}_{ij}}^{l}, \kappa_{\breve{d}_{ij}}^{u} \right]^2 + \left[\delta_{\breve{d}_{ij}}^{l}, \delta_{\breve{d}_{ij}}^{u} \right]^2 \right) \right)^{\omega_i} \right)^{\nu_j}} \right)
$$

$$
= \left(\frac{\bigoplus_{j=1}^{\alpha_1} \nu_j \left(\bigoplus_{i=1}^{\alpha_2+1} \omega_i \mathcal{M}_{\breve{d}_{ij}} \right)}{} \sqrt{1 - \prod_{j=1}^{\alpha_1} \left(\prod_{i=1}^{\alpha_2+1} \left(1 - \left[\kappa_{\breve{d}_{ij}}^{l}, \kappa_{\breve{d}_{ij}}^{u} \right]^2 \right)^{\omega_i} \right)^{\nu_j}}, \sqrt{\prod_{j=1}^{\alpha_1} \left(\prod_{i=1}^{\alpha_2+1} \left(1 - \left[\kappa_{\breve{d}_{ij}}^{l}, \kappa_{\breve{d}_{ij}}^{u} \right]^2 \right)^{\omega_i} \right)^{\nu_j} - \prod_{j=1}^{\alpha_1} \left(\prod_{i=1}^{\alpha_2+1} \left(1 - \left(\left[\kappa_{\breve{d}_{ij}}^{l}, \kappa_{\breve{d}_{ij}}^{u} \right]^2 + \left[\delta_{\breve{d}_{ij}}^{l}, \delta_{\breve{d}_{ij}}^{u} \right]^2 \right) \right)^{\omega_i} \right)^{\nu_j}} \right).
$$

For $m = \alpha_1 + 1$ and $n = \alpha_2 + 1$, we have

$$
\bigoplus_{j=1}^{\alpha_1+1} \nu_j \left(\bigoplus_{i=1}^{\alpha_2+1} \omega_i \mathcal{M}_{\breve{d}_{ij}} \right) = \bigoplus_{j=1}^{\alpha_1+1} \nu_j \left(\bigoplus_{i=1}^{\alpha_2} \omega_i \mathcal{M}_{\breve{d}_{ij}} \oplus \omega_{\alpha_2+1} \mathcal{M}_{\breve{d}_{(\alpha_2+1)j}} \right)
$$

$$
= \bigoplus_{j=1}^{\alpha_1+1} \bigoplus_{i=1}^{\alpha_2} \nu_j \omega_i \mathcal{M}_{\breve{d}_{ij}} \bigoplus_{j=1}^{\alpha_1+1} \nu_j \omega_{\alpha_2+1} \mathcal{M}_{\breve{d}_{(\alpha_2+1)j}}
$$

$$
= \left(\sqrt{1 - \prod_{j=1}^{\alpha_1+1} \left(\prod_{i=1}^{\alpha_2} \left(1 - \left[\kappa_{\breve{d}_{ij}}^{l}, \kappa_{\breve{d}_{ij}}^{u} \right]^2 \right)^{\omega_i} \right)^{\nu_j}} \oplus \sqrt{1 - \prod_{j=1}^{\alpha_1+1} \left(\left(1 - \left[\kappa_{\breve{d}_{(\alpha_2+1)j}}^{l}, \kappa_{\breve{d}_{(\alpha_2+1)j}}^{u} \right]^2 \right)^{\omega_{\alpha_2+1}} \right)^{\nu_j}}, \sqrt{\prod_{j=1}^{\alpha_1+1} \left(\prod_{i=1}^{\alpha_2} \left(1 - \left[\kappa_{\breve{d}_{ij}}^{l}, \kappa_{\breve{d}_{ij}}^{u} \right]^2 \right)^{\omega_i} \right)^{\nu_j} - \prod_{j=1}^{\alpha_1+1} \left(\prod_{i=1}^{\alpha_2} \left(1 - \left(\left[\kappa_{\breve{d}_{ij}}^{l}, \kappa_{\breve{d}_{ij}}^{u} \right]^2 + \left[\delta_{\breve{d}_{ij}}^{l}, \delta_{\breve{d}_{ij}}^{u} \right]^2 \right) \right)^{\omega_i} \right)^{\nu_j}} \oplus \sqrt{\prod_{j=1}^{\alpha_1+1} \left(\left(1 - \left[\kappa_{\breve{d}_{(\alpha_2+1)j}}^{l}, \kappa_{\breve{d}_{(\alpha_2+1)j}}^{u} \right]^2 \right)^{\omega_{\alpha_2+1}} \right)^{\nu_j} - \prod_{j=1}^{\alpha_1+1} \left((1 - \left(\left[\kappa_{\breve{d}_{(\alpha_2+1)j}}^{l}, \kappa_{\breve{d}_{(\alpha_2+1)j}}^{u} \right]^2 + \left[\delta_{\breve{d}_{(\alpha_2+1)j}}^{l}, \delta_{\breve{d}_{(\alpha_2+1)j}}^{u} \right]^2 \right)^{\omega_{\alpha_2+1}} \right)^{\nu_j}} \right)
$$

$$
= \left(\sqrt{1 - \prod_{j=1}^{\alpha_1+1} \left(\prod_{i=1}^{\alpha_2+1} \left(1 - \left[\kappa_{\breve{d}_{ij}}^{l}, \kappa_{\breve{d}_{ij}}^{u} \right]^2 \right)^{\omega_i} \right)^{\nu_j}}, \sqrt{\prod_{j=1}^{\alpha_1+1} \left(\prod_{i=1}^{\alpha_2+1} \left(1 - \left[\kappa_{\breve{d}_{ij}}^{l}, \kappa_{\breve{d}_{ij}}^{u} \right]^2 \right)^{\omega_i} \right)^{\nu_j} - \prod_{j=1}^{\alpha_1+1} \left(\prod_{i=1}^{\alpha_2+1} \left(1 - \left(\left[\kappa_{\breve{d}_{ij}}^{l}, \kappa_{\breve{d}_{ij}}^{u} \right]^2 + \left[\delta_{\breve{d}_{ij}}^{l}, \delta_{\breve{d}_{ij}}^{u} \right]^2 \right) \right)^{\omega_i} \right)^{\nu_j}} \right).
$$

Therefore, it holds for $m = \alpha_1 + 1$ and $n = \alpha_2 + 1$. So, we can say that it holds $\forall$ m, n.

Example 3.1. Let $\mathcal{R}=\{\mathcal{R}_1,\mathcal{R}_2,\mathcal{R}_3,\mathcal{R}_4\}$ be a team of experts with the assumed weights $\omega_i=(0.1,0.2,0.4,0.3)^T$. The team of specialists designates the good looks of a community under contemplated features $A=\{e_1=\text{Parkland},\ e_2=\text{safety arrangement}\}$ with their compatible sub-parameters, Parkland $=e_1=\{e_{11}=\text{with grass}, e_{12}=\text{without grass}\}$ Safety arrangement $=e_2=\{e_{21}=\text{guards}, e_{22}=\text{cameras}\}$. Let $A=e_1\times e_2$ be a set of sub-attributes

$$A=e_1\times e_2=\{e_{11},e_{12}\}\times\{e_{21},e_{22}\}=\{(e_{11},e_{21}),(e_{11},e_{22}),(e_{12},e_{21}),(e_{12},e_{22})\}.$$

$A=\{\breve{d}_1,\breve{d}_2,\breve{d}_3,\breve{d}_4\}$ be a collection of multi sub-parameters with their weights $\nu_j=(0.3,0.1,0.2,0.4)^T$. The assessment standards for each substitute in terms of IVPFHSN $(\mathcal{M},A)=\left(\left[\kappa^l_{\breve{d}_{ij}},\kappa^u_{\breve{d}_{ij}}\right],\left[\delta^l_{\breve{d}_{ij}},\delta^u_{\breve{d}_{ij}}\right]\right)_{3\times 4}$ given as:

$$(\mathcal{M},A)=\begin{bmatrix}([0.4,0.5],[0.2,0.5]) & ([0.5,0.6],[0.7,0.8]) & ([0.4,0.6],[0.2,0.5]) & ([0.2,0.4],[0.2,0.6])\\ ([0.2,0.7],[0.2,0.6]) & ([0.4,0.5],[0.1,0.6]) & ([0.2,0.3],[0.4,0.8]) & ([0.2,0.5],[0.4,0.7])\\ ([0.3,0.5],[0.1,0.4]) & ([0.2,0.7],[0.4,0.6]) & ([0.4,0.7],[0.3,0.7]) & ([0.5,0.7],[0.2,0.4])\\ ([0.4,0.6],[0.1,0.7]) & ([0.3,0.7],[0.4,0.5]) & ([0.3,0.6],[0.3,0.5]) & ([0.3,0.6],[0.3,0.5])\end{bmatrix}$$

By using the above theorem, we have

$$\text{IVPFHSIWA}\left(\mathcal{M}_{\breve{d}_{11}},\mathcal{M}_{\breve{d}_{12}},\dots\dots\dots,\mathcal{M}_{\breve{d}_{34}}\right)$$

$$=\left(\frac{\sqrt{1-\prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(1-\left[\kappa^l_{\breve{d}_{ij}},\kappa^u_{\breve{d}_{ij}}\right]^2\right)^{\omega_i}\right)^{\nu_j}},}{\sqrt{\prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(1-\left[\kappa^l_{\breve{d}_{ij}},\kappa^u_{\breve{d}_{ij}}\right]^2\right)^{\omega_i}\right)^{\nu_j}-\prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(1-\left(\left[\kappa^l_{\breve{d}_{ij}},\kappa^u_{\breve{d}_{ij}}\right]^2+\left[\delta^l_{\breve{d}_{ij}},\delta^u_{\breve{d}_{ij}}\right]^2\right)\right)^{\omega_i}\right)^{\nu_j}}}\right)$$

$$=\left(\frac{\sqrt{1-\left(\begin{matrix}\left\{\begin{matrix}[0.75,0.84]^{0.1}[0.51,0.96]^{0.2}\\ [0.75,0.91]^{0.4}[0.64,0.84]^{0.3}\end{matrix}\right\}^{0.3}\left\{\begin{matrix}[0.64,0.75]^{0.1}[0.75,0.84]^{0.2}\\ [0.51,0.96]^{0.4}[0.51,0.91]^{0.3}\end{matrix}\right\}^{0.1}\\ \left\{\begin{matrix}[0.64,0.84]^{0.1}[0.91,0.96]^{0.2}\\ [0.51,0.84]^{0.4}[0.64,0.91]^{0.3}\end{matrix}\right\}^{0.2}\left\{\begin{matrix}[0.84,0.96]^{0.1}[0.75,0.96]^{0.2}\\ [0.51,0.75]^{0.4}[0.64,0.91]^{0.3}\end{matrix}\right\}^{0.4}\end{matrix}\right)},}{\sqrt{\begin{matrix}\left(\begin{matrix}\left\{\begin{matrix}[0.75,0.84]^{0.1}[0.51,0.96]^{0.2}\\ [0.75,0.91]^{0.4}[0.64,0.84]^{0.3}\end{matrix}\right\}^{0.3}\left\{\begin{matrix}[0.64,0.75]^{0.1}[0.75,0.84]^{0.2}\\ [0.51,0.96]^{0.4}[0.51,0.91]^{0.3}\end{matrix}\right\}^{0.1}\\ \left\{\begin{matrix}[0.64,0.84]^{0.1}[0.91,0.96]^{0.2}\\ [0.51,0.84]^{0.4}[0.64,0.91]^{0.3}\end{matrix}\right\}^{0.2}\left\{\begin{matrix}[0.84,0.96]^{0.1}[0.75,0.96]^{0.2}\\ [0.51,0.75]^{0.4}[0.64,0.91]^{0.3}\end{matrix}\right\}^{0.4}\end{matrix}\right)-\\ \left(\begin{matrix}\left\{\begin{matrix}(1-[0.2,0.5])^{0.1}(1-[0.08,0.85])^{0.2}\\ (1-[0.1,0.41])^{0.4}(1-[0.25,0.85])^{0.3}\end{matrix}\right\}^{0.3}\left\{\begin{matrix}(1-[0.74,1])^{0.1}(1-[0.17,0.61])^{0.2}\\ (1-[0.2,0.85])^{0.4}(1-[0.25,0.74])^{0.3}\end{matrix}\right\}^{0.1}\\ \left\{\begin{matrix}(1-[0.2,0.61])^{0.1}(1-[0.2,0.73])^{0.2}\\ (1-[0.25,0.98])^{0.4}(1-[0.18,0.61])^{0.3}\end{matrix}\right\}^{0.2}\left\{\begin{matrix}(1-[0.08,0.52])^{0.1}(1-[0.2,0.74])^{0.2}\\ (1-[0.29,0.65])^{0.4}(1-[0.18,0.61])^{0.3}\end{matrix}\right\}^{0.4}\end{matrix}\right)\end{matrix}}}\right)$$

$$
=\left(\begin{array}{c}
\sqrt{1-\left(\begin{array}{cc}
\begin{Bmatrix}[0.9716,0.9827][0.8740,0.9919]\\ [0.913,0.9630][0.8747,0.9490]\end{Bmatrix}^{0.3} & \begin{Bmatrix}[0.9564,0.9716][0.9441,0.9657]\\ [0.7639,0.9838][0.8171,0.9721]\end{Bmatrix}^{0.1}\\
\begin{Bmatrix}[0.9564,0.9827][0.9813,0.9919]\\ [0.7639,0.9326][0.8747,0.9721]\end{Bmatrix}^{0.2} & \begin{Bmatrix}[0.9827,0.9959][0.9441,0.9919]\\ [0.7639,0.8913][0.8747,0.9721]\end{Bmatrix}^{0.4}
\end{array}\right)},\\
\sqrt{\begin{array}{c}\left(\begin{array}{cc}
\begin{Bmatrix}[0.9716,0.9827][0.8740,0.9919]\\ [0.8670,0.9630][0.8747,0.9490]\end{Bmatrix}^{0.3} & \begin{Bmatrix}[0.9564,0.9716][0.9441,0.9657]\\ [0.7639,0.9838][0.8187,0.9721]\end{Bmatrix}^{0.1}\\
\begin{Bmatrix}[0.9564,0.9827][0.9813,0.9919]\\ [0.7639,0.9326][0.8747,0.9721]\end{Bmatrix}^{0.2} & \begin{Bmatrix}[0.9827,0.9959][0.9441,0.9919]\\ [0.7639,0.8913][0.8747,0.9721]\end{Bmatrix}^{0.4}
\end{array}\right)-\\
\left(\begin{array}{cc}
\begin{Bmatrix}[0.9330,0.9779][0.6843,0.9835]\\ [0.8097,0.9587][0.5660,0.9173]\end{Bmatrix}^{0.3} & \begin{Bmatrix}[0,0.8740][0.8283,0.9634]\\ [0.4682,0.9146][0.6676,0.9173]\end{Bmatrix}^{0.1}\\
\begin{Bmatrix}[0.9101,0.9779][0.7696,0.9564]\\ [0.2091,0.8913][0.7539,0.9422]\end{Bmatrix}^{0.2} & \begin{Bmatrix}[0.9292,0.9917][0.7638,0.9564]\\ [0.6571,0.8720][0.8539,0.9422]\end{Bmatrix}^{0.4}
\end{array}\right)\end{array}}
\end{array}\right)
$$

$= ([0.3523, 0.6102], [0.7923, 0.2895]).$

3.1. Properties of IVPFHSIWA operator

3.1.1. (Idempotency) If $\mathcal{M}_{\breve{d}_{ij}} = \mathcal{M}_{\breve{d}_k} = \left(\left[\kappa^l_{\breve{d}_{ij}}, \kappa^u_{\breve{d}_{ij}}\right], \left[\delta^l_{\breve{d}_{ij}}, \delta^u_{\breve{d}_{ij}}\right]\right)$ $\forall\, i, j$, then,

$$\text{IVPFHSIWA}\,(\mathcal{M}_{\breve{d}_{11}}, \mathcal{M}_{\breve{d}_{12}}, \ldots\ldots\ldots, \mathcal{M}_{\breve{d}_{nm}}) = \mathcal{M}_{\breve{d}_k}.$$

Proof. As we know that all $\mathcal{M}_{\breve{d}_{ij}} = \mathcal{M}_{\breve{d}_k} = \left(\left[\kappa^l_{\breve{d}_{ij}}, \kappa^u_{\breve{d}_{ij}}\right], \left[\delta^l_{\breve{d}_{ij}}, \delta^u_{\breve{d}_{ij}}\right]\right)$. Then, we have

IVPFHSIWA $(\mathcal{M}_{\breve{d}_{11}}, \mathcal{M}_{\breve{d}_{12}}, \ldots\ldots\ldots, \mathcal{M}_{\breve{d}_{nm}})$

$$
=\left(\begin{array}{c}
\sqrt{1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\left[\kappa^l_{\breve{d}_{ij}}, \kappa^u_{\breve{d}_{ij}}\right]^2\right)^{\omega_i}\right)^{\nu_j}},\\
\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\left[\kappa^l_{\breve{d}_{ij}}, \kappa^u_{\breve{d}_{ij}}\right]^2\right)^{\omega_i}\right)^{\nu_j}-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\left(\left[\kappa^l_{\breve{d}_{ij}}, \kappa^u_{\breve{d}_{ij}}\right]^2+\left[\delta^l_{\breve{d}_{ij}}, \delta^u_{\breve{d}_{ij}}\right]^2\right)\right)^{\omega_i}\right)^{\nu_j}}
\end{array}\right)
$$

$$
=\left(\begin{array}{c}
\sqrt{1-\left(\left(1-\left[\kappa^l_{\breve{d}_{ij}}, \kappa^u_{\breve{d}_{ij}}\right]^2\right)^{\sum_{i=1}^{n}\omega_i}\right)^{\sum_{j=1}^{m}\nu_j}},\\
\sqrt{\left(\left(1-\left[\kappa^l_{\breve{d}_{ij}}, \kappa^u_{\breve{d}_{ij}}\right]^2\right)^{\sum_{i=1}^{n}\omega_i}\right)^{\sum_{j=1}^{m}\nu_j}-\left(\left(1-\left(\left[\kappa^l_{\breve{d}_{ij}}, \kappa^u_{\breve{d}_{ij}}\right]^2+\left[\delta^l_{\breve{d}_{ij}}, \delta^u_{\breve{d}_{ij}}\right]^2\right)\right)^{\sum_{i=1}^{n}\omega_i}\right)^{\sum_{j=1}^{m}\nu_j}}
\end{array}\right).
$$

As $\sum_{j=1}^{m}\nu_j = 1$ and $\sum_{i=1}^{n}\omega_i = 1$, then

$$
=\left(\sqrt{1-\left(1-\left[\kappa^l_{\breve{d}_{ij}}, \kappa^u_{\breve{d}_{ij}}\right]^2\right)}, \sqrt{1-\left[\kappa^l_{\breve{d}_{ij}}, \kappa^u_{\breve{d}_{ij}}\right]^2-\left(1-\left(\left[\kappa^l_{\breve{d}_{ij}}, \kappa^u_{\breve{d}_{ij}}\right]^2+\left[\delta^l_{\breve{d}_{ij}}, \delta^u_{\breve{d}_{ij}}\right]^2\right)\right)}\right)
$$

$= \left(\left[\kappa^{l}_{\breve{d}_{ij}}, \kappa^{u}_{\breve{d}_{ij}}\right], \left[\delta^{l}_{\breve{d}_{ij}}, \delta^{u}_{\breve{d}_{ij}}\right]\right)$

$= \mathcal{M}_{\breve{d}_k}$.

3.1.2. (Boundedness) Let $\mathcal{M}_{\breve{d}_{ij}}$ be a collection of IVPFHSNs where $\mathcal{M}^{-}_{\breve{d}_{ij}} = \left(\min_j \min_i \left\{\left[\kappa^{l}_{\breve{d}_{ij}}, \kappa^{u}_{\breve{d}_{ij}}\right]\right\}, \max_j \max_i \left\{\left[\delta^{l}_{\breve{d}_{ij}}, \delta^{u}_{\breve{d}_{ij}}\right]\right\}\right)$ and $\mathcal{M}^{+}_{\breve{d}_{ij}} = \left(\max_j \max_i \left\{\left[\kappa^{l}_{\breve{d}_{ij}}, \kappa^{u}_{\breve{d}_{ij}}\right]\right\}, \min_j \min_i \left\{\left[\delta^{l}_{\breve{d}_{ij}}, \delta^{u}_{\breve{d}_{ij}}\right]\right\}\right)$.
Then

$$\mathcal{M}^{-}_{\breve{d}_{ij}} \leq \mathrm{IVPFHSIWA}\left(\mathcal{M}_{\breve{d}_{11}}, \mathcal{M}_{\breve{d}_{12}}, \dots \dots \dots, \mathcal{M}_{\breve{d}_{nm}}\right) \leq \mathcal{M}^{+}_{\breve{d}_{ij}}.$$

Proof. As we know that $\mathcal{M}_{\breve{d}_{ij}} = \left\langle\left[\kappa^{l}_{\breve{d}_{ij}}, \kappa^{u}_{\breve{d}_{ij}}\right], \left[\delta^{l}_{\breve{d}_{ij}}, \delta^{u}_{\breve{d}_{ij}}\right]\right\rangle$ be an IVPFHSN, then

$$\min_j \min_i \left\{\left[\kappa^{l}_{\breve{d}_{ij}}, \kappa^{u}_{\breve{d}_{ij}}\right]^2\right\} \leq \left[\kappa^{l}_{\breve{d}_{ij}}, \kappa^{u}_{\breve{d}_{ij}}\right]^2 \leq \max_j \max_i \left\{\left[\kappa^{l}_{\breve{d}_{ij}}, \kappa^{u}_{\breve{d}_{ij}}\right]^2\right\}$$

$$\Rightarrow 1 - \max_j \max_i \left\{\left[\kappa^{l}_{\breve{d}_{ij}}, \kappa^{u}_{\breve{d}_{ij}}\right]^2\right\} \leq 1 - \left[\kappa^{l}_{\breve{d}_{ij}}, \kappa^{u}_{\breve{d}_{ij}}\right]^2 \leq 1 - \min_j \min_i \left\{\left[\kappa^{l}_{\breve{d}_{ij}}, \kappa^{u}_{\breve{d}_{ij}}\right]^2\right\}$$

$$\Leftrightarrow \left(1 - \max_j \max_i \left\{\left[\kappa^{l}_{\breve{d}_{ij}}, \kappa^{u}_{\breve{d}_{ij}}\right]^2\right\}\right)^{\omega_i} \leq \left(1 - \left[\kappa^{l}_{\breve{d}_{ij}}, \kappa^{u}_{\breve{d}_{ij}}\right]^2\right)^{\omega_i} \leq \left(1 - \min_j \min_i \left\{\left[\kappa^{l}_{\breve{d}_{ij}}, \kappa^{u}_{\breve{d}_{ij}}\right]^2\right\}\right)^{\omega_i}$$

$$\Leftrightarrow \left(1 - \max_j \max_i \left\{\left[\kappa^{l}_{\breve{d}_{ij}}, \kappa^{u}_{\breve{d}_{ij}}\right]^2\right\}\right)^{\sum_{i=1}^{n}\omega_i} \leq \prod_{i=1}^{n}\left(1 - \left[\kappa^{l}_{\breve{d}_{ij}}, \kappa^{u}_{\breve{d}_{ij}}\right]^2\right)^{\omega_i}$$
$$\leq \left(1 - \min_j \min_i \left\{\left[\kappa^{l}_{\breve{d}_{ij}}, \kappa^{u}_{\breve{d}_{ij}}\right]^2\right\}\right)^{\sum_{i=1}^{n}\omega_i}$$

$$\Leftrightarrow \left(1 - \max_j \max_i \left\{\left[\kappa^{l}_{\breve{d}_{ij}}, \kappa^{u}_{\breve{d}_{ij}}\right]^2\right\}\right)^{\sum_{j=1}^{n}\nu_j} \leq \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - \left[\kappa^{l}_{\breve{d}_{ij}}, \kappa^{u}_{\breve{d}_{ij}}\right]^2\right)^{\omega_i}\right)^{\nu_j}$$
$$\leq \left(1 - \min_j \min_i \left\{\left[\kappa^{l}_{\breve{d}_{ij}}, \kappa^{u}_{\breve{d}_{ij}}\right]^2\right\}\right)^{\sum_{j=1}^{n}\nu_j}$$

$$\Leftrightarrow 1 - \max_j \max_i \left\{\left[\kappa^{l}_{\breve{d}_{ij}}, \kappa^{u}_{\breve{d}_{ij}}\right]^2\right\} \leq \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - \left[\kappa^{l}_{\breve{d}_{ij}}, \kappa^{u}_{\breve{d}_{ij}}\right]^2\right)^{\omega_i}\right)^{\nu_j} \leq 1 - \min_j \min_i \left\{\left[\kappa^{l}_{\breve{d}_{ij}}, \kappa^{u}_{\breve{d}_{ij}}\right]^2\right\}$$

$$\Leftrightarrow \min_j \min_i \left\{\left[\kappa^{l}_{\breve{d}_{ij}}, \kappa^{u}_{\breve{d}_{ij}}\right]^2\right\} \leq 1 - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - \left[\kappa^{l}_{\breve{d}_{ij}}, \kappa^{u}_{\breve{d}_{ij}}\right]^2\right)^{\omega_i}\right)^{\nu_j} \leq \max_j \max_i \left\{\left[\kappa^{l}_{\breve{d}_{ij}}, \kappa^{u}_{\breve{d}_{ij}}\right]^2\right\}. \tag{3.1}$$

$$\Leftrightarrow \min_j \min_i \left\{\left[\kappa^{l}_{\breve{d}_{ij}}, \kappa^{u}_{\breve{d}_{ij}}\right]\right\} \leq \sqrt{1 - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - \left[\kappa^{l}_{\breve{d}_{ij}}, \kappa^{u}_{\breve{d}_{ij}}\right]^2\right)^{\omega_i}\right)^{\nu_j}} \leq \max_j \max_i \left\{\left[\kappa^{l}_{\breve{d}_{ij}}, \kappa^{u}_{\breve{d}_{ij}}\right]\right\}. \tag{3.2}$$

Similarly,

$$\min_j \min_i \left\{\left[\kappa^{l}_{\breve{d}_{ij}}, \kappa^{u}_{\breve{d}_{ij}}\right]^2 + \left[\delta^{l}_{\breve{d}_{ij}}, \delta^{u}_{\breve{d}_{ij}}\right]^2\right\} \leq \left[\kappa^{l}_{\breve{d}_{ij}}, \kappa^{u}_{\breve{d}_{ij}}\right]^2 + \left[\delta^{l}_{\breve{d}_{ij}}, \delta^{u}_{\breve{d}_{ij}}\right]^2$$

$$\leq \max_j \max_i \left\{\left[\kappa^{l}_{\breve{d}_{ij}}, \kappa^{u}_{\breve{d}_{ij}}\right]^2 + \left[\delta^{l}_{\breve{d}_{ij}}, \delta^{u}_{\breve{d}_{ij}}\right]^2\right\}$$

$$\Rightarrow 1 - \max_j \max_i \left\{\left[\kappa^{l}_{\breve{d}_{ij}}, \kappa^{u}_{\breve{d}_{ij}}\right]^2 + \left[\delta^{l}_{\breve{d}_{ij}}, \delta^{u}_{\breve{d}_{ij}}\right]^2\right\} \leq 1 - \left(\left[\kappa^{l}_{\breve{d}_{ij}}, \kappa^{u}_{\breve{d}_{ij}}\right]^2 + \left[\delta^{l}_{\breve{d}_{ij}}, \delta^{u}_{ij}\right]^2\right)$$

$$\leq 1 - \min_j \min_i \left\{\left[\kappa^{l}_{\breve{d}_{ij}}, \kappa^{u}_{\breve{d}_{ij}}\right]^2 + \left[\delta^{l}_{\breve{d}_{ij}}, \delta^{u}_{\breve{d}_{ij}}\right]^2\right\}$$

$$\Leftrightarrow \left(1 - \max_j \max_i \left\{\left[\kappa^{l}_{\breve{d}_{ij}}, \kappa^{u}_{\breve{d}_{ij}}\right]^2 + \left[\delta^{l}_{\breve{d}_{ij}}, \delta^{u}_{\breve{d}_{ij}}\right]^2\right\}\right)^{\omega_i} \leq \left(1 - \left(\left[\kappa^{l}_{\breve{d}_{ij}}, \kappa^{u}_{\breve{d}_{ij}}\right]^2 + \left[\delta^{l}_{\breve{d}_{ij}}, \delta^{u}_{\breve{d}_{ij}}\right]^2\right)\right)^{\omega_i}$$

$$\leq \left(1 - \min_j \min_i \left\{\left[\kappa^{l}_{\breve{d}_{ij}}, \kappa^{u}_{\breve{d}_{ij}}\right]^2 + \left[\delta^{l}_{\breve{d}_{ij}}, \delta^{u}_{\breve{d}_{ij}}\right]^2\right\}\right)^{\omega_i}$$

$$\Leftrightarrow \left(1 - \max_j \max_i \left\{\left[\kappa^{l}_{\breve{d}_{ij}}, \kappa^{u}_{\breve{d}_{ij}}\right]^2 + \left[\delta^{l}_{\breve{d}_{ij}}, \delta^{u}_{\breve{d}_{ij}}\right]^2\right\}\right)^{\sum_{i=1}^{n} \omega_i} \leq \prod_{i=1}^{n} \left(1 - \left(\left[\kappa^{l}_{\breve{d}_{ij}}, \kappa^{u}_{\breve{d}_{ij}}\right]^2 + \left[\delta^{l}_{\breve{d}_{ij}}, \delta^{u}_{\breve{d}_{ij}}\right]^2\right)\right)^{\omega_i}$$

$$\leq \left(1 - \min_j \min_i \left\{\left[\kappa^{l}_{\breve{d}_{ij}}, \kappa^{u}_{\breve{d}_{ij}}\right]^2 + \left[\delta^{l}_{\breve{d}_{ij}}, \delta^{u}_{\breve{d}_{ij}}\right]^2\right\}\right)^{\sum_{i=1}^{n} \omega_i}$$

$$\Leftrightarrow \left(1 - \max_j \max_i \left\{\left[\kappa^{l}_{\breve{d}_{ij}}, \kappa^{u}_{\breve{d}_{ij}}\right]^2 + \left[\delta^{l}_{\breve{d}_{ij}}, \delta^{u}_{\breve{d}_{ij}}\right]^2\right\}\right)^{\sum_{j=1}^{n} \nu_j}$$

$$\leq \prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(1 - \left(\left[\kappa^{l}_{\breve{d}_{ij}}, \kappa^{u}_{\breve{d}_{ij}}\right]^2 + \left[\delta^{l}_{\breve{d}_{ij}}, \delta^{u}_{\breve{d}_{ij}}\right]^2\right)\right)^{\omega_i}\right)^{\nu_j}$$

$$\leq \left(1 - \min_j \min_i \left\{\left[\kappa^{l}_{\breve{d}_{ij}}, \kappa^{u}_{\breve{d}_{ij}}\right]^2 + \left[\delta^{l}_{\breve{d}_{ij}}, \delta^{u}_{\breve{d}_{ij}}\right]^2\right\}\right)^{\sum_{j=1}^{n} \nu_j}$$

$$\Leftrightarrow 1 - \max_j \max_i \left\{\left[\kappa^{l}_{\breve{d}_{ij}}, \kappa^{u}_{\breve{d}_{ij}}\right]^2 + \left[\delta^{l}_{\breve{d}_{ij}}, \delta^{u}_{\breve{d}_{ij}}\right]^2\right\} \leq \prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(1 - \left(\left[\kappa^{l}_{\breve{d}_{ij}}, \kappa^{u}_{\breve{d}_{ij}}\right]^2 + \left[\delta^{l}_{\breve{d}_{ij}}, \delta^{u}_{\breve{d}_{ij}}\right]^2\right)\right)^{\omega_i}\right)^{\nu_j}$$

$$\leq 1 - \min_j \min_i \left\{\left[\kappa^{l}_{\breve{d}_{ij}}, \kappa^{u}_{\breve{d}_{ij}}\right]^2 + \left[\delta^{l}_{\breve{d}_{ij}}, \delta^{u}_{\breve{d}_{ij}}\right]^2\right\}$$

$$\Leftrightarrow \min_j \min_i \left\{\left[\kappa^{l}_{\breve{d}_{ij}}, \kappa^{u}_{\breve{d}_{ij}}\right]^2 + \left[\delta^{l}_{\breve{d}_{ij}}, \delta^{u}_{\breve{d}_{ij}}\right]^2\right\} \leq 1 - \prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(1 - \left(\left[\kappa^{l}_{\breve{d}_{ij}}, \kappa^{u}_{\breve{d}_{ij}}\right]^2 + \left[\delta^{l}_{\breve{d}_{ij}}, \delta^{u}_{\breve{d}_{ij}}\right]^2\right)\right)^{\omega_i}\right)^{\nu_j} \leq \max_j \max_i \left\{\left[\kappa^{l}_{\breve{d}_{ij}}, \kappa^{u}_{\breve{d}_{ij}}\right]^2 + \left[\delta^{l}_{\breve{d}_{ij}}, \delta^{u}_{\breve{d}_{ij}}\right]^2\right\} \tag{3.3}$$

Subtracting inequality (3.3) from (3.1).

$$\Leftrightarrow \min_j \min_i \left\{\left[\delta^l_{\breve{d}_{ij}}, \delta^u_{\breve{d}_{ij}}\right]^2\right\} \le \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\left[\kappa^l_{\breve{d}_{ij}}, \kappa^u_{\breve{d}_{ij}}\right]^2\right)^{\omega_i}\right)^{\nu_j} - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\left(\left[\kappa^l_{\breve{d}_{ij}}, \kappa^u_{\breve{d}_{ij}}\right]^2+\left[\delta^l_{\breve{d}_{ij}}, \delta^u_{\breve{d}_{ij}}\right]^2\right)\right)^{\omega_i}\right)^{\nu_j} \le \max_j \max_i \left\{\left[\delta^l_{\breve{d}_{ij}}, \delta^u_{\breve{d}_{ij}}\right]^2\right\}$$

$$\Leftrightarrow \min_j \min_i \left\{\left[\delta^l_{\breve{d}_{ij}}, \delta^u_{\breve{d}_{ij}}\right]\right\} \le \sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\left[\kappa^l_{\breve{d}_{ij}}, \kappa^u_{\breve{d}_{ij}}\right]^2\right)^{\omega_i}\right)^{\nu_j} - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\left(\left[\kappa^l_{\breve{d}_{ij}}, \kappa^u_{\breve{d}_{ij}}\right]^2+\left[\delta^l_{\breve{d}_{ij}}, \delta^u_{\breve{d}_{ij}}\right]^2\right)\right)^{\omega_i}\right)^{\nu_j}} \le \max_j \max_i \left\{\left[\delta^l_{\breve{d}_{ij}}, \delta^u_{\breve{d}_{ij}}\right]\right\} \tag{3.4}$$

Let IVPFHSIWA $(\mathcal{M}_{\breve{d}_{11}}, \mathcal{M}_{\breve{d}_{12}}, \dots\dots\dots, \mathcal{M}_{\breve{d}_{nm}}) = \left\langle\left[\kappa^l_{\breve{d}_k}, \kappa^u_{\breve{d}_k}\right], \left[\delta^l_{\breve{d}_k}, \delta^u_{\breve{d}_k}\right]\right\rangle = \mathcal{M}_{\breve{d}_k}$, from(3.2) and (3.4): $\min_j \min_i \left\{\left[\kappa^l_{\breve{d}_{ij}}, \kappa^u_{\breve{d}_{ij}}\right]\right\} \le \mathcal{M}_{\breve{d}_k} \le \max_j \max_i \left\{\left[\kappa^l_{\breve{d}_{ij}}, \kappa^u_{\breve{d}_{ij}}\right]\right\}$ and $\min_j \min_i \left\{\left[\delta^l_{\breve{d}_{ij}}, \delta^u_{\breve{d}_{ij}}\right]\right\} \le \mathcal{M}_{\breve{d}_k} \le \max_j \max_i \left\{\left[\delta^l_{\breve{d}_{ij}}, \delta^u_{\breve{d}_{ij}}\right]\right\}$ correspondingly.

Using Eq (2.1).

$$S(\mathcal{M}_{\breve{d}_k}) = \frac{\left(\kappa^l_{\breve{d}_k}\right)^2+\left(\kappa^u_{\breve{d}_k}\right)^2-\left(\delta^l_{\breve{d}_k}\right)^2-\left(\delta^u_{\breve{d}_k}\right)^2}{2} \le \max_j \max_i \left\{\left[\kappa^l_{\breve{d}_{ij}}, \kappa^u_{\breve{d}_{ij}}\right]\right\} - \min_j \min_i \left\{\left[\delta^l_{\breve{d}_{ij}}, \delta^u_{\breve{d}_{ij}}\right]\right\}$$

$$= S\left(\mathcal{M}^-_{\breve{d}_k}\right)$$

$$S(\mathcal{M}_{\breve{d}_k}) = \frac{\left(\kappa^l_{\breve{d}_k}\right)^2+\left(\kappa^u_{\breve{d}_k}\right)^2-\left(\delta^l_{\breve{d}_k}\right)^2-\left(\delta^u_{\breve{d}_k}\right)^2}{2} \ge \min_j \min_i \left\{\left[\kappa^l_{\breve{d}_{ij}}, \kappa^u_{\breve{d}_{ij}}\right]\right\} - \max_j \max_i \left\{\left[\delta^l_{\breve{d}_{ij}}, \delta^u_{\breve{d}_{ij}}\right]\right\} = S\left(\mathcal{M}^+_{\breve{d}_k}\right).$$

Then, by order relation between two IVPFHSNs, we have

$$\mathcal{M}^-_{\breve{d}_{ij}} \le \text{IVPFHSIWA}\left(\mathcal{M}_{\breve{d}_{11}}, \mathcal{M}_{\breve{d}_{12}}, \dots\dots\dots, \mathcal{M}_{\breve{d}_{nm}}\right) \le \mathcal{M}^+_{\breve{d}_{ij}}.$$

3.1.3. (Homogeneity) Prove that IVPFHSIWA $(\beta\mathcal{M}_{\breve{d}_{11}}, \beta\mathcal{M}_{\breve{d}_{12}}, \dots\dots\dots, \beta\mathcal{M}_{\breve{d}_{nm}}) = \beta$ IVPFHSIWA $\left(\mathcal{M}_{\breve{d}_{11}}, \mathcal{M}_{\breve{d}_{12}}, \dots\dots\dots, \mathcal{M}_{\breve{d}_{nm}}\right)$ for any $\beta > 0$.

Proof. Let $\mathcal{M}_{\breve{d}_{ij}}$ be an IVPFHSN and $\beta > 0$, then

$$\beta\mathcal{M}_{\breve{d}_{ij}} = \left(\left(\sqrt{1-\left(1-\left[\kappa^l_{\breve{d}_{ij}}, \kappa^u_{\breve{d}_{ij}}\right]^2\right)^{\beta}}, \sqrt{\left(1-[\kappa^l_{\breve{d}_{ij}}, \kappa^u_{\breve{d}_{ij}}]^2\right)^{\beta} - [1-\left([\kappa^l_{\breve{d}_{ij}}, \kappa^u_{\breve{d}_{ij}}]^2+[\delta^l_{\breve{d}_{ij}}, \delta^u_{\breve{d}_{ij}}]^2\right)^{\beta}}\right)\right).$$

So,

$\beta\mathcal{M}_{\breve{d}_{11}}, \beta\mathcal{M}_{\breve{d}_{12}}, \dots\dots\dots, \beta\mathcal{M}_{\breve{d}_{nm}}$

$$= \left(\frac{\sqrt{1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\left[\kappa^{l}_{\breve{d}_{ij}},\kappa^{u}_{\breve{d}_{ij}}\right]^{2}\right)^{\beta\omega_i}\right)^{\nu_j}},}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\left[\kappa^{l}_{\breve{d}_{ij}},\kappa^{u}_{\breve{d}_{ij}}\right]^{2}\right)^{\beta\omega_i}\right)^{\nu_j}-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\left(\left[\kappa^{l}_{\breve{d}_{ij}},\kappa^{u}_{\breve{d}_{ij}}\right]^{2}+\left[\delta^{l}_{\breve{d}_{ij}},\delta^{u}_{\breve{d}_{ij}}\right]^{2}\right)\right)^{\beta\omega_i}\right)^{\nu_j}}} \right)$$

$$= \left(\frac{\sqrt{1-\left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\left[\kappa^{l}_{\breve{d}_{ij}},\kappa^{u}_{\breve{d}_{ij}}\right]^{2}\right)^{\omega_i}\right)^{\nu_j}\right)^{\beta}},}{\sqrt{\left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\left[\kappa^{l}_{\breve{d}_{ij}},\kappa^{u}_{\breve{d}_{ij}}\right]^{2}\right)^{\omega_i}\right)^{\nu_j}\right)^{\beta}-\left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\left(\left[\kappa^{l}_{\breve{d}_{ij}},\kappa^{u}_{\breve{d}_{ij}}\right]^{2}+\left[\delta^{l}_{\breve{d}_{ij}},\delta^{u}_{\breve{d}_{ij}}\right]^{2}\right)\right)^{\omega_i}\right)^{\nu_j}\right)^{\beta}}} \right)$$

$= \beta \text{ IVPFHSIWA } \left(\mathcal{M}_{\breve{d}_{11}}, \mathcal{M}_{\breve{d}_{12}}, \dots\dots\dots, \mathcal{M}_{\breve{d}_{nm}}\right).$

Also, the geometric interactional aggregation operator with some important results and properties for IVPFHSS using the interactional operational laws for IVPFHSNs is given as follows.

Definition 3.3. Let $\mathcal{M}_{\breve{d}_k} = \left(\left[\kappa^{l}_{\breve{d}_k},\kappa^{u}_{\breve{d}_k}\right],\left[\delta^{l}_{\breve{d}_k},\delta^{u}_{\breve{d}_k}\right]\right)$ be a collection of IVPFHSNs, and ω_i and ν_j be the weights for specialists and multi sub-attributes, disparately, with certain circumstances $\omega_i > 0$, $\sum_{i=1}^{n}\omega_i = 1$; $\nu_j > 0$, $\sum_{j=1}^{m}\nu_j = 1$. Then, the IVPFHSIWA operator is defined as IVPFHSIWG: $\Psi^n \longrightarrow \Psi$

$$\text{IVPFHSIWG}\left(\mathcal{M}_{\breve{d}_{11}}, \mathcal{M}_{\breve{d}_{12}}, \dots\dots\dots, \mathcal{M}_{\breve{d}_{nm}}\right) = \otimes_{j=1}^{m}\left(\otimes_{i=1}^{n}\left(\mathcal{M}_{\breve{d}_{ij}}\right)^{\omega_i}\right)^{\nu_j}.$$

Theorem 3.2. Let $\mathcal{M}_{\breve{d}_{ij}} = \left(\left[\kappa^{l}_{\breve{d}_{ij}},\kappa^{u}_{\breve{d}_{ij}}\right],\left[\delta^{l}_{\breve{d}_{ij}},\delta^{u}_{\breve{d}_{ij}}\right]\right)$ is a set of IVPFHSNs, and the resulting value after aggregation is also an IVPFHSN.

$\text{IVPFHSIWG }(\mathcal{M}_{\breve{d}_{11}}, \mathcal{M}_{\breve{d}_{12}}, \dots\dots\dots, \mathcal{M}_{\breve{d}_{nm}})$

$$= \left(\frac{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\left[\delta^{l}_{\breve{d}_{ij}},\delta^{u}_{\breve{d}_{ij}}\right]^{2}\right)^{\omega_i}\right)^{\nu_j}-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\left(\left[\kappa^{l}_{\breve{d}_{ij}},\kappa^{u}_{\breve{d}_{ij}}\right]^{2}+\left[\delta^{l}_{\breve{d}_{ij}},\delta^{u}_{\breve{d}_{ij}}\right]^{2}\right)\right)^{\omega_i}\right)^{\nu_j}},}{\sqrt{1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\left[\delta^{l}_{\breve{d}_{ij}},\delta^{u}_{\breve{d}_{ij}}\right]^{2}\right)^{\omega_i}\right)^{\nu_j}}} \right).$$

ω_i and ν_j be the weights for experts and attributes disparately, with assumed conditions $\omega_i > 0$, $\sum_{i=1}^{n}\omega_i = 1$; $\nu_j > 0$, $\sum_{j=1}^{m}\nu_j = 1$.

Proof. The IVPFHSIWG operator can be demonstrated through the mathematical induction principle outlined below.

For $n = 1$, we get $\omega_1 = 1$. Then, we have

$$\mathrm{IVPFHSIWG}\left(\mathcal{M}_{\breve{d}_{11}}, \mathcal{M}_{\breve{d}_{12}}, \ldots\ldots\ldots, \mathcal{M}_{\breve{d}_{1m}}\right) = \otimes_{j=1}^{m}\left(\mathcal{M}_{\breve{d}_{1j}}\right)^{\nu_j}$$

$$\mathrm{IVPFHSIWG}\left(\mathcal{M}_{\breve{d}_{11}}, \mathcal{M}_{\breve{d}_{12}}, \ldots\ldots\ldots, \mathcal{M}_{\breve{d}_{nm}}\right)$$

$$= \left(\begin{array}{c}\sqrt{\prod_{j=1}^{m}\left(1-\left[\delta^{l}_{\breve{d}_{1j}}, \delta^{u}_{\breve{d}_{1j}}\right]^{2}\right)^{\nu_j} - \prod_{j=1}^{m}\left(1-\left(\left[\kappa^{l}_{\breve{d}_{1j}}, \kappa^{u}_{\breve{d}_{1j}}\right]^{2} + \left[\delta^{l}_{\breve{d}_{1j}}, \delta^{u}_{\breve{d}_{1j}}\right]^{2}\right)\right)^{\nu_j}}, \\ \sqrt{1-\prod_{j=1}^{m}\left(1-\left[\delta^{l}_{\breve{d}_{1j}}, \delta^{u}_{\breve{d}_{1j}}\right]^{2}\right)^{\nu_j}}\end{array}\right)$$

$$= \left(\begin{array}{c}\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{1}\left(1-\left[\delta^{l}_{\breve{d}_{ij}}, \delta^{u}_{\breve{d}_{ij}}\right]^{2}\right)^{\omega_i}\right)^{\nu_j} - \prod_{j=1}^{m}\left(\prod_{i=1}^{1}\left(1-\left(\left[\kappa^{l}_{\breve{d}_{ij}}, \kappa^{u}_{\breve{d}_{ij}}\right]^{2} + \left[\delta^{l}_{\breve{d}_{ij}}, \delta^{u}_{\breve{d}_{ij}}\right]^{2}\right)\right)^{\omega_i}\right)^{\nu_j}}, \\ \sqrt{1-\prod_{j=1}^{m}\left(\prod_{i=1}^{1}\left(1-\left[\delta^{l}_{\breve{d}_{ij}}, \delta^{u}_{\breve{d}_{ij}}\right]^{2}\right)^{\omega_i}\right)^{\nu_j}}\end{array}\right).$$

For $m = 1$, we get $\nu_1 = 1$. Then, we have

$$\mathrm{IVPFHSIWG}\left(\mathcal{M}_{\breve{d}_{11}}, \mathcal{M}_{\breve{d}_{21}}, \ldots\ldots\ldots, \mathcal{M}_{\breve{d}_{n1}}\right) = \otimes_{i=1}^{n}\left(\mathcal{M}_{\breve{d}_{i1}}\right)^{\omega_i}$$

$$= \left(\begin{array}{c}\sqrt{\prod_{i=1}^{n}\left(1-\left[\delta^{l}_{\breve{d}_{i1}}, \delta^{u}_{\breve{d}_{i1}}\right]^{2}\right)^{\omega_i} - \prod_{i=1}^{n}\left(1-\left(\left[\kappa^{l}_{\breve{d}_{i1}}, \kappa^{u}_{\breve{d}_{i1}}\right]^{2} + \left[\delta^{l}_{\breve{d}_{i1}}, \delta^{u}_{\breve{d}_{i1}}\right]^{2}\right)\right)^{\omega_i}}, \\ \sqrt{1-\prod_{i=1}^{n}\left(1-\left[\delta^{l}_{\breve{d}_{i1}}, \delta^{u}_{\breve{d}_{i1}}\right]^{2}\right)^{\omega_i}}\end{array}\right)$$

$$= \left(\begin{array}{c}\sqrt{\prod_{j=1}^{1}\left(\prod_{i=1}^{n}\left(1-\left[\delta^{l}_{\breve{d}_{ij}}, \delta^{u}_{\breve{d}_{ij}}\right]^{2}\right)^{\omega_i}\right)^{\nu_j} - \prod_{j=1}^{1}\left(\prod_{i=1}^{n}\left(1-\left(\left[\kappa^{l}_{\breve{d}_{ij}}, \kappa^{u}_{\breve{d}_{ij}}\right]^{2} + \left[\delta^{l}_{\breve{d}_{ij}}, \delta^{u}_{\breve{d}_{ij}}\right]^{2}\right)\right)^{\omega_i}\right)^{\nu_j}}, \\ \sqrt{1-\prod_{j=1}^{1}\left(\prod_{i=1}^{n}\left(1-\left[\delta^{l}_{\breve{d}_{ij}}, \delta^{u}_{\breve{d}_{ij}}\right]^{2}\right)^{\omega_i}\right)^{\nu_j}}\end{array}\right).$$

The preceding theorem has been demonstrated to hold for the cases where $n = 1$ and $m = 1$.
To demonstrate that it also holds true for $m = \alpha_1 + 1, n = \alpha_2$ and $m = \alpha_1, n = \alpha_2 + 1$, the following is considered.

$$\otimes_{j=1}^{\alpha_1+1}\left(\otimes_{i=1}^{\alpha_2}\left(\mathcal{M}_{\breve{d}_{ij}}\right)^{\omega_i}\right)^{\nu_j}$$

$$= \left(\sqrt{\prod_{j=1}^{\alpha_1+1}\left(\prod_{i=1}^{\alpha_2}\left(1-\left[\delta^{l}_{\breve{d}_{ij}},\delta^{u}_{\breve{d}_{ij}}\right]^2\right)^{\omega_i}\right)^{\nu_j} - \prod_{j=1}^{\alpha_1+1}\left(\prod_{i=1}^{\alpha_2}\left(1-\left(\left[\kappa^{l}_{\breve{d}_{ij}},\kappa^{u}_{\breve{d}_{ij}}\right]^2+\left[\delta^{l}_{\breve{d}_{ij}},\delta^{u}_{\breve{d}_{ij}}\right]^2\right)\right)^{\omega_i}\right)^{\nu_j}}, \atop \sqrt{1-\prod_{j=1}^{\alpha_1+1}\left(\prod_{i=1}^{\alpha_2}\left(1-\left[\delta^{l}_{\breve{d}_{ij}},\delta^{u}_{\breve{d}_{ij}}\right]^2\right)^{\omega_i}\right)^{\nu_j}} \right)$$

$$\otimes_{j=1}^{\alpha_1}\left(\otimes_{i=1}^{\alpha_2+1}\left(\mathcal{M}_{\breve{d}_{ij}}\right)^{\omega_i}\right)^{\nu_j}$$

$$= \left(\sqrt{\prod_{j=1}^{\alpha_1}\left(\prod_{i=1}^{\alpha_2+1}\left(1-\left[\delta^{l}_{\breve{d}_{ij}},\delta^{u}_{\breve{d}_{ij}}\right]^2\right)^{\omega_i}\right)^{\nu_j} - \prod_{j=1}^{\alpha_1}\left(\prod_{i=1}^{\alpha_2+1}\left(1-\left(\left[\kappa^{l}_{\breve{d}_{ij}},\kappa^{u}_{\breve{d}_{ij}}\right]^2+\left[\delta^{l}_{\breve{d}_{ij}},\delta^{u}_{\breve{d}_{ij}}\right]^2\right)\right)^{\omega_i}\right)^{\nu_j}}, \atop \sqrt{1-\prod_{j=1}^{\alpha_1}\left(\prod_{i=1}^{\alpha_2+1}\left(1-\left[\delta^{l}_{\breve{d}_{ij}},\delta^{u}_{\breve{d}_{ij}}\right]^2\right)^{\omega_i}\right)^{\nu_j}} \right).$$

For $m=\alpha_1+1$ and $n=\alpha_2+1$, we have

$$\otimes_{j=1}^{\alpha_1+1}\left(\otimes_{i=1}^{\alpha_2+1}\left(\mathcal{M}_{\breve{d}_{ij}}\right)^{\omega_i}\right)^{\nu_j} = \otimes_{j=1}^{\alpha_1+1}\left(\left(\otimes_{i=1}^{\alpha_2}\left(\mathcal{M}_{\breve{d}_{ij}}\right)^{\omega_i}\otimes\left(\mathcal{M}_{\breve{d}_{(\alpha_2+1)j}}\right)^{\omega_{\alpha_2+1}}\right)\right)^{\nu_j}$$

$$= \otimes_{j=1}^{\alpha_1+1}\left(\otimes_{i=1}^{\alpha_2}\left(\mathcal{M}_{\breve{d}_{ij}}\right)^{\omega_i}\right)^{\nu_j}\otimes_{j=1}^{\alpha_1+1}\left(\mathcal{M}_{\breve{d}_{(\alpha_2+1)j}}\right)^{\nu_j\omega_{\alpha_2+1}}$$

$$= \left(\sqrt{\prod_{j=1}^{\alpha_1+1}\left(\prod_{i=1}^{\alpha_2}\left(1-\left[\delta^{l}_{\breve{d}_{ij}},\delta^{u}_{\breve{d}_{ij}}\right]^2\right)^{\omega_i}\right)^{\nu_j} - \prod_{j=1}^{\alpha_1+1}\left(\prod_{i=1}^{\alpha_2}\left(1-\left(\left[\kappa^{l}_{\breve{d}_{ij}},\kappa^{u}_{\breve{d}_{ij}}\right]^2+\left[\delta^{l}_{\breve{d}_{ij}},\delta^{u}_{\breve{d}_{ij}}\right]^2\right)\right)^{\omega_i}\right)^{\nu_j}} \otimes \sqrt{\prod_{j=1}^{\alpha_1+1}\left(\left(1-\left[\delta^{l}_{\breve{d}_{(\alpha_2+1)j}},\delta^{u}_{\breve{d}_{(\alpha_2+1)j}}\right]^2\right)^{\omega_{\alpha_2+1}}\right)^{\nu_j} - \prod_{j=1}^{\alpha_1+1}\left(\left(1-\left(\left[\kappa^{l}_{\breve{d}_{(\alpha_2+1)j}},\kappa^{u}_{\breve{d}_{(\alpha_2+1)j}}\right]^2+\left[\delta^{l}_{\breve{d}_{(\alpha_2+1)j}},\delta^{u}_{\breve{d}_{(\alpha_2+1)j}}\right]^2\right)\right)^{\omega_{\alpha_2+1}}\right)^{\nu_j}}, \atop \sqrt{1-\prod_{j=1}^{\alpha_1+1}\left(\prod_{i=1}^{\alpha_2}\left(1-\left[\delta^{l}_{\breve{d}_{ij}},\delta^{u}_{\breve{d}_{ij}}\right]^2\right)^{\omega_i}\right)^{\nu_j}} \otimes \sqrt{1-\prod_{j=1}^{\alpha_1+1}\left(\left(1-\left[\delta^{l}_{\breve{d}_{(\alpha_2+1)j}},\delta^{u}_{\breve{d}_{(\alpha_2+1)j}}\right]^2\right)^{\omega_{\alpha_2+1}}\right)^{\nu_j}} \right)$$

$$= \left(\sqrt{\prod_{j=1}^{\alpha_1+1}\left(\prod_{i=1}^{\alpha_2+1}\left(1-\left[\delta^{l}_{\breve{d}_{ij}},\delta^{u}_{\breve{d}_{ij}}\right]^2\right)^{\omega_i}\right)^{\nu_j} - \prod_{j=1}^{\alpha_1+1}\left(\prod_{i=1}^{\alpha_2+1}\left(1-\left(\left[\kappa^{l}_{\breve{d}_{ij}},\kappa^{u}_{\breve{d}_{ij}}\right]^2+\left[\delta^{l}_{\breve{d}_{ij}},\delta^{u}_{\breve{d}_{ij}}\right]^2\right)\right)^{\omega_i}\right)^{\nu_j}}, \atop \sqrt{1-\prod_{j=1}^{\alpha_1+1}\left(\prod_{i=1}^{\alpha_2+1}\left(1-\left[\delta^{l}_{\breve{d}_{ij}},\delta^{u}_{\breve{d}_{ij}}\right]^2\right)^{\omega_i}\right)^{\nu_j}} \right).$$

It holds for $m=\alpha_1+1$ and $n=\alpha_2+1$. So, it also holds $\forall\, m, n$.

Example 3.2. Let $\mathcal{R}=\{\mathcal{R}_1,\mathcal{R}_2,\mathcal{R}_3,\mathcal{R}_4\}$ be a team of experts with the assumed weights $\omega_i=(0.1, 0.2, 0.4, 0.3)^T$. The team of specialists designates the good looks of a community under contemplated features $A=\{e_1=\text{Parkland},\ e_2=\text{safety arrangement}\}$ with their compatible sub-parameters, Parkland $= e_1 = \{e_{11} = \text{with grass}, e_{12} = \text{without grass}\}$ Safety arrangement $= e_2 = \{e_{21} = \text{guards}, e_{22} = \text{cameras}\}$. Let A = $e_1 \times e_2$ be a set of sub-attributes

$$A = e_1 \times e_2 = \{e_{11}, e_{12}\} \times \{e_{21}, e_{22}\} = \{(e_{11}, e_{21}), (e_{11}, e_{22}), (e_{12}, e_{21}), (e_{12}, e_{22})\}$$

A = $\{\breve{d}_1, \breve{d}_2, \breve{d}_3, \breve{d}_4\}$ be a collection of multi sub-parameters with their weights $\nu_j = (0.3, 0.1, 0.2, 0.4)^T$. The assessment standards for each substitute in terms of IVPFHSN $(\mathcal{M}, A) = \left(\left[\kappa^l_{\breve{d}_{ij}}, \kappa^u_{\breve{d}_{ij}}\right], \left[\delta^l_{\breve{d}_{ij}}, \delta^u_{\breve{d}_{ij}}\right]\right)_{3\times 4}$ given as:

$$(\mathcal{M}, A) = \begin{bmatrix} ([0.3, 0.8], [0.4, 0.5]) & ([0.4, 0.6], [0.3, 0.7]) & ([0.5, 0.8], [0.5, 0.6]) & ([0.4, 0.9], [0.3, 0.7]) \\ ([0.1, 0.5], [0.2, 0.3]) & ([0.3, 0.8], [0.5, 0.7]) & ([0.2, 0.4], [0.2, 0.3]) & ([0.3, 0.8], [0.6, 0.7]) \\ ([0.2, 0.9], [0.2, 0.3]) & ([0.5, 0.7], [0.2, 0.6]) & ([0.2, 0.4], [0.2, 0.8]) & ([0.3, 0.8], [0.5, 0.8]) \end{bmatrix}$$

By using the above theorem, we have

$$\text{IVPFHSIWG}\left(\mathcal{M}_{\breve{d}_{11}}, \mathcal{M}_{\breve{d}_{12}}, \ldots\ldots\ldots, \mathcal{M}_{\breve{d}_{nm}}\right)$$

$$= \left(\sqrt{\prod_{j=1}^{4}\left(\prod_{i=1}^{3}\left(1-\left[\delta^l_{\breve{d}_{ij}}, \delta^u_{\breve{d}_{ij}}\right]^2\right)^{\omega_i}\right)^{\nu_j} - \prod_{j=1}^{4}\left(\prod_{i=1}^{3}\left(1-\left(\left[\kappa^l_{\breve{d}_{ij}}, \kappa^u_{\breve{d}_{ij}}\right]^2 + \left[\delta^l_{\breve{d}_{ij}}, \delta^u_{\breve{d}_{ij}}\right]^2\right)\right)^{\omega_i}\right)^{\nu_j}}, \sqrt{1-\prod_{j=1}^{4}\left(\prod_{i=1}^{3}\left(1-\left[\delta^l_{\breve{d}_{ij}}, \delta^u_{\breve{d}_{ij}}\right]^2\right)^{\omega_i}\right)^{\nu_j}} \right)$$

$$\left(\sqrt{\begin{pmatrix} \begin{Bmatrix} [0.75, 0.96]^{0.1}[0.64, 0.96]^{0.2} \\ [0.84, 0.99]^{0.4}[0.51, 0.91]^{0.3} \end{Bmatrix}^{0.3} \begin{Bmatrix} [0.36, 0.51]^{0.1}[0.64, 0.99]^{0.2} \\ [0.64, 0.84]^{0.4}[0.75, 0.84]^{0.3} \end{Bmatrix}^{0.1} \\ \begin{Bmatrix} [0.75, 0.96]^{0.1}[0.36, 0.84]^{0.2} \\ [0.51, 0.91]^{0.4}[0.75, 0.91]^{0.3} \end{Bmatrix}^{0.2} \begin{Bmatrix} [0.64, 0.96]^{0.1}[0.51, 0.84]^{0.2} \\ [0.84, 0.96]^{0.4}[0.75, 0.91]^{0.3} \end{Bmatrix}^{0.4} \end{pmatrix} - \begin{pmatrix} \begin{Bmatrix} (1-[0.2, 0.5])^{0.1}(1-[0.08, 0.85])^{0.2} \\ (1-[0.1, 0.41])^{0.4}(1-[0.25, 0.85])^{0.3} \end{Bmatrix}^{0.3} \begin{Bmatrix} (1-[0.74, 1])^{0.1}(1-[0.17, 0.61])^{0.2} \\ (1-[0.2, 0.85])^{0.4}(1-[0.25, 0.74])^{0.3} \end{Bmatrix}^{0.1} \\ \begin{Bmatrix} (1-[0.2, 0.61])^{0.1}(1-[0.2, 0.73])^{0.2} \\ (1-[0.25, 0.98])^{0.4}(1-[0.18, 0.61])^{0.3} \end{Bmatrix}^{0.2} \begin{Bmatrix} (1-[0.08, 0.52])^{0.1}(1-[0.2, 0.74])^{0.2} \\ (1-[0.29, 0.65])^{0.4}(1-[0.18, 0.61])^{0.3} \end{Bmatrix}^{0.4} \end{pmatrix}}, \sqrt{1-\begin{pmatrix} \begin{Bmatrix} [0.75, 0.96]^{0.1}[0.64, 0.96]^{0.2} \\ [0.84, 0.99]^{0.4}[0.51, 0.91]^{0.3} \end{Bmatrix}^{0.3} \begin{Bmatrix} [0.36, 0.51]^{0.1}[0.64, 0.99]^{0.2} \\ [0.64, 0.84]^{0.4}[0.75, 0.84]^{0.3} \end{Bmatrix}^{0.1} \\ \begin{Bmatrix} [0.75, 0.96]^{0.1}[0.36, 0.84]^{0.2} \\ [0.51, 0.91]^{0.4}[0.75, 0.91]^{0.3} \end{Bmatrix}^{0.2} \begin{Bmatrix} [0.64, 0.96]^{0.1}[0.51, 0.84]^{0.2} \\ [0.84, 0.96]^{0.4}[0.75, 0.91]^{0.3} \end{Bmatrix}^{0.4} \end{pmatrix}} \right)$$

$$= \left(\sqrt{\begin{pmatrix} [0.6772, 0.9564]^{0.3}[0.6337, 0.8258]^{0.1} \\ [0.5550, 0.9004]^{0.2}[0.7151, 0.9198]^{0.4} \end{pmatrix} - \begin{pmatrix} [0.2926, 0.8458]^{0.3}[0, 0.7064]^{0.1} \\ [0.1104, 0.7854]^{0.2}[0.3516, 0.7793]^{0.4} \end{pmatrix}}, \sqrt{1-\begin{pmatrix} [0.6772, 0.9564]^{0.3}[0.6337, 0.8258]^{0.1} \\ [0.5550, 0.9004]^{0.2}[0.7151, 0.9198]^{0.4} \end{pmatrix}} \right)$$

$= ([0.8128, 0.3530], [0.2886, 0.5825])$.

Now, we will debate the anticipated properties for the IVPFHSIWG operator.

3.2. Properties of IVPFHSIWG operator

3.2.1. (Idempotency) If $\mathcal{M}_{\check{d}_{ij}} = \mathcal{M}_{\check{d}_k} = \left(\left[\kappa^l_{\check{d}_{ij}}, \kappa^u_{\check{d}_{ij}}\right], \left[\delta^l_{\check{d}_{ij}}, \delta^u_{\check{d}_{ij}}\right]\right)\ \forall\, i, j$, then,

$$\text{IVPFHSIWG}\,(\mathcal{M}_{\check{d}_{11}}, \mathcal{M}_{\check{d}_{12}}, \dots \dots \dots, \mathcal{M}_{\check{d}_{nm}}) = \mathcal{M}_{\check{d}_k}.$$

Proof. As we know that all $\mathcal{M}_{\check{d}_{ij}} = \mathcal{M}_{\check{d}_k} = \left(\left[\kappa^l_{\check{d}_{ij}}, \kappa^u_{\check{d}_{ij}}\right], \left[\delta^l_{\check{d}_{ij}}, \delta^u_{\check{d}_{ij}}\right]\right)$, then, we have

$$\text{IVPFHSIWG}\,(\mathcal{M}_{\check{d}_{11}}, \mathcal{M}_{\check{d}_{12}}, \dots \dots \dots, \mathcal{M}_{\check{d}_{nm}})$$

$$= \left(\begin{array}{c} \sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\left[\delta^l_{\check{d}_{ij}}, \delta^u_{\check{d}_{ij}}\right]^2\right)^{\omega_i}\right)^{\nu_j} - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\left(\left[\kappa^l_{\check{d}_{ij}}, \kappa^u_{\check{d}_{ij}}\right]^2 + \left[\delta^l_{\check{d}_{ij}}, \delta^u_{\check{d}_{ij}}\right]^2\right)\right)^{\omega_i}\right)^{\nu_j}}, \\ \sqrt{1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\left[\delta^l_{\check{d}_{ij}}, \delta^u_{\check{d}_{ij}}\right]^2\right)^{\omega_i}\right)^{\nu_j}} \end{array} \right)$$

$$= \left(\begin{array}{c} \sqrt{\left(\left(1-\left[\delta^l_{\check{d}_{ij}}, \delta^u_{\check{d}_{ij}}\right]^2\right)^{\sum_{i=1}^{n}\omega_i}\right)^{\sum_{j=1}^{m}\nu_j} - \left(\left(1-\left(\left[\kappa^l_{\check{d}_{ij}}, \kappa^u_{\check{d}_{ij}}\right]^2 + \left[\delta^l_{\check{d}_{ij}}, \delta^u_{\check{d}_{ij}}\right]^2\right)\right)^{\sum_{i=1}^{n}\omega_i}\right)^{\sum_{j=1}^{m}\nu_j}}, \\ \sqrt{1-\left(\left(1-\left[\delta^l_{\check{d}_{ij}}, \delta^u_{\check{d}_{ij}}\right]^2\right)^{\sum_{i=1}^{n}\omega_i}\right)^{\sum_{j=1}^{m}\nu_j}} \end{array} \right).$$

As $\sum_{j=1}^{m}\nu_j = 1$ and $\sum_{i=1}^{n}\omega_i = 1$, then we have

$$= \left(\sqrt{\left(1-\left[\delta^l_{\check{d}_{ij}}, \delta^u_{\check{d}_{ij}}\right]^2\right) - \left(1-\left(\left[\kappa^l_{\check{d}_{ij}}, \kappa^u_{\check{d}_{ij}}\right]^2 + \left[\delta^l_{\check{d}_{ij}}, \delta^u_{\check{d}_{ij}}\right]^2\right)\right)}, \sqrt{1-\left(1-\left[\delta^l_{\check{d}_{ij}}, \delta^u_{\check{d}_{ij}}\right]^2\right)}\right)$$

$$= \left(\left[\kappa^l_{\check{d}_{ij}}, \kappa^u_{\check{d}_{ij}}\right], \left[\delta^l_{\check{d}_{ij}}, \delta^u_{\check{d}_{ij}}\right]\right)$$

$$= \mathcal{M}_{\check{d}_k}.$$

3.2.2. (Boundedness) Let $\mathcal{M}_{\check{d}_{ij}}$ be a collection of IVPFHSNs where $\mathcal{M}^-_{\check{d}_{ij}} = \left(\min_j \min_i \left\{\left[\kappa^l_{\check{d}_{ij}}, \kappa^u_{\check{d}_{ij}}\right]\right\}, \max_j \max_i \left\{\left[\delta^l_{\check{d}_{ij}}, \delta^u_{\check{d}_{ij}}\right]\right\}\right)$ and $\mathcal{M}^+_{\check{d}_{ij}} = \left(\max_j \max_i \left\{\left[\kappa^l_{\check{d}_{ij}}, \kappa^u_{\check{d}_{ij}}\right]\right\}, \min_j \min_i \left\{\left[\delta^l_{\check{d}_{ij}}, \delta^u_{\check{d}_{ij}}\right]\right\}\right)$, then

$$\mathcal{M}^-_{\check{d}_{ij}} \le \text{IVPFHSIWG}\left(\mathcal{M}_{\check{d}_{11}}, \mathcal{M}_{\check{d}_{12}}, \dots \dots \dots, \mathcal{M}_{\check{d}_{nm}}\right) \le \mathcal{M}^+_{\check{d}_{ij}}.$$

Proof. As we know that $\mathcal{M}_{\check{d}_{ij}} = \left(\left[\kappa^l_{\check{d}_{ij}}, \kappa^u_{\check{d}_{ij}}\right], \left[\delta^l_{\check{d}_{ij}}, \delta^u_{\check{d}_{ij}}\right]\right)$ be an IVPFHSN, then

$$\min_j \min_i \left\{\left[\delta^l_{\breve{d}_{ij}}, \delta^u_{\breve{d}_{ij}}\right]^2\right\} \leq \left[\delta^l_{\breve{d}_{ij}}, \delta^u_{\breve{d}_{ij}}\right]^2 \leq \max_j \max_i \left\{\left[\delta^l_{\breve{d}_{ij}}, \delta^u_{\breve{d}_{ij}}\right]^2\right\}$$

$$\Rightarrow 1 - \max_j \max_i \left\{\left[\delta^l_{\breve{d}_{ij}}, \delta^u_{\breve{d}_{ij}}\right]^2\right\} \leq 1 - \left[\delta^l_{\breve{d}_{ij}}, \delta^u_{\breve{d}_{ij}}\right]^2 \leq 1 - \min_j \min_i \left\{\left[\delta^l_{\breve{d}_{ij}}, \delta^u_{\breve{d}_{ij}}\right]^2\right\}$$

$$\Leftrightarrow \left(1 - \max_j \max_i \left\{\left[\delta^l_{\breve{d}_{ij}}, \delta^u_{\breve{d}_{ij}}\right]^2\right\}\right)^{\omega_i} \leq \left(1 - \left[\delta^l_{\breve{d}_{ij}}, \delta^u_{\breve{d}_{ij}}\right]^2\right)^{\omega_i} \leq \left(1 - \min_j \min_i \left\{\left[\delta^l_{\breve{d}_{ij}}, \delta^u_{\breve{d}_{ij}}\right]^2\right\}\right)^{\omega_i}$$

$$\Leftrightarrow \left(1 - \max_j \max_i \left\{\left[\delta^l_{\breve{d}_{ij}}, \delta^u_{\breve{d}_{ij}}\right]^2\right\}\right)^{\sum_{i=1}^n \omega_i} \leq \prod_{i=1}^n \left(1 - \left[\delta^l_{\breve{d}_{ij}}, \delta^u_{\breve{d}_{ij}}\right]^2\right)^{\omega_i}$$
$$\leq \left(1 - \min_j \min_i \left\{\left[\delta^l_{\breve{d}_{ij}}, \delta^u_{\breve{d}_{ij}}\right]^2\right\}\right)^{\sum_{i=1}^n \omega_i}$$

$$\Leftrightarrow \left(1 - \max_j \max_i \left\{\left[\delta^l_{\breve{d}_{ij}}, \delta^u_{\breve{d}_{ij}}\right]^2\right\}\right)^{\sum_{j=1}^n \nu_j} \leq \prod_{j=1}^m \left(\prod_{i=1}^n \left(1 - \left[\delta^l_{\breve{d}_{ij}}, \delta^u_{\breve{d}_{ij}}\right]^2\right)^{\omega_i}\right)^{\nu_j}$$
$$\leq \left(1 - \min_j \min_i \left\{\left[\delta^l_{\breve{d}_{ij}}, \delta^u_{\breve{d}_{ij}}\right]^2\right\}\right)^{\sum_{j=1}^n \nu_j}$$

$$\Leftrightarrow 1 - \max_j \max_i \left\{\left[\delta^l_{\breve{d}_{ij}}, \delta^u_{\breve{d}_{ij}}\right]^2\right\} \leq \prod_{j=1}^m \left(\prod_{i=1}^n \left(1 - \left[\delta^l_{\breve{d}_{ij}}, \delta^u_{\breve{d}_{ij}}\right]^2\right)^{\omega_i}\right)^{\nu_j} \leq 1 - \min_j \min_i \left\{\left[\delta^l_{\breve{d}_{ij}}, \delta^u_{\breve{d}_{ij}}\right]^2\right\}$$

$$\Leftrightarrow \min_j \min_i \left\{\left[\delta^l_{\breve{d}_{ij}}, \delta^u_{\breve{d}_{ij}}\right]^2\right\} \leq 1 - \prod_{j=1}^m \left(\prod_{i=1}^n \left(1 - \left[\delta^l_{\breve{d}_{ij}}, \delta^u_{\breve{d}_{ij}}\right]^2\right)^{\omega_i}\right)^{\nu_j} \leq \max_j \max_i \left\{\left[\delta^l_{\breve{d}_{ij}}, \delta^u_{\breve{d}_{ij}}\right]^2\right\}$$

$$\Leftrightarrow \min_j \min_i \left\{\left[\delta^l_{\breve{d}_{ij}}, \delta^u_{\breve{d}_{ij}}\right]\right\} \leq \sqrt{1 - \prod_{j=1}^m \left(\prod_{i=1}^n \left(1 - \left[\delta^l_{\breve{d}_{ij}}, \delta^u_{\breve{d}_{ij}}\right]^2\right)^{\omega_i}\right)^{\nu_j}} \leq \max_j \max_i \left\{\left[\delta^l_{\breve{d}_{ij}}, \delta^u_{\breve{d}_{ij}}\right]\right\} \tag{3.5}$$

Similarly, we can prove that

$$\min_j \min_i \left\{\left[\kappa^l_{\breve{d}_{ij}}, \kappa^u_{\breve{d}_{ij}}\right]\right\} \leq$$
$$\sqrt{\prod_{j=1}^m \left(\prod_{i=1}^n \left(1 - \left[\delta^l_{\breve{d}_{ij}}, \delta^u_{\breve{d}_{ij}}\right]^2\right)^{\omega_i}\right)^{\nu_j} - \prod_{j=1}^m \left(\prod_{i=1}^n \left(1 - \left(\left[\kappa^l_{\breve{d}_{ij}}, \kappa^u_{\breve{d}_{ij}}\right]^2 + \left[\delta^l_{\breve{d}_{ij}}, \delta^u_{\breve{d}_{ij}}\right]^2\right)\right)^{\omega_i}\right)^{\nu_j}} \leq$$
$$\max_j \max_i \left\{\left[\kappa^l_{\breve{d}_{ij}}, \kappa^u_{\breve{d}_{ij}}\right]\right\}. \tag{3.6}$$

Let IVPFHSIWG $\left(\mathcal{M}_{\breve{d}_{11}}, \mathcal{M}_{\breve{d}_{12}}, \ldots\ldots\ldots, \mathcal{M}_{\breve{d}_{nm}}\right) = \left\langle\left[\kappa^l_{\breve{d}_k}, \kappa^u_{\breve{d}_k}\right], \left[\delta^l_{\breve{d}_k}, \delta^u_{\breve{d}_k}\right]\right\rangle = \mathcal{M}_{\breve{d}_k}$, then (3.5) and (3.6) can be arranged as follows:

$\min_j \min_i \left\{\left[\kappa^l_{\breve{d}_{ij}}, \kappa^u_{\breve{d}_{ij}}\right]\right\} \leq \mathcal{M}_{\breve{d}_k} \leq \max_j \max_i \left\{\left[\kappa^l_{\breve{d}_{ij}}, \kappa^u_{\breve{d}_{ij}}\right]\right\}$ and $\min_j \min_i \left\{\left[\delta^l_{\breve{d}_{ij}}, \delta^u_{\breve{d}_{ij}}\right]\right\} \leq \mathcal{M}_{\breve{d}_k} \leq \max_j \max_i \left\{\left[\delta^l_{\breve{d}_{ij}}, \delta^u_{\breve{d}_{ij}}\right]\right\}$ respectively.
Using Eq (2.1).

$$S(\mathcal{M}_{\breve{d}_k}) = \frac{\left(\kappa^l_{\breve{d}_k}\right)^2 + \left(\kappa^u_{\breve{d}_k}\right)^2 - \left(\delta^l_{\breve{d}_k}\right)^2 - \left(\delta^u_{\breve{d}_k}\right)^2}{2} \leq \max_j \max_i \left\{\left[\kappa^l_{\breve{d}_{ij}}, \kappa^u_{\breve{d}_{ij}}\right]\right\} - \min_j \min_i \left\{\left[\delta^l_{\breve{d}_{ij}}, \delta^u_{\breve{d}_{ij}}\right]\right\} = S\left(\mathcal{M}^-_{\breve{d}_{ij}}\right)$$

$$S(\mathcal{M}_{\breve{d}_k}) = \frac{\left(\kappa^l_{\breve{d}_k}\right)^2 + \left(\kappa^u_{\breve{d}_k}\right)^2 - \left(\delta^l_{\breve{d}_k}\right)^2 - \left(\delta^u_{\breve{d}_k}\right)^2}{2} \geq \min_j \min_i \left\{\left[\kappa^l_{\breve{d}_{ij}}, \kappa^u_{\breve{d}_{ij}}\right]\right\} - \max_j \max_i \left\{\left[\delta^l_{\breve{d}_{ij}}, \delta^u_{\breve{d}_{ij}}\right]\right\} = S\left(\mathcal{M}^+_{\breve{d}_{ij}}\right).$$

Then, by order relation between two IVPFHSNs, we have

$$\mathcal{M}^-_{\breve{d}_{ij}} \leq \text{IVPFHSIWG}\left(\mathcal{M}_{\breve{d}_{11}}, \mathcal{M}_{\breve{d}_{12}}, \dots \dots \dots, \mathcal{M}_{\breve{d}_{nm}}\right) \leq \mathcal{M}^+_{\breve{d}_{ij}}.$$

3.2.3. (Homogeneity) Prove that $\text{IVPFHSIWG}\left(\beta\mathcal{M}_{\breve{d}_{11}}, \beta\mathcal{M}_{\breve{d}_{12}}, \dots \dots \dots, \beta\mathcal{M}_{\breve{d}_{nm}}\right) = \beta\, \text{IVPFHSIWG}\left(\mathcal{M}_{\breve{d}_{11}}, \mathcal{M}_{\breve{d}_{12}}, \dots \dots \dots, \mathcal{M}_{\breve{d}_{nm}}\right)$ $\beta > 0$.
Proof. Let $\mathcal{M}_{\breve{d}_{ij}}$ be an IVPFHSN and $\beta > 0$, then

$$\beta\mathcal{M}_{\breve{d}_{ij}} = \left(\sqrt{1 - \left(1 - \left[\kappa^l_{\breve{d}_{ij}}, \kappa^u_{\breve{d}_{ij}}\right]^2\right)^{\beta}}, \sqrt{\left(1 - [\kappa^l_{\breve{d}_{ij}}, \kappa^u_{\breve{d}_{ij}}]^2\right)^{\beta} - [1 - \left([\kappa^l_{\breve{d}_{ij}}, \kappa^u_{\breve{d}_{ij}}]^2 + [\delta^l_{\breve{d}_{ij}}, \delta^u_{\breve{d}_{ij}}]^2\right)^{\beta}}\right).$$

So,

$\text{IVPFHSIWG}\left(\beta\mathcal{M}_{\breve{d}_{11}}, \beta\mathcal{M}_{\breve{d}_{12}}, \dots \dots \dots, \beta\mathcal{M}_{\breve{d}_{nm}}\right)$

$$= \left(\begin{array}{c} \sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - \left[\delta^l_{\breve{d}_{ij}}, \delta^u_{\breve{d}_{ij}}\right]^2\right)^{\beta\omega_i}\right)^{\nu_j} - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - \left(\left[\kappa^l_{\breve{d}_{ij}}, \kappa^u_{\breve{d}_{ij}}\right]^2 + \left[\delta^l_{\breve{d}_{ij}}, \delta^u_{\breve{d}_{ij}}\right]^2\right)\right)^{\beta\omega_i}\right)^{\nu_j}}, \\ \sqrt{1 - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - \left[\delta^l_{\breve{d}_{ij}}, \delta^u_{\breve{d}_{ij}}\right]^2\right)^{\beta\omega_i}\right)^{\nu_j}} \end{array}\right)$$

$$= \left(\begin{array}{c} \sqrt{\left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - \left[\delta^l_{\breve{d}_{ij}}, \delta^u_{\breve{d}_{ij}}\right]^2\right)^{\omega_i}\right)^{\nu_j}\right)^{\beta} - \left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - \left(\left[\kappa^l_{\breve{d}_{ij}}, \kappa^u_{\breve{d}_{ij}}\right]^2 + \left[\delta^l_{\breve{d}_{ij}}, \delta^u_{\breve{d}_{ij}}\right]^2\right)\right)^{\omega_i}\right)^{\nu_j}\right)^{\beta}}, \\ \sqrt{1 - \left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - \left[\delta^l_{\breve{d}_{ij}}, \delta^u_{\breve{d}_{ij}}\right]^2\right)^{\omega_i}\right)^{\nu_j}\right)^{\beta}} \end{array}\right)$$

$= \beta\, \text{IVPFHSIWG}\left(\mathcal{M}_{\breve{d}_{11}}, \mathcal{M}_{\breve{d}_{12}}, \dots \dots \dots, \mathcal{M}_{\breve{d}_{nm}}\right).$

4. The proposed MCGDM approach based on our developed operators

To verify the effectiveness of the proposed interaction AOs, a DM technique has been proposed to solve the MCGDM problems. Additionally, a statistical analysis will be conducted to demonstrate the practicality of the proposed methodology.

4.1. Proposed MCGDM approach

Suppose we have a set of s alternatives $\mathfrak{I} = \{\mathfrak{I}^1, \mathfrak{I}^2, \mathfrak{I}^3, \ldots, \mathfrak{I}^s\}$ and a set of r decision-makers $\mathcal{U} = \{\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3, \ldots, \mathcal{U}_r\}$. The specialists' weights are denoted by $\omega_i = (\omega_1, \omega_2, \omega_3, \ldots, \omega_n)^T$, where $\omega_i > 0$, $\sum_{i=1}^{n} \omega_i = 1$. Let $\mathfrak{L} = \{e_1, e_2, e_3, \ldots, e_m\}$ be a set of parameters, and $\mathfrak{L}' = \{(e_{1\rho} \times e_{2\rho} \times \ldots \times e_{m\rho})\ \forall\, \rho \in \{1, 2, \ldots, t\}\}$ be a set of multi sub-attributes with weights $\nu = (\nu_1, \nu_2, \nu_3, \ldots, \nu_n)^T$, where $\nu_i > 0, \sum_{i=1}^{n} \nu_i = 1$, known as $\mathfrak{L}' = \{\check{d}_\partial : \partial \in \{1, 2, \ldots, m\}\}$. The team of experts $\{\kappa^i : i = 1, 2, \ldots, n\}$ evaluate the substitutes $\{\mathfrak{H}^{(z)} : z = 1, 2, \ldots, s\}$ under the selected sub-attributes $\{\check{d}_\partial : \partial = 1, 2, \ldots, k\}$ as IVPFHSNs. We denote this as $\left(\mathfrak{I}^{(z)}_{\check{d}_{ik}}\right)_{n\times m} = \left(\left[\kappa^l_{\check{d}_{ik}}, \kappa^u_{\check{d}_{ik}}\right], \left[\delta^l_{\check{d}_{ik}}, \delta^u_{\check{d}_{ik}}\right]\right)_{n*m}$, where $0 \le \kappa^l_{\check{d}_{ik}}, \kappa^u_{\check{d}_{ik}}, \delta^l_{\check{d}_{ik}}, \delta^u_{\check{d}_{ik}} \le 1$ and $0 \le \left(\kappa^u_{\check{d}_{ik}}\right)^2 + \left(\delta^u_{\check{d}_{ik}}\right)^2 \le 1\ \forall$ i, k. and $0 \le \kappa^l_{\check{d}_{ik}}, \kappa^u_{\check{d}_{ik}}, \delta^l_{\check{d}_{ik}}, \delta^u_{\check{d}_{ik}} \le 1\ \forall\, i, k$. The decision-makers deliver their judgments in the form of IVPFHSNs Θ_k for each substitute. The stepwise algorithm involves established operators given as follows:

Step 1. According to the expert's opinion, obtain a decision matrix in IVPFHSNs for each alternative.

$$\left(\mathfrak{I}^{(z)}_{\check{d}_{ik}}\right)_{n\times m} = \left(\left[\kappa^l_{\check{d}_{ik}}, \kappa^u_{\check{d}_{ik}}\right], \left[\delta^l_{\check{d}_{ik}}, \delta^u_{\check{d}_{ik}}\right]\right)_{n*m}$$

$$= \begin{bmatrix} \left(\left[\kappa^l_{\check{d}_{11}}, \kappa^u_{\check{d}_{11}}\right], \left[\delta^l_{\check{d}_{11}}, \delta^u_{\check{d}_{11}}\right]\right) & \left(\left[\kappa^l_{\check{d}_{12}}, \kappa^u_{\check{d}_{12}}\right], \left[\delta^l_{\check{d}_{12}}, \delta^u_{\check{d}_{12}}\right]\right) & \cdots & \left(\left[\kappa^l_{\check{d}_{1m}}, \kappa^u_{\check{d}_{1m}}\right], \left[\delta^l_{\check{d}_{1m}}, \delta^u_{\check{d}_{1m}}\right]\right) \\ \left(\left[\kappa^l_{\check{d}_{21}}, \kappa^u_{\check{d}_{21}}\right], \left[\delta^l_{\check{d}_{21}}, \delta^u_{\check{d}_{21}}\right]\right) & \left(\left[\kappa^l_{\check{d}_{22}}, \kappa^u_{\check{d}_{22}}\right], \left[\delta^l_{\check{d}_{22}}, \delta^u_{\check{d}_{22}}\right]\right) & & \left(\left[\kappa^l_{\check{d}_{2n}}, \kappa^u_{\check{d}_{2n}}\right], \left[\delta^l_{\check{d}_{2m}}, \delta^u_{\check{d}_{2m}}\right]\right) \\ & \vdots & \ddots & \vdots \\ \left(\left[\kappa^l_{\check{d}_{n1}}, \kappa^u_{\check{d}_{n1}}\right], \left[\delta^l_{\check{d}_{n1}}, \delta^u_{\check{d}_{n1}}\right]\right) & \left(\left[\kappa^l_{\check{d}_{n2}}, \kappa^u_{\check{d}_{n2}}\right], \left[\delta^l_{\check{d}_{n2}}, \delta^u_{\check{d}_{n2}}\right]\right) & \cdots & \left(\left[\kappa^l_{\check{d}_{nm}}, \kappa^u_{\check{d}_{nm}}\right], \left[\delta^l_{\check{d}_{nm}}, \delta^u_{\check{d}_{nm}}\right]\right) \end{bmatrix}.$$

Step 2. Use normalization rules to convert cost-type characteristics into benefit types and establish normalization decision metrics.

$$\mathcal{M}_{\check{d}_{ik}} = \begin{cases} \mathcal{M}^c_{\check{d}_{ij}} = \left(\left[\delta^l_{\check{d}_{ik}}, \delta^u_{\check{d}_{ik}}\right], \left[\kappa^l_{\check{d}_{ik}}, \kappa^u_{\check{d}_{ik}}\right]\right)_{n\times m} & \text{cost type parameter} \\ \mathcal{M}_{\check{d}_{ij}} = \left(\left[\kappa^l_{\check{d}_{ik}}, \kappa^u_{\check{d}_{ik}}\right], \left[\delta^l_{\check{d}_{ik}}, \delta^u_{\check{d}_{ik}}\right]\right)_{n\times m} & \text{benefit type parameter} \end{cases}.$$

Step 3. Calculate the aggregated values for each alternative via developed IVPFHSIWA and IVPFHSIWG.

Step 4. Compute the score values for each alternative.

Step 5. Examine the ranking of the alternatives.

4.2. Sustainable thermal power equipment supplier selection and their environmental impacts

Sustainable thermal power equipment supplier selection can significantly impact the environmental performance of thermal power plants. By working with suppliers who prioritize sustainability, power plant operators can reduce the environmental impact of their operations and create a more sustainable supply chain. The following are some environmental impacts of sustainable thermal power equipment supplier selection:

- Energy efficiency: Sustainable thermal power equipment suppliers can provide equipment that is designed to be more energy-efficient. This can help reduce the energy needed to operate the power plant, reducing greenhouse gas emissions and lowering the plant's environmental impact.
- Emissions reduction: Sustainable thermal power equipment suppliers can provide equipment designed to emit lower levels of greenhouse gases and other pollutants. This might involve using cleaner fuels, improved combustion systems, or advanced emission control technologies.
- Waste reduction and management: Sustainable thermal power equipment suppliers can provide equipment designed to generate less waste or recycle or reuse waste generated by the power plant. This can help to reduce the environmental impact of the plant's waste management practices.
- Water usage reduction: Sustainable thermal power equipment suppliers can provide equipment designed to minimize the water needed for cooling. This might involve using closed-loop cooling systems, water-efficient equipment designs, or water-saving technologies.
- Social and environmental responsibility: Sustainable thermal power equipment suppliers can demonstrate social and environmental commitment. This might involve implementing sustainable business practices, ethical sourcing, and responsible waste management.

Sustainable thermal power equipment supplier selection is critical to creating a more sustainable thermal power industry. By considering factors such as energy efficiency, emissions reduction, waste reduction and management, water usage reduction, and social and environmental responsibility, power plant operators can select suppliers who prioritize sustainability and reduce the environmental impact of their operations. It benefits the environment and helps improve the power plant’s efficiency, sustainability, and reputation.

The question of sustainable supplier selection (SSS) is both logical and authentic. Key issues include top supplier supply chain mobility, large-scale, low-cost production, and initiative. The inclusion of environmentally friendly prototypes and other sustainable and modern structures in the SSS development indicates that proper SSS is multifaceted and multifaceted. SSS refers to various beneficial achievements to the atmosphere or the people and is frequently mentioned in the literature as “sustainable supplier selection”. This is a complicated, multidimensional topic with unpredictable morals, and the assessment procedure desires to consider many observations. From these facts, vendor assortment is frequently referred to as an “orientation” subject in the works, where multidisciplinary DM approaches are broadly used. With Pakistan’s improvement and opening up, numerous power plant tasks have been constructed to see the mandate for electricity for community and commercial growth. To come across the prerequisite of this construction, a structure of community proposals and tenders has been used since 1985 to purchase thermal power generation apparatus. The assortment of thermal power equipment (TPE) suppliers is a significant part of the trying and order administration of thermal power apparatus. It is also a primary state for thermal power plants (TPP) steady and continuing growth. Figure 1 displays the portion of the energy produced from dissimilar cradles. With the massive escalation in the usage of fossil fuels and growing environment trash, the expressions “green growth” and “sustainable development” have grown into leading in the universal dissertation.

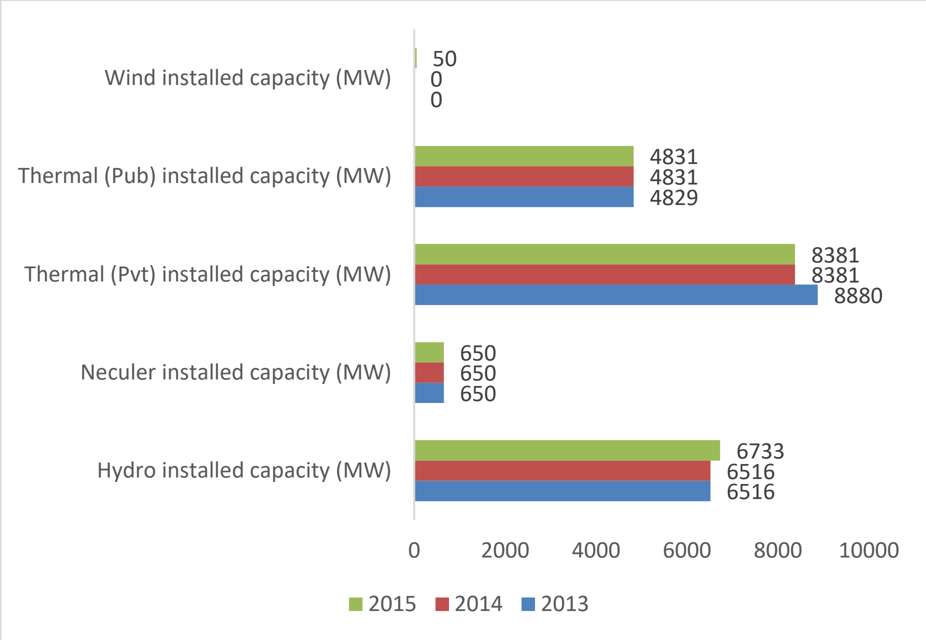

Figure 1. Share in electricity generation.

The water is mainly intense in a TPP; the hot water produces condensation, the steam turbine is exchanged, and the generator is activated. The high temperature and pressure condensation are liable for the spin of the turbine and then transmitted to the generator to produce electricity. The condenser summarizes the suspension through the organization cycle and returns it to a formerly warm point. TPP is separated into two classifications permitting their fuel: fossil fuel (FF) power plants and biomass fuel power plants. Fossil fuel TPP produces energy by sweltering FF, such as coal, natural gas, or oil. They are manufactured on balance and can precede indeterminately. The power plant practices steam turbines, while natural gas-fired services use gas turbines. Numerous biomass power generation amenities burn the surplus of firewood, farming, or structure wood. Biomass fuel is openly used in burning power plant reservoirs to deliver energy for comparable kinds of FF steam generators. According to a prime mover, TPP is separated into steam, gas, and combined cycle power plants.

The vibrant density produced by the growing steam enterprises at the turbine edges of the steam power plant. Practically all non-hydroelectric power plants use this technique. Around 80% of the sphere's energy is produced done steam turbines. A gas turbine power plant contains three leading portions: compressor, combustion system, and turbine: mutual cycle power plants use gas and steam machines to produce energy. The new gas turbine heat is focused on the adjacent steam turbine, which creates extra energy. Most of Pakistan's energy originates from TPP, which uses possessions such as oil, gas, and coal. Selected are mutual cycles, although others are steam and gas turbines. There are 49 thermal power plants in Punjab, Sindh, and Balochistan. Thermal power interpretations for 61% of Pakistan's energy. Pakistan can install 16,599 MW of energy. Guddu has an ability of 2402 MW, TPS Muzaffargarh has a capability of 1350 MW, Kot Addu has a power of 1638 MW, and Hubco Balochistan has an aptitude of 1200 MW. Accomplished in 1960 with a volume of 195 MW, NGPS Multan is the oldest. Pakistan has finalized three biomass-driven power generation amenities with a whole ability of 67 MW [52]. The nominal power of several thermal power places is exposed in Figure 2. Thermal power is the crucial cradle of energy in Pakistan, and the choice of green suppliers of TPE is serious for the charming and long-term improvement of TPP. Therefore, choosing the right green supplier of TPE with green fabrication attentiveness in an atmosphere of energy maintenance and discharge bargain and indorsing sustainable growth is of extreme prominence for the long-term development of the corporation and the usefulness of Pakistan's power business. What are the elementary ideologies of SCM? In 1997, Anderson et al. [53] circulated an article titled “Seven Principles of Supply Chain Management”. SCM was a comparatively novel thought then, but this article makes an exceptional contract of assigning the basics of SCM in one curved. After over 20 years, the exertion was acknowledged as “classic” research and republished in 2010. The script has acknowledged over 300 certifications in academic works and trade journals.

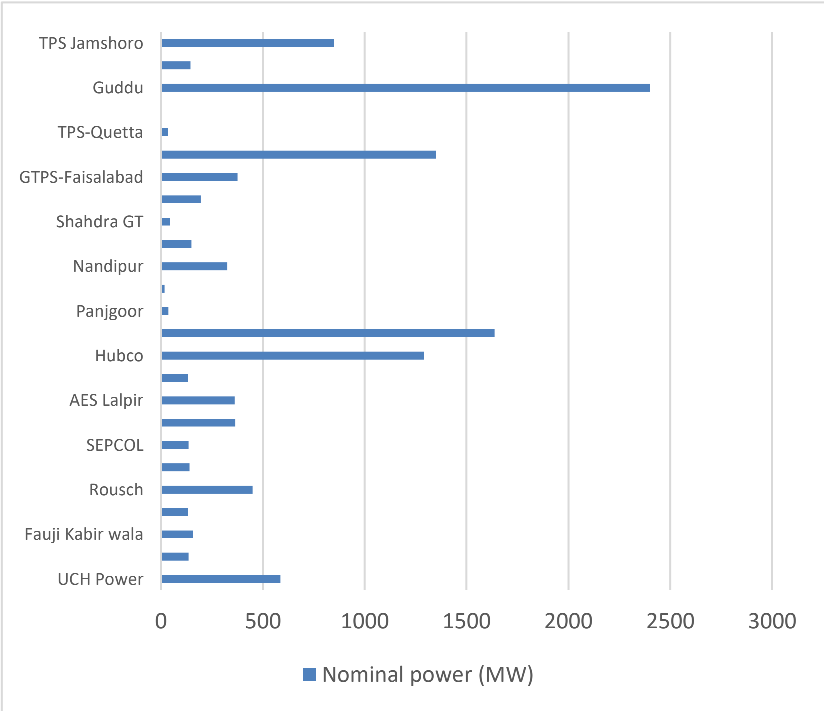

Figure 2. Nominal power of different thermal power stations [52].

The study of SSS can be separated into expressive and methodical models. Descriptive study aspects at the significant parameters of supplier valuation and assortment. Selecting an ecological supplier is an imperative MCDM delinquent [54]. The MCDM approach categorizes probable substitutes and chooses the finest replacement using an explicit method built on proven DM facts derived from different factors. This assessment is gradually becoming an exciting subject in science, systems science, and management science. Selecting a supplier is MCGDM anxiety that needs several deliberate features comprising cost, delivery time, ecological influence, etc. Enlightening a corporation's ecological influence must be a significant feature of its organization configuration and corporate intentions in mandate to be prosperous. Dickson [55] acknowledged 23 features contractors consider it connected to problems in choosing dissimilar suppliers. They initiated that the most meaningful metrics were time, efficacy, budget, and distribution. Wind et al. [56] discussed numerous facets elaborate in the enactment assessment of other suppliers. Ho et al. [57] introduced multi-criterion selection criteria of international journals from 2000 to 2008, revised all skills and committed that the maximum mutual geographies used to extend supplier productivity were delivery, worth, budget, etc. Weber et al. [58] studied 74 magazines on SSS in investigational study models and acknowledged numerous approaches that have appeared in the investigation over the past 25 years. They determined that record approaches are linear weighting, regression models, and some optimization algorithms. A modern summary of the supplier employment and assortment procedure can originate here. Amid et al. [59] studied vague parameters in the supplier assessment hypothesis. Jolai et al. [60] proposed a fuzzy MCDM technique to obtain cumulative scores from various dealers and then planned the best proper mechanism employing subordinate impartial development methods. Sevkli et al. [61] confirmed the weight of their fuzzy linear program design by an investigational arrangement technique for dealer assortment. Environmental contests are more applicable in industrial zones due to environmental variation and global warming [62]. In recent epochs, green supply chain management study has expected more and more consideration to diminish air contamination, educate sentience and protect the atmosphere [63]. Classifying and choosing related low-carbon suppliers is necessary to generate a sustainable supply chain and consequential conservational interferences.

As mentioned earlier, the MCDM delinquent denotes a supplier assessment procedure since, at the DM phase, rigor and several factors need to be investigated and certified [64]. The MCDM method has been used to assess and achieve low-carbon suppliers but also undertakes that feature information is convincing and precise [65]. Luckily, fast financial development and an energetic corporate environment mark it challenging for decision-makers to deliver consistent exploration or a priori facts that contain conflicts of social perceptive. Tong and Wang [66] recently used encouragement IF operators to resolve the problem of selecting a low-carbon supplier. The parable that greenery will lead to lower sales and sophisticated operating costs is over. Many companies now feel the need to combine their customers' environmental protection strategies with their SSCM and will not be able to fulfill the desire to convert to current income. In a healthy environment, there is a link between sustainability and economic incentives for the growth of different companies. The business development concept inspires SSCM and has identified areas where work can be changed to increase profits. Green logistics can sustain much lower production, such as CO_2 and CO. Consumption, including non-fossil energy, reduces smog, affecting our environment and stimulating respiration. Due to rapid growth, many fossil fuels are destroying the environment. For marine life, for example, air travel also affects the mix due to the use of diesel. The literature looks for aspects that identify sustainable suppliers for TPE compatible with altered researchers.

Supplier selection is a critical process in thermal power equipment, as the choice of supplier can significantly impact the power plant’s environmental performance and efficiency. The following are some important factors to consider when selecting a supplier for thermal power equipment:

- ❖ Reliability and quality: The reliability and quality of the supplier's equipment are crucial factors to consider. Power plant operators should look for suppliers with a proven track record of providing reliable, high-quality equipment that meets the plant's specifications and requirements.
- ❖ Cost-effectiveness: The cost-effectiveness of the supplier's equipment is an important consideration. Power plant operators should look for suppliers that offer competitive pricing without compromising on the quality and performance of the equipment.
- ❖ Technical expertise: The supplier's technical expertise is an important factor to consider. Power plant operators should look for suppliers that have a deep understanding of the equipment and the specific needs of the power plant and can provide expert advice and support throughout the equipment's life cycle.
- ❖ Environmental performance: The environmental performance of the supplier's equipment is a crucial consideration. Power plant operators should look for suppliers that prioritize sustainability, offer energy-efficient equipment, low emissions, and minimize waste and water usage.
- ❖ Customer service and support: The supplier’s customer service and support level is an important factors to consider. Power plant operators should look for suppliers that offer comprehensive

support services, including maintenance, repair, and replacement services, to certify the reliable and efficient procedure of the equipment.

- ❖ Supply chain sustainability: The sustainability of the supplier's supply chain is an important consideration. Power plant operators should look for suppliers that prioritize ethical and sustainable sourcing practices and have transparent supply chains that minimize environmental impacts.
- ❖ Performance: The performance of supplier selection in the thermal equipment plant industry can have a significant impact on the environmental and financial performance of the power plant. Choosing the right suppliers can help power plant operators to reduce their environmental impact, improve efficiency, and reduce operating costs. On the other hand, choosing the wrong suppliers can lead to equipment failures, environmental violations, and other costly and time-consuming problems.
- ❖ Equipment efficiency: The efficiency of thermal power equipment is typically measured using the heat rate, the fuel required to generate a unit of electricity. A lower heat rate indicates higher efficiency, as less fuel is required to produce the same amount of electricity. The efficiency of thermal power equipment can be improved through various means, including upgrading existing equipment, optimizing operations, and operating thermal power equipment at its optimal performance level can help improve efficiency. This can include reducing excess air in the combustion process, maintaining proper fuel-to-air ratios, optimizing steam pressure and temperature and using cleaner fuels, implementing energy recovery systems, and incorporating advanced technologies.
- ❖ Quality: Quality can be an important factor in selecting suppliers for thermal power equipment. High-quality equipment can help to ensure the efficient and reliable operation of the power plant, while low-quality equipment can lead to equipment failures, downtime, and costly repairs. When selecting suppliers for thermal power equipment, they must consider the quality of the equipment they offer and their quality management systems and processes. Suppliers with a proven track record of providing high-quality equipment and certifications, such as ISO 9001 or other quality standards, can help ensure that the equipment meets the required quality standards. In addition, it is important to consider the supplier's quality control processes and testing procedures. Suppliers that conduct thorough testing of their equipment and have robust quality control processes in place can help to ensure that the equipment is of high quality and meets the required performance standards. Overall, selecting suppliers prioritizing quality can help ensure the efficient and reliable operation of thermal power equipment, minimize equipment failures and downtime, and reduce the risk of costly repairs.

In summary, selecting the right supplier for thermal power equipment involves considering factors such as reliability and quality, cost-effectiveness, technical expertise, environmental performance, customer service and support, supply chain sustainability, performance, equipment efficiency, and quality. By prioritizing these factors, power plant operators can ensure that they choose a supplier that can provide high-quality, reliable, and environmentally-friendly equipment that meets the plant’s specific needs. In this article, Table 1 presents ranking values for sustainable suppliers. Zeng et al. [67] established PF-confident AOs to address low-carbon supplier choices. As shown in the introduction, the easing of IVPFHSS MD and NMD boundaries consents for broader possibility, constructing IVPFHSS superior to IVIFS, IVPFS, IVIFSS, IVPFSS, and IVIFHSS in the description of unpredictable and confusing facts. In the context of IVPFHSS, it is essential and suitable to systematically examine the substance of an assortment of suppliers of TPE.

Table 1. Criteria for the assortment of the suitable supplier in TPE.

	Criteria	Sub-criteria
$\mathbf{d_1}$	Performance	Actual production time, scheduled time
$\mathbf{d_2}$	Equipment efficiency	75% < OEE < 85% (acceptable), 85% < OEE < 95% (good).
$\mathbf{d_3}$	Quality	Good

Suppose $\mathfrak{I}^1$, $\mathfrak{I}^2$, $\mathfrak{I}^3$ and $\mathfrak{I}^4$ be a collection of alternatives. The characteristic of supplier selection is listed as follows: $\mathfrak{L} = \{\mathrm{d}_1 = \text{Performance}, \mathrm{d}_2 = \text{Equipment efficiency}, \mathrm{d}_3 = \text{Quality}\}$. The conforming sub-attributes of the deliberated parameters, Performance = d_1 = $\{\mathrm{d}_{11} = \text{Actual production time}, \mathrm{d}_{12} = \text{scheduled time}\}$, Equipment efficiency = d_2 = $\{\mathrm{d}_{21} = 75\% < \text{OEE} < 85\%\ (\text{acceptable}), \mathrm{d}_{22} = 85\% < \text{OEE} < 95\%\ (\text{good})\}$, Quality = d_3 = $\{\mathrm{d}_{31} = \text{Good}\}$. Let $\mathfrak{L}' = \mathrm{d}_1 \times \mathrm{d}_2 \times \mathrm{d}_3$ be a set of sub-attributes. $\mathfrak{L}' = \mathrm{d}_1 \times \mathrm{d}_2 \times \mathrm{d}_3 = \{\mathrm{d}_{11}, \mathrm{d}_{12}\} \times \{\mathrm{d}_{21}, \mathrm{d}_{22}\} \times \{\mathrm{d}_{31}\}$ $= \begin{Bmatrix} (\mathrm{d}_{11}, \mathrm{d}_{21}, \mathrm{d}_{31}), (\mathrm{d}_{11}, \mathrm{d}_{21}, \mathrm{d}_{32}), \\ (\mathrm{d}_{12}, \mathrm{d}_{21}, \mathrm{d}_{31}), (\mathrm{d}_{12}, \mathrm{d}_{21}, \mathrm{d}_{32}) \end{Bmatrix}$, $\mathfrak{L}' = \{\check{\mathrm{d}}_1, \check{\mathrm{d}}_2, \check{\mathrm{d}}_3, \check{\mathrm{d}}_4\}$ be a set of all sub-attributes with weights $(0.3, 0.1, 0.2, 0.4)^{\mathrm{T}}$. Let $\{\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3, \mathcal{U}_4\}$ be a team of specialists with weights $(0.1, 0.2, 0.4, 0.3)^{\mathrm{T}}$. To assess the finest substitute, specialists make available their predilections in IVPFHSNs. The flowchart of the proposed model is presented in Figure 3.

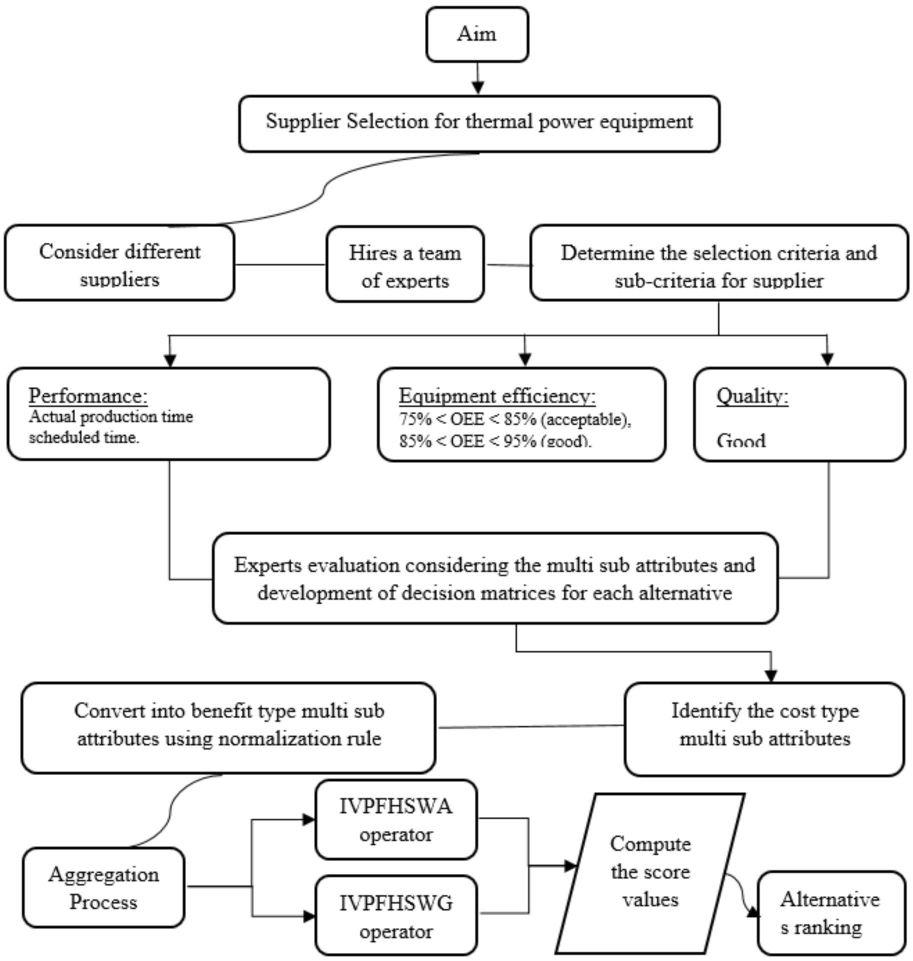

Figure 3. Flowchart of proposed MCGDM model.

4.3. By IVPFHSIWA operator

Step 1. The decision-makers opinion on IVPFHSNs is given in Tables 2–5.

Table 2. Decision matrix for $\Im^1$ in the form of IVPFHSN.

	$\check{\mathbf{d}}_1$	$\check{\mathbf{d}}_2$	$\check{\mathbf{d}}_3$	$\check{\mathbf{d}}_4$
$\mathcal{U}_1$	$([0.4,0.5],[0.2,0.5])$	$([0.7,0.8],[0.5,0.6])$	$([0.4,0.6],[0.2,0.5])$	$([0.2,0.4],[0.2,0.6])$
$\mathcal{U}_2$	$([0.2,0.7],[0.2,0.6])$	$([0.1,0.6],[0.4,0.5])$	$([0.2,0.3],[0.4,0.8])$	$([0.2,0.5],[0.4,0.7])$
$\mathcal{U}_3$	$([0.3,0.5],[0.1,0.4])$	$([0.4,0.6],[0.2,0.7])$	$([0.4,0.7],[0.3,0.7])$	$([0.5,0.7],[0.2,0.4])$
$\mathcal{U}_4$	$([0.4,0.6],[0.3,0.7])$	$([0.4,0.5],[0.3,0.7])$	$([0.3,0.6],[0.3,0.5])$	$([0.3,0.6],[0.3,0.5])$

Table 3. Decision matrix for $\Im^2$ in the form of IVPFHSN.

	$\check{\mathbf{d}}_1$	$\check{\mathbf{d}}_2$	$\check{\mathbf{d}}_3$	$\check{\mathbf{d}}_4$
$\mathcal{U}_1$	$([0.3,0.6],[0.5,0.6])$	$([0.2,0.7],[0.5,0.7])$	$([0.2,0.7],[0.4,0.5])$	$([0.6,0.7],[0.5,0.8])$
$\mathcal{U}_2$	$([0.3,0.5],[0.5,0.8])$	$([0.1,0.4],[0.4,0.5])$	$([0.1,0.5],[0.3,0.7])$	$([0.4,0.5],[0.3,0.6])$
$\mathcal{U}_3$	$([0.2,0.6],[0.1,0.4])$	$([0.1,0.2],[0.2,0.9])$	$([0.4,0.7],[0.3,0.7])$	$([0.5,0.8],[0.2,0.6])$
$\mathcal{U}_4$	$([0.2,0.3],[0.3,0.8])$	$([0.3,0.5],[0.2,0.8])$	$([0.3,0.7],[0.2,0.6])$	$([0.1,0.7],[0.3,0.6])$

Table 4. Decision matrix for $\Im^3$ in the form of IVPFHSN.

	$\check{\mathbf{d}}_1$	$\check{\mathbf{d}}_2$	$\check{\mathbf{d}}_3$	$\check{\mathbf{d}}_4$
$\mathcal{U}_1$	$([0.3,0.4],[0.2,0.7])$	$([0.3,0.4],[0.4,0.6])$	$([0.5,0.6],[0.4,0.5])$	$([0.3,0.4],[0.3,0.6])$
$\mathcal{U}_2$	$([0.4,0.6],[0.3,0.7])$	$([0.3,0.5],[0.2,0.3])$	$([0.3,0.5],[0.5,0.8])$	$([0.2,0.6],[0.2,0.4])$
$\mathcal{U}_3$	$([0.2,0.4],[0.3,0.4])$	$([0.3,0.5],[0.3,0.7])$	$([0.3,0.7],[0.3,0.8])$	$([0.1,0.3],[0.5,0.6])$
$\mathcal{U}_4$	$([0.3,0.7],[0.3,0.7])$	$([0.3,0.5],[0.2,0.4])$	$([0.2,0.5],[0.3,0.6])$	$([0.3,0.4],[0.3,0.7])$

Table 5. Decision matrix for $\Im^4$ in the form of IVPFHSN.

	$\check{\mathbf{d}}_1$	$\check{\mathbf{d}}_2$	$\check{\mathbf{d}}_3$	$\check{\mathbf{d}}_4$
$\mathcal{U}_1$	$([0.3,0.5],[0.2,0.6])$	$([0.2,0.6],[0.4,0.7])$	$([0.2,0.5],[0.3,0.6])$	$([0.5,0.7],[0.6,0.8])$
$\mathcal{U}_2$	$([0.2,0.7],[0.3,0.8])$	$([0.1,0.5],[0.4,0.7])$	$([0.5,0.7],[0.4,0.5])$	$([0.2,0.5],[0.3,0.4])$
$\mathcal{U}_3$	$([0.2,0.5],[0.1,0.6])$	$([0.2,0.5],[0.1,0.5])$	$([0.2,0.4],[0.2,0.7])$	$([0.3,0.5],[0.1,0.5])$
$\mathcal{U}_4$	$([0.2,0.4],[0.5,0.8])$	$([0.2,0.5],[0.5,0.8])$	$([0.2,0.7],[0.3,0.6])$	$([0.2,0.5],[0.4,0.5])$

Step 2. Use the normalization rule to convert the cost type parameters to benefit type parameters and obtain the normalized Pythagorean fuzzy hypersoft decision matrices in Tables 6–9.

Table 6. Normalized IVPFHS decision matrix for $\Im^1$.

	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$
$\mathbf{x}_1$	$([.4,.5],[.2,.5])$	$([.5,.6],[.7,.8])$	$([.4,.6],[.2,.5])$	$([.2,.4],[.2,.6])$
$\mathbf{x}_2$	$([.2,.7],[.2,.6])$	$([.4,.5],[.1,.6])$	$([.2,.3],[.4,.8])$	$([.2,.5],[.4,.7])$
$\mathbf{x}_3$	$([.3,.5],[.1,.4])$	$([.2,.7],[.4,.6])$	$([.4,.7],[.3,.7])$	$([.5,.7],[.2,.4])$
$\mathbf{x}_4$	$([.4,.6],[.3,.7])$	$([.3,.7],[.4,.5])$	$([.3,.6],[.3,.5])$	$([.3,.6],[.3,.5])$

Table 7. Normalized IVPFHS decision matrix for $\mathfrak{J}^2$.

	$\mathbf{e_1}$	$\mathbf{e_2}$	$\mathbf{e_3}$	$\mathbf{e_4}$
$\mathbf{x_1}$	$([.3,.6],[.5,.6])$	$([.5,.7],[.2,.7])$	$([.2,.7],[.4,.5])$	$([.6,.7],[.5,.8])$
$\mathbf{x_2}$	$([.3,.5],[.5,.8])$	$([.4,.5],[.1,.4])$	$([.1,.5],[.3,.7])$	$([.4,.5],[.3,.6])$
$\mathbf{x_3}$	$([.2,.6],[.1,.4])$	$([.2,.9],[.1,.2])$	$([.4,.7],[.3,.7])$	$([.5,.8],[.2,.6])$
$\mathbf{x_4}$	$([.2,.3],[.3,.8])$	$([.2,.8],[.3,.5])$	$([.3,.7],[.2,.6])$	$([.1,.7],[.3,.6])$

Table 8. Normalized IVPFHS decision matrix for $\mathfrak{J}^3$.

	$\mathbf{e_1}$	$\mathbf{e_2}$	$\mathbf{e_3}$	$\mathbf{e_4}$
$\mathbf{x_1}$	$([.3,.4],[.2,.7])$	$([.4,.6],[.3,.4])$	$([.5,.6],[.4,.5])$	$([.3,.4],[.3,.6])$
$\mathbf{x_2}$	$([.4,.6],[.3,.7])$	$([.2,.3],[.3,.5])$	$([.3,.5],[.5,.8])$	$([.2,.6],[.2,.4])$
$\mathbf{x_3}$	$([.2,.4],[.3,.4])$	$([.3,.7],[.3,.5])$	$([.3,.7],[.3,.8])$	$([.1,.3],[.5,.6])$
$\mathbf{x_4}$	$([.3,.7],[.3,.7])$	$([.2,.4],[.3,.5])$	$([.2,.5],[.3,.6])$	$([.3,.4],[.3,.7])$

Table 9. Normalized IVPFHS decision matrix for $\mathfrak{J}^4$.

	$\mathbf{e_1}$	$\mathbf{e_2}$	$\mathbf{e_3}$	$\mathbf{e_4}$
$\mathbf{x_1}$	$([.3,.5],[.2,.6])$	$([.4,.7],[.2,.6])$	$([.2,.5],[.3,.6])$	$([.5,.7],[.6,.8])$
$\mathbf{x_2}$	$([.2,.7],[.3,.8])$	$([.4,.7],[.1,.5])$	$([.5,.7],[.4,.5])$	$([.2,.5],[.3,.4])$
$\mathbf{x_3}$	$([.2,.5],[.1,.6])$	$([.1,.5],[.2,.5])$	$([.2,.4],[.2,.7])$	$([.3,.5],[.1,.5])$
$\mathbf{x_4}$	$([.2,.4],[.5,.8])$	$([.5,.8],[.2,.6])$	$([.2,.7],[.3,.6])$	$([.2,.5],[.4,.5])$

Step 3. Communicate the planned IVPFHSIWA operator to the attained information. We will attain the opinion of the decision-makers such as: $\Theta_1 = ([0.3523, 0.6102], [0.7923, 0.2895])$, $\Theta_2 = ([0.3442, 0.6781], [0.7350, 0.3050])$, $\Theta_3 = ([0.2665, 0.5249], [0.8512, 0.3470])$, $\Theta_4 = ([0.3018, 0.5618], [0.8273, 0.2851])$.

Step 4. Use the score function $S = \frac{\left(\kappa^{l}_{\check{d}_{ij}}\right)^2 + \left(\kappa^{u}_{\check{d}_{ij}}\right)^2 - \left(\delta^{l}_{\check{d}_{ij}}\right)^2 - \left(\delta^{u}_{\check{d}_{ij}}\right)^2}{2}$ for the IVPFHSS to estimate the score values for all alternatives. $S(\Theta_1) = -0.1028$, $S(\Theta_2) = -0.0275$, $S(\Theta_3) = -0.2492$, and $S(\Theta_4) = -0.1795$.

Step 5. Alternatives ranking; $S(\Theta_2) > S(\Theta_1) > S(\Theta_4) > S(\Theta_3)$. Which displays that $\mathfrak{J}^2$ is the finest alternate. So, $\mathfrak{J}^2 > \mathfrak{J}^1 > \mathfrak{J}^4 > \mathfrak{J}^3$.

4.4. By IVPFHSIWG operator

Step 1. Obtain IVPFHS decision matrices (Tables 2–5).
Step 2. Normalize the IVPFHS decision matrices (Tables 6–9).
Step 3. Communicate the planned IVPFHSIWG operator to the taught information. We will acquire the opinion of the decision-makers, such as: $\Theta_1 = ([0.8128, 0.3530], [0.2886, 0.5825])$, $\Theta_2 = ([0.7909, 0.3494], [0.2990, 0.6119])$, $\Theta_3 = ([0.7670, 0.2709], [0.3476, 0.6416])$, $\Theta_4 = ([0.8049, 0.2830], [0.3038, 0.5935])$.
Step 4. Use the score function $S = \frac{\left(\kappa^{l}_{\check{d}_{ij}}\right)^2 + \left(\kappa^{u}_{\check{d}_{ij}}\right)^2 - \left(\delta^{l}_{\check{d}_{ij}}\right)^2 - \left(\delta^{u}_{\check{d}_{ij}}\right)^2}{2}$ IVPFHSS to compute the score values for all substitutes, such as $S(\Theta_1) = 0.1419$, $S(\Theta_2) = 0.1813$, $S(\Theta_3) = 0.0646$, and $S(\Theta_4) = 0.1417$.

Step 5. Alternatives ranking; $S(\Theta_2) > S(\Theta_1) > S(\Theta_4) > S(\Theta_3)$. Which displays that $\mathfrak{I}^2$ is the best substitute. So, $\mathfrak{I}^2 > \mathfrak{I}^1 > \mathfrak{I}^4 > \mathfrak{I}^3$.

5. Comparative studies

To authenticate the proposed scheme's efficacy, an assessment of the projected model and prevalent approaches is scheduled in the consequent section.

5.1. Supremacy of the planned method

The scheduled approach is both effective and persuasive. A novel MCGDM model based on IVPFHSIWA and IVPFHSIWG operators is developed in the IVPFHSS framework. Our proposed model is more sophisticated than existing methods and can significantly improve MCGDM problems. The developed model is versatile and adaptable, allowing it to handle inconsistencies, obligations, and changing outputs. Different models have distinct classification methods, and the proposed technique can adjust rankings and probabilities directly. Systematic studies and evaluations determined that the current method's consequences are more reliable than prevailing techniques. The emergence of several hybrid structures of FS, including IVFS, IVIFS, IVPFS, IVIFSS, IVIFHSS, and IVPFSS, is attributed to the promising settings of IVPFHSS. It is easier to integrate incomplete and uncertain data in the DM process and present the constituent information more logically. False and misleading data are often mixed in the DM process. Therefore, our proposed methodology is expected to be more effective, significant, advanced, and refined than many hybrid FS configurations. A feature analysis comparing the proposed approach and some prevailing models is presented in Table 10.

Table 10. Feature analysis of different models with a planned model.

	Fuzzy information	Aggregated attributes information	Aggregated information in intervals form	Interaction aggregated information
IVFS [3]	√	×	√	×
IVIFWA [50]	√	×	√	×
IVIFWG [51]	√	×	√	×
IVPFWA [25]	√	×	√	×
IVPFWG [27]	√	×	√	×
IFSWA [33]	√	√	×	×
IFSWG [33]	√	√	×	×
IVIFSWA [36]	√	√	√	×
IVIFSWG [36]	√	√	√	×
IFHSWA [43]	√	√	×	×
IFHSWG [43]	√	√	×	×
IVPFSWA [38]	√	√	√	×
IVPFSWG [38]	√	√	√	×
IVPFHSWA [45]	√	√	√	×
IVPFHSWG [45]	√	√	√	×
Proposed IVPFHSIWA	√	√	√	√
Proposed IVPFHSIWG	√	√	√	√

5.2. Comparative analysis

To demonstrate the efficacy of the planned technique, we compared its outcomes with prevailing approaches for IVPFS, IVIFSS, and IVPFSS. The findings of this comparison are displayed in Table 11. Wang and Liu [50] introduced the IVIFWA approach in their previous research, while Xu and Chen [51] proposed the IVIFWG operators. However, these approaches have certain limitations in determining alternative parameter values. These operators are also unsuitable when $MD + NMD > 1$, as determined by experts. Similarly, Zulqarnain et al. [36] proved that AOs for IVIFSS could not support the decision-makers assortment when $\mathrm{MD} + \mathrm{NMD} > 1$. The IVPFWA operator proposed by Peng and Yang [25] and the IVPFWG operator introduced by Rahman et al. [27] cannot handle the parametric values of alternatives. Zulqarnain et al. [38] developed the IVPFSWA and IVPFSWG operators to address this issue by dealing with the parameterized values of alternatives in interval form. Additionally, if $\kappa^{u}_{\mathrm{d}_{ij}} + \delta^{u}_{\mathrm{d}_{ij}} \leq 1$, the IVPFSS can be condensed to IVIFSS. Zulqarnain et al. [45] presented AOs for IVPFHSS and utilized them to solve MCDM problems involving the parameterized values of sub-attributes. However, in some cases, standing AOs may produce unattractive outcomes. To report these composite concerns, we developed interactive AOs for IVPFHSS capable of handling multi-sub attributes consistent with the standing AOs. The IVPFHSS is a generalization of IVPFSS and an extension of IVIFHSS. Thus, based on the above details, the operators proposed in this article are highly substantial, stable, and effective. Table 11 equates the proposed model with the predominant models.

Table 11. Comparison of scheduled operators with some prevalent operators.

AO	$\mathfrak{J}^1$	$\mathfrak{J}^2$	$\mathfrak{J}^3$	$\mathfrak{J}^4$	Alternatives ranking	Optimal choice
IVIFWA[50]	0.3681	0.4573	0.3509	0.2146	$\mathfrak{J}^2 > \mathfrak{J}^1 > \mathfrak{J}^3 > \mathfrak{J}^4$	$\mathfrak{J}^2$
IVIFWG [51]	0.3104	0.3952	0.2914	0.2753	$\mathfrak{J}^2 > \mathfrak{J}^1 > \mathfrak{J}^3 > \mathfrak{J}^4$	$\mathfrak{J}^2$
IVPFWA [25]	0.0154	0.0251	0.0198	0.0247	$\mathfrak{J}^2 > \mathfrak{J}^4 > \mathfrak{J}^3 > \mathfrak{J}^1$	$\mathfrak{J}^2$
IVPFWG [27]	0.0364	0.0856	0.0786	0.0475	$\mathfrak{J}^2 > \mathfrak{J}^3 > \mathfrak{J}^1 > \mathfrak{J}^4$	$\mathfrak{J}^2$
IVIFSWA [36]	0.0235	0.0723	0.0584	0.0253	$\mathfrak{J}^2 > \mathfrak{J}^3 > \mathfrak{J}^1 > \mathfrak{J}^4$	$\mathfrak{J}^2$
IVIFSWG [36]	0.2365	0.7234	0.5840	0.6525	$\mathfrak{J}^2 > \mathfrak{J}^4 > \mathfrak{J}^3 > \mathfrak{J}^1$	$\mathfrak{J}^2$
IVPFSWA [38]	0.0377	0.0834	0.0113	0.0141	$\mathfrak{J}^2 > \mathfrak{J}^1 > \mathfrak{J}^4 > \mathfrak{J}^3$	$\mathfrak{J}^2$
IVPFSWG [38]	0.0524	0.0754	0.0241	0/0114	$\mathfrak{J}^2 > \mathfrak{J}^1 > \mathfrak{J}^3 > \mathfrak{J}^4$	$\mathfrak{J}^2$
IVPFHSWA [45]	0.0578	0.0599	0.0266	–0.0382	$\mathfrak{J}^2 > \mathfrak{J}^1 > \mathfrak{J}^3 > \mathfrak{J}^4$	$\mathfrak{J}^2$
IVPFHSWG [45]	0.0654	0.0752	0.0241	0.0114	$\mathfrak{J}^2 > \mathfrak{J}^1 > \mathfrak{J}^3 > \mathfrak{J}^4$	$\mathfrak{J}^2$
IVPFHSIWA	–0.1028	–0.0275	–0.2492	–0.1795	$\mathfrak{J}^2 > \mathfrak{J}^1 > \mathfrak{J}^4 > \mathfrak{J}^3$	$\mathfrak{J}^2$
IVPFHSIWG	0.1419	0.1813	0.0646	0.1417	$\mathfrak{J}^2 > \mathfrak{J}^1 > \mathfrak{J}^4 > \mathfrak{J}^3$	$\mathfrak{J}^2$

So, we have the right to be startled by the manipulation and unpredictability of the DM process for the predominant operators. Planned supplies for this technique-associated exploit have a minor impact on contrary causes. In this way, it reduces the association of unpredictable and presumed particulars in the magnification of DM. Figure 4 spectacles the graphical demo of the comparison analysis.

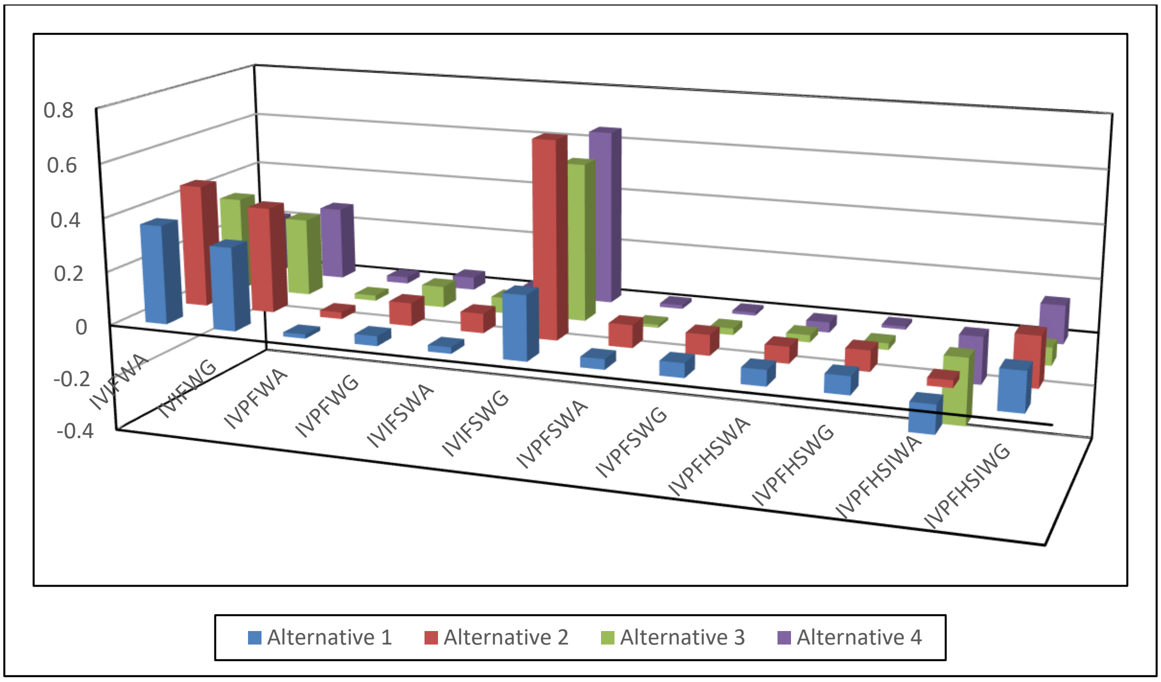

Figure 4. Comparative analysis of the planned method with prevailing models.

6. Conclusions

The assessment of alternatives advocated by DMs is habitually conveyed by stringent checks disturbing DM exploration. An IVPFHSS is a robust mathematical structure for communicating indefinite and unreliable data in real-life surroundings to improve these boundaries. Also, DM is a robust system that expands the ventures of classifying the most helpful alternative. It takes seriously how much existent approach is needed to isolate statistical decision-makers. The most operative methodology in DM is to pay close devotion and attention to your objectives. This research endorses a novel approach for selecting sustainable suppliers under the IVPFHSS setting. A structure that reports on the complication of sustainable suppliers specified in real life. Lack of consideration for interactions and complex scenarios between attributes can impede some challenging implications in MCGDM. Mathematical modeling for supplier selection can overlook certain effects when integrating objectives under financial, superior, and welfare constraints. Surveys should be limited to optimal decisions and assessing decision requirements. In flexible DM, the expert's evaluation of alternative data provided by the expert is often uncertain, irregular, and imprudent, which can be accommodated by using an IVPFHSS to handle this uncertain information. The primary aim of this study is to introduce operational laws for the interactive IVPFHSS setting. Based on the established operational laws, we propose the IVPFHSIWA and IVPFHSIWG operators for IVPFHSS with their desired properties. Furthermore, based on the validated operators, a DM approach has been premeditated to address MCGDM obstacles. The anticipated methodology allows evaluating and selecting green thermal power equipment suppliers with limited or scarce computational data, and IVPFHSN can overwhelm hesitation. A mathematical example is presented to verify that the operator advocating for solving DM agendas has a more representative system. An in-depth examination of several existing techniques is provided. Based on the results achieved, it can be resolute that the scheme anticipated in this study is the most robust and operative approach to address the challenges of MCGDM. Future studies will emphasize defining Einstein AOs, Einstein-ordered AOs, distance, and similarity measures with compatible features. Also, it can be prolonged to interval-valued q-rung orthopair fuzzy hypersoft sets with essential operations and several AOs with their DM approaches. We can also integrate interval-valued q-rung orthopair fuzzy hypersoft numbers with other MCGDM techniques and further engage in practical application in problems of medical diagnosis, material selection, pattern recognition, information fusion, supply chain management, etc. Moreover, several topological, algebraic, and ordered structures can be extant for interval-valued q-rung orthopair fuzzy hypersoft sets with their DM approaches.

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Conflict of interest

The authors declare no competing interests.

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