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Research article

The C^3 parametric eighth-degree interpolation spline function

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Abstract: The C^3 parametric interpolation spline function is presented this paper, which has the similar properties of the classical cubic Hermite interpolation spline with additional flexibility and high approximation rates. Moreover, a group of eighth-degree bases with three parameters is constructed. Then, the interpolation spline function is defined based on the proposed basis functions. And the interpolation error and the technique for determining the optimal interpolation are also given. The results show that when the interpolation conditions remain unchanged, the proposed interpolation spline functions retain C^3 continuity, and the shape of the curve can be controlled by the parameters. When the optimal values of parameters are chosen, the interpolation spline function can achieve higher approximation rates.

Keywords: cubic Hermite spline; interpolated function; approximation; C^3 continuity
Mathematics Subject Classification: 65D07

1. Introduction

In CAGD and CAD, it is always important to be able to adjust and control the shape of designs. The original method is to alter the position of the control points to meet the requirements of modification, which is out dated. In recent decades, it is feasible to modify the shape by introducing parameters, such as Bézier curve with parameters in [1–3], B-spline curve with parameters in [4–6]. However, these methods are complicated in the design and computation processes. The cubic splines are widely applied because of its simplicity and convenience. Nevertheless, the limitations are the inflexibility and low degree of continuity in [7–9]. In order to take full advantages and overcome the shortcomings of the cubic splines, many authors have studied Hermite splines with parameters, such as piecewise rational cubic Hermite interpolation splines with parameters in [10,11], piecewise Quartic Hermite interpolation splines with parameters in [12], and piecewise cubic trigonometric Hermite interpolation splines with parameters in [13].

Although these interpolation splines can control the shape of the spline with the given interpolation condition and have C^1 continuity, it is essential to impose certain constraints on the parameters to make the spline C^2 continuous. Furthermore, in order to simultaneously solve the shortcomings of interpolation splines in shape control and continuity, the papers [14,15] proposed quintic Hermite interpolation splines with parameters and cubic trigonometric Hermite interpolation splines with parameters respectively. These two types of Hermite interpolation splines can not only regulate the shape of the spline via the parameters, but also automatically possesses C^2 continuity. Nonetheless, in some practical engineering, interpolation splines with C^3 continuity are required. For example, in the process of acceleration and turning of automatic vehicles such as cars and trains, in order to make passengers feel comfortable, it is necessary to maintain the uniform change of acceleration, while C^2 continuity can merely maintain the uniform change of speed. Therefore, it is necessary to consider the continuity of acceleration (i.e., C^3continuity) when designing highway and railway routes [16]. The article [17] constructed a class of seventh-degree interpolation splines with C^3 continuity, but the shape of the interpolation curve can only be adjusted globally. If the curve shape needs to be modified locally, this kind of spline is insufficient.

The eighth-degree polynomial interpolation splines with three parameters are constructed in this paper, which has the following characteristics:

(1) The interpolation function reach C^3 continuity;

(2) Given the interpolated function, the method of choosing the parameter values is provided so that the interpolation function has a high degree of approximation.

2. The Eighth-degree interpolation basis functions

Definition 1. For any given parameter α, β, γ and $0 \le t \le 1$, the following four functions with variable *t*

$$\begin{cases} f_0(t) = a_0(t)\alpha + b_0(t) + \gamma e(t) \\ f_1(t) = a_1(t)\alpha + b_1(t) + \gamma e(t) \\ g_0(t) = c_0(t)\beta + d_0(t) + \gamma e(t) \\ g_1(t) = c_1(t)\beta + d_1(t) + \gamma e(t) \end{cases} \tag{1}$$

are called eighth-degree interpolation basis functions with parameters, where

$$\begin{cases} a_0(t) = t^2 - 10t^4 + 20t^5 - 15t^6 + 4t^7 \\ b_0(t) = 1 - 35t^4 + 84t^5 - 70t^6 + 20t^7 \\ a_1(t) = 5t^4 - 14t^5 + 13t^6 - 4t^7 \\ b_1(t) = 35t^4 - 84t^5 + 70t^6 - 20t^7 \\ c_0(t) = t^3 - 4t^4 + 6t^5 - 4t^6 + t^7 \\ d_0(t) = t - 20t^4 + 45t^5 - 36t^6 + 10t^7 \\ c_1(t) = -t^4 + 3t^5 - 3t^6 + t^7 \\ d_1(t) = -15t^4 + 39t^5 - 34t^6 + 10t^7 \\ e(t) = t^4 - 4t^5 + 6t^6 - 4t^7 + t^8 \end{cases}$$

Also, when $\gamma = 0$, Equation (1) degenerates to the seventh-degree Hermite basis functions in Eq (1) in the paper [16].

Theorem 1. The eighth-degree interpolation basis functions have the properties of end-points as follows:

(i) $\begin{cases} f_0(0) = 1, f_1(1) = 0, g_0(0) = 0, g_1(1) = 0 \\ f_0(0) = 0, f_1(1) = 1, g_0(0) = 0, g_1(1) = 0 \end{cases}$'

(ii) $\begin{cases} f_0{}'(0) = 0, f_1{}'(1) = 0, g_0{}'(0) = 1, g_1{}'(1) = 0 \\ f_0{}'(0) = 0, f_1{}'(1) = 0, g_0{}'(0) = 0, g_1{}'(1) = 1 \end{cases}$'

(iii) $\begin{cases} f_0{}''(0) = 2\alpha, f_1{}''(0) = 0, g_0{}''(0) = 0, g_1{}''(1) = 0 \\ f_0{}''(0) = 0, f_1{}''(1) = 2\alpha, g_0{}''(0) = 0, g_1{}''(1) = 0 \end{cases}$'

(iv) $\begin{cases} f_0{}'''(0) = 0, f_1{}'''(1) = 0, g_0{}'''(0) = 6\beta, g_1{}'''(1) = 0 \\ f_0{}'''(0) = 0, f_1{}'''(1) = 0, g_0{}'''(0) = 0, g_1{}'''(1) = 6\beta \end{cases}$.

3. Parametric eighth-degree interpolation spline functions

3.1. Definition of the eighth-degree interpolation spline function

Based on the eighth-degree basis functions (1), we can also define the corresponding interpolation spline function.

Definition 2. Let the function $y = f(x)$ be defined on the interval $[a, b]$, and $a = x_0 < x_1 < \cdots < x_n = b$ be a subdivision of the interval $[a, b]$. Denote $h_i = x_{i+1} - x, t = \frac{x - x_i}{h_i}$, then the following functions on the interval $[x_i, x_{i+1}] (i = 0,1, \cdots, n - 1)$

$$s_i(x) = f_0(t, \alpha_i, \gamma_i) y_i + f_1(t, \alpha_i, \gamma_i) y_{i+1} + g_0(t, \beta_i, \gamma_i) h_i m_i + g_1(t, \beta_i, \gamma_i) h_i m_{i+1} \qquad (2)$$

are the eighth-degree interpolation spline function to the interpolated function $y = f(x)$,where $f_i(t)$ and $g_i(t) (i = 0,1)$ are the basis function expressed in Eq (1), and $\alpha_i, \beta_i, \gamma_i$ are the parameters.

3.2. Error estimation of interpolation spline function

When the interpolation spline function satisfies certain conditions, the error estimation of the function (3) can be discussed.

According to Theorem 1 and the function (2), we can get that for any parameter $\alpha_i, \beta_i, \gamma_i \in R$, the eighth-degree interpolation spline function interpolates the given data (x_i, y_i, m_i) and always maintains C^1 continuity.

Furthermore, we can obtain that $s_i''(x_{i+1}) = \frac{h_i \alpha_{i+1}}{h_{i+1}\alpha_i} s_{i+1}''(x_{i+1})$ and $s_i'''(x_{i+1}) = \frac{h_i \beta_{i+1}}{h_{i+1}\beta_i} s_{i+1}'''(x_{i+1})$, which show that the interpolation spline function (2) is G^2 and G^3 continuous.

Particularly, for any parameter γ_i, when $\alpha_i = \alpha_{i+1}$, $\beta_i = \beta_{i+1}$ and $h_i = h_{i+1}$, the interpolation spline function (2) is C^3 continuous.

When the interpolation spline function satisfies certain conditions, the error estimation of the function (3) can be discussed.

Theorem 2. Assume $y = f(x)$ be a function with continuously 4th derivatives on the interval $[a, b]$, and$a = x_0 < x_1 < \cdots < x_n = b$ be a subdivision of the interval $[a, b]$. Denote $h_i = x_{i+1} - x, t = \frac{x - x_i}{h_i}$, $f(x_i) = y_i$, $f'(x_i) = m_i$. For $\alpha_i, \beta_i, \gamma_i \in R$, $[x_i, x_{i+1}](i = 0,1,\cdots,n-1)$, the error estimation of interpolation function (2) can be represented in the form

$$R_i(x) = f(x) - s_i(x) = \frac{f^{(4)}(\xi_i) - s_i^{(4)}(\xi_i)}{4!}(x - x_i)^2(x - x_{i+1})^2 \tag{3}$$

for some $\xi_i \in [x_i, x_{i+1}]$.The error term (3) have the following useful bounds on its magnitude

$$|R_i(x)| \le \frac{M_i + A_i}{2^4 4!} h_i^4,$$

where $M_i = \max_{x_i \le x \le x_{i+1}} |f(x)|$, $A_i = \max_{x_i \le x \le x_{i+1}} |s_i(x)|$.

Proof. According to Theorem 1, it can be obtained by simple calculation

$$\begin{cases} R_i(x_i) = f(x_i) - s_i(x_i) = 0 \\ R_i(x_{i+1}) = f(x_{i+1}) - s_i(x_{i+1}) = 0 \\ R_i'(x_i) = f'(x_i) - s_i'(x_i) = 0 \\ R_i'(x_{i+1}) = f'(x_{i+1}) - s_i'(x_{i+1}) = 0 \end{cases}. \tag{4}$$

That is to say the Eq (4) is trivially satisfied if x coincides with one of the interpolation points $x_i(i = 0,1,\cdots,n-1)$. We need to be concerned only with the case where x does not coincide with one of the interpolation points. Let

$$R_i(x) = f(x) - s_i(x) = k(x)(x - x_i)^2(x - x_{i+1})^2, \tag{5}$$

where $k(x)$ is the selected function.

Keeping x fixed, consider $u \in (x_i, x_{i+1})$ given $(i = 0,1,\cdots,n-1)$ by

$$\varphi(u) = f(u) - s_i(u) - k(x)(u - x_i)^2 (u - x_{i+1})^2 . \tag{6}$$

By the assumption on $f(x)$, the $\varphi(u)$ has also continuously 4$^{\text{th}}$ derivatives. Obviously, $\varphi(u)$ has at least five zeros on the interval $[x_i, x_{i+1}]$. Then by Rolle's the theorem the derivative $\varphi'(u)$ has at least four zeros. Repeating the argument, by induction we deduce that the derivative $\varphi^{(4)}(u)$ has at least one zero in $[x_i, x_{i+1}]$, which we denote by ξ_i. For this zero we have that

$$\varphi^{(4)}(\xi_i) = f^{(4)}(\xi_i) - s_i^{(4)}(\xi_i) - k(x) \cdot 4! = 0 . \tag{7}$$

Namely

$$k(x) = \frac{f^{(4)}(\xi_i) - s_i^{(4)}(\xi_i)}{4!} . \tag{8}$$

From this we obtain (4).

For $x \in [x_i, x_{i+1}](i = 0,1,\cdots,n-1)$, we have $(x - x_i)^2 (x - x_{i+1})^2 \le \frac{h_i^4}{2^4}$. Then the differentiable function $f(x)$ can be estimated by

$$|R_i(x)| \le \frac{M_i + A_i}{2^4 4!} h_i^4 ,$$

where $M_i = \max\limits_{x_i \le x \le x_{i+1}} |f(x)|$, $A_i = \max\limits_{x_i \le x \le x_{i+1}} |s_i(x)|$.

The interpolation function $s_i(x)$ converges to the interpolated function $f(x)$ on the interval $[x_i, x_{i+1}]$ as h_i approaches zero.

4. The optimal interpolation function

Because the interpolation basis function contains different parameters, the shape of the interpolation spline function will change globally or locally when the parameters take different values.

Example 1. Considering the interpolated function

$$f(x) = \frac{1}{1+x^2}, -5 \le x \le 5$$

with equidistant interpolation points $x_i = -5 + 2i (i = 0,1,\cdots,5)$. The graph of the corresponding eight interpolation function is shown in Figure 1, where the solid line is the interpolated function, all parameters of the blue line are set as 0, and all parameters of the red line are set as 0 except $\gamma_2 = 8$. We can see that the shape of the interpolation function in the third segment has changed.

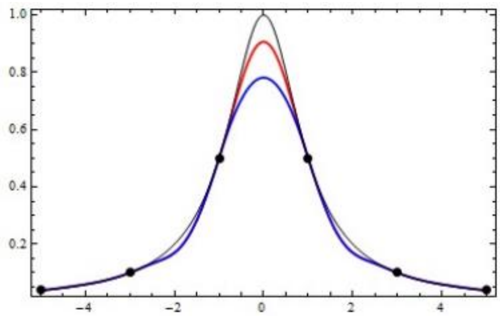

Figure 1. Locally adjustable eighth-degree Hermite interpolation functions.

Obviously, the interpolation function should be able to approximate the interpolated function very well when the parameters are set appropriately. So how do you measure the effectiveness? In general, the smaller the overall interpolation error, the better the interpolation is. Usually, the overall interpolation error between the piecewise interpolation functions$s_i(x)$ and the interpolated function $y = f(x)$ can be expressed as

$$e(\alpha_i, \beta_i, \gamma_i) = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} (s_i(x) - f(x))^2 \mathrm{d}x. \tag{9}$$

In order to obtain the optimal interpolation effect of the parametric eight-degree Hermite spline interpolation function, it is necessary to determine the appropriate value of the parameters $\alpha_i, \beta_i, \gamma_i$ to minimize the overall interpolation error, and then there is an optimization model

$$\begin{cases} min\, e\,(\alpha_i, \beta_i, \gamma_i) = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} (s_i(x) - f(x))^2 \mathrm{d}x \\ s.t. \qquad \alpha_i, \beta_i, \gamma_i \in \mathrm{R} \end{cases}. \tag{10}$$

On the interval $[x_i, x_{i+1}] (i = 0,1,\cdots,n-1)$, take the partial derivatives with respect to α_i, β_i and γ_i, and let them equal to 0, the following system of equations is obtained

$$\begin{cases} \frac{\partial e(\alpha_i,\beta_i,\gamma_i)}{\partial \alpha_i} = 0, \\ \frac{\partial e(\alpha_i,\beta_i,\gamma_i)}{\partial \beta_i} = 0, \qquad i = 0,1,\cdots,n-1 \\ \frac{\partial e(\alpha_i,\beta_i,\gamma_i)}{\partial \gamma_i} = 0, \end{cases}. \tag{11}$$

The optimal values $\alpha_i, \beta_i, \gamma_i (i = 0, 1, \cdots, n-1)$ are solved by Eq (11).

Example 2. For the interpolation conditions given in Example 1, the optimal parameter values and interval interpolation errors of each segment can be obtained by solving Eq (11) as shown in Table 1 below. The graph of the eighth-degree interpolation function and the interpolated function is shown in Figure 2.

Table 1. The optimal parameter values and interval interpolation errors of each segment.

Interval	α_i	β_i	γ_i	Interval interpolation errors
[−5, −3]	0.738 522 93	−1.930 292 72	−0.927 999 19	$1.039\,85 \times 10^{-7}$
[−3, −1]	1.668 589 48	−1.712 593 25	−1.132 861 38	$1.772\,22 \times 10^{-7}$
[−1, 1]	2.773 918 45	−6.814 726 78	33.882 831 54	$1.289\,05 \times 10^{-7}$
[1, 3]	1.668 589 47	−1.712 593 25	3.747 156 76	$1.768\,85 \times 10^{-7}$
[3, 5]	0.738 522 89	−1.930 292 69	24.029 255 75	$5.033\,19 \times 10^{-8}$

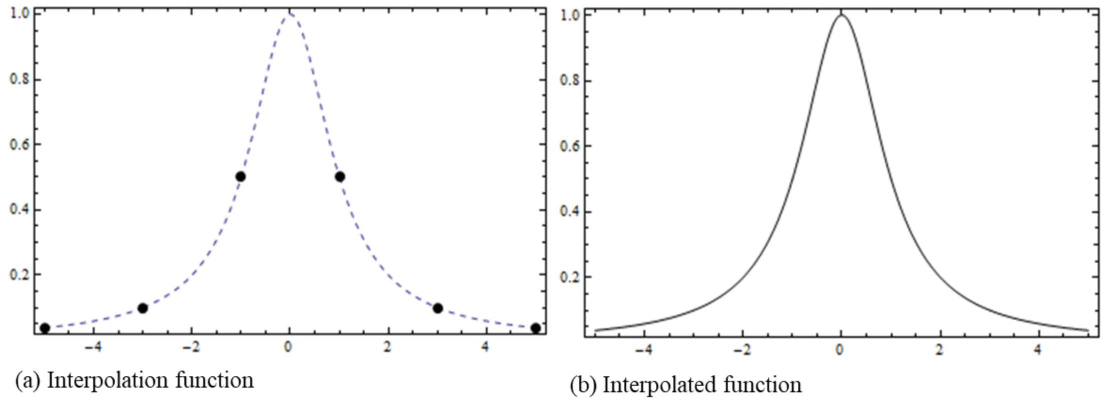

Figure 2. Graphs of the eighth-degree interpolation spline function and the interpolated function.

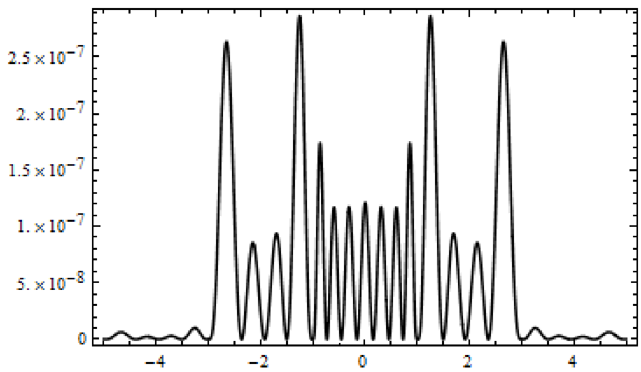

Figure 3. The overall error curve of the eighth-degree interpolation function.

By adding the interpolation errors of each interval in Table 1, the overall interpolation error of the interpolation function to the interpolated function is obtained, and the overall interpolation error is 6.373×10^{-7}. Under the same interpolation conditions, the quintic Hermite interpolation [15], cubic trigonometric Hermite interpolation spline [16], piecewise seventh-degree Hermite interpolation spline [17] and parametric eighth-degree Hermite interpolation spline proposed in this paper are respectively used to interpolate the function of Example 1 The optimal values of parameters and overall interpolation errors of various methods are shown in Table 2.

It can be seen from Figure 3 that the best interpolation function almost coincides with the interpolated function, which indicates that the parametric eighth-degree interpolation function has a better interpolation effect when the parameters are properly chosen. As shown in Table 2 the overall interpolation error of the parametric eighth-degree interpolation function is smaller than that of the quintic Hermite interpolation spline function [14] and the cubic trigonometric Hermite interpolation spline function [15]. This is because when approximating the interpolated function, the parametric eighth-degree Hermite interpolation function achieves C^3 continuity, which has better approximation than the C^2 quartic Hermite interpolation spline function [14] and the cubic trigonometric Hermite interpolation spline function [15].

Table 2. Comparison of the overall interpolation errors of various methods.

Method	The overall interpolation errors
The quintic hermite interpolation [14]	4.551×10^{-1}
Cubic trigonometric hermite interpolation spline [15]	2.974×10^{-1}
Piecewise seventh-degree hermite interpolation spline [16]	2.032×10^{-2}
Ours	6.373×10^{-7}

5. Conclusions

The parametric eighth-degree interpolation splines constructed in this paper inherit the main properties of the classical cubic Hermite splines and overcome the shortcomings. Compared with the classical cubic Hermite spline, which is C^1 continuous, the presented spline curve can reach C3 continuity. Therefore, it is appropriate for applications requiring high degree of smoothness. Furthermore, when the control points and their tangent vectors are given, the shape of the constructed spline curve can be adjusted globally and locally by adapting different parameters, addressing the issue of inflexibility of the shape of the classical cubic Hermite spline curve. Therefore, when the problems arise that interpolation effect should be adjusted properly in practical engineering designs, the proposed splines are very valuable tools. In addition, the eighth-degree parametric splines still adopt the piecewise polynomial form, which not only have a relatively simple expression, but also stays in line with the standard Bézier curve, B-spline curve and other polynomial parameter curves in CAD/CAM system.

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Conflict of interest

The authors declare no conflict of interest.

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