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Research article

Lie n-centralizers of generalized matrix algebras

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Abstract: In this paper, we introduce the notion of Lie n-centralizers. We then give a description of Lie n-centralizers on a generalized matrix algebra and present the necessary and sufficient conditions for a Lie n-centralizer to be proper. As applications, we determine generalized Lie n-derivations on a generalized matrix algebra and Lie n-centralizers of some operator algebras.

Keywords: Lie n-centralizer; generalized matrix algebra; generalized Lie n-derivation
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1. Introduction

Let R be a unital commutative ring, A be an algebra over R and $Z(A)$ be the center of A. Let $[x, y] = xy - yx$ denote the Lie product of elements $x, y \in A$. An R-linear map $\phi : A \to A$ is called a *left (right) centralizer* if $\phi(xy) = \phi(x)y(\phi(xy) = x\phi(y))$ holds for all $x, y \in A$. Further, an R-linear map $\phi : A \to A$ is called a *Lie centralizer* if $\phi([x, y]) = [\phi(x), y]$ for all $x, y \in A$. It is easy to prove that ϕ is a Lie centralizer on A if and only if $\phi([x, y]) = [x, \phi(y)]$ for all $x, y \in A$. Suppose that λ is an element of $Z(A)$ and $\tau : A \to Z(A)$ is a linear map vanishing at commutators $[x, y]$ for all $x, y \in A$. Then, the linear map $\phi : A \to A$ satisfying $\phi(a) = \lambda a + \tau(a)$ is a Lie centralizer and is called the *proper Lie centralizer*. However, not every Lie centralizer is necessarily a proper Lie centralizer. Recently, the structure of Lie centralizers on triangular algebras and generalized matrix algebras has been studied by many mathematicians. In 2020, Jabeen studied Lie centralizers on generalized matrix algebras and obtained the necessary and sufficient conditions for a Lie centralizer to be proper (see [1]). Fošner and Jing investigated the additivity of Lie centralizers on triangular rings and characterized both centralizers and Lie centralizers on triangular rings and nest algebras in [2]. Liu gave a description of nonlinear Lie centralizers for a certain class of generalized matrix algebras in [3]. Some special Lie centralizers on triangular algebras and generalized matrix algebras were studied in [4–7]. Fadaee et al. extended the results of Jabeen to Lie triple centralizers and characterized generalized Lie triple derivations on generalized matrix algebras in [8]. Accordingly, we can further develop the definition of Lie n-centralizers. Let us define the following sequence of polynomials:

$$\begin{aligned} p_1(x_1) &= x_1, \\ p_2(x_1, x_2) &= [p_1(x_1), x_2] = [x_1, x_2], \\ p_3(x_1, x_2, x_3) &= [p_2(x_1, x_2), x_3] = [[x_1, x_2], x_3], \\ &\cdots\cdots \\ p_n(x_1, x_2, \ldots, x_n) &= [p_{n-1}(x_1, x_2, \ldots, x_{n-1}), x_n]. \end{aligned}$$

The polynomial $p_n(x_1, x_2, \ldots, x_n)$ is said to be an *$(n-1)$-th commutator* ($n \geq 2$). A *Lie n-centralizer* is an R-linear map $\phi : A \to A$ which satisfies the rule

$$\phi(p_n(x_1, x_2, \ldots, x_n)) = p_n(\phi(x_1), x_2, \ldots, x_n)$$

for all $x_1, x_2, \ldots, x_n \in A$. If there exists an element $\lambda \in Z(A)$ and an R-linear map $\tau : A \to Z(A)$ vanishing on each $(n-1)$-th commutator $p_n(x_1, x_2, \ldots, x_n)$ such that $\phi(x) = \lambda x + \tau(x)$ for all $x \in A$, then the Lie n-centralizer ϕ is called a *proper Lie n-centralizer*.

In this paper, we extend the results of Jabeen [1] and Fadaee et al. [8] and give the necessary and sufficient conditions for a Lie n-centralizer to be proper on a generalized matrix algebra.

Let A be an algebra. An R-linear map $L : A \to A$ is a *Lie derivation* if $L([x, y]) = [L(x), y] + [x, L(y)]$ holds for all $x, y \in A$. An R-linear map $G : A \to A$ is a *generalized Lie derivation* with an associated Lie derivation L on A if $G([x, y]) = [G(x), y] + [x, L(y)]$ holds for all $x, y \in A$. A *Lie n-derivation* is an R-linear map $\Psi : A \to A$ which satisfies the rule

$$\Psi(p_n(x_1, x_2, \ldots, x_n)) = \sum_{k=1}^{n} p_n(x_1, \ldots, x_{k-1}, \Psi(x_k), x_{k+1}, \ldots, x_n)$$

for all $x_1, x_2, \ldots, x_n \in A$. One can give the definition of generalized Lie n-derivations in an analogous manner. An R-linear map $\Phi : A \to A$ is called a *generalized Lie n-derivation* if there exists a Lie n-derivation Ψ such that

$$\Phi(p_n(x_1, x_2, \ldots, x_n)) = p_n(\Phi(x_1), x_2, \ldots, x_n) + \sum_{k=2}^{n} p_n(x_1, \ldots, x_{k-1}, \Psi(x_k), x_{k+1}, \ldots, x_n)$$

for all $x_1, x_2, \ldots, x_n \in A$. We say that Ψ is an associated Lie n-derivation of Φ. They are part of an important class of maps on algebras. It is easily checked that G is a generalized Lie derivation with an associated Lie derivation L if and only if $G - L$ is a Lie centralizer. Therefore, if we characterize Lie centralizers and Lie derivations, then we can get the characterization of a generalized Lie derivation on an algebra. Likewise, there is a similar relationship between a Lie n-derivation Ψ and a generalized Lie n-derivation Φ, that is, Φ is a generalized Lie n-derivation with an associated Lie n-derivation Ψ if and only if $\Phi - \Psi$ is a Lie n-centralizer (Lemma 4.1). We can describe generalized Lie n-derivations by Lie n-centralizers.

In this paper, we set out the preliminaries in Section 2. We then characterize the structure of a Lie n-centralizer ϕ (Theorem 3.1) and obtain the necessary and sufficient conditions for ϕ to be proper (Theorem 3.3). In Section 4, we use the results obtained to determine generalized Lie n-derivations (Theorem 4.2) and apply our results to some other algebras (Theorem 4.3).

2. Preliminaries

A Morita context consists of two R-algebras A and B, two bimodules M and N, where M is an (A, B)-bimodule and N is a (B, A)-bimodule, and two bimodule homomorphisms called the pairings $\Phi_{MN} : M \otimes_B N \to A$ and $\Psi_{NM} : N \otimes_A M \to B$ satisfying the following commutative diagrams:

$$\begin{tikzcd}
M \otimes_B N \otimes_A M \arrow[r, "\Phi_{MN}\otimes I_M"] \arrow[d, "I_M\otimes\Psi_{NM}"'] & A \otimes_A M \arrow[d, "\cong"] \\
M \otimes_B B \arrow[r, "\cong"] & M
\end{tikzcd}$$

and

$$\begin{tikzcd}
N \otimes_A M \otimes_B N \arrow[r, "\Psi_{NM}\otimes I_N"] \arrow[d, "I_N\otimes\Phi_{MN}"'] & B \otimes_B N \arrow[d, "\cong"] \\
N \otimes_A A \arrow[r, "\cong"] & N.
\end{tikzcd}$$

If $(A, B, M, N, \Phi_{MN}, \Psi_{NM})$ is a Morita context, then the set

$$G = \left\{ \begin{pmatrix} a & m \\ n & b \end{pmatrix} : a \in A, m \in M, n \in N, b \in B \right\}$$

forms an algebra under matrix-like addition and multiplication, where at least one of the two bimodules M and N is distinct from zero. Such an algebra is called a *generalized matrix algebra* and is usually denoted by $G = \begin{pmatrix} A & M \\ N & B \end{pmatrix}$. Obviously, when $M = 0$ or $N = 0$, G exactly degenerates to the so-called *triangular algebra*. For a detailed introduction on generalized matrix algebras, we refer the reader to [9].

If A and B are unital algebras with unities 1_A and 1_B, respectively, then $\begin{pmatrix} 1_A & 0 \\ 0 & 1_B \end{pmatrix}$ is the unity of the generalized matrix algebra G. Set $e = \begin{pmatrix} 1_A & 0 \\ 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 0 & 1_B \end{pmatrix}$. Then, G can be written as $G = eGe \oplus eGf \oplus fGe \oplus fGf$, where eGe is a subalgebra of G isomorphic to A, fGf is a subalgebra of G isomorphic to B, eGf is an (eGe, fGf)-bimodule isomorphic to the bimodule M, and fGe is an (fGf, eGe)-bimodule isomorphic to the bimodule N.

Let D be a unital algebra with an idempotent $e \neq 0, 1$ and let f denote the idempotent $1 - e$. In this case D can be represented in the so-called Peirce decomposition form $D = eDe \oplus eDf \oplus fDe \oplus fDf$. The following property was introduced by Benkovič and Širovnik in [10].

$$\begin{aligned} exe \cdot eDf = 0 = fDe \cdot exe \Rightarrow exe = 0, \\ eDf \cdot fxf = 0 = fxf \cdot fDe \Rightarrow fxf = 0. \end{aligned} \tag{2.1}$$

Some specific examples of unital algebras with nontrivial idempotents having the property (2.1) are triangular algebras, matrix algebras and prime algebras with nontrivial idempotents. It is worth mentioning that generalized matrix algebras can be regarded as special unital algebras with nontrivial idempotents having the property (2.1) (see [9]). Therefore, (2.1) can be rewritten as follows on the generalized matrix algebra $G = \begin{pmatrix} A & M \\ N & B \end{pmatrix}$.

$$\begin{aligned} &a \in A, \quad aM = 0 \quad \text{and} \quad Na = 0 \Rightarrow a = 0, \\ &b \in B, \quad Mb = 0 \quad \text{and} \quad bN = 0 \Rightarrow b = 0. \end{aligned} \tag{2.2}$$

If G is a generalized matrix algebra satisfying the property (2.2), then the result [11, Proposition 2.1] tells us that the center of G is

$$Z(G) = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : am = mb, na = bn \ \text{ for all } \ m \in M, n \in N \right\}.$$

Define two natural projections $\pi_A : G \to A$ and $\pi_B : G \to B$ by $\pi_A\left(\begin{pmatrix} a & m \\ n & b \end{pmatrix}\right) = a$ and $\pi_B\left(\begin{pmatrix} a & m \\ n & b \end{pmatrix}\right) = b$. It is easy to see that $\pi_A(Z(G))$ is a subalgebra of $Z(A)$ and that $\pi_B(Z(G))$ is a subalgebra of $Z(B)$. According to [11, Proposition 2.1], there exists a unique algebraic isomorphism $\eta : \pi_A(Z(G)) \to \pi_B(Z(G))$ such that $am = m\eta(a)$ and $na = \eta(a)n$ for all $a \in \pi_A(Z(G)), m \in M, n \in N$.

Let S be a subset of an algebra D. We set

$$Z_{n-1}(S) = \{a \in S | p_n(a, a_1, \ldots, a_{n-1}) = 0 \quad \text{for all} \ \ a_1, \ldots, a_{n-1} \in S\}.$$

3. Lie n-centralizers

Theorem 3.1. *Let $G = \begin{pmatrix} A & M \\ N & B \end{pmatrix}$ be a generalized matrix algebra over a commutative ring R. If an R-linear map $\phi : G \to G$ is a Lie n-centralizer, then ϕ has the form*

$$\phi \begin{pmatrix} a & m \\ n & b \end{pmatrix} = \begin{pmatrix} f_1(a) + k_1(b) & g_2(m) \\ h_3(n) & f_4(a) + k_4(b) \end{pmatrix},$$

where $f_1 : A \to A$, $k_1 : B \to Z_{n-1}(A)$, $g_2 : M \to M$, $h_3 : N \to N$, $f_4 : A \to Z_{n-1}(B)$ and $k_4 : B \to B$ are R-linear maps satisfying the following conditions:

(i) *f_1 is a Lie n-centralizer on A, $p_n(f_4(a), b_1, \ldots, b_{n-1}) = 0$, $f_4(p_n(a_1, a_2, \ldots, a_n)) = 0$, and $f_1(mn) - k_1(nm) = g_2(m)n = mh_3(n)$ for all $a, a_1, \ldots, a_n \in A$, $b_1, b_2, \ldots, b_{n-1} \in B$, $m \in M$, $n \in N$.*

(ii) *k_4 is a Lie n-centralizer on B, $p_n(k_1(b), a_1, \ldots, a_{n-1}) = 0$, $k_1(p_n(b_1, b_2, \ldots, b_n)) = 0$, and $k_4(nm) - f_4(mn) = ng_2(m) = h_3(n)m$ for all $a_1, \ldots, a_{n-1} \in A$, $b, b_1, \ldots, b_n \in B$, $m \in M$, $n \in N$.*

(iii) *$g_2(am) = ag_2(m) = f_1(a)m - mf_4(a)$, and $g_2(mb) = g_2(m)b = mk_4(b) - k_1(b)m$ for all $a \in A, m \in M, b \in B$.*

(iv) *$h_3(na) = h_3(n)a = nf_1(a) - f_4(a)n$, and $h_3(bn) = bh_3(n) = k_4(b)n - nk_1(b)$ for all $a \in A, n \in N, b \in B$.*

Proof. Assume that ϕ has the form

$$\phi\begin{pmatrix} a & m \\ n & b \end{pmatrix} = \begin{pmatrix} f_1(a) + g_1(m) + h_1(n) + k_1(b) & f_2(a) + g_2(m) + h_2(n) + k_2(b) \\ f_3(a) + g_3(m) + h_3(n) + k_3(b) & f_4(a) + g_4(m) + h_4(n) + k_4(b) \end{pmatrix},$$

where $f_1 : A \to A$, $f_2 : A \to M$, $f_3 : A \to N$, $f_4 : A \to B$; $g_1 : M \to A$, $g_2 : M \to M$, $g_3 : M \to N$, $g_4 : M \to B$; $h_1 : N \to A$, $h_2 : N \to M$, $h_3 : N \to N$, $h_4 : N \to B$, and $k_1 : B \to A$, $k_2 : B \to M$, $k_3 : B \to N$, $k_4 : B \to B$ are R-linear maps. Since ϕ is a Lie n-centralizer, we have

$$\phi(p_n(X_1, X_2, \ldots, X_n)) = p_n(\phi(X_1), X_2, \ldots, X_n) \tag{3.1}$$

for all $X_1, X_2, \ldots, X_n \in G$.

Let us choose $X_1 = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$, $X_2 = \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}$, $X_3 = \ldots = X_n = \begin{pmatrix} 0 & 0 \\ 0 & 1_B \end{pmatrix}$ in (3.1). Then, we get

$$\begin{aligned}\begin{pmatrix} g_1(am) & g_2(am) \\ g_3(am) & g_4(am) \end{pmatrix} =& \phi(p_n(X_1, X_2, \ldots, X_n)) = p_n(\phi(X_1), X_2, \ldots, X_n) \\ =& p_n\left(\begin{pmatrix} f_1(a) & f_2(a) \\ f_3(a) & f_4(a) \end{pmatrix}, \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & 0 \\ 0 & 1_B \end{pmatrix}\right) \\ =& \begin{pmatrix} 0 & f_1(a)m - mf_4(a) \\ 0 & 0 \end{pmatrix}.\end{aligned}$$

Comparing both sides, we get $g_2(am) = f_1(a)m - mf_4(a)$ and $g_1(am) = g_3(am) = g_4(am) = 0$ for all $a \in A$ and $m \in M$. Now, if we set $a = 1_A$, then we find that

$$g_1(m) = g_3(m) = g_4(m) = 0 \text{ and } g_2(m) = f_1(1_A)m - mf_4(1_A) \tag{3.2}$$

for all $m \in M$. Similarly, taking $X_1 = \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}$, $X_2 = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$, $X_3 = \ldots = X_n = \begin{pmatrix} 0 & 0 \\ 0 & 1_B \end{pmatrix}$ in (3.1), we have $g_2(am) = ag_2(m)$ for all $a \in A, m \in M$.

If we take $X_1 = \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}$, $X_2 = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}$, $X_3 = \ldots = X_n = \begin{pmatrix} 0 & 0 \\ 0 & 1_B \end{pmatrix}$ and $X_1 = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}$, $X_2 = \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}$, $X_3 = \ldots = X_n = \begin{pmatrix} 0 & 0 \\ 0 & 1_B \end{pmatrix}$ in (3.1), respectively, then we obtain

$$\begin{pmatrix} 0 & g_2(mb) \\ 0 & 0 \end{pmatrix} = \phi(p_n(X_1, X_2, \ldots, X_n)) = p_n(\phi(X_1), X_2, \ldots, X_n) = \begin{pmatrix} 0 & g_2(m)b \\ 0 & 0 \end{pmatrix}$$

and

$$\begin{aligned}\begin{pmatrix} 0 & -g_2(mb) \\ 0 & 0 \end{pmatrix} =& \phi(p_n(X_1, X_2, \ldots, X_n)) = p_n(\phi(X_1), X_2, \ldots, X_n) \\ =& \begin{pmatrix} 0 & k_1(b)m - mk_4(b) \\ 0 & 0 \end{pmatrix}.\end{aligned}$$

Hence, $g_2(mb) = g_2(m)b = mk_4(b) - k_1(b)m$ for all $m \in M, b \in B$. In particular, we have

$$g_2(m) = mk_4(1_B) - k_1(1_B)m \tag{3.3}$$

for all $m \in M$.

Setting $X_1 = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$, $X_2 = \begin{pmatrix} 0 & 0 \\ n & 0 \end{pmatrix}$, $X_3 = \ldots = X_n = \begin{pmatrix} 1_A & 0 \\ 0 & 0 \end{pmatrix}$ in (3.1), we get

$$\begin{aligned}\begin{pmatrix} -h_1(na) & -h_2(na) \\ -h_3(na) & -h_4(na) \end{pmatrix} =& \phi(p_n(X_1, X_2, \ldots, X_n)) = p_n(\phi(X_1), X_2, \ldots, X_n) \\ =& \begin{pmatrix} 0 & 0 \\ f_4(a)n - nf_1(a) & 0 \end{pmatrix}.\end{aligned}$$

Comparing both sides, we have $h_3(na) = nf_1(a) - f_4(a)n$ and $h_1(na) = h_2(na) = h_4(na) = 0$ for all $a \in A, n \in N$. Putting $a = 1_A$ leads to

$$h_1(n) = h_2(n) = h_4(n) = 0 \ \text{ and } \ h_3(n) = nf_1(1_A) - f_4(1_A)n \tag{3.4}$$

for all $n \in N$. Similarly, considering $X_1 = \begin{pmatrix} 0 & 0 \\ n & 0 \end{pmatrix}$, $X_2 = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$, $X_3 = \ldots = X_n = \begin{pmatrix} 1_A & 0 \\ 0 & 0 \end{pmatrix}$ in (3.1), we find $h_3(na) = h_3(n)a$ for all $a \in A, n \in N$.

Let us consider $X_1 = \begin{pmatrix} 0 & 0 \\ n & 0 \end{pmatrix}$, $X_2 = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}$, $X_3 = \ldots = X_n = \begin{pmatrix} 1_A & 0 \\ 0 & 0 \end{pmatrix}$ and $X_1 = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}$, $X_2 = \begin{pmatrix} 0 & 0 \\ n & 0 \end{pmatrix}$, $X_3 = \ldots = X_n = \begin{pmatrix} 1_A & 0 \\ 0 & 0 \end{pmatrix}$ in (3.1), respectively. Then, we arrive at $h_3(bn) = bh_3(n)$ and $h_3(bn) = k_4(b)n - nk_1(b)$ for all $n \in N, b \in B$. In particular, we obtain

$$h_3(n) = k_4(1_B)n - nk_1(1_B) \tag{3.5}$$

for all $n \in N$.

Let $X_1 = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$, $X_2 = \begin{pmatrix} 0 & 0 \\ 0 & b_1 \end{pmatrix}$, $X_3 = \begin{pmatrix} 0 & 0 \\ 0 & b_2 \end{pmatrix}, \ldots, X_n = \begin{pmatrix} 0 & 0 \\ 0 & b_{n-1} \end{pmatrix}$ in (3.1). Then, we deduce that

$$\begin{aligned} 0 =& \phi(p_n(X_1, X_2, \ldots, X_n)) = p_n(\phi(X_1), X_2, \ldots, X_n) \\ =& \begin{pmatrix} 0 & f_2(a)b_1b_2\ldots b_{n-1} \\ (-1)^{n-1}b_{n-1}\ldots b_2b_1f_3(a) & p_n(f_4(a), b_1, \ldots, b_{n-1}) \end{pmatrix}\end{aligned}$$

for all $a \in A$, $b_1, b_2, \ldots, b_{n-1} \in B$. It follows that

$$f_2(a)b_1b_2\ldots b_{n-1} = (-1)^{n-1}b_{n-1}\ldots b_2b_1f_3(a) = 0 \ \text{ and } \ p_n(f_4(a), b_1, \ldots, b_{n-1}) = 0.$$

If we take $b_1 = b_2 = \ldots = b_{n-1} = 1_B$, then we have

$$f_2(a) = f_3(a) = 0 \tag{3.6}$$

for all $a \in A$.

If $X_1 = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}$, $X_2 = \begin{pmatrix} a_1 & 0 \\ 0 & 0 \end{pmatrix}$, $X_3 = \begin{pmatrix} a_2 & 0 \\ 0 & 0 \end{pmatrix}, \ldots, X_n = \begin{pmatrix} a_{n-1} & 0 \\ 0 & 0 \end{pmatrix}$ in (3.1), then we arrive at

$$\begin{aligned} 0 =& \phi(p_n(X_1, X_2, \ldots, X_n)) = p_n(\phi(X_1), X_2, \ldots, X_n) \\ =& \begin{pmatrix} p_n(k_1(b), a_1, \ldots, a_{n-1}) & (-1)^{n-1}a_{n-1}\ldots a_1k_2(b) \\ k_3(b)a_1\ldots a_{n-1} & 0 \end{pmatrix}.\end{aligned}$$

Hence, $k_3(b)a_1\dots a_{n-1} = (-1)^{n-1}a_{n-1}\dots a_1k_2(b) = 0$ and $p_n(k_1(b),a_1,\dots,a_{n-1}) = 0$ for all $b\in B, a_1,\dots,a_{n-1}\in A$. Taking $a_1 = \dots = a_{n-1} = 1_A$, we see that $k_2(b) = k_3(b) = 0$ for all $b\in B$.

Assume that $X_1 = \begin{pmatrix} a_1 & 0\\ 0 & 0\end{pmatrix}, X_2 = \begin{pmatrix} a_2 & 0\\ 0 & 0\end{pmatrix},\dots, X_n = \begin{pmatrix} a_n & 0\\ 0 & 0\end{pmatrix}$ in (3.1), and then we get from (3.6) that

$$\begin{aligned}\begin{pmatrix} f_1(p_n(a_1,a_2,\dots,a_n)) & 0\\ 0 & f_4(p_n(a_1,a_2,\dots,a_n))\end{pmatrix} =&\phi(p_n(X_1,X_2,\dots,X_n))\\ =& p_n(\phi(X_1),X_2,\dots,X_n) = \begin{pmatrix} p_n(f_1(a_1),a_2,\dots,a_n) & 0\\ 0 & 0\end{pmatrix}\end{aligned}$$

for all $a_1,a_2,\dots,a_n\in A$. From the above relation, we deduce that f_1 is a Lie n-centralizer on A and $f_4(p_n(a_1,a_2,\dots,a_n)) = 0$ for all $a_1,a_2,\dots,a_n \in A$. Similarly, setting $X_1 = \begin{pmatrix} 0 & 0\\ 0 & b_1\end{pmatrix}$, $X_2 = \begin{pmatrix} 0 & 0\\ 0 & b_2\end{pmatrix},\dots, X_n = \begin{pmatrix} 0 & 0\\ 0 & b_n\end{pmatrix}$ in (3.1), we obtain that k_4 is a Lie n-centralizer on B and $k_1(p_n(b_1,b_2,\dots,b_n)) = 0$ for all $b_1,b_2,\dots,b_n\in B$.

Let us take $X_1 = \begin{pmatrix} 0 & m\\ 0 & 0\end{pmatrix}, X_2 = \dots = X_{n-1} = \begin{pmatrix} 0 & 0\\ 0 & 1_B\end{pmatrix}, X_n = \begin{pmatrix} 0 & 0\\ n & 0\end{pmatrix}$ in (3.1). Then, we have

$$\begin{aligned}\begin{pmatrix} f_1(mn)-k_1(nm) & 0\\ 0 & f_4(mn)-k_4(nm)\end{pmatrix} =&\phi(p_n(X_1,X_2,\dots,X_n))\\ =& p_n(\phi(X_1),X_2,\dots,X_n) = \begin{pmatrix} g_2(m)n & 0\\ 0 & -ng_2(m)\end{pmatrix}.\end{aligned}$$

It follows that $f_1(mn)-k_1(nm) = g_2(m)n$ and $k_4(nm)-f_4(mn) = ng_2(m)$ for all $m\in M$ and $n\in N$. Similarly, taking $X_1 = \begin{pmatrix} 0 & 0\\ n & 0\end{pmatrix}$, $X_2 = \dots = X_{n-1} = \begin{pmatrix} 1_A & 0\\ 0 & 0\end{pmatrix}$, $X_n = \begin{pmatrix} 0 & m\\ 0 & 0\end{pmatrix}$ in (3.1), we obtain that $k_4(nm)-f_4(mn) = h_3(n)m$ and $f_1(mn)-k_1(nm) = mh_3(n)$ for all $m\in M$ and $n\in N$. □

In the case that G satisfies (2.2), we will show in the next corollary that the conditions $f_4(p_n(a_1,a_2,\dots,a_n)) = 0$ and $k_1(p_n(b_1,b_2,\dots,b_n)) = 0$ can be omitted, and $k_1 : B\to Z(A)$ and $f_4 : A\to Z(B)$ hold.

Corollary 3.2. *Let* $G = \begin{pmatrix} A & M\\ N & B\end{pmatrix}$ *satisfy*

$$a\in A,\quad aM = 0\quad \textit{and}\quad Na = 0\Rightarrow a = 0,$$

$$b\in B,\quad Mb = 0\quad \textit{and}\quad bN = 0\Rightarrow b = 0.$$

Suppose that an R-linear map $\phi : G\to G$ *is a Lie n-centralizer, and then* ϕ *has the form*

$$\phi\begin{pmatrix} a & m\\ n & b\end{pmatrix} = \begin{pmatrix} f_1(a)+k_1(b) & g_2(m)\\ h_3(n) & f_4(a)+k_4(b)\end{pmatrix},$$

where $f_1 : A\to A$, $k_1 : B\to Z(A)$, $g_2 : M\to M$, $h_3 : N\to N$, $f_4 : A\to Z(B)$ *and* $k_4 : B\to B$ *are R-linear maps satisfying the following conditions:*

(i) f_1 *is a Lie n-centralizer on A, and* $f_1(mn) - k_1(nm) = g_2(m)n = mh_3(n)$ *for all* $m \in M, n \in N$.

(ii) k_4 *is a Lie n-centralizer on B, and* $k_4(nm) - f_4(mn) = ng_2(m) = h_3(n)m$ *for all* $m \in M, n \in N$.

(iii) $g_2(am) = ag_2(m) = f_1(a)m - mf_4(a)$*, and* $g_2(mb) = g_2(m)b = mk_4(b) - k_1(b)m$ *for all* $a \in A, m \in M, b \in B$.

(iv) $h_3(na) = h_3(n)a = nf_1(a) - f_4(a)n$*, and* $h_3(bn) = bh_3(n) = k_4(b)n - nk_1(b)$ *for all* $a \in A, n \in N, b \in B$.

Proof. Since ϕ is a Lie n-centralizer, it follows that ϕ satisfies Theorem 3.1. First, we claim that

$$g_2(p_n(a_1, a_2, \ldots, a_n)m) = p_n(f_1(a_1), a_2, \ldots, a_n)m \tag{3.7}$$

for all $a_1, a_2, \ldots, a_n \in A$ and $m \in M$. In fact, we can proceed by induction with n. If $n = 2$, then we can get from $g_2(am) = ag_2(m) = f_1(a)m - mf_4(a)$ that

$$\begin{aligned} g_2([a_1, a_2]m) =& g_2(a_1a_2m) - g_2(a_2a_1m) \\ =& f_1(a_1)a_2m - a_2mf_4(a_1) - a_2(f_1(a_1)m - mf_4(a_1)) \\ =& [f_1(a_1), a_2]m. \end{aligned}$$

This shows that (3.7) is true for $n = 2$. We now assume that $g_2(p_{n-1}(a_1, a_2, \ldots, a_{n-1})m) = p_{n-1}(f_1(a_1), a_2, \ldots, a_{n-1})m$. Then,

$$\begin{aligned} & g_2(p_n(a_1, a_2, \ldots, a_n)m) \\ & = g_2(p_{n-1}(a_1, a_2, \ldots, a_{n-1})a_nm - a_np_{n-1}(a_1, a_2, \ldots, a_{n-1})m) \\ & = p_{n-1}(f_1(a_1), a_2, \ldots, a_{n-1})a_nm - a_ng_2(p_{n-1}(a_1, a_2, \ldots, a_{n-1})m) \\ & = p_{n-1}(f_1(a_1), a_2, \ldots, a_{n-1})a_nm - a_np_{n-1}(f_1(a_1), a_2, \ldots, a_{n-1})m \\ & = p_n(f_1(a_1), a_2, \ldots, a_n)m. \end{aligned}$$

Next, according to $g_2(am) = f_1(a)m - mf_4(a)$ and (3.7), we have

$$\begin{aligned} & f_1(p_n(a_1, a_2, \ldots, a_n))m - mf_4(p_n(a_1, a_2, \ldots, a_n)) \\ & = g_2(p_n(a_1, a_2, \ldots, a_n)m) \\ & = p_n(f_1(a_1), a_2, \ldots, a_n)m \end{aligned}$$

for all $a_1, a_2, \ldots, a_n \in A$ and $m \in M$. Since f_1 is a Lie n-centralizer on A, we have $f_1(p_n(a_1, a_2, \ldots, a_n)) = p_n(f_1(a_1), a_2, \ldots, a_n)$. This implies that $Mf_4(p_n(a_1, a_2, \ldots, a_n)) = 0$. Similarly, we obtain $f_4(p_n(a_1, a_2, \ldots, a_n))N = 0$ for all $a_1, a_2, \ldots, a_n \in A$. Finally, we arrive at $f_4(p_n(a_1, a_2, \ldots, a_n)) = 0$ from the hypothesis. In an analogous way, we can easily get that $k_1(p_n(b_1, b_2, \ldots, b_n)) = 0$ for all $b_1, b_2, \ldots, b_n \in B$.

According to the condition (iii) of Theorem 3.1, we have

$$\begin{aligned} f_1(a)mb - mbf_4(a) =& g_2(amb) = (f_1(a)m - mf_4(a))b \\ =& f_1(a)mb - mf_4(a)b \end{aligned}$$

for all $a \in A, m \in M, b \in B$. It follows that $M(bf_4(a) - f_4(a)b) = 0$. Similarly, by the argument above and the condition (iv) of Theorem 3.1, we get $(bf_4(a) - f_4(a)b)N = 0$. Therefore, $bf_4(a) - f_4(a)b = 0$. This yields that $f_4(a) \in Z(B)$ for all $a \in A$. In a similar way, we can deduce that $k_1(b) \in Z(A)$ for all $b \in B$. ◻

Now we give the necessary and sufficient conditions for a Lie n-centralizer on a generalized matrix algebra to be proper.

Theorem 3.3. *Let* $G = \begin{pmatrix} A & M \\ N & B \end{pmatrix}$ *be a generalized matrix algebra over a commutative ring* R. *Suppose that* G *satisfies the following conditions:*

$$a \in A, \quad aM = 0 \quad \text{and} \quad Na = 0 \Rightarrow a = 0,$$

$$b \in B, \quad Mb = 0 \quad \text{and} \quad bN = 0 \Rightarrow b = 0.$$

If an R*-linear map* $\phi : G \to G$ *is a Lie* n*-centralizer, then the following statements are equivalent:*

(i) ϕ *is a proper Lie* n*-centralizer, that is,* $\phi(X) = \lambda X + \theta(X)$ *for all* $X \in G$*, where* $\lambda \in Z(G)$ *and* $\theta : G \to Z(G)$ *is a linear map which annihilates all* $(n-1)$*-th commutators.*

(ii) $f_4(A) \subseteq \pi_B(Z(G))$*, and* $k_1(B) \subseteq \pi_A(Z(G))$.

(iii) $f_4(1_A) \in \pi_B(Z(G))$*, and* $k_1(1_B) \in \pi_A(Z(G))$.

Proof. According to Corollary 3.2, ϕ has the following form:

$$\phi \begin{pmatrix} a & m \\ n & b \end{pmatrix} = \begin{pmatrix} f_1(a) + k_1(b) & g_2(m) \\ h_3(n) & f_4(a) + k_4(b) \end{pmatrix},$$

where $f_1 : A \to A$, $k_1 : B \to Z(A)$, $g_2 : M \to M$, $h_3 : N \to N$, $f_4 : A \to Z(B)$ and $k_4 : B \to B$ are linear maps with the properties mentioned in Corollary 3.2.

(i)$\Rightarrow$(ii). Suppose that ϕ is a proper Lie n-centralizer on G. Then, there exists an element $\lambda = \begin{pmatrix} a_1 & 0 \\ 0 & \eta(a_1) \end{pmatrix} \in Z(G)$ and a linear map $\theta : G \to Z(G)$ such that $\phi(X) = \lambda X + \theta(X)$ for all $X \in G$, where $a_1 \in \pi_A(Z(G))$. Now, let us take $X = \begin{pmatrix} 0 & am \\ na & 0 \end{pmatrix} \in G$ and $\theta(X) = \begin{pmatrix} a_2 & 0 \\ 0 & \eta(a_2) \end{pmatrix}$, $a_2 \in \pi_A(Z(G))$, and then we have

$$\phi(X) = \begin{pmatrix} 0 & g_2(am) \\ h_3(na) & 0 \end{pmatrix} = \begin{pmatrix} 0 & f_1(a)m - mf_4(a) \\ nf_1(a) - f_4(a)n & 0 \end{pmatrix}$$

and

$$\begin{aligned} \phi(X) = \lambda X + \theta(X) &= \begin{pmatrix} a_1 & 0 \\ 0 & \eta(a_1) \end{pmatrix} \begin{pmatrix} 0 & am \\ na & 0 \end{pmatrix} + \begin{pmatrix} a_2 & 0 \\ 0 & \eta(a_2) \end{pmatrix} \\ &= \begin{pmatrix} a_2 & a_1 am \\ \eta(a_1)na & \eta(a_2) \end{pmatrix} \end{aligned}$$

for all $a_1, a_2 \in \pi_A(Z(G))$, $a \in A$, $m \in M$, $n \in N$. Comparing the above relations, we conclude that $f_1(a)m - mf_4(a) = a_1 am$ and $nf_1(a) - f_4(a)n = \eta(a_1)na = na_1 a$. Thus,

$$(f_1(a) - a_1 a)m = mf_4(a) \qquad \text{and} \qquad n(f_1(a) - a_1 a) = f_4(a)n$$

for all $a_1 \in \pi_A(Z(G))$, $a \in A$, $m \in M$, $n \in N$. By the definition of $Z(G)$, we obtain $f_4(a) \in \pi_B(Z(G))$ for all $a \in A$.

If we choose $X = \begin{pmatrix} 0 & mb \\ bn & 0 \end{pmatrix}$ and $\theta(X) = \begin{pmatrix} a_3 & 0 \\ 0 & \eta(a_3) \end{pmatrix}$, $a_3 \in \pi_A(Z(G))$, then we arrive at

$$\phi(X) = \begin{pmatrix} 0 & g_2(mb) \\ h_3(bn) & 0 \end{pmatrix} = \begin{pmatrix} 0 & mk_4(b) - k_1(b)m \\ k_4(b)n - nk_1(b) & 0 \end{pmatrix}$$

and

$$\phi(X) = \lambda X + \theta(X) = \begin{pmatrix} a_3 & a_1 mb \\ \eta(a_1)bn & \eta(a_3) \end{pmatrix}$$

for all $a_1, a_3 \in \pi_A(Z(G))$, $m \in M$, $n \in N$, $b \in B$. Combining the last two equations, we find that $mk_4(b) - k_1(b)m = a_1 mb = m\eta(a_1)b$ and $k_4(b)n - nk_1(b) = \eta(a_1)bn$. It follows that

$$m(k_4(b) - \eta(a_1)b) = k_1(b)m \qquad \text{and} \qquad (k_4(b) - \eta(a_1)b)n = nk_1(b)$$

for all $a_1 \in \pi_A(Z(G))$, $m \in M$, $n \in N$, $b \in B$. Hence, $k_1(b) \in \pi_A(Z(G))$ for all $b \in B$.

(ii)⇒(iii) It is clear.

(iii)⇒ (i) According to the hypothesis, we define

$$\lambda = \begin{pmatrix} f_1(1_A) - \eta^{-1}(f_4(1_A)) & 0 \\ 0 & k_4(1_B) - \eta(k_1(1_B)) \end{pmatrix}.$$

We claim that $\lambda \in Z(G)$. Indeed, using (3.2)–(3.5), we get

$$\begin{aligned} f_1(1_A)m - \eta^{-1}(f_4(1_A))m &= g_2(m) = mk_4(1_B) - m\eta(k_1(1_B)), \\ nf_1(1_A) - n\eta^{-1}(f_4(1_A)) &= h_3(n) = k_4(1_B)n - \eta(k_1(1_B))n \end{aligned}$$

for all $m \in M$, $n \in N$. It follows that $\lambda \in Z(G)$.

Suppose that $\theta(X) = \phi(X) - \lambda X$ for all $X \in G$. We assert that $\theta(X) \in Z(G)$. Applying Corollary 3.2 yields that

$$\begin{aligned} \theta(X) = &\begin{pmatrix} f_1(a) - f_1(1_A)a + \eta^{-1}(f_4(1_A))a & 0 \\ 0 & f_4(a) \end{pmatrix} \\ &+ \begin{pmatrix} k_1(b) & 0 \\ 0 & k_4(b) - k_4(1_B)b + \eta(k_1(1_B))b \end{pmatrix}. \end{aligned}$$

Moreover, according to Corollary 3.2, we get

$$\begin{aligned} &(f_1(a) - f_1(1_A)a + \eta^{-1}(f_4(1_A))a)m - mf_4(a) \\ =&f_1(a)m - mf_4(a) + amf_4(1_A) - f_1(1_A)am \\ =&g_2(am) - g_2(am) = 0, \end{aligned}$$

$$\begin{aligned} &n(f_1(a) - f_1(1_A)a + \eta^{-1}(f_4(1_A))a) - f_4(a)n \\ =&nf_1(a) - f_4(a)n + f_4(1_A)na - nf_1(1_A)a \\ =&h_3(na) - h_3(n)a = 0, \end{aligned}$$

$$
\begin{aligned}
& m(k_4(b) - k_4(1_B)b + \eta(k_1(1_B))b) - k_1(b)m \\
=& mk_4(b) - k_1(b)m + k_1(1_B)mb - mk_4(1_B)b \\
=& g_2(mb) - g_2(m)b = 0
\end{aligned}
$$

and

$$
\begin{aligned}
& (k_4(b) - k_4(1_B)b + \eta(k_1(1_B))b)n - nk_1(b) \\
=& k_4(b)n - nk_1(b) + bnk_1(1_B) - k_4(1_B)bn \\
=& h_3(bn) - h_3(bn) = 0.
\end{aligned}
$$

From the above expressions, we have

$$
\begin{pmatrix} f_1(a) - f_1(1_A)a + \eta^{-1}(f_4(1_A))a & 0 \\ 0 & f_4(a) \end{pmatrix} \in Z(G)
$$

and

$$
\begin{pmatrix} k_1(b) & 0 \\ 0 & k_4(b) - k_4(1_B)b + \eta(k_1(1_B))b \end{pmatrix} \in Z(G).
$$

Thus, $\theta(X) \in Z(G)$ for all $X \in G$.

Finally, by the fact that ϕ is a Lie n-centralizer and $\phi(X) = \lambda X + \theta(X)$, we obtain

$$
\begin{aligned}
\theta(p_n(X_1, X_2, \ldots, X_n)) &= \phi(p_n(X_1, X_2, \ldots, X_n)) - \lambda p_n(X_1, X_2, \ldots, X_n) \\
&= p_n(\phi(X_1), X_2, \ldots, X_n) - \lambda p_n(X_1, X_2, \ldots, X_n) \\
&= p_n(\lambda X_1 + \theta(X_1), X_2, \ldots, X_n) - \lambda p_n(X_1, X_2, \ldots, X_n) \\
&= 0
\end{aligned}
$$

for all $X_1, X_2, \ldots, X_n \in G$. □

Theorem 3.4. *Let $G = \begin{pmatrix} A & M \\ N & B \end{pmatrix}$ be a generalized matrix algebra over a commutative ring R. Suppose that G satisfies the following conditions:*

$$
a \in A, \quad aM = 0 \quad \text{and} \quad Na = 0 \Rightarrow a = 0,
$$

$$
b \in B, \quad Mb = 0 \quad \text{and} \quad bN = 0 \Rightarrow b = 0.
$$

If we assume that

(i) $\pi_B(Z(G)) = Z(B)$ *or* $p_n(A, A, \ldots, A) = A$,

(ii) $\pi_A(Z(G)) = Z(A)$ *or* $p_n(B, B, \ldots, B) = B$,

then an R-linear map $\phi : G \to G$ is a Lie n-centralizer if and only if ϕ is proper.

Proof. Let ϕ be a Lie n-centralizer. Suppose that $\pi_B(Z(G)) = Z(B)$, and then it follows from Corollary 3.2 that $f_4(A) \subseteq Z(B) = \pi_B(Z(G))$. That is, $f_4(A) \subseteq \pi_B(Z(G))$. If $p_n(A, A, \ldots, A) = A$, then we can get $f_4(A) = f_4(p_n(A, A, \ldots, A)) = 0$ from the proof of Corollary 3.2. Therefore, $f_4(A) \subseteq \pi_B(Z(G))$. Similarly, by the condition (ii), we have $k_1(B) \subseteq \pi_A(Z(G))$. It follows from Theorem 3.3 that ϕ is proper. The converse is clear. □

4. Applications

In this section, we refer to some applications of Theorem 3.4. First, we characterize generalized Lie n-derivations on generalized matrix algebras. Let D be an algebra. An R-linear map $\psi : D \to D$ is called a *Jordan derivation* if it satisfies $\psi(x \circ y) = \psi(x) \circ y + x \circ \psi(y)$ for all $x, y \in D$. We say that a Jordan derivation $\psi : D \to D$ is a singular Jordan derivation according to the decomposition $D = eDe + eDf + fDe + fDf$ if $\psi(eDe + fDf) = 0$, $\psi(eDf) \subseteq fDe$, $\psi(fDe) \subseteq eDf$. Benkovič and Eremita in [12] introduced the following useful condition:

$$[x, D] \in Z(D) \Rightarrow x \in Z(D) \quad \text{for all} \quad x \in D. \tag{4.1}$$

Note that (4.1) is equivalent to the condition that there do not exist nonzero central inner derivations of D. The usual examples of algebras satisfying (4.1) are commutative algebras, prime algebras, and triangular algebras. To prove our result, we need the following lemma.

Lemma 4.1. *Let D be an algebra. The linear map Φ is a generalized Lie n-derivation with an associated Lie n-derivation Ψ if and only if $\Phi - \Psi$ is a Lie n-centralizer.*

Proof. Suppose that $\Phi - \Psi$ is a Lie n-centralizer. Set $\phi = \Phi - \Psi$. It follows that

$$\begin{aligned}\Phi(p_n(x_1, x_2, \ldots, x_n)) =& \Psi(p_n(x_1, x_2, \ldots, x_n)) + \phi(p_n(x_1, x_2, \ldots, x_n))\\ =& p_n(\Psi(x_1), x_2, \ldots, x_n) + p_n(x_1, \Psi(x_2), \ldots, x_n)\\ &+ \ldots + p_n(x_1, x_2, \ldots, \Psi(x_n)) + p_n(\phi(x_1), x_2, \ldots, x_n)\\ =& p_n(\Phi(x_1), x_2, \ldots, x_n) + p_n(x_1, \Psi(x_2), \ldots, x_n)\\ &+ \ldots + p_n(x_1, x_2, \ldots, \Psi(x_n))\end{aligned}$$

for all $x_1, x_2, \ldots, x_n \in D$. Hence, Φ is a generalized Lie n-derivation with an associated Lie n-derivation Ψ. The converse is clear. $\square$

According to [13, Theorem 2.1], we have the following result.

Theorem 4.2. *Let $G = \begin{pmatrix} A & M \\ N & B \end{pmatrix}$ be an $(n-1)$-torsion free generalized matrix algebra satisfying the following conditions:*

$$\begin{aligned} &a \in A, \quad aM = 0 \quad \textit{and} \quad Na = 0 \Rightarrow a = 0,\\ &b \in B, \quad Mb = 0 \quad \textit{and} \quad bN = 0 \Rightarrow b = 0.\end{aligned}$$

Let us assume that

(i) $\pi_A(Z(G)) = Z(A)$ *and* $\pi_B(Z(G)) = Z(B)$.

(ii) *Either A or B contains no central ideals.*

(iii) *Either A or B satisfies* (4.1) *when* $n \geq 3$.

Then, every generalized Lie n-derivation $\Phi : G \to G$ with an associated Lie n-derivation Ψ is of the form $\Phi(X) = \lambda X + d(X) + \psi(X) + \gamma(X)$, where $\lambda \in Z(G)$, $d : G \to G$ is a derivation, $\psi : G \to G$ is a singular Jordan derivation, and $\gamma : G \to Z(G)$ is a linear map that vanishes on $p_n(G, G, \ldots, G)$.

Proof. By Lemma 4.1, $\phi = \Phi - \Psi$ is a Lie n-centralizer on G. According to Theorem 3.4, we have $\phi(X) = \lambda X + \theta(X)$ for all $X \in G$, where $\lambda \in Z(G)$, and $\theta : G \to Z(G)$ is a linear map which annihilates all $(n-1)$-th commutators. It follows from [13, Theorem 2.1] that $\Psi = d+\psi+\tau$, where d is a derivation, ψ is a singular Jordan derivation, and $\tau : G \to Z(G)$ is a linear map such that $\tau(p_n(G, G, \ldots, G)) = 0$. Define $\gamma = \theta + \tau$. It follows that $\gamma : G \to Z(G)$ is a linear map satisfying $\gamma(p_n(G, G, \ldots, G)) = 0$ and

$$\begin{aligned}\Phi(X) =& \Psi(X) + \phi(X)\\ =& d(X) + \psi(X) + \tau(X) + \lambda X + \theta(X)\\ =& \lambda X + d(X) + \psi(X) + \gamma(X)\end{aligned}$$

for all $X \in G$. □

In view of [9] and [14], we obtain the following

Theorem 4.3. *Let G be any of the following algebras:*

(i) $M_n(A)$ $(n \geq 2)$*, the full matrix algebra over A, where A is a 2-torsion free unital algebra.*

(ii) $T_n(A)$ $(n \geq 2)$*, the upper triangular matrix algebra over A, where A is a 2-torsion free unital algebra.*

(iii) $B_{n\bar{k}}(A)$ $(n \geq 3)$*, the block upper triangular matrix algebra defined over A with* $B_{n\bar{k}}(A) \neq M_n(A)$*.*

(iv) *Standard operator algebra on a complex Banach space.*

(v) *Factor von Neumann algebra.*

(vi) *Nontrivial nest algebra on a complex Hilbert space.*

Then, an R-linear map $\phi : G \to G$ *is a Lie n-centralizer if and only if* ϕ *is proper.*

5. Conclusions

This paper gives the notion of Lie n-centralizers and characterizes the structure of a Lie n-centralizer ϕ on a generalized matrix algebra. The necessary and sufficient conditions for ϕ to be proper are obtained. Using the results obtained, we can determine generalized Lie n-derivations on a generalized matrix algebra and Lie n-centralizers on some other algebras.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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