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Research article

Regularity and higher integrability of weak solutions to a class of non-Newtonian variation-inequality problems arising from American lookback options

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Abstract: This paper presents the proofs of the higher integrability and regularity of weak solutions to a class of variation-inequality problems that are formulated by a non-Newtonian parabolic operator. After obtaining the gradient estimate, the higher order integrability of the weak solution is analyzed. We also examine the internal regularity estimate of the weak solution by utilizing a test function of the difference type.

Keywords: variation-inequality problems; non-Newtonian parabolic operator; higher integrability; regularity
Mathematics Subject Classification: 35K99, 97M30

1. Introduction

The American lookback option is a crucial aspect of option pricing whose value can be determined through a variation-inequality approach. This option allows investors to observe the risk assets' lowest price $S_t, t \in [0, T]$ during the time frame $[0, T]$ and purchase them at that price before the transaction time t. If the option is exercised on the expiry date, its value is calculated as follows [1–3]:

$$V(S_T, J_T, T) = S_T - J_T,$$

where $J_t = \min_{t\in[0,t]} S_t$. American look-back options allow investors to exercise their options at any point during the time interval $[0, T]$. This means that the value of the option, denoted by $V(S_t, J_t, T)$, is greater than or equal to the difference between the stock price at time t, S_t and the minimum stock price within the time interval (J_t), which is represented by

$$V(S_t, J_t, T) \geq S_t - J_t.$$

According to the literature [4], the option value $V(S, J, t)$ at any given time is governed by a variation inequality:

$$\begin{cases} (\partial_t V + \frac{1}{2}\sigma^2 S^2 \partial_{SS} V + rS\partial_S V - rV) \times (V - S + J) = 0, & J \geq 0, S \geq J, t \in [0, T], \\ \partial_t V + \frac{1}{2}\sigma^2 S^2 \partial_{SS} V + rS\partial_S V - rV \geq 0, & J \geq 0, S \geq J, t \in [0, T], \\ V - S + J \geq 0, & J \geq 0, S \geq J, t \in [0, T], \\ V(S, J, T) = S - J, & J \geq 0, S \geq J, \end{cases} \tag{1}$$

where r represents the risk-free interest rate of the financial market, and σ represents the volatility of option-linked stocks.

The theoretical study of variation-inequality has increasingly gained the attention of scholars. In 2022, wu developed a fourth-order p-Laplacian Kirchhoff operator, given by

$$L\phi = \partial_t \phi - \Delta\left((1 + \lambda \|\Delta\phi\|^{p(x)}_{L^{p(x)}(\Omega)})|\Delta\phi|^{p(x)-2}\Delta\phi\right) + \gamma\phi$$

and investigated a variation-inequality problem [5]:

$$\begin{cases} \min\{L\phi, \phi - \phi_0\} = 0, & (x, t) \in \Omega_T, \\ \phi(0, x) = \phi_0(x), & x \in \Omega, \\ \phi(t, x) = 0, & (x, t) \in \partial\Omega \times (0, T), \end{cases} \tag{2}$$

where $\Omega_T = \Omega \times (0, T)$, Ω is a N-dimensional domain with $N \geq 2$, where ϕ_0 is a given function and γ is a positive constant.The Leray-Schauder principle, a penalty function, and inequality amplification techniques were used to establish the existence, stability and uniqueness of the weak solution. Li and Bi [6] examined a 2-D variation-inequality system and proved the existence of weak solutions by analyzing upper and lower solutions of the auxiliary problem. The issues of existence for variation-inequality problems have been extensively studied in [7,8], with relevant literature reviewed. Additionally, uniqueness and stability of weak solutions for variation-inequality problems have been proven in [9–11]. Further results on solvability and well-posedness can be found in [12,13] and related references. However, research regarding regularity and higher integrability of this type of problem appears to be less explored.

We investigate a kind of variation-inequality problem

$$\begin{cases} Lu \geq 0, & (x, t) \in \Omega_T, \\ u - u_0 \geq 0, & (x, t) \in \Omega_T, \\ Lu(u - u_0) = 0, & (x, t) \in \Omega_T, \\ u(0, x) = u_0(x), & x \in \Omega, \\ u(t, x) = \frac{\partial u}{\partial \nu} = 0, & (x, t) \in \partial\Omega \times (0, T) \end{cases} \tag{3}$$

defined on $\Omega \times (0, T)$, featuring a non-Newtonian parabolic operator (4) in which

$$Lu = \partial_t u - \operatorname{div}(|\nabla u|^{p-2}\nabla u) - f, \ p > 2. \tag{4}$$

Here, $u_0 : \Omega \to R$ is measurable, and $u_0 \in H^1(\Omega)$.

We aim to investigate the regularity and higher integrability of weak solutions to problem (1). Our approach involves several steps. First, we establish the existence of Δu by utilizing the weak solution of an auxiliary problem. Next, we analyze the first-order spatial gradient estimation and the time gradient estimation of u by employing the weak solution of the variation-inequality. We then combine the results obtained from both these analyses to obtain higher-order integrability using the inequality amplification technique. Final, we construct the weak solution using the difference operator and approximate the estimate of the partial derivative using its estimate. Through this process, we are able to obtain the regularity of the variational inequality problem.

2. Statement of the problem and some preliminaries

Firstly, we present the weak solution of the variation-inequality. This solution's existence can be found in various literatures [5,6]. To do so, we provide a set of maximal monotone maps given by

$$G = \{u|u(x) = 0, x > 0; u(x) \in [0, -M_0], \quad x = 0\} \tag{5}$$

where M_0 is a positive constant.

Using a standard energy method, it can be shown that variation-inequality (3) has a unique solution if and only if,

(a) $u \in L^\infty(0,T,H^1(\Omega))$, $\partial_t u \in L^\infty(0,T,L^2(\Omega))$ and $\xi \in G$ for any $(x,t) \in \Omega_T$,
(b) $u(x,t) \geq u_0(x)$, $u(x,0) = u_0(x)$ for any $(x,t) \in \Omega_T$,
(c) for every test-function $\varphi \in C^1(\bar{\Omega}_T)$, there admits the equalities

$$\int\int_{\Omega_T} \partial_t u \cdot \varphi + |\nabla u|^{p-2}\nabla u \nabla\varphi \mathrm{d}x\mathrm{d}t + \int\int_{\Omega_T} f\varphi \mathrm{d}x\mathrm{d}t = \int\int_{\Omega_T} \xi \cdot \varphi \mathrm{d}x\mathrm{d}t \tag{6}$$

and

$$\int\int_{\Omega_T} \partial_t u \cdot \varphi - \mathrm{div}(|\nabla u|^{p-2}\nabla u) \cdot \varphi \mathrm{d}x\mathrm{d}t + \int\int_{\Omega_T} f\varphi \mathrm{d}x\mathrm{d}t = \int\int_{\Omega_T} \xi \cdot \varphi \mathrm{d}x\mathrm{d}t. \tag{7}$$

To analyze the regularity of variation-inequality (3), we need to introduce the following operators and their corresponding results. Readers can refer to literature [14] for proofs of some of these results.

The difference operator of $u(x,t)$ in the e_i direction, denoted by $\Delta_h^i u(x,t)$, is given by

$$\Delta_h^i u(x,t) = \frac{u(x+he_i,t) - u(x,t)}{h}$$

where e_i is the unit vector in the x_i direction.

Lemma 2.1. *[14] (1) Assume that Δ_h^{i*} is the conjugate operator of Δ_h^i satisfies $\Delta_h^{i*} = -\Delta_{-h}^i$. Then, one gets*

$$\int_{R_n} f(x)\Delta_h^i g(x)\mathrm{d}x = \int_{R_n} g(x)\Delta_h^{i*} f(x)\mathrm{d}x.$$

(2) D_j and Δ_h^i are interchangeable, in other words,

$$D_j\Delta_h^i f(x) = \Delta_h^i D_j f(x), i = 1,2,\cdots,N, j = 1,2,\cdots,N.$$

(3) Suppose T_h^i represents the displacement operator in the x_i direction. Then,

$$\Delta_h^i f(x)g(x) = f(x)\Delta_h^i g(x) + T_h^i g(x)\Delta_h^i f(x).$$

(4) If $u \in W^{1,p}(\Omega)$, then for any $\Omega' \subset\subset \Omega$, we have $\|\Delta_h^i u\|_{L^p(\Omega')} \leq \|D_i u\|_{L^p(\Omega)}$.
(5) Let h be small enough. If $\|\Delta_h^i u\|_{L^p(\Omega)} \leq C$, then $\|D_i u\|_{L^p(\Omega)} \leq C$.

Using Holder inequalities and combining with Lemma 2.1 (4), it is clear to verify that

$$\int_\Omega |D_i u \Delta_h^i u| \mathrm{d}x \le \int_\Omega |D_i u|^2 \mathrm{d}x, \ \int_\Omega |D_i u \Delta_h^{i*} u| \mathrm{d}x \le \int_\Omega |D_i u|^2 \mathrm{d}x. \tag{8}$$

Lemma 2.2. *Let* $u \in H^2(\Omega)$*, and h be small enough. If* $\int_\Omega |D_i u \Delta_h^i u| \mathrm{d}x \le C$ *, then*

$$\int_\Omega |\Delta_h^i u|^2 \mathrm{d}x \le C. \tag{9}$$

Proof. Using Taylor expansion method, there is an $\theta \in [0, 1]$, such that

$$u(x + he_i, t) = u(x, t) + D_i u(x, t)h + \frac{1}{2} D_i^2 u(x + \theta he_i, t)h^2.$$

Rearranging the above equation, we have

$$D_i u(x, t) = \frac{u(x + he_i, t) - u(x, t)}{h} - \frac{1}{2} D_i^2 u(x + \theta he_i, t)h.$$

Since $u \in H^2(\Omega)$,

$$\int_\Omega |D_i u \Delta_h^i u| \mathrm{d}x \ge \int_\Omega |\Delta_h^i u|^2 \mathrm{d}x + |\Omega| \mathrm{O}(h).$$

Then, if h is small enough, (9) holds. □

Lemma 2.3. *Assume* $u \in H^2(\Omega)$*, and m is a positive integer. Let h be small enough. If* $|\int_\Omega D_i[(\Delta_h^i u)^m] \Delta_h^i [(D_i u)^m] \mathrm{d}x| \le C$*, then we have*

$$\int_\Omega |\Delta_h^i [(\Delta_h^i u)^m]|^2 \mathrm{d}x \le C, \ \int_\Omega |D_i[(\Delta_h^i u)^m]|^2 \mathrm{d}x \le C.$$

Proof. Since $D_i u(x, t) = \Delta_h^i u + \mathrm{O}(h)$, carrying out binomial expansion to $[\Delta_h^i u + \mathrm{O}(h)]^m$ gives $(D_i u)^m = (\Delta_h^i u)^m + \mathrm{O}(h)$, so

$$|\int_\Omega D_i[(\Delta_h^i u)^m] \Delta_h^i [(D_i u)^m] \mathrm{d}x| = |\int_\Omega D_i[(\Delta_h^i u)^m] \Delta_h^i [(\Delta_h^i u)^m] \mathrm{d}x| + \mathrm{O}(h).$$

If h is small enough, using Lemma 2.2 obtains

$$|\int_\Omega D_i[(\Delta_h^i u)^m] \Delta_h^i [(\Delta_h^i u)^m] \mathrm{d}x| \ge \int_\Omega |\Delta_h^i [(D_i u)^m]|^2 \mathrm{d}x + \mathrm{O}(h).$$

Hence, the first part of Lemma 2.3 is proved from which the second part is an immediate result. □

3. Exisence of Δu

First, we consider the existence of Δu and use ε to construct a penalty function $\beta_\varepsilon(\cdot)$ to control the variation-inequality (3). The penalty map $\beta_\varepsilon : \mathbb{R} \to \mathbb{R}_-$ satisfies

$$\beta_\varepsilon(x) = 0 \text{ if } x > \varepsilon, \ \beta_\varepsilon(x) \in [-M_0, 0) \text{ if } x \in [0, \varepsilon]. \tag{10}$$

Furthermore, we use the following auxiliary problem to approach the variation-inequality (3):

$$\begin{cases} Lu_\varepsilon = -\beta_\varepsilon(u_\varepsilon - u_0), & (x,t) \in \Omega_T, \\ u_\varepsilon(x,0) = u_{0\varepsilon}(x), & x \in \Omega, \\ u_\varepsilon(x,t) = \varepsilon, & (x,t) \in \partial\Omega_T, \end{cases} \tag{11}$$

where

$$L_\varepsilon u_\varepsilon = \partial_t u_\varepsilon - \mathrm{div}((|\nabla u_\varepsilon|^2 + \varepsilon)^{\frac{p-2}{2}} \nabla u_\varepsilon) + f.$$

Using a similar method as in [5,6], we can find a solution u_ε for problem (11) that satisfies $u_\varepsilon \in L^\infty(0,T; W^{1,p}(\Omega))$, $\partial_t u_\varepsilon \in L^\infty(0,T; L^2(\Omega))$, and the identity

$$\int_\Omega (\partial_t u_\varepsilon \cdot \varphi + (|\nabla u_\varepsilon|^2 + \varepsilon)^{\frac{p-2}{2}} \nabla u_\varepsilon \nabla\varphi + f\varphi)\mathrm{d}x = -\int_\Omega \beta_\varepsilon(u_\varepsilon - u_0)\varphi \mathrm{d}x \tag{12}$$

with $\varphi \in C^1(\bar{\Omega}_T)$. Additionally, for any $\varepsilon \in (0,1)$,

$$u_{0\varepsilon} \le u_\varepsilon \le |u_0|_\infty + \varepsilon, \ u_{\varepsilon_1} \le u_{\varepsilon_2} \text{ for } \varepsilon_1 \le \varepsilon_2. \tag{13}$$

By choosing u_ε as the test function in Eq (12), we can follow a similar approach as in [5] to obtain the following inequality:

$$\|\nabla u_\varepsilon\|_{L^\infty(0,T;L^p(\Omega))} \le \|(|\nabla u_\varepsilon^m|^2 + \varepsilon)^{\frac{p-2}{2}} |\nabla u_\varepsilon^m|^2\|_{L^\infty(0,T;L^1(\Omega))} \le C.$$

This implies that for any $\varepsilon \in (0,1)$, there exists a subsequence of $\{u_\varepsilon\}_{0<\varepsilon<1}$ (which we still denote by $\{u_\varepsilon\}_{0<\varepsilon<1}$) and a function v, such that:

$$\Delta u_\varepsilon \to v \text{ in } L^\infty(0,T; L^p(\Omega)) \text{ as } \varepsilon \to 0.$$

Indeed, from [13], we know that $\{u_\varepsilon, \varepsilon \in (0,1)\}$ is a bounded sequence, which allows us to extract a subsequence without loss of generality, denoted again by $\{u_\varepsilon, \varepsilon \in (0,1)\}$, that converges almost everywhere in Ω_T to some function u:

$$u_\varepsilon \to u \text{ a.e.in } \Omega_T \text{ as } \varepsilon \to 0. \tag{14}$$

Now, we verify $v = \Delta u$. By suing integral by part, one gets

$$\int\int_{\Omega_T} \Delta u_\varepsilon \phi \mathrm{d}x = -\int\int_{\Omega_T} u_\varepsilon \Delta\phi \mathrm{d}x.$$

Thus, from Eq (14) we have for any $\phi \in C_0^\infty(\Omega)$,

$$\int\int_{\Omega_T} \Delta u_\varepsilon \phi \mathrm{d}x = -\int\int_{\Omega_T} u_\varepsilon \Delta\phi \mathrm{d}x \to -\int\int_{\Omega_T} u\Delta\phi \mathrm{d}x = \int\int_{\Omega_T} \Delta u\phi \mathrm{d}x.$$

This implies that $v = \Delta u$ a.e. in Ω_T.

4. Higher integrability of the gradient

This section gives several gradient estimates of u. First, choosing u as a test function in (6) gives

$$\int\int_{\Omega_T} \partial_t u \cdot u + |\nabla u|^p \mathrm{d}x\mathrm{d}t + \int\int_{\Omega_T} fu\mathrm{d}x\mathrm{d}t = \int\int_{\Omega_T} \xi \cdot u\mathrm{d}x\mathrm{d}t. \tag{15}$$

Since $\int\int_{\Omega_T} \partial_t u \cdot u\mathrm{d}x\mathrm{d}t = \int_0^T \int_\Omega \partial_t u^2\mathrm{d}x\mathrm{d}t = \int_\Omega |u(,T)|^2\mathrm{d}x - \int_\Omega |u_0|^2\mathrm{d}x$, one, from (15) can get that

$$\int\int_{\Omega_T} |\nabla u|^p \mathrm{d}x\mathrm{d}t \le \int\int_{\Omega_T} \xi \cdot u\mathrm{d}x\mathrm{d}t + \int_\Omega |u_0|^2\mathrm{d}x. \tag{16}$$

Applying Holder and Young inequalities,

$$\int_0^T \int_\Omega \xi \cdot \Delta u\mathrm{d}x\mathrm{d}t \le (p-1)/pM_0^{p/(p-1)}T|\Omega| + \frac{1}{p}\int\int_{\Omega_T} |\Delta u|^p\mathrm{d}x\mathrm{d}t. \tag{17}$$

Combining (16) and (17), it is inferred that (note that $p \ge 1$),

$$\int_0^T \int_\Omega |\nabla u|^p \mathrm{d}x\mathrm{d}t \le \int_\Omega |\nabla u_0|^2\mathrm{d}x + (p-1)/pM_0^{p/(p-1)}T|\Omega|. \tag{18}$$

Second, letting $\partial_t u$ be a test function in (6), we can find that

$$\int\int_{\Omega_T} |\partial_t u|^2 + |\nabla u|^{p-2}\nabla u\nabla\partial_t u\mathrm{d}x\mathrm{d}t + \int\int_{\Omega_T} f\partial_t u\mathrm{d}x\mathrm{d}t = \int\int_{\Omega_T} \xi \cdot \partial_t u\mathrm{d}x\mathrm{d}t. \tag{19}$$

Using differential transformation technology, one gets

$$\begin{aligned}&\int\int_{\Omega_T} |\nabla u|^{p-2}\nabla u\nabla\partial_t u\mathrm{d}x\mathrm{d}t\\ &= \tfrac{1}{p}\int\int_{\Omega_T} |\nabla u|^p\mathrm{d}x\mathrm{d}t = \tfrac{1}{p}\|\nabla u(\cdot,T)\|_{L^p(\Omega)} - \tfrac{1}{p}\|\nabla u_0\|_{L^p(\Omega)}.\end{aligned} \tag{20}$$

Applying Holder and Young inequalities with parameters $(1/2, 1/2)$,

$$\left|\int\int_{\Omega_T} f\partial_t u\mathrm{d}x\mathrm{d}t\right| \le C(f,|\Omega|,T) + \frac{1}{8}\int\int_{\Omega_T} |\partial_t u|^2\mathrm{d}x\mathrm{d}t, \tag{21}$$

$$\left|\int\int_{\Omega_T} \xi \cdot \partial_t u\mathrm{d}x\mathrm{d}t\right| \le C(M_0,|\Omega|,T) + \frac{1}{8}\int\int_{\Omega_T} |\partial_t u|^2\mathrm{d}x\mathrm{d}t. \tag{22}$$

Combining (19)–(22) and dropping the term $\frac{1}{p}\|\nabla u(\cdot,T)\|_{L^p(\Omega)}$, one can get

$$\frac{3}{4}\int\int_{\Omega_T} |\partial_t u|^2\mathrm{d}x\mathrm{d}t \le \frac{1}{p}\|\nabla u_0\|_{L^p(\Omega)} + C(M_0,f,|\Omega|,T). \tag{23}$$

Final, we choose φ= $\mathrm{div}(|\nabla u|^{p-2}\nabla u)$ in (7) to arrive at

$$\begin{aligned}&\int\int_{\Omega_T} \partial_t u \cdot \mathrm{div}(|\nabla u|^{p-2}\nabla u) + |\mathrm{div}(|\nabla u|^{p-2}\nabla u)|^2\mathrm{d}x\mathrm{d}t\\ &+ \int\int_{\Omega_T} f\mathrm{div}(|\nabla u|^{p-2}\nabla u)\mathrm{d}x\mathrm{d}t = \int\int_{\Omega_T} \xi \cdot \mathrm{div}(|\nabla u|^{p-2}\nabla u)\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{24}$$

Applying integral by part and joining with (20),

$$\begin{aligned}&\int\int_{\Omega_T}\partial_t u\cdot \mathrm{div}(|\nabla u|^{p-2}\nabla u)\mathrm{d}x\mathrm{d}t\\&=\int\int_{\Omega_T}|\nabla u|^{p-2}\nabla u\nabla\partial_t u\mathrm{d}x\mathrm{d}t=\frac{1}{p}\|\nabla u(\cdot,T)\|_{L^p(\Omega)}-\frac{1}{p}\|\nabla u_0\|_{L^p(\Omega)}.\end{aligned}\tag{25}$$

Applying Holder and Young inequalities to $\int\int_{\Omega_T} f\mathrm{div}(|\nabla u|^{p-2}\nabla u)\mathrm{d}x\mathrm{d}t$ and $\int\int_{\Omega_T}\xi\cdot\mathrm{div}(|\nabla u|^{p-2}\nabla u)\mathrm{d}x\mathrm{d}t$,

$$\left|\int\int_{\Omega_T} f\mathrm{div}(|\nabla u|^{p-2}\nabla u)\mathrm{d}x\mathrm{d}t\right|\le C(f,|\Omega|,T)+\frac{1}{8}\int\int_{\Omega_T}|\mathrm{div}(|\nabla u|^{p-2}\nabla u)|^2\mathrm{d}x\mathrm{d}t,\tag{26}$$

$$\int\int_{\Omega_T}\xi\cdot\mathrm{div}(|\nabla u|^{p-2}\nabla u)\mathrm{d}x\mathrm{d}t\le C(M_0,|\Omega|,T)+\frac{1}{8}\int\int_{\Omega_T}|\mathrm{div}(|\nabla u|^{p-2}\nabla u)|^2\mathrm{d}x\mathrm{d}t.\tag{27}$$

Substituting (26) and (27) to (27) and dropping the term $\frac{1}{p}\|\nabla u(\cdot,T)\|_{L^p(\Omega)}$,

$$\frac{3}{4}\int\int_{\Omega_T}|\mathrm{div}(|\nabla u|^{p-2}\nabla u)|^2\mathrm{d}x\mathrm{d}t\le\frac{1}{p}\|\nabla u_0\|_{L^p(\Omega)}+C(M_0,f,|\Omega|,T).\tag{28}$$

Theorem 4.1. *If* $u_0\in W_0^{1,p}(\Omega)$ *and* $f\in L^1(0,T;L^2(\Omega))$*, then*

$$\|\mathrm{div}(|\nabla u|^{p-2}\nabla u)\|^2_{L^2(\Omega_T)}\le\frac{4}{3p}\|\nabla u_0\|_{L^p(\Omega)}+C(M_0,f,|\Omega|,T).$$

It should be pointed out that the higher-order term $\mathrm{div}(|\nabla u|^{p-2}\nabla u)$ in the non-Newtonian parabolic operator Lu has not been explained before and is substituted into Eq (24). While the final result displays the boundedness of $\mathrm{div}(|\nabla u|^{p-2}\nabla u)$, a more reasonable proof may be required. Thus, we provide such a proof in Section 5.

5. Regularity of solution

This section considers the regularity estimate of weak solutions. We draw inspiration from literature [14] and introduce the difference operator Δ_h^i and its conjugate Δ_h^{*i} into the test function. Since $u\in L^\infty(0,T;W^{1,p}(\Omega))$, we have

$$\varphi=\Delta_h^{i*}(|\Delta_h^i u|^{p-2}\Delta_h^i u)\in L^\infty(0,T;W^{1,p}(\Omega)).$$

By choosing φ as a test function in (7), we get:

$$\begin{aligned}&\int\int_{\Omega_T}\partial_t u\cdot\Delta_h^{i*}(|\Delta_h^i u|^{p-2}\Delta_h^i u)+|\nabla u|^{p-2}\nabla u\cdot\nabla\Delta_h^{i*}(|\Delta_h^i u|^{p-2}\Delta_h^i u)\mathrm{d}x\mathrm{d}t\\&+\int\int_{\Omega_T} f\cdot\Delta_h^{i*}(|\Delta_h^i u|^{p-2}\Delta_h^i u)\mathrm{d}x\mathrm{d}t=\int_0^t\int_\Omega\xi\cdot\Delta_h^{i*}(|\Delta_h^i u|^{p-2}\Delta_h^i u)\mathrm{d}x\mathrm{d}t.\end{aligned}\tag{29}$$

We first consider $\int\int_{\Omega_T}\partial_t u\cdot\Delta_h^{i*}(|\Delta_h^i u|^{p-2}\Delta_h^i u)\mathrm{d}x\mathrm{d}t$. Using Lemma 2.1 (1) and some differential transformation technologies

$$\begin{aligned}&\int\int_{\Omega_T}\partial_t u\cdot\Delta_h^{i*}(|\Delta_h^i u|^{p-2}\Delta_h^i u)\mathrm{d}x\mathrm{d}t\\&=\int_{\Omega_T}\partial_t(\Delta_h^i u)(|\Delta_h^i u|^{p-2}\Delta_h^i u)\mathrm{d}x\mathrm{d}t=\frac{1}{p}\int_{\Omega_T}\partial_t|\Delta_h^i u|^p\mathrm{d}x\mathrm{d}t\\&=\frac{1}{p}\int_\Omega|\Delta_h^i u(x,T)|^2\mathrm{d}x-\frac{1}{p}\int_\Omega|\Delta_h^i u_0|^2\mathrm{d}x.\end{aligned}\tag{30}$$

Substituting (30) into (29) and removing the non negative term $\frac{1}{p}\int_\Omega |\Delta_h^i u(x,T)|^2 \mathrm{d}x$, one gets

$$\begin{aligned}&\int\int_{\Omega_T} |\nabla u|^{p-2}\nabla u\cdot\nabla\Delta_h^{i*}(|\Delta_h^i u|^{p-2}\Delta_h^i u)\mathrm{d}x\mathrm{d}t+\int\int_{\Omega_T} f\cdot\Delta_h^{i*}(|\Delta_h^i u|^{p-2}\Delta_h^i u)\mathrm{d}x\mathrm{d}t\\ &\le\int\int_{\Omega_T}\xi\cdot\Delta_h^{i*}(|\Delta_h^i u|^{p-2}\Delta_h^i u)\mathrm{d}x\mathrm{d}t+\frac{1}{p}\int_\Omega|\Delta_h^i u_0|^2\eta^2\mathrm{d}x.\end{aligned}\tag{31}$$

Using the commutative properties of Δ_h^{i*} and ∇

$$\int\int_{\Omega_T}\left||\nabla u|^{p-2}\nabla u\cdot\nabla\Delta_h^{i*}(|\Delta_h^i u|^{p-2}\Delta_h^i u)\right|\mathrm{d}x\mathrm{d}t=\int_0^T\int_\Omega\Delta_h^i(|\nabla u|^{p-2}\nabla u)\cdot\nabla(|\Delta_h^i u|^{p-2}\Delta_h^i u)\mathrm{d}x\mathrm{d}t.$$

It follows from Lemma 2.4 that if h is small enough,

$$\int\int_{\Omega_T}\left||\nabla u|^{p-2}\nabla u\cdot\nabla\Delta_h^{i*}(|\Delta_h^i u|^{p-2}\Delta_h^i u)\right|\mathrm{d}x\mathrm{d}t\ge\int_0^T\int_\Omega|\Delta_h^i(|\Delta_h^i u|^{p-2}\Delta_h^i u)|^2\mathrm{d}x\mathrm{d}t.\tag{32}$$

So, combining (31) and (32) and applying Lemma 2.1 (5), inequality (31) can be rewritten as

$$\begin{aligned}&\int_0^T\int_\Omega|D_i(|\Delta_h^i u|^{p-2}\Delta_h^i u)|^2\mathrm{d}x\mathrm{d}t\\ &\le\int_0^t\int_\Omega\xi\cdot\Delta_h^{i*}(|\Delta_h^i u|^{p-2}\Delta_h^i u)\mathrm{d}x\mathrm{d}t+\frac{1}{p}\int_\Omega|\Delta_h^i u_0|^2\mathrm{d}x-\int\int_{\Omega_T} f\cdot\Delta_h^{i*}(|\Delta_h^i u|^{p-2}\Delta_h^i u)\mathrm{d}x\mathrm{d}t.\end{aligned}\tag{33}$$

Applying Holder and Young inequalities,

$$\begin{aligned}&\int_0^T\int_\Omega f\Delta_h^{i*}(|\Delta_h^i u|^{p-2}\Delta_h^i u)\mathrm{d}x\mathrm{d}t\\ &\le 2\int_0^T\int_\Omega f^2\mathrm{d}x\mathrm{d}t+\frac{1}{8}\int_0^T\int_\Omega\left[\Delta_h^{i*}(|\Delta_h^i u|^{p-2}\Delta_h^i u)\right]^2\mathrm{d}x\mathrm{d}t\\ &\le 2\int_0^T\int_\Omega f^2\mathrm{d}x\mathrm{d}t+\frac{1}{8}\int_0^T\int_\Omega\left[D_i(|\Delta_h^i u|^{p-2}\Delta_h^i u)\right]^2\mathrm{d}x\mathrm{d}t,\end{aligned}\tag{34}$$

$$\begin{aligned}&\int_0^t\int_\Omega\xi\cdot\Delta_h^{i*}(|\Delta_h^i u|^{p-2}\Delta_h^i u)\mathrm{d}x\mathrm{d}t\\ &\le C(M_0,|\Omega_0|,T)+\frac{1}{8}\int_0^T\int_\Omega\left[\Delta_h^{i*}(|\Delta_h^i u|^{p-2}\Delta_h^i u)\right]^2\mathrm{d}x\mathrm{d}t\\ &\le C(M_0,|\Omega_0|,T)+\frac{1}{8}\int_0^T\int_\Omega\left[D_i(|\Delta_h^i u|^{p-2}\Delta_h^i u)\right]^2\mathrm{d}x\mathrm{d}t.\end{aligned}\tag{35}$$

Inserting (34) and (35) into (33), it is inferred that

$$\int_0^T\int_\Omega|D_i(|\Delta_h^i u|^{p-2}\Delta_h^i u)|^2\mathrm{d}x\mathrm{d}t\le C(M_0,|\Omega_0|,T)+4\int_0^T\int_\Omega f^2\mathrm{d}x\mathrm{d}t+\frac{2}{p}\int_\Omega|\Delta_h^i u_0|^2\mathrm{d}x.$$

From Lemma 2.1 (4), we have $\int_\Omega|\Delta_h^i u_0|^2\mathrm{d}x\le\int_\Omega|D_i u_0|^2\mathrm{d}x$, so one can use lemma 2.1 (5) to arrive at

$$\int_0^T\int_\Omega|D_i(|D_i u|^{p-2}D_i u)|^2\mathrm{d}x\mathrm{d}t\le C(M_0,|\Omega_0|,T)+4\int_0^T\int_\Omega f^2\mathrm{d}x\mathrm{d}t+\frac{2}{p}\int_\Omega|D_i u_0|^2\mathrm{d}x.\tag{36}$$

Adding the above formula from 1 to N, we summarize the following result.

Theorem 5.1. *If* $u_0\in W_0^{1,p}(\Omega)$ and $f\in L^1(0,T;L^2(\Omega))$*, then*

$$\|\mathrm{div}(|\nabla u|^{p-2}\nabla u)\|^2_{L^2(\Omega_T)}\le C(M_0,|\Omega_0|,T)+4\int_0^T\int_\Omega f^2\mathrm{d}x\mathrm{d}t+\frac{2}{p}\int_\Omega|\nabla u_0|^2\mathrm{d}x.$$

Using Poincare inequality twice, it can be easily verified that

$$\|\nabla u\|_{L(0,T;L^{2p-2}(\Omega))}\le C_{\mathrm{poincare}}\|\mathrm{div}(|\nabla u|^{p-2}\nabla u)\|^2_{L^2(\Omega_T)},$$

$$\|u\|_{L(0,T;L^{2p-2}(\Omega))}\le C_{\mathrm{poincare}}\|\nabla u\|_{L(0,T;L^{2p-2}(\Omega))},$$

where C_{poincare} is the poincare parameter, such that we obtain the following theorem.

Theorem 5.2. *For* $u_0\in W_0^{1,p}(\Omega)$ and $f\in L^1(0,T;L^2(\Omega))$*, we have* $u\in L(0,T;W^{1,2p-2}(\Omega))$.

6. Conclusions

In this paper, we investigate the variation-inequality problem (3) featuring a non-Newtonian operator. First, we establish the norm boundedness of the gradient ∇u_ε based on the weak solution of the auxiliary problem. We then utilize weak limit to prove the existence of the gradient of the solution to the variation-inequality (3). Second, we analyze the higher order integrability of the solutions of variation-inequality (3). To achieve this, we use the weak Eq (6) of variation-inequality (3), which is fundamental in the study of higher order integrability, to obtain the gradient estimate of the solutions (18) and (23). Further, we select the higher order term of the non-Newtonian parabolic operator as the test function and combine the gradient estimations (18) and (23) to establish the higher order integrability of the weak solution. Last, from the perspective of regularity, we obtain high-order gradient estimates for the variational inequality (3).

It is important to note that using second-order spatial partial derivatives as test functions to analyze regularity and higher-order integrability may not meet the conditions for weak solutions. To avoid discussing the existence and rationality of the test function, we construct it using the difference function. This leads to the derivation of integral inequality (31), which is crucial for proving regularity. The regularity of weak solutions is estimated using Holder and Young inequalities.

There are still some points worth discussing in this article. If we introduce the cutoff factor η in formula (29) to construct the test function as follows:

$$\varphi = \Delta_h^{i*}(\eta^2|\Delta_h^i u|^{p-2}\Delta_h^i u) \in L^\infty(0,T;W^{1,p}(\Omega)).$$

Here, $\eta \in C_0^\infty(\Omega)$ is a cutoff factor on $\Omega' \subset\subset \Omega$ satisfying:

$$0 \leq \eta \leq 1, \eta = 1 \text{ in } \Omega', \text{dist}(\text{supp}\eta, \Omega) \geq 2d, d = \text{dist}(\Omega', \Omega).$$

By following the proof in Section 5, we obtain the following estimate:

$$\begin{aligned}&\|\text{div}(|\nabla u|^{p-2}\nabla u)\|^2_{L^2(\Omega_T)}\\ &\leq C(M_0, |\Omega_0|, T) + C\|\nabla u\|^2_{L^{2p-2}(\Omega_T)} + 4\int_0^T\int_\Omega f^2\text{d}x\text{d}t + \tfrac{2}{p}\int_\Omega |\nabla u_0|^2\text{d}x.\end{aligned}$$

If $p = 2$, we can easily obtain the following:

$$\|\Delta u\|^2_{L^2(\Omega_T)} \leq C(M_0, |\Omega_0|, T) + C\|\nabla u\|^2_{L^2(\Omega_T)} + 4\int_0^T\int_\Omega f^2\text{d}x\text{d}t + \int_\Omega |\nabla u_0|^2\text{d}x.$$

In this case, it is easy to deduce that $u \in L(0, T, H^k(\Omega))$, where k is a positive integer satisfying $k \geq 2$. However, when $p \geq 2$, it is impossible to obtain a similar result as in the case of $p = 2$ because we cannot prove $\|\Delta u\||^2_{L^{2p-2}(\Omega_T)} \leq \|\text{div}(|\nabla u|^{p-2}\nabla u)\|^2_{L^2(\Omega_T)}$.

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Conflict of interest

The author declares no conflict of interest.

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