Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(6): 14592–14608.
DOI: 10.3934/math.2023746
Received: 08 October 2022
Revised: 17 March 2023
Accepted: 06 April 2023
Published: 20 April 2023

Research article

Krasnoselskii-type results for equiexpansive and equicontractive operators with application in radiative transfer equations

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Abstract: In this paper, we study the results of fixed points for the operator equations of type $x = H(Fx, x)$ using the idea of measure of noncompactness and assuming that the operator F is k -set contractive (strictly k -set contractive, or a continuous) and the family $\{H(u, .) : u\}$ is equiexpansive or equicontractive. The obtained results are generalization of Krasnoselskii type fixed point results. Some examples are given to elaborate new concepts. We use the main result to find the existence of solutions for the stationary radiative transfer equation in a channel. We demonstrate our theory with an example by comparison of an approximate solution with the exact solution.

Keywords: Krasnoselskii; equiexpansive; equicontractive; fixed points; radiative transfer
Mathematics Subject Classification: 47H10, 54H25

1. Introduction

One of the most emerging and applicable branches of nonlinear mathematical analysis is fixed point theory. A renowned principle states that every operator equation can be altered into a fixed point equation problem and vice versa. Essentially fixed point theory is divided into two branches, where the first one is metric fixed point theory and the second is topological fixed point theory [1–5]. In this article we are intending to develop a theory which is not only based upon topological fixed point theory but also contains an essence of metric fixed point theory as well. It is based on a blend of two the most celebrated results of fixed point theory, the first is the Banach contraction principle and the second is the Schauder fixed point theorem.

In 1958, whilst studying the existence theory of neutral and perturbed differential equations, Krasnoselskii found that the solution of these types of equations can be expressed as a sum of a compact operator and a contractive operator. This ignited an idea in Krasnoselskii's mind to prove his famous fixed point result for the sum of compact and contractive operators[6], some related results can be seen in [7–10].

Astrophysicists analyze the light coming from objects similar to a star's atmosphere of vastly different physical conditions by solving the radiative transfer equation as a tool [11,12]. The physical phenomenon of energy transfer in the form of electromagnetic radiation is called radiative transfer. The transmission of radiation through a medium is affected by many factors like absorption, emission and scattering processes. At the end we apply our main result to obtain the existence of solution of the radiative transfer equation [12].

Krasnoselskii [6] generalized the Banach contraction principle and Schauder fixed point theorem in the following way.

Theorem 1.1 Suppose Ω is a non-empty closed and convex subset of a Banach space Z . Let F_1 and F_2 map Ω into Z such that
(1) $F_1\varkappa + F_2 y \in \Omega$ for all $\varkappa, y$ in Ω;
(2) F_1 is continuous and compact ;
(3) F_2 is contraction mapping.
Then there is $\varkappa \in \Omega$ such that $F_1\varkappa + F_2\varkappa = \varkappa$.

Taking F_1 to be the zero operator, we obtain the Banach contraction principle and by taking F_2 to be the zero operator, we obtain the Schauder fixed point theorem. In the Krasnoselskii fixed point theorem there are two operators: one is compact and the other is contraction. So, the theorem applies to contraction and compact operators only. There are also expansive and noncompact operators (k-set contraction and condensing) for which the fixed point investigation of Krasnoselskii type can be studied. For applications and the physical significance of noncompact operators, see [13]. Also, for generalization, there is a need to study the fixed point results for the operator H of two variables the one special case of which is $H(\varkappa, y) = F_1\varkappa + F_2 y$. This type of work can be seen in [14–18]. For example, Nashed and Wong [17] proved the following result.

Theorem 1.2 Suppose Ω is the convex closed and bounded subset of a Banach space Z. Consider a mapping H from $\Omega \times \Omega$ into Ω such that
(1) $\|H(\varkappa, y_1) - H(\varkappa, y_2)\| \leq \gamma \|y_1 - y_2\| \qquad \gamma \in [0, 1)$;
and for all $y \in Z$,
(2) $\|H(\varkappa_1, y) - H(\varkappa_2, y)\| \leq \|F_1\varkappa_1 - F_1\varkappa_2\|$,
where F_1 maps Ω into Ω and is completely continuous. Then there is $\varkappa \in \Omega$ such that $H(\varkappa, \varkappa) = \varkappa$.

Remark 1.1 Theorem 1.1 is the particular case of Theorem 1.2. This can be seen by defining $H(\varkappa, y) = F_1\varkappa + F_2 y$.

To generalize the concept of compact operators we need a measure of non compactness which gives the deviation from relative compactness for a given set. Sadovskii [[13], p. 500], using the idea of measure of a noncompactness, generalized the Schauder fixed point theorem by introducing condensing operators, i.e., a more general class than the compact operators. The following result is due to Sadovskii.

Theorem 1.3 [13] (Sadovskii) Suppose Ω is the nonempty convex closed bounded subset of a Banach space Z and $T : \Omega \longrightarrow \Omega$ is condensing. Then, T has a fixed point.

Xiang and Yuan [[19], see also in [20]] proved the result as a variant of the Banach contraction principle. They proved that there exists a unique fixed point for the expansive operators in a complete metric space. The result can be stated as follows:

Theorem 1.4 Let Z be a complete metric space, Ω be a non-empty closed subset and T be an expansive mapping of Ω into Z such that $\Omega \subseteq T(\Omega)$. Then there is a unique $\varkappa \in \Omega$ such that $T\varkappa = \varkappa$.

Remark 1.2 In the Banach contraction principle $T(\Omega) \subseteq \Omega$ and T is a contraction, while in the above result, T is an expansive and $\Omega \subseteq T(\Omega)$.

To generalize and find variants of Krasnoselskii's fixed point theorem, we need the following definitions and results.

Definition 1.1 [[15] , see also in ([21], p. 497)] The family $\{H(\varkappa, .) : \varkappa\}$ is called equicontractive if there is an $\alpha \in [0, 1)$ such that

$$\|H(\varkappa, y_1) - H(\varkappa, y_2)\| \leq \alpha \|y_1 - y_2\|$$

for all $(\varkappa, y_1)$, $(\varkappa, y_2)$ in the domain of H.

Similarly we define the following.

Definition 1.2 [22] The family $\{H(\varkappa, .) : \varkappa\}$ is known as equiexpansive if there exists $h > 1$ such that

$$\|H(\varkappa, y_1) - H(\varkappa, y_2)\| \geq h \|y_1 - y_2\|$$

for all $(\varkappa, y_1)$, $(\varkappa, y_2)$ in the domain of H.

The next example demonstrates the above definition.

Example 1.1 Consider $Z = \mathbb{R}$ with subset $\Omega = [0, 1]$ with standard metric $d(\varkappa, y) = |\varkappa - y|$ for all $\varkappa, y \in \Omega$. Define the mappings $H_1, H_2 : \Omega \times \Omega \to \mathbb{R}$ by

$$H_1(\varkappa, y) = \frac{\arctan \varkappa}{\pi + \arctan \varkappa} + \frac{1 + y}{1 - y} \text{ for all } \varkappa, y \in \Omega$$

and

$$H_2(\varkappa, y) = \frac{\arctan y}{\pi + \arctan y} + \frac{1 + \varkappa}{1 - \varkappa} \text{ for all } \varkappa, y \in \Omega$$

Then, clearly

$$|H_2(\varkappa, y_1) - H_2(\varkappa, y_2)| \leq \frac{1}{2} |y_1 - y_2|$$

and

$$|H_1(\varkappa, y_1) - H_1(\varkappa, y_2)| \geq 2 |y_1 - y_2| .$$

That is, H_2 is equicontractive and H_1 is equiexpansive.

Definition 1.3 [23] Let $B(Z)$ be the family of all bounded subsets of Banach space Z. A mapping $\mu : B(Z) \longrightarrow [0, +\infty)$ is said to be a measure of noncompactness defined on Z if it satisfies the following properties:

(1) Regularity: for any $F_1 \in B(Z), \mu(F_1) = 0, \Leftrightarrow F_1$ is pre-compact;
(2) Invariant under closure: for any $F_1 \in B(Z), \mu(F_1) = \mu\left(\overline{F_1}\right)$;
(3) Semi-additivity: for $F_1, F_2 \in B(Z), \mu(F_1 \cup F_2) = \max\{\mu(F_1), \mu(F_2)\}$.

We can also deduce the following;

(4) Monotonicity: for $F_1, F_2 \in B(Z)$ $F_1 \subseteq F_2$ implies $\mu(F_1) \leq \mu(F_2)$;
(5) Algebraic semi-additivity: $F_1, F_2 \in B(Z)$, $\mu(F_1 + F_2) \leq \mu(F_1) + \mu(F_2)$.

A number of articles related to the fixed point theory using the measure of noncompactness can be seen in [24–28] and the references therein.

Definition 1.4 [19] Consider a subset Ω of the Banach space Z. A mapping T of Ω into Z is called k-set contractive if
(1) T is bounded and continuous
(2) $\mu(T(F_1)) \leq k\mu(F_1)$ for any bounded subset F_1 of Ω.
T is called strictly k-set contractive if
(1) T is k-set contractive;
(2) for all bounded subsets F_1 of Ω with $\mu(F_1) \neq 0$, we have $\mu(T(F_1)) < k\mu(F_1)$.
T is called a condensing map if T is strictly 1-set contractive.

Definition 1.5 [19] Let Ω be the subset of a metric space Z. A mapping T of Ω into Z is called expansive if there is a constant $h > 1$ such that

$$d(T\varkappa, Ty) \geq hd(\varkappa, y)$$

for all $\varkappa, y \in \Omega$.

Definition 1.6 [6] Consider a metric space Z and $T : Z \to Z$. Then, we state that T is a contraction mapping if there is a number δ such that $0 < \delta < 1$ and

$$d(T\varkappa, Ty) \leq \delta d(\varkappa, y)$$

for all $\varkappa, y \in Z$.

Theorem 1.5 [6] (Banach) Any contraction mapping of a complete metric space Z into Z has a unique fixed point in Z.

Lemma 1.1 [19] Suppose Ω is the subset of a Banach space Z and F_1 is a Lipschitizian mapping of Ω into Z such that

$$\|F_1\varkappa - F_1y\| \leq k\|\varkappa - y\|$$

for $\varkappa, y \in \Omega$. Then we have that $\mu(F_1(S)) \leq k\mu(S)$ for every bounded subset S of Ω.

2. Main results

This section is devoted to the existence of a solution of operator equations of the type $H(Fs, s) = s$, where F is a $k - set$ contractive or condensing self operator on a given Banach space and H is either an equicontractive or equiexpansive operator. As a special case one can take $H = S + T$ where S and T are operators in which one is contractive/expansive and the other is compact type. In this way some generalized variants of the Krasnoselskii theorem are proved. These results will generalize the results of Xiang and colleagues [18–20,29] and Krasnoselskii and Schauder [[6], p. 25 and p. 31].

Suppose Ω is the subset of a Banach space Z and consider the mapping $H : F_1(\Omega) \times \Omega \longrightarrow Z$. We, in this paper, study the existence of a solution of equations $\varkappa = H(F_1\varkappa, \varkappa)$ by taking different assumptions on F_1, H and Ω and using the measure of noncompactness. We assume the following to prove our fixed-point results;

(i) the family $\{H(u, .) : u \in F_1(\Omega)\}$ is equiexpansive or equicontractive.

(ii) F_1 is a k-set contractive operator of Ω into Z or strictly a k-set contractive operator.

(iii) $\Omega \subseteq H(u, \Omega)$ for each fixed $u \in F_1(\Omega)$ with equiexpansive family while $H(F_1(\Omega), \Omega) \subseteq \Omega$ with equicontractive family.

2.1. Results for equiexpansive mappings

Theorem 2.1 Suppose Ω is the nonempty convex closed and bounded subset of the Banach space Z. Consider a mapping H of $F_1(\Omega) \times \Omega$ into Z such that $\Omega \subseteq H(u, \Omega)$ for $u \in F_1(\Omega)$ and F_1 is a k-set contractive mapping of Ω into Z with the following:
(1) the family $\{H(u, .) : u \in F_1(\Omega)\}$ is equiexpansive with $h > k + 1$;
(2) $\|H(u, y) - H(u', y)\| \leq \|u - u'\|$ for all $u, u' \in F_1(\Omega)$ and $y \in \Omega$;
Then there is $s \in \Omega$ such that $H(F_1 s, s) = s$.

Proof. Define $M : \Omega \to Z$ by $M(y) = H(u, y)$ for $u \in F_1(\Omega)$. Then M is expansive because

$$\|M(y_1) - M(y_2)\| = \|H(u, y_1) - H(u, y_2)\| \geq h\,\|y_1 - y_2\|,$$

and $\Omega \subseteq H(u, \Omega) = M(\Omega)$. From Theorem 1.4, there is a unique point in Ω say Gu such that $Gu = H(u, Gu)$. Now

$$\begin{aligned}
\|Gu - Gu'\| &= \|H(u, Gu) - H(u', Gu')\| \\
&= \|(H(u', Gu) - H(u', Gu')) - (H(u', Gu) - H(u, Gu))\| \\
&\geq \|H(u', Gu) - H(u', Gu')\| - \|H(u', Gu) - H(u, Gu)\| \\
&\geq h\,\|Gu - Gu'\| - \|u - u'\|.
\end{aligned}$$

This means that

$$\|Gu - Gu'\| \leq \frac{1}{h-1}\,\|u - u'\| \qquad h > 1,$$

which indicates that G is continuous from $F_1(\Omega)$ into Ω. Also $G \circ F_1$ is a continuous from Ω into Ω. From the above inequality and Lemma 1.1 and since F_1 is a k-set contractive mapping, we have

$$\mu(G \circ F_1(N)) = \mu(G(F_1(N))) \leq \frac{1}{h-1}\mu(F_1(N)) \leq \frac{k}{h-1}\mu(N) < \mu(N)$$

for all $N \subseteq \Omega$. Hence $G \circ F_1$ is condensing and by Theorem 1.3, there is $s \in \Omega$ such that $G \circ F_1(s) = s$ or $G(F_1 s) = s$. Since for $F_1 s \in F_1(\Omega)$ there is a unique $G(F_1 s) \in \Omega$ such that $H(F_1 s, G(F_1 s)) = G(F_1 s)$ and also $G(F_1 s) = s$; therefore, $H(F_1 s, s) = s$. □

Corollary 2.1 Suppose Ω is the nonempty convex closed and bounded subset of the Banach space Z. H is the mapping of $F_1(\Omega) \times \Omega$ into Z such that $\Omega \subseteq H(u, \Omega)$ for $u \in F_1(\Omega)$ and
(1) $F_1 : \Omega \longrightarrow Z$ is continuous such that $F_1(\Omega)$ resides in a compact subset of Z;
(2) the family $\{H(u, .) : u \in F_1(\Omega)\}$ is equiexpansive with $h > k + 1$;
(3) $\|H(u, y) - H(u', y)\| \leq \|u - u'\|$ for all $u, u' \in F_1(\Omega)$ and $y \in \Omega$;
Then there is $s \in \Omega$ such that $H(F_1 s, s) = s$.

Proof. From (1) we have that $F_1(\Omega) \subseteq C$, where C is the compact subset of the Banach space Z. Therefore $\mu(F_1(N)) \leq \mu(C) = 0$ for all $N \subseteq \Omega$. This shows that $\mu(F_1(N)) \leq k\mu(N)$ and F_1 is a k-set contractive mapping. Hence using Theorem 2.1, we get the required result. □

Remark 2.1 Theorem 2.2 in [19] is the special case of the above corollary with the assumption that Ω is a bounded set. This can be shown by defining $H(u, y) = u + F_2 y$ with $u = F_1 \varkappa \in F_1(\Omega)$, where F_1 is a continuous map from Ω into Z and F_2 is an expansive map from Ω into Z.

Corollary 2.2 Suppose Ω is the nonempty bounded convex and closed subset of the Banach space Z. Consider the mappings F_1 and F_2 such that
(1) $F_1 : \Omega \longrightarrow Z$ is a k-set contractive mapping;
(2) $F_2 : \Omega \longrightarrow Z$ is an expansive mapping with $h > k + 1$;
(3) $\Omega \subseteq u + F_2(\Omega)$ for $u \in F_1(\Omega)$.
Then there is $s \in \Omega$ such that $F_1 s + F_2 s = s$.

Proof. Define $H(u, y) = u + F_2 y$ with $u = F_1 \varkappa \in F_1(\Omega)$. Here $H : F_1(\Omega) \times \Omega \longrightarrow Z$ is a mapping and Ω is the nonempty bounded convex and closed subset of the Banach space Z. We show that all conditions of the Theorem 2.1 are satisfied. Since $H(u, \Omega) = u + F_2(\Omega)$ and from (3) $\Omega \subseteq u + F_2(\Omega)$, $\Omega \subseteq H(u, \Omega)$. Now, from (2), since F_2 is an expansive mapping for $h > k + 1$,

$$\|H(u, y_1) - H(u, y_2)\| = \|F_2 y_1 - F_2 y_2\| \geq h \|y_1 - y_2\|.$$

This shows that the family $\{H(u, .) : u \in F_1(\Omega)\}$ is equiexpansive with $h > k + 1$. Also $F_1 : \Omega \longrightarrow Z$ is a k-set contractive mapping and

$$\|H(u, y) - H(u', y)\| = \|u - u'\| \leq \|u - u'\|.$$

Hence there is $s \in \Omega$ such that $s = H(F_1 s, s) = F_1 s + F_2 s$. □

Corollary 2.3 Suppose Ω is the nonempty bounded convex and closed subset of the Banach space Z. Consider the mappings F_1 and F_2 such that
(1) $F_1 : \Omega \longrightarrow \Omega \subseteq Z$ is a nonexpansive and k-set contractive mapping;
(2) $F_2 : \Omega \longrightarrow Z$ is an expansive mapping with $h > k + 1$;
(3) $\Omega \subseteq F_1 u + F_2(\Omega)$ for $F_1 u \in F_1(\Omega)$.
Then there is $s \in \Omega$ such that $(F_1 \circ F_1)s + F_2 s = s$.

Proof. Define $H(u, y) = F_1 u + F_2 y$ with $F_1 u \in F_1(\Omega)$ (Theorem 2.1). Since $H(u, \Omega) = F_1 u + F_2(\Omega)$ and from (3) $\Omega \subseteq F_1 u + F_2(\Omega)$ for $F_1 u \in F_1(\Omega)$; therefore, $\Omega \subseteq H(u, \Omega)$. Now since F_2 is an expansive mapping therefore,

$$\|H(u, y_1) - H(u, y_2)\| = \|F_2 y_1 - F_2 y_2\| \geq h \|y_1 - y_2\|.$$

Also from (1), we have

$$\|H(u, y) - H(u', y)\| = \|F_1 u - F_1 u'\| \leq \|u - u'\|.$$

Hence there is $s \in \Omega$ such that $s = H(F_1 s, s) = (F_1 \circ F_1)s + F_2 s$. □

Corollary 2.4 Suppose Ω is the nonempty bounded convex and closed subset of the Banach space Z. Consider the mappings F_1 and F_2 such that
(1) $F_1 : \Omega \longrightarrow \Omega \subseteq Z$ is nonexpansive and $F_1(\Omega)$ resides in a compact subset of Z.;
(2) $F_2 : \Omega \longrightarrow Z$ is an expansive mapping with $h > k + 1$;
(3) $\Omega \subseteq F_1 u + F_2(\Omega)$ for $F_1 u \in F_1(\Omega)$.
Then there is $s \in \Omega$ such that $(F_1 \circ F_1)s + F_2 s = s$.

Proof. We use similar arguments as in the above corollaries. □

Remark 2.2 Theorem 2.1 is the generalized form of Theorems 2.2 and 2.6 in [19] and a variant of the Krasnoselskii fixed point theorem.

2.2. Results for equicontractive mappings

Theorem 2.2 Suppose Ω is the nonempty bounded closed and convex subset of a Banach space Z. Let F_1 be a continuous mapping of Ω into Z and H be a mapping of $\Omega \times \Omega$ into Ω such that
(1) $\mu(H(N, \Omega)) < \mu(N)$ for all $N \subseteq \Omega$ with $\mu(N) \neq 0$;
(2) the family $\{H(\varkappa, .) : \varkappa \in \Omega\}$ is equicontractive;
(3) $\|H(\varkappa, y) - H(\varkappa', y)\| \leq \|F_1\varkappa - F_1\varkappa'\|$ for all $\varkappa, \varkappa' \in \Omega$ and $y \in \Omega$;
Then there is $s \in \Omega$ such that $H(s, s) = s$.

Proof. Define $M : \Omega \to \Omega$ by $M(y) = H(\varkappa, y)$ for $\varkappa \in \Omega$. Then, M is a contraction and there is a unique point in Ω say $G\varkappa$ such that $G\varkappa = H(\varkappa, G\varkappa)$. Now,

$$\begin{aligned}
\|G(\varkappa) - G(\varkappa')\| &= \|H(\varkappa, G\varkappa) - H(\varkappa', G\varkappa')\| \\
&= \|(H(\varkappa', G\varkappa) - H(\varkappa', G\varkappa')) - (H(\varkappa', G\varkappa) - H(\varkappa, G\varkappa))\| \\
&\leq \|H(\varkappa', G\varkappa) - H(\varkappa', G\varkappa')\| + \|H(\varkappa', G\varkappa) - H(\varkappa, G\varkappa)\| \\
&\leq \alpha \|G\varkappa - G\varkappa'\| + \|F_1\varkappa - F_1\varkappa'\|.
\end{aligned}$$

This means that

$$\|G\varkappa - G\varkappa'\| \leq \frac{1}{1-\alpha} \|F_1\varkappa - F_1\varkappa'\|.$$

Continuity of F_1 implies that $G : \Omega \longrightarrow \Omega$ is a continuous function. Since $N \subseteq \Omega$, $G(N) \subseteq G(\Omega) \subseteq \Omega$ and $H(N, G(N))) \subseteq H(N, \Omega))$. Hence $\mu(H(N, G(N))) \leq \mu(H(N, \Omega))$. Now, from (1), we have

$$\mu(G(N)) = \mu(H(N, G(N))) \leq \mu(H(N, \Omega)) < \mu(N).$$

Hence $G : \Omega \longrightarrow \Omega$ is condensing. By Theorem 1.3, there exists $s \in \Omega$ such that $Gs = s$. As for $s \in \Omega$, there is a unique $Gs \in \Omega$ such that $H(s, Gs) = Gs$ and $Gs = s$; therefore, $H(s, s) = s$. □

Corollary 2.5 Suppose Ω is the nonempty bounded closed and convex subset of a Banach space Z. Consider the mappings F_1 and F_2 of Ω into Z such that
(1) $F_1\varkappa + F_2 y \in \Omega$ for all $\varkappa, y$ in Ω;
(2) F_1 is a condensing mapping;
(3) F_2 is a contraction and $F_2(\Omega)$ lies in a compact subset of Z.
Then there is $\varkappa \in \Omega$ such that $F_1\varkappa + F_2\varkappa = \varkappa$.

Proof. Define $H(\varkappa, y) = F_1\varkappa + F_2 y$. Obviously $H : \Omega \times \Omega \longrightarrow \Omega$ and

$$\mu(H(N, \Omega) = \mu(F_1(N) + F_2(\Omega)) \leq \mu(F_1(N)) + \mu(F_2(\Omega))) \leq \mu(F_1(N)) < \mu(N).$$

Now using (3),

$$\|H(\varkappa, y_1) - H(\varkappa, y_2)\| = \|F_2 y_1 - F_2 y_2\| \leq \alpha \|y_1 - y_2\|,$$

and, also,

$$\|H(\varkappa, y) - H(\varkappa', y)\| = \|F_1\varkappa - F_1\varkappa'\| \leq \|F_1\varkappa - F_1\varkappa'\|.$$

Hence there exists $\varkappa \in \Omega$ such that $F_1\varkappa + F_2\varkappa = \varkappa$. □

Remark 2.3 Theorem 2.2 and the above corollary are variants of the Krasnoselskii fixed point theorem.

Corollary 2.6 Suppose Ω is the nonempty bounded closed and convex subset of a Banach space Z. Suppose $F_1 : \Omega \longrightarrow Z$ is a continuous mapping and $F_2 : \Omega \longrightarrow Z$ is a contraction mapping such that
(1) $F_1\varkappa + F_2 y \in \Omega$;
(2) $\mu(F_1(N) + F_2(\Omega)) < \mu(N)$ for all $N \subseteq \Omega$ with $\mu(N) \neq 0$.
Then there is $s \in \Omega$ such that $F_1 s + F_2 s = s$.

Proof. Define $H(\varkappa, y) = F_1\varkappa + F_2 y$ (Theorem 2.2). Now since F_2 is a contraction mapping,

$$\|H(\varkappa, y_1) - H(\varkappa, y_2)\| = \|F_2 y_1 - F_2 y_2\| \leq \alpha \|y_1 - y_2\|.$$

Also,

$$\|H(\varkappa, y) - H(\varkappa', y)\| = \|F_1\varkappa - F_1\varkappa'\| \leq \|F_1\varkappa - F_1\varkappa'\|,$$

and using (2),

$$\mu(H(N, \Omega)) = \mu(F_1(N) + F_2(\Omega)) < \mu(N).$$

Hence there is $s \in \Omega$ such that $s = H(s, s) = F_1 s + F_2 s$. □

Theorem 2.3 Suppose Ω is the nonempty bounded closed convex subset of a Banach space Z and $F_1 : \Omega \longrightarrow Z$ is continuous. Let $H : F_1(\Omega) \times \Omega \longrightarrow \Omega$ be the mapping such that
(1) $\mu(H(F_1(N), \Omega)) < \mu(N)$ for all $N \subseteq \Omega$ with $\mu(N) \neq 0$;
(2) the family $\{H(u, .) : u \in F_1(\Omega)\}$ is equicontractive;
(3) $\|H(u, y) - H(u', y)\| \leq \|u - u'\|$ for all $u, u' \in F_1(\Omega)$ and $y \in \Omega$;
Then there is $s \in \Omega$ such that $H(F_1 s, s) = s$.

Proof. Let $u \in F_1(\Omega)$, and define a map $M : \Omega \longrightarrow \Omega$ by $M(y) = H(u, y)$. Clearly M is a contraction and there is a unique point in Ω say Gu such that $Gu = H(u, Gu)$. Now

$$\begin{aligned} \|G(u) - G(u')\| &= \|H(u, Gu) - H(u', Gu')\| \\ &= \|(H(u', Gu) - H(u', Gu')) - (H(u', Gu) - H(u, Gu))\| \\ &\leq \|H(u', Gu) - H(u', Gu')\| + \|H(u', Gu) - H(u, Gu)\| \\ &\leq \alpha \|Gu - Gu'\| + \|u - u'\|. \end{aligned}$$

This means that

$$\|Gu - Gu'\| \leq \frac{1}{1-\alpha} \|u - u'\| \qquad \alpha \in (0, 1),$$

which indicates that G is continuous from $F_1(\Omega)$ into Ω and $G \circ F_1$ is a continuous from Ω into Ω.

Also for $N \subseteq \Omega$ and using (1), we get

$$\mu(G \circ F_1(N)) = \mu(G(F_1(N))) = \mu(H(F_1(N), G(F_1(N))) \leq \mu(H(F_1(N), \Omega) < \mu(N).$$

Thus $G \circ F_1$ is a condensing mapping of Ω into Ω and by Theorem 1.3, there is $s \in \Omega$ such that $G(F_1 s) = s$. Hence, $s = G(F_1 s) = H(F_1 s, G(F_1 s)) = H(F_1 s, s)$. □

Corollary 2.7 Suppose Ω is the nonempty bounded closed and convex subset of a Banach space Z. Consider the mappings F_1 and F_2 of Ω into Z such that
(1) $F_1\varkappa + F_2 y \in \Omega$;
(2) $F_2 : \Omega \longrightarrow Z$ is a contraction with $k < 1$ and $F_2(\Omega)$ resides in a compact subset of Z;
(3) $F_1 : \Omega \longrightarrow Z$ is strictly a $(1-k)$-set contractive mapping;.
Then $\varkappa = F_1\varkappa + F_2\varkappa$ has a solution in Ω.

Proof. Define $H(u,y) = u + F_2 y$ with $u = F_1\varkappa \in F_1(\Omega)$ (Theorem 2.3). Using (1) $H(u,y) \in \Omega$ for all $u = F_1\varkappa, \varkappa \in \Omega$ and $y \in \Omega$. Now since F_2 is contraction mapping, $\|H(u,y_1) - H(u,y_2)\| = \|F_2 y_1 - F_2 y_2\| \leq k\,\|y_1 - y_2\|$. Also, $\|H(u,y) - H(u',y)\| = \|u - u'\| \leq \|u - u'\|$. Now, for $N \subseteq \Omega$,

$$\mu(H(F_1(N),\Omega)) = \mu(F_1(N) + F_2(\Omega)) \leq \mu(F_1(N)) + \mu(F_2(\Omega)) < (1-k)\mu(N) < \mu(N).$$

Hence, there is $s \in \Omega$ such that $s = H(F_1 s, s) = F_1 s + F_2 s$. □

Remark 2.4 If we take $F_2 = O$ with $k = 0$ in the above corollary, we obtain Sadovskii fixed point theorem.

Theorem 2.4 Suppose Ω is the nonempty bounded closed and convex subset of a Banach space Z. Let $H : F_1(\Omega) \times \Omega \longrightarrow \Omega$ be a mapping such that
(1) $\|H(u,y) - H(u',y)\| \leq \|u - u'\|$;
(2) the family $\{H(u,.) : u \in F_1(\Omega)\}$ is equicontractive with $k < 1$;
(3) $F_1 : \Omega \longrightarrow Z$ is strictly a $(1-k)$-set contractive mapping;
Under these conditions, $\varkappa = H(F_1\varkappa, \varkappa)$ has a solution in Ω.

Proof. Consider a mapping $M : \Omega \longrightarrow \Omega$ by $M(y) = H(u,y)$ for $u \in F_1(\Omega)$. Clearly M is a contraction and there is a unique point in Ω say Gu such that $Gu = H(u, Gu)$. Now

$$\begin{aligned}
\|G(u) - G(u')\| &= \|H(u,Gu) - H(u',Gu')\| \\
&= \|(H(u',Gu) - H(u',Gu')) - (H(u',Gu) - H(u,Gu))\| \\
&\leq \|H(u',Gu) - H(u',Gu')\| + \|H(u',Gu) - H(u,Gu)\| \\
&\leq k\,\|Gu - Gu'\| + \|u - u'\|.
\end{aligned}$$

This means that

$$\|Gu - Gu'\| \leq \frac{1}{1-k}\,\|u - u'\| \qquad k \in (0,1),$$

which indicates that G is continuous from $F_1(\Omega)$ into Ω and $G \circ F_1$ is a continuous from Ω into Ω.

Also, from (3), the above inequality and Lemma 1.1, we have

$$\mu(G \circ F_1(N)) = \mu(G(F_1(N))) \leq \frac{1}{1-k}\mu(F_1(N)) < \frac{1-k}{1-k}\mu(N) = \mu(N)$$

for all $N \subseteq \Omega$. Hence $G \circ F_1$ is a condensing mapping of Ω into Ω and by Theorem 1.3, there is $s \in \Omega$ such that $G(F_1 s) = s$. Hence $s = G(F_1 s) = H(F_1 s, G(F_1 s)) = H(F_1 s, s)$. □

Corollary 2.8 Suppose Ω is the nonempty bounded closed and convex subset of a Banach space Z. Consider the mappings F_1 and F_2 of Ω into Z such that
(1) $F_1\varkappa + F_2 y \in \Omega$;

(2) $F_2 : \Omega \longrightarrow Z$ is a contraction with $k < 1$;
(3) F_1 is strictly a $(1-k)$-set contractive mapping;
Then $\varkappa = F_1\varkappa + F_2\varkappa$ has a solution in Ω.

Proof. Define $H(u, y) = u + F_2 y$ with $u = F_1\varkappa \in F_1(\Omega)$. Using (1), $H(u, y) \in \Omega$ for all $u = F_1\varkappa, \varkappa \in \Omega$ and $y \in \Omega$. Now, since F_2 is a contraction mapping, $\|H(u, y_1) - H(u, y_2)\| = \|F_2 y_1 - F_2 y_2\| \leq k\|y_1 - y_2\|$. Also, $\|H(u, y) - H(u', y)\| = \|u - u'\| \leq \|u - u'\|$. Hence, there is $s \in \Omega$ such that $s = H(F_1 s, s) = F_1 s + F_2 s$. □

Remark 2.5 If we take $F_2 = O$ and $k = 0$, (in the above corollary) we obtain the Sadovskii fixed point theorem. Taking $F_1 = O$, we get the Banach contraction principle.

Corollary 2.9 Suppose Ω is the nonempty bounded closed and convex subset of a Banach space Z. Consider the mappings F_1 and F_2 of Ω into Z such that
(1) $F_1\varkappa + F_2 y \in \Omega$;
(2) $F_2 : \Omega \longrightarrow Z$ is a contraction with $k < 1$;
(3) F_1 is compact and continuous.
Then $\varkappa = F_1\varkappa + F_2\varkappa$ has a solution in Ω.

Proof. Since every compact operator is strictly $(1-k)$-set contractive, using the above corollary we get the required result. □

Remark 2.6 Taking $F_2 = O$, we obtain the Schauder fixed point theorem and with $F_1 = O$ we obtain the Banach contraction principle.

3. Application

Astrophysicists analyze the light coming from objects similar to a star's atmosphere with vastly different physical conditions by solving the radiative transfer equation as a tool. In this section we prove an existence result for the radiative transfer equation of a specific type. A quantity that describes the field of radiation is known as the diffuse radiance/emission variable ξ. All other terms can be considered as a source terms. The most generalized form is given by

$$\mu \frac{\partial \xi}{\partial x}(\tau, \varkappa, v) = \xi(\tau, \varkappa, v) - J(\tau, \varkappa, v),$$

here ξ is the diffuse radiation and J is the source function.

Consider a class of stationary radiative transfer equations in a channel:

$$v_3 \frac{\partial \xi}{\partial \varkappa}(\tau, \varkappa, v) + \sigma(\varkappa, v)\xi(\tau, \varkappa, v) - \lambda\xi(\tau, \varkappa, v) = \int_{U^2} r(\varkappa, v, u, \xi(\tau, \varkappa, u))\, du \tag{1}$$

in $(0, 1) \times (0, 1) \times U^2, \lambda \in \mathbb{C}$, where U is a unit sphere in $\mathbb{R}^3$ and $\xi(\tau, \varkappa, v)$ is an unknown complex function (energy density function) with

$$\xi(\tau, \tau, v) = \xi^{\tau}(v) \tag{2}$$

where

$$\begin{aligned} \xi^{\tau}(v) &= (1-\tau)^{\frac{1}{\theta}}\xi(0,0,v)|_{U^2} + \tau^{\frac{1}{\theta}}\xi(1,1,v)|_{U^2} \\ &= (1-\tau)^{\frac{1}{\theta}} M^1(\xi(1,1,v)|_{U^2}) + \tau^{\frac{1}{\theta}} M^2(\xi(0,0,v)|_{U^2}) \end{aligned}$$

with $0<\theta\le 1$. Clearly for $\theta=1$, we get the problem (4.3) + (4.4) in [29], and for $\theta=1$ and $\tau=0$, we get problem (4.1) + (4.2) of [29] [and, also, see [11]]. In fact, we generalize the problem and present an implicit way of finding solution of the problem (1) + (2).

To prove the existence of solutions of (1) + (2), we consider the following assumptions.

$(A1)$: $r\in C\left((0,1)\times U^2\times U^2\times\mathbb{C}\right)$, and $r(y,v,u,\xi)=0$ for every $(y,v,u,\xi)\in(0,1)\times U^2\times U^2\times\mathbb{C}$ with $v_3\le\frac{1}{2}$; there is some $a\in L^1\left((0,1)\times U^2\right)$ and $q>0$ such that

$$|r(\varkappa,v,u,\xi)-r(\varkappa,v,u,\zeta)|\le a(\varkappa,u)|\xi-\zeta|,$$

and $|r(\varkappa,v,u,\xi)|\le qa(\varkappa,u)$ for every $(\varkappa,v,u,\xi)\in(0,1)\times U^2\times U^2\times\mathbb{C}$.

$(A2)$: $M^i : C\left(U^2\right)\to C\left(U^2\right)$, with $M^i(0)=0$ satisfying

$$\left|M^i(\zeta)-M^i(\xi)\right|\le q|\zeta-\xi|$$

on U^2 for every $\xi,\zeta\in C\left(U^2\right)$, $i=1,2$.

$(A3)$: $q+\sup\limits_{v\in U^2}\int_0^1|\sigma(y,v)|\,dy+|\lambda|+\int_0^1\int_{U^2}a(\varkappa,u)\,dudy\le\gamma<1$ and $2\gamma\le q+1$.

We use Theorem 2.1 to prove the next existence result for the solution of radiative transfer equation (1).

Theorem 3.1 Suppose $(A1)$, $(A2)$ and $(A3)$ are satisfied; then, there exists a solution $\xi\in C\left([0,1]\times[0,1]\times U^2\right)$ of (1) + (2).

Proof. Let $E=\left\{\xi\in C\left([0,1]\times[0,1]\times U^2\right):\xi(\tau,\varkappa,v)=0 \text{ for } v_3\le\frac{1}{2}\right\}$ be the Banach space with the norm $\|\xi\|=\sup\left\{|\xi(\tau,\varkappa,v)|:(\tau,\varkappa,v)\in[0,1]\times[0,1]\times U^2\right\}$. Let $K=\{\zeta\in E:\|\zeta\|\le q_1\}$ be the convex and closed subset of E. We define $H:S(K)\times K\to K$ by

$$\begin{aligned} H(S\xi_1,\xi_2) &= -bv_3\xi_1^{\tau}+b\int_{\tau}^{\varkappa}\sigma(y,v)\xi_1(\tau,y,v)\,dy-\lambda b\int_{\tau}^{\varkappa}\xi_1(\tau,y,v)\,dy \\ &\quad -b\int_{\tau}^{\varkappa}\int_{U^2}r(\varkappa,v,u,\xi_1(\tau,\varkappa,u))\,dudy+(1+bv_3)\xi_2(\tau,\varkappa,v), \end{aligned}$$

where

$$\begin{aligned} S(\xi) &= -bv_3\xi^{\tau}+b\int_{\tau}^{\varkappa}\sigma(y,v)\xi(\tau,y,v)\,dy \\ &\quad -\lambda b\int_{\tau}^{\varkappa}\xi(\tau,y,v)\,dy-b\int_{\tau}^{\varkappa}\int_{U^2}r(\varkappa,v,u,\xi(\tau,\varkappa,u))\,dudy. \end{aligned}$$

Clearly, every $\zeta \in E$ satisfying $\zeta = H(S\zeta, \zeta)$ is the solution of problem (1) + (2). Now, we show that H satisfies all conditions of Theorem 2.1. For this, first consider

$$\begin{aligned}\left|H(S\xi, \xi_1) - H(S\xi, \xi_2)\right| &= \left|(1+bv_3)\,\xi_1(\tau, \varkappa, v) - (1+bv_3)\,\xi_2(\tau, \varkappa, v)\right| \\ &= \left|(1+bv_3)\left(\xi_1(\tau, \varkappa, v) - \xi_2(\tau, \varkappa, v)\right)\right| \\ &\geq \left(1+\frac{b}{2}\right)\left|\xi_1(\tau, \varkappa, v) - \xi_2(\tau, \varkappa, v)\right|\end{aligned}$$

for $\xi_1, \xi_2 \in K$. Therefore we get

$$\left\|H(S\xi, \xi_1) - H(S\xi, \xi_2)\right\| \geq \left(1+\frac{b}{2}\right)\left\|\xi_1 - \xi_2\right\|;$$

hence $H(\zeta, \cdot)$ is equiexpansive. Now, consider

$$\begin{aligned}\left|H(S\xi_1, \zeta) - H(S\xi_2, \zeta)\right| &= \Bigg| bv_3\left(\xi_2^\tau - \xi_1^\tau\right) + b\int_\tau^\varkappa \sigma(y, v)\left(\xi_1(\tau, y, v) - \xi_2(\tau, y, v)\right)dy \\ &\quad -\lambda b\int_\tau^\varkappa \left(\xi_1(\tau, y, v) - \xi_2(\tau, y, v)\right)dy \\ &\quad -b\int_\tau^\varkappa\int_{U^2}\left(r(\varkappa, v, u, \xi_1(\tau, \varkappa, u)) - r(\varkappa, v, u, \xi_2(\tau, \varkappa, u))\right)dudy\Bigg| \\ &= \left|S\xi_1 - S\xi_2\right|;\end{aligned}$$

this implies

$$\left\|H(S\xi_1, \zeta) - H(S\xi_2, \zeta)\right\| \leq \left\|S\xi_1 - S\xi_2\right\|,$$

which shows that H satisfies (2) of Theorem 2.1.
Now, we will show that S is a k-set contraction; for this consider

$$\left|S\xi_1 - S\xi_2\right| \leq b\begin{pmatrix} q\left\|\xi_1 - \xi_2\right\| + \int_0^1 |\sigma(y, v)|\left|\xi_1(\tau, y, v) - \xi_2(\tau, y, v)\right|dy \\ +|\lambda|\int_0^1 \left|\xi_1(\tau, y, v) - \xi_2(\tau, y, v)\right|dy \\ +\int_0^1\int_{U^2}\left|r(\varkappa, v, u, \xi_1(\tau, \varkappa, u)) - r(\varkappa, v, u, \xi_2(\tau, \varkappa, u))\right|dudy \end{pmatrix}$$

$$\leq \quad b\begin{pmatrix} q\left\|\xi_1 - \xi_2\right\| + \int_0^1 |\sigma(y, v)|\left|\xi_1(\tau, y, v) - \xi_2(\tau, y, v)\right|dy \\ +|\lambda|\int_0^1 \left|\xi_1(\tau, y, v) - \xi_2(\tau, y, v)\right|dy \\ +\int_0^1\int_{U^2} a(\varkappa, u)\left|\xi_1(\tau, \varkappa, u) - \xi_2(\tau, \varkappa, u)\right|dudy \end{pmatrix}$$

$$\leq \quad b\left(q+\int\limits_{0}^{1}|\sigma(y,v)|\,dy+|\lambda|+\int\limits_{0}^{1}\int\limits_{U^2}a(\varkappa,u)\,dudy\right)\left\|\xi_1-\xi_2\right\|.$$

Clearly, from above we have

$$\left\|S\xi_1-S\xi_2\right\|\leq\gamma b\left\|\xi_1-\xi_2\right\|,$$

which means that S is a γb-set contraction (also for $\gamma=\frac{1}{2}$). Hence, all conditions of Theorem 2.1 are satisfied to obtain $\zeta\in K$ such that $\zeta=H(S\zeta,\zeta)$, a solution of (1) + (2). □

In the following example, we consider the special case of (1) without parameter τ.

Example 3.1 Consider the following radiative transfer equation:

$$\xi(x,u)=\alpha\xi(x,u)+b\int\limits_{0}^{x}\sigma(y,u)\xi(y,u)\,dy-\lambda b\int\limits_{0}^{x}\xi(y,u)\,dy-b\int\limits_{0}^{x}\int\limits_{0}^{1}r(y,u,)\,\xi(y,u)\,dudy.$$

For a more simplified case let $\sigma(y,u)=\lambda$ and $b=\frac{1}{10}$; then it reduces to

$$\xi(x,u)=\alpha\xi(x,u)-\int\limits_{0}^{x}\int\limits_{0}^{1}b\cdot r(y,u,)\,\xi(y,u)\,dudy$$

with source term $b\int\limits_{0}^{x}\int\limits_{0}^{1}r(y,u,)\,\xi(y,u)\,dudy$. To reduce the complexities, we consider the following form:

$$\xi(x,u)=1+xe^{-u}-bx\sin(\xi(x,u))-\int\limits_{0}^{x}\int\limits_{0}^{1}b\frac{\sin(\xi(x,u))\sqrt{1+\xi(y,s)}}{(1+ye^{-s})^{\frac{1}{2}}}dsdy.$$

The exact solution of the above equation is

$$\xi(x,u)=1+xe^{-u}.$$

Since

$$\left|b\frac{\sin(\xi(x,u))\sqrt{1+\xi(x,u)}}{(1+xe^{-u})^{\frac{1}{2}}}-b\frac{\sin(\xi_1(x,u))\sqrt{1+\xi_1(x,u)}}{(1+xe^{-u})^{\frac{1}{2}}}\right|\leq b\left|\xi(y,s)-\xi_1(y,s)\right|$$

for $(x,u)\in[0,1]\times[0,1]$, the above integral equation satisfies all conditions of the above theorem; now, we take the initial guess

$$\xi_0(x,u)=1+x.$$

The iterative process is as follows:

$$\xi_{n+1}(x,u)=1+xe^{-u}-bx\sin(\xi_n(x,u))-\int\limits_{0}^{x}\int\limits_{0}^{1}b\frac{\sin(\xi_0(x,u))\sqrt{\xi_0(y,s)}}{(1+ye^{-s})^{\frac{1}{2}}}dsdy \text{ for } n=0,1,2,3,...$$

Taking initial guess $\xi_0(x, u) = 1+x$, the approximate solution is obtained after two iterations. The graph of the exact solution and the approximate solution are given below in Figures 1 and 2, respectively.

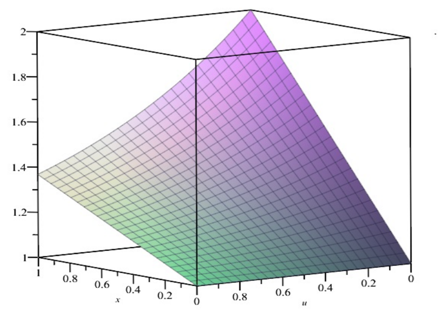

Figure 1. Exact solution.

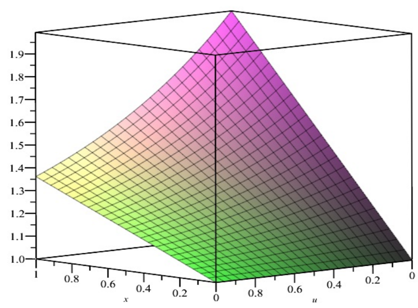

Figure 2. Approximate solution.

We calculated the absolute error with the formula

$$Error = \left|\xi(x, u) - \xi_2(x, u)\right| \text{ for } (x, u) \in [0, 1] \times [0, 1].$$

The graph of the absolute error is given below in Figure 3; clearly, no significant error has occurred.

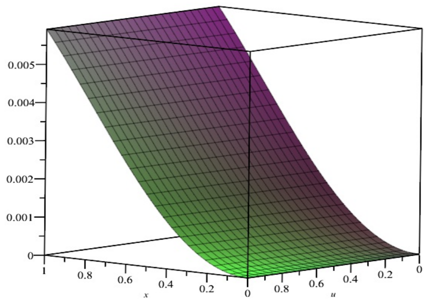

Figure 3. Absolute error.

4. Conclusions

Some new variants of Krasnoselskii fixed point theorems are proved using the notions of equicontractive and equiexpansive mappings. The tool of the measure of noncompactness is used to weaken the compactness of the operator. An application for the existence of solutions of radiative transfer equations has been established. An example is given as a special case of radiative transfer equation.

Acknowledgement

The authors A. Mukheimer and T. Abdeljawad would like to thank Prince Sultan University for paying APC and for support through the TAS research group. The authors N. Ahmad and N. Mehmood would like to acknowledge the Higher Education Commission of Pakistan for providing support under the project No: 20-15549/NRPU.

Conflict of interest

From all authors, there is no conflict of interest.

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