Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(6): 14572–14591.
DOI: 10.3934/math.2023745
Received: 04 February 2023
Revised: 11 April 2023
Accepted: 11 April 2023
Published: 20 April 2023

Research article

Novel algorithms to approximate the solution of nonlinear integro-differential equations of Volterra-Fredholm integro type

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Abstract: This study is devoted to examine the existence and uniqueness behavior of a nonlinear integro-differential equation of Volterra-Fredholm integral type in continues space. Then, we examine its solution by modification of Adomian and homotopy analysis methods numerically. Initially, the proposed model is reformulated into an abstract space, and the existence and uniqueness of solution is constructed by employing Arzela-Ascoli and Krasnoselskii fixed point theorems. Furthermore, suitable conditions are developed to prove the proposed model's continues behavior which reflects the stable generation. At last, three test examples are presented to verify the established theoretical concepts.

Keywords: boundary value problem; Arzela-Ascoli theorem; Krasnosel'skii theorem; nonlinear integro equation
Mathematics Subject Classification: 34A45, 34B15, 45J05, 65L10

1. Introduction

In the theory of ordinary and fractional calculus, boundary value problems for differential equations play an important role in the context of integral equations. They often occur in approximation models of the real-world problems, for example, in physics, material sciences, fractional calculus theory, ecology and epidemiology (see [1–5]). Also, it motivates the in-depth study of these types of integral models aiming to prove existence or/and uniqueness of their solutions.

Fredholm, Volterra, and integro-differential equations have important properties and are frequently used in many areas of mathematics. Particularly, nonlinear integro-differential equations which cannot be expressed as the solution of a linear combination of the functions and their derivatives. These equations are often difficult to solve and require numerical methods to approximate the solutions and they have been studied and examined by many researchers. Thus, these types of integral problems appear in many mathematical models, computational algorithms, engineering problems, and physics as well as fractional calculus theory (cf. other articles [6–11]).

On the other hand, the Adomian decomposition and its modification are used frequently in many branches of applied mathematics, especially in integral equation theory. Therefore, many scholars, including Wazwaz and his students, have studied these numerical algorithms to tackle some difficult problems and find effective results. These approaches have also been applied to the numerical solution of Abel's integral equations, the Bagley-Torvik equations, the Fredholm and Volterra integral equations, the integro equations, and numbers that are involved in an important position in applied mathematics to obtain meaningful relations and representations in previous articles, see [12–17] and closed references therein.

On the other hands, the exploration and solution of integro differential equation of nonlinear Volterra and Fredholm types have attracted more and more attention by using homotopy analysis methods. Over the years, this method has been proposed to find solution of linear and nonlinear integral equations, for example, see [18–23].

Among the already known findings and results in the study of BVPs that include the construction of integro-differential equations are those obtained in previous studies. For instance, in this paper, we will consider a nonlinear integro-differential Eq (1.1) and solving it by using modified Adomian decomposition method (MADM) and homotopy analysis methods (HAM),

$$\begin{aligned}&\theta\psi^{''}(\zeta) + \mathcal{A}(\zeta)\psi^{'}(\zeta) + \mathcal{B}(\zeta)\psi(\zeta)\\ =&f(\zeta) + \lambda_1 \int_{a_0}^{\zeta} K_1(\zeta, y)[\psi(y)]^p dy + \lambda_2 \int_{a_0}^{a_1} K_2(\zeta, y)[\psi(y)]^p dy, \text{ for } \zeta \in [a_0, a_1],\end{aligned} \tag{1.1}$$

with the following conditions

$$\psi(a_0) = \eta_1, \quad \psi(a_1) = \eta_2, \tag{1.2}$$

where $p \geq 0$ and $\eta_1, \eta_2 \in \mathbb{R}$, $\theta, \lambda_1, \lambda_2$ are non zero real parameters, and the functions $\mathcal{A}, \mathcal{B}, f$ and the disjoint kernels K_1, K_2 are known functions satisfying certain conditions to be assigned in the next section. Note that $\zeta \mapsto \psi(\zeta)$ is the sought function to be determined in the space $C^2([a_0, a_1], \mathbb{R})$.

The rest of our study is arranged as follows: In Section 2, we recall the main concepts, and existence and uniqueness of the solution. Section 3 describes the methods of solution of (1.1) by the algorithms proposed in this article in detail in Subsections 3.1 and 3.2, respectively. Section 4 describes the numerical results and analysis. Finally, Section 5 gives the conclusion of our study.

2. Basic tools and existence of solutions

In this section, we briefly review some basic elements of the Volterra-Fredholm integral equations and integro-differential equations. For a comprehensive study on these topics, we refer the interested reader to [24–29].

Definition 2.1. *[24] Let X be a Banach space and let $\mathcal{T} : X \to X$ be a self-operator, i.e. $\mathcal{T}(x) = x$ for all $x \in X$. Then, the formula for $\mathcal{T}$ is simply $\mathcal{T}(x) = x$ for all $x \in X$.*

First, we state the contraction mapping concept.

Definition 2.2. *[25] A mapping $\tau : \mathscr{M} \to \mathscr{M}$ is contraction mapping or contraction defined on the Banach space $(\mathscr{M}, \mathrm{d})$, if there exists a constant $\imath$ with $0 \le \imath < 1$, such that $\mathrm{d}(f(\zeta), f(\mathrm{y})) \le \imath \mathrm{d}(\zeta, \mathrm{y})$ $\forall \zeta, \mathrm{y} \in \mathscr{M}$.*

Next, we consider the Banach contraction principle.

Theorem 2.1. *[26] if (X, d) is a complete metric space and $T : X \to X$ is a function, such that there exists a constant $0 \le k < 1$ for which*

$$d(T(x), T(y)) \le k \cdot d(x, y),$$

for all $x, y \in X$, then T has exactly one fixed point, i.e. there exists a unique $x_0 \in X$ such that $T(x_0) = x_0$.

Theorem 2.2. *[29] Suppose that $g(\zeta) = lim_{\ell\to\infty} g_\ell(\zeta)$ on $\Im = [a, b]$, where $g, g_1, g_2, \ldots$ are all Riemann integrable functions on $\Im$. If $\{g_n(\zeta)\}_{\ell=1}^{\infty}$ is uniformly bounded on $\Im$, then one can have*

$$\int_{a_0}^{a_1} g(\tau)\mathrm{d}\tau = \lim_{\ell\to\infty} \left(\int_{a_0}^{a_1} g_\ell(\tau)\mathrm{d}\tau \right)$$

and

$$\lim_{\ell\to\infty} \left(\int_{a_0}^{a_1} \left| g_\ell(\tau) - g(\tau) \right| \mathrm{d}\tau \right) = 0.$$

In the following theorem, we recall the Arzela-Ascoli theorem.

Theorem 2.3. *[28] Suppose that a sequence $\{f_\ell\}_{\ell=0}^{\infty}$ is bounded and equicontinuous in $C[a_0, a_1]$. Then $\{f_\ell\}_{\ell=0}^{\infty}$ has a subsequence, which is a uniformly convergent.*

Another important theorem in our study is the Krasnoselskii fixed point theorem, which is stated as follows.

Theorem 2.4. *[27] Let $\mathscr{M}$ be bounded, closed and convex subset in a Banach space X. Let $\mathcal{A}, \mathcal{B} : \mathscr{M} \to \mathscr{M}$ be two operators satisfying the following conditions:*

1) $\mathcal{A}$ is continuous and compact;

2) $\mathcal{A}\zeta + \mathcal{B}y \in \mathcal{M}$, $\forall \zeta, y \in \mathcal{M}$;

3) $\mathcal{B}$ *is a contraction.*

Then, there exists $z \in \mathcal{M}$ *such that* $\mathcal{A}z + \mathcal{B}z = z$.

Let us briefly recall the following concepts that will be involved in proving the next theorem of existence and uniqueness of the solutions.
Main postulates:

(1) The functions $\mathcal{A}, \mathcal{B} \in C(\Im, \mathbb{R})$.
(2) The $f \in C^2(\Im, \mathbb{R})$, where f is a known free function.
(3) The known kernels $(\zeta, y) \mapsto K_i(\zeta, y), i = 1, 2$ are continuous in for all $\zeta, y \in \Im$ with values in $\mathbb{R}$.
(4) For each $\zeta \in \Im$, $\gamma_i > 0$ and $i = 1, 2$, one have

$$\left(\int_{a_0}^{a_1} (K_i(\zeta, y))^2 \, dy\right)^{\frac{1}{2}} \leq \gamma_i.$$

(5) $(\alpha + h_1|\lambda_1|C_1^*(1) + |\lambda_2|C_2^*(1)) \leq |\theta|$, where

$$\begin{aligned}
\alpha &= (a_1 - a_0)\left(\|\mathcal{A}\|_\infty + (a_1 - a_0)\|\mathcal{B}\|_\infty\right), \\
C_i^*(m) &= \binom{p}{m} \frac{\gamma_i (a_1 - a_0)^{2m+\frac{1}{2}} (d^*(m))^{\frac{1}{2}}}{(2p - 2m + 1)^{\frac{1}{2}}}, \quad \text{for } i = 1, 2, \\
d^*(m) &= \left\{\eta_2^{2p-2m} + \eta_2^{2p-2m-1}\eta_1 + \ldots + \eta_1^{2p-2m}\right\}.
\end{aligned}$$

(6) $(\alpha + |\lambda_1|\Lambda_1 + |\lambda_2|\Lambda_2) \leq |\theta|$, where

$$\begin{aligned}
\Lambda_1 &= \sum_{m=1}^{p} \frac{e(m) h^m C_1^*(m)}{(a_1 - a_0)^{3m-3}}, \\
\Lambda_2 &= \sum_{m=1}^{p} \frac{e(m) C_2^*(m)}{(a_1 - a_0)^{3m-3}}.
\end{aligned}$$

Above $\Im$ is a closed interval of $[a_0, a_1]$. In other words, $\Im = [a_0, a_1]$, $e(m)$ is a finite positive constants depends on m, and $e(1) = 1$.

Theorem 2.5. *Assume that conditions (1)–(3) holds. Then, the following nonlinear Volterra-Fredholm integral equations (NVFIE)*

$$\begin{aligned}
&\theta g(\zeta) + \int_{a_0}^{a_1} \left[\mathcal{W}(\zeta, \tau) - \lambda_1 \int_{a_0}^{\zeta} R_1(\zeta, y; 1) H_2(y, \tau) dy - \lambda_2 \int_{a_0}^{a_1} R_2(\zeta, y; 1) H_2(y, \tau) dy\right] g(\tau) d\tau = \mathcal{F}(\zeta) \\
&+ \lambda_1 \int_{a_0}^{\zeta} \sum_{m=2}^{p} R_1(\zeta, y; m) \left(\int_{a_0}^{a_1} H_2(y, \tau) g(\tau) d\tau\right)^m dy + \lambda_2 \int_{a_0}^{a_1} \sum_{m=2}^{p} R_2(\zeta, y; m) \left(\int_{a_0}^{a_1} H_2(y, \tau) g(\tau) d\tau\right)^m dy, \quad (2.1)
\end{aligned}$$

is equivalent to the boundary value problems (1.1) and (1.2)*, where*

$$\mathrm{g}(\zeta) := \psi^{''}(\zeta), \tag{2.2}$$

$$\mathscr{W}(\zeta,\tau) := \frac{1}{(a_1-a_0)}\begin{cases} \mathscr{W}_1(\zeta,\tau) = (\tau-a_0)(\mathcal{A}(\zeta)-(a_1-\zeta)\mathcal{B}(\zeta)), & a_0\le\tau\le\zeta, \\ \\ \mathscr{W}_2(\zeta,\tau) = (\tau-a_1)(\mathcal{A}(\zeta)-(a_0-\zeta)\mathcal{B}(\zeta)), & \zeta\le\tau\le a_1, \end{cases} \tag{2.3}$$

$$R_i(\zeta,\mathrm{y};m) := \binom{p}{m}\frac{K_i(x,y)}{(b-a)^p}\left[\eta_1(a_1-\mathrm{y})+\eta_2(\mathrm{y}-a_0)\right]^{p-m}, \quad i=1,2, \tag{2.4}$$

$$H_2(\mathrm{y},\tau) := \begin{cases} (a_1-\mathrm{y})(a_0-\tau), & a_0\le\tau\le\mathrm{y}, \\ \\ (a_0-\mathrm{y})(a_1-\tau), & \mathrm{y}\le\tau\le a_1, \end{cases} \tag{2.5}$$

$$\mu(\zeta) := \frac{1}{(a_1-a_0)}(\eta_1\left[-\mathcal{A}(\zeta)+(a_1-\zeta)\mathcal{B}(\zeta)\right]+\eta_2\left[\mathcal{A}(\zeta)+(\zeta-a_0)\mathcal{B}(\zeta)\right]), \tag{2.6}$$

$$\mathscr{F}(\zeta) := f(\zeta)-\mu(\zeta)+\lambda_1\int_{a_0}^{\zeta}R_1(\zeta,\mathrm{y};0)\mathrm{dy}+\lambda_2\int_{a_0}^{a_1}R_2(\zeta,\mathrm{y};0)\mathrm{dy}. \tag{2.7}$$

Proof. Let $\psi^{''}(\zeta) = \mathrm{g}(\zeta)$, where the function $\zeta\mapsto\mathrm{g}(\zeta)$ is an element of $C(\Im,\mathbb{R})$. Therefore,

$$\psi^{'}(\zeta) = \psi^{'}(a_0)+\int_{a_0}^{\zeta}\mathrm{g}(\tau)\mathrm{d}\tau \tag{2.8}$$

and

$$\psi(\zeta) = \eta_1+(\zeta-a_0)\psi^{'}(a_0)+\int_{a_0}^{\zeta}(\zeta-\tau)\mathrm{g}(\tau)\mathrm{d}\tau. \tag{2.9}$$

By using $\zeta = a_1$ in (2.9), and then using its result in (2.8) and (2.9), we obtain

$$\psi^{'}(\zeta) = \frac{1}{(a_1-a_0)}\left[(\eta_2-\eta_1)+\int_{a_0}^{a_1}H_1(\zeta,\tau)\mathrm{g}(\tau)\mathrm{d}\tau\right], \tag{2.10}$$

$$\psi(\zeta) = \frac{1}{(a_1-a_0)}\left[\eta_1(a_1-\zeta)+\eta_2(\zeta-a_0)+\int_{a_0}^{a_1}H_2(\zeta,\tau)\mathrm{g}(\tau)\mathrm{d}\tau\right], \tag{2.11}$$

where

$$H_1(\zeta,\tau) := \begin{cases} (\tau-a_0), & a_0\le\tau\le\zeta, \\ (\tau-a_1), & \zeta\le\tau\le a_1, \end{cases}$$

$$H_2(\zeta,\tau) := \begin{cases} (a_1-\zeta)(a_0-\tau), & a_0\le\tau\le\zeta, \\ (a_0-\zeta)(a_1-\tau), & \zeta\le\tau\le a_1. \end{cases}$$

More generally,

$$[\psi(\zeta)]^p = \frac{1}{(a_1-a_0)^p}\sum_{m=0}^{p}\binom{p}{m}[\eta_1(a_1-\zeta)+\eta_2(\zeta-a_0)]^{p-m}\left(\int_{a_0}^{a_1} H_2(\zeta,\tau)\mathrm{g}(\tau)\mathrm{d}\tau\right)^m. \tag{2.12}$$

By using the assumption that $\psi''(\zeta) := \mathrm{g}(\zeta)$, (2.10)–(2.12) in (1.1), it follows that

$$\begin{aligned}
&\mu\mathrm{g}(\zeta) + \frac{\mathcal{A}(\zeta)}{a_1-a_0}(\eta_2-\eta_1) + \frac{1}{a_1-a_0}\int_{a_0}^{a_1}[\mathcal{A}(\zeta)H_1(\zeta,\tau)+\mathcal{B}(\zeta)H_2(\zeta,\tau)]\,\mathrm{g}(\tau)\mathrm{d}\tau + \frac{\mathcal{B}(\zeta)}{a_1-a_0}((a_1-\zeta)\eta_1+(\zeta-a_0)\eta_2)\\
&= f(\zeta) + \frac{\lambda_1}{(a_1-a_0)^p}\int_{a_0}^{\zeta}\sum_{m=0}^{p}\binom{p}{m}K_1(\zeta,\mathrm{y})[(a_1-\zeta)\eta_1+(\zeta-a_0)\eta_2]^{p-m}\left(\int_{a_0}^{a_1} H_2(\mathrm{y},\tau)\mathrm{g}(\tau)\mathrm{d}\tau\right)^m\mathrm{dy}\\
&\quad + \frac{\lambda_2}{(a_1-a_0)^p}\int_{a_0}^{a_1}\sum_{m=0}^{p}\binom{p}{m}K_2(\zeta,\mathrm{y})[(a_1-\zeta)\eta_1+(\zeta-a_0)\eta_2]^{p-m}\left(\int_{a_0}^{a_1} H_2(\mathrm{y},\tau)\mathrm{g}(\tau)\mathrm{d}\tau\right)^m\mathrm{dy}.
\end{aligned}$$

Simplifying the last identity, it follows that

$$\begin{aligned}
&\theta\mathrm{g}(\zeta) + \int_{a_0}^{a_1}\left[\mathscr{W}(\zeta,\tau) - \lambda_1\int_{a_0}^{\zeta} R_1(\zeta,\mathrm{y};1)H_2(\mathrm{y},\tau)\mathrm{dy} - \lambda_2\int_{a_0}^{a_1} R_2(\zeta,\mathrm{y};1)H_2(\mathrm{y},\tau)\mathrm{dy}\right]\mathrm{g}(\tau)\mathrm{d}\tau\\
&=\mathscr{F}(\zeta) + \lambda_1\int_{a_0}^{\zeta}\sum_{m=2}^{p}R_1(\zeta,\mathrm{y};m)\left(\int_{a_0}^{a_1} H_2(\mathrm{y},\tau)\mathrm{g}(\tau)\mathrm{d}\tau\right)^m\mathrm{dy} + \lambda_2\int_{a_0}^{a_1}\sum_{m=2}^{p}R_2(\zeta,\mathrm{y};m)\left(\int_{a_0}^{a_1} H_2(\mathrm{y},\tau)\mathrm{g}(\tau)\mathrm{d}\tau\right)^m\mathrm{dy},
\end{aligned} \tag{2.13}$$

where $R_i(\zeta,\mathrm{y};m), i = 1,2,\ H(\zeta,\tau),\ \mu(\zeta)$ and $\mathscr{F}(\zeta)$ are as (2.4)–(2.7) in the statement of the theorem. A straight forward calculation can give the converse of the theorem. Hence, the proof is done. □

To (2.13) has a continuous solution, we need the conditions (1)–(4) to be satisfied as stated in the following theorem.

Theorem 2.6. *Assume that the conditions (1)–(4) hold, then the* NVFIE (2.13) *possesses continuous solution.*

Proof. Suppose that $\Gamma_r := \left\{\mathrm{g} \in C(\Im,\mathbb{R}) : \|\mathrm{g}\|_\infty = \sup_{\zeta\in\Im}|\mathrm{g}(\zeta)| \le r\right\}$ for which the radius $r > 0$ is a finite solution for

$$|\lambda_1|\sum_{m=1}^{p}(h_1 r)^m C_1^*(m) + |\lambda_2|\sum_{m=1}^{p} r^m C_2^*(m) + (\alpha - |\theta|)\,r + \|\mathscr{F}\|_\infty = 0,$$

where h_1 is an upper bound of $\left|\mathscr{W}_2(\zeta,\tau)\right|$. Let $\mathrm{g}_1, \mathrm{g}_2 \in \Gamma_r$ and

$$(\mathscr{T}\mathrm{g}_1)(\zeta)=\frac{1}{\theta}\mathscr{F}(\zeta)-\frac{1}{\theta}\int_{a_0}^{a_1}\Big[\mathscr{W}(\zeta,\tau)-\lambda_1\int_{a_0}^{\zeta}R_1(\zeta,\mathrm{y};1)H_2(\mathrm{y},\tau)\mathrm{dy}-\lambda_2\int_{a_0}^{a_1}R_2(\zeta,\mathrm{y};1)H_2(\mathrm{y},\tau)\mathrm{dy}\Big]\mathrm{g}(\tau)\mathrm{d}\tau$$

and

$$(\mathscr{W}\mathrm{g}_2)(\zeta)=\frac{\lambda_1}{\theta}\int_{a_0}^{\zeta}\sum_{m=2}^{p}R_1(\zeta,\mathrm{y};m)\left(\int_{a_0}^{a_1}H_2(\mathrm{y},\tau)\mathrm{g}(\tau)\mathrm{d}\tau\right)^m\mathrm{dy}+\frac{\lambda_2}{\theta}\int_{a_0}^{a_1}\sum_{m=2}^{p}R_2(\zeta,\mathrm{y};m)\left(\int_{a_0}^{a_1}H_2(\mathrm{y},\tau)\mathrm{g}(\tau)\mathrm{d}\tau\right)^m\mathrm{dy}.$$

Now, we see that

$$\begin{aligned}
|(\mathscr{T}\mathrm{g}_1)(\zeta)| &\le \frac{1}{|\theta|}|\mathscr{F}(\zeta)|+\frac{r}{|\theta|}\int_{a_0}^{a_1}|\mathscr{W}(\zeta,\tau)|\mathrm{d}\tau+\frac{|\lambda_1|r}{|\theta|}\int_{a_0}^{a_1}\int_{a_0}^{\zeta}|R_1(\zeta,\mathrm{y};1)||H_2(\mathrm{y},\tau)|\mathrm{dyd}\tau\\
&+\frac{|\lambda_2|r}{|\theta|}\int_{a_0}^{a_1}\int_{a_0}^{a_1}|R_2(\zeta,\mathrm{y};1)||H_2(\mathrm{y},\tau)|\mathrm{dyd}\tau\\
&\le \frac{1}{|\theta|}|\mathscr{F}(\zeta)|+\frac{\alpha_r}{|\theta|}+\frac{h_1|\lambda_1|pr}{|\theta|(a_1-a_0)^{p-3}}\int_{a_0}^{a_1}\frac{|K_1(\zeta,\mathrm{y})|}{|(\eta_1-\eta_2)\mathrm{y}+(\eta_2a_1-\eta_1a_0)|^{1-p}}\mathrm{dy}\\
&+\frac{|\lambda_2|pr}{|\theta|(a_1-a_0)^{p-3}}\int_{a_0}^{a_1}\frac{|K_2(\zeta,\mathrm{y})|}{|(\eta_1-\eta_2)\mathrm{y}+(\eta_2a_1-\eta_1a_0)|^{1-p}}\mathrm{dy}\\
&\le \frac{1}{|\theta|}|\mathscr{F}(\zeta)|+\frac{\alpha r}{|\theta|}+h_1|\lambda_1|\frac{p(a_1-a_0)^{\frac{5}{2}}(\mathrm{d}^*(1))^{\frac{1}{2}}r}{|\theta|(2p-1)^{\frac{1}{2}}}\left(\int_{a_0}^{a_1}(K_1(\zeta,\mathrm{y}))^2\mathrm{dy}\right)^{\frac{1}{2}}\\
&+|\lambda_2|\frac{p(a_1-a_0)^{\frac{5}{2}}(\mathrm{d}^*(1))^{\frac{1}{2}}r}{|\theta|(2p-1)^{\frac{1}{2}}}\left(\int_{a_0}^{a_1}(K_2(\zeta,\mathrm{y}))^2\mathrm{dy}\right)^{\frac{1}{2}}\\
&\le \frac{1}{|\theta|}\|\mathscr{F}(\zeta)\|_\infty+\frac{1}{|\theta|}\Big[\alpha+(h_1|\lambda_1|C_1^*(1)+|\lambda_2|C_2^*(1))\Big]r.
\end{aligned}\tag{2.14}$$

By using similar arguments, it follows that

$$\begin{aligned}
|(\mathscr{W}\mathrm{g}_2)(\zeta)| &\le \frac{|\lambda_1|}{|\theta|}\int_{a_0}^{\zeta}\sum_{m=2}^{p}|R_1(\zeta,\mathrm{y};m)|\left(\int_{a_0}^{a_1}|H_2(\mathrm{y},\tau)\mathrm{g}(\tau)|\mathrm{d}\tau\right)^m\mathrm{dy}\\
&+\frac{|\lambda_2|}{|\theta|}\int_{a_0}^{a_1}\sum_{m=2}^{p}|R_2(\zeta,\mathrm{y};m|)\left(\int_{a_0}^{a_1}|H_2(\mathrm{y},\tau)\mathrm{g}(\tau)|\mathrm{d}\tau\right)^m\mathrm{dy}\\
&\le |\lambda_1|\sum_{m=2}^{p}\binom{p}{m}\frac{(a_1-a_0)^{2m+\frac{1}{2}}(\mathrm{d}^*(m))^{\frac{1}{2}}(h_1r)^m}{|\theta|(2p-2m+1)^{\frac{1}{2}}}\left(\int_{a_0}^{a_1}(K_1(\zeta,\mathrm{y}))^2\mathrm{dy}\right)^{\frac{1}{2}}
\end{aligned}$$

$$+ |\lambda_2| \sum_{m=2}^{p} \binom{p}{m} \frac{(a_1 - a_0)^{2m+\frac{1}{2}} (d^*(m))^{\frac{1}{2}} r^m}{|\theta|(2p - 2m + 1)^{\frac{1}{2}}} \left(\int_{a_0}^{a_1} (K_2(\zeta, \mathrm{y}))^2 \mathrm{dy} \right)^{\frac{1}{2}}$$

$$\leq \frac{1}{|\theta|} \left(|\lambda_1| \sum_{m=2}^{p} (h_1 r)^m C_1^*(m) + |\lambda_2| \sum_{m=2}^{p} r^m C_2^*(m) \right). \tag{2.15}$$

In view of (2.14) and (2.15), we can deduce

$$\|\mathscr{T}(\mathrm{g}_1) + \mathscr{W}(\mathrm{g}_2)\|_\infty \leq \|\mathscr{T}(\mathrm{g}_1)\|_\infty + \|\mathscr{W}(\mathrm{g}_2)\|_\infty$$

$$\leq \frac{1}{|\theta|} \|\mathscr{F}(\zeta)\|_\infty + \frac{r}{|\theta|} (\alpha + (h_1|\lambda_1|C_1^*(1) + |\lambda_2|C_2^*(1)) + \frac{1}{|\theta|} \left(|\lambda_1| \sum_{m=2}^{p} C_1^*(m)(h_1 r)^m + |\lambda_2| \sum_{m=2}^{p} C_2^*(m) r^m \right),$$

or simply,

$$\|\mathscr{T}(\mathrm{g}_1) + \mathscr{W}(\mathrm{g}_2)\|_\infty \leq \frac{1}{|\theta|} \|\mathscr{F}(\zeta)\|_\infty + \frac{1}{|\theta|} \left(|\lambda_1| \sum_{m=1}^{p} C_1^*(m)(h_1 r)^m + |\lambda_2| \sum_{m=1}^{p} C_2^*(m) r^m \right) + \frac{\alpha r}{|\theta|} = r,$$

it follows that $\mathscr{T}(\mathrm{g}_1) + \mathscr{W}(\mathrm{g}_2) \in \Gamma_r$ for each $\mathrm{g}_1, \mathrm{g}_2 \in \Gamma_r$.

On the other hand, suppose that $\zeta_1, \zeta_2 \in \Im$ with $\zeta_1 < \zeta_2$. Note that the functions $\mathscr{F}$, $\mathscr{W}_1$ and $\mathscr{W}_2$ are continuous in ζ according to postulates (1)–(3), then it will follow that

$$\left|(\mathscr{T}\mathrm{g}_1)(\zeta_2) - (\mathscr{T}\mathrm{g}_1)(\zeta_1)\right| \leq \frac{1}{|\theta|} |\mathscr{F}(\zeta_2) - \mathscr{F}(\zeta_1)| + \frac{r}{|\theta|(a_1 - a_0)} \int_{a_0}^{\zeta_1} |\mathscr{W}_1(\zeta_2, \tau) - \mathscr{W}_1(\zeta_1, \tau)| \mathrm{d}\tau$$

$$+ \frac{r}{|\theta|(a_1 - a_0)} \int_{a_0}^{\zeta_1} |\mathscr{W}_2(\zeta_2, \tau) - \mathscr{W}_2(\zeta_1, \tau)| \mathrm{d}\tau + \int_{a_0}^{\zeta_1} |\mathscr{W}_1(\zeta_2, \tau) - \mathscr{W}_2(\zeta_1, \tau)| \mathrm{d}\tau$$

$$+ \frac{h_1 |\lambda_1| pr}{|\theta|(a_1 - a_0)^{p-3}} \int_{a_0}^{a_1} \frac{|K_1(\zeta_2, \mathrm{y}) - K_1(\zeta_1, \mathrm{y})|}{|(\eta_1 - \eta_2)\mathrm{y} + (\eta_2 a_1 - \eta_1 a_0)|^{1-p}} \mathrm{dy}$$

$$+ \frac{|\lambda_2| pr}{|\theta|(a_1 - a_0)^{p-3}} \int_{a_0}^{a_1} \frac{\left|K_2(\zeta_2, \mathrm{y}) - K_2(\zeta_1, \mathrm{y})\right|}{|(\eta_1 - \eta_2)\mathrm{y} + (\eta_2 a_1 - \eta_1 a_0)|^{1-p}} \mathrm{dy}. \tag{2.16}$$

As the right-hand side of Eq (2.16) is independent from $u \in \Gamma_r$, it tends to zero as $\zeta_2 - \zeta_1 \to 0$. This implies that $|(\mathscr{T}\mathrm{g}_1)(\zeta_2) - (\mathscr{T}\mathrm{g}_1)(\zeta_1)| \to 0$ as $\zeta_2 \to \zeta_1$.

Similarly, one can have

$$\left|(\mathscr{W}\mathrm{g}_2)(\zeta_2) - (\mathscr{W}\mathrm{g}_2)(\zeta_1)\right| \leq \frac{|\lambda_1|}{|\theta|} \sum_{m=2}^{p} \binom{p}{m} (a_1 - a_0)^{3m-p} (h_1 r)^m \int_{a_0}^{a_1} \frac{|K_1(\zeta_2, \mathrm{y})| - K_1(\zeta_1, \mathrm{y})|}{|(\eta_1 - \eta_2)\mathrm{y} + (\eta_2 a_1 - \eta_1 a_0)|^{2m-2p}} \mathrm{dy}$$

$$+ \frac{|\lambda_2|}{|\theta|} \sum_{m=2}^{p} \binom{p}{m} (a_1 - a_0)^{3m-p} r^m \int_{a_0}^{a_1} \frac{|K_2(\zeta_2, \mathrm{y}) - K_2(\zeta_1, \mathrm{y})|}{|(\eta_1 - \eta_2)\mathrm{y} + (\eta_2 a_1 - \eta_1 a_0)|^{2m-2p}} \mathrm{dy} \tag{2.17}$$

and again it tends to zero whereas $\zeta_2 - \zeta_1$ tends to zero. Hence, the set $(\mathscr{T} + \mathscr{W})\Gamma_r$ is equicontinuous. Furthermore, $\mathscr{T}\mathrm{g}_1, \mathscr{W}\mathrm{g}_2 \in C(\Im, \mathbb{R})$, and consequently, $\mathscr{T} + \mathscr{W}$ is an operator on Γ_r.

Now, we suppose that g, g^* are two functions of Γ_r. So,

$$\|\mathscr{T}(\mathrm{g}) - \mathscr{T}(\mathrm{g}^*)\|_\infty \leq \frac{1}{|\theta|}\left(\alpha + h_1|\lambda_1|C_1^*(1) + |\lambda_2|C_2^*(1)\right)\|\mathrm{g} - \mathrm{g}^*\|_\infty. \tag{2.18}$$

Therefore, $\mathscr{T}$ is a contraction mapping on Γ_r due to postulate (5) and $\|\mathscr{T}(\mathrm{g}) - \mathscr{T}(\mathrm{g}^*)\|_\infty \leq \|\mathrm{g} - \mathrm{g}^*\|_\infty$.

Let g_ℓ be a sequence, such that $\mathrm{g}_\ell \to \mathrm{g}$ in $C[\Im, \mathbb{R}]$. Then, for each $\mathrm{g}_\ell, \mathrm{g} \in \Gamma_r$ and $\zeta \in \Im$, we have

$$\begin{aligned}\left|(\mathscr{W}\mathrm{g}_\ell)(x) - (\mathscr{W}\mathrm{g})(x)\right| \leq{}& \frac{|\lambda_1|}{|\theta|}\int_{a_0}^{\zeta}\sum_{m=2}^{p}|R_1(\zeta, \mathrm{y}; m)|\left[\left(\int_{a_0}^{a_1} H_2(\mathrm{y}, \tau)\mathrm{g}_\ell(\tau)\mathrm{d}\tau\right)^m - \left(\int_{a_0}^{a_1} H_2(\mathrm{y}, \tau)\mathrm{g}(\tau)\mathrm{d}\tau\right)^m\right]\mathrm{dy}\\
&+ \frac{|\lambda_2|}{|\theta|}\int_{a_0}^{a_1}\sum_{m=2}^{p}|R_2(\zeta, \mathrm{y}; m)|\left[\left(\int_{a_0}^{a_1} H_2(\mathrm{y}, \tau)\mathrm{g}_\ell(\tau)\mathrm{d}\tau\right)^m - \left(\int_{a_0}^{a_1} H_2(\mathrm{y}, \tau)\mathrm{g}(\tau)\mathrm{d}\tau\right)^m\right]\mathrm{dy}\\
\leq{}& \frac{|\lambda_1|}{|\theta|}\int_{a_0}^{\zeta}\sum_{m=2}^{p}|R_1(\zeta, \mathrm{y}; m)|e(m)\left(\int_{a_0}^{a_1} H_2(\mathrm{y}, \tau)\left|\mathrm{g}_\ell(t) - \mathrm{g}(t)\right|\mathrm{d}\tau\right)\mathrm{dy}\\
&+ \frac{|\lambda_2|}{|\theta|}\int_{a_0}^{a_1}\sum_{m=2}^{p}|R_2(\zeta, \mathrm{y}; m)|e(m)\left(\int_{a_0}^{a_1} H_2(\mathrm{y}, \tau)\left|\mathrm{g}_\ell(\tau) - \mathrm{g}(\tau)\right|\mathrm{d}\tau\right)\mathrm{dy}.\end{aligned}$$

By applying the Arzela bounded convergence theorem on it, it follows that

$$\begin{aligned}\lim_{\ell\to\infty}\left|(\mathscr{W}\mathrm{g}_\ell)(x) - (\mathscr{W}\mathrm{g})(\zeta)\right| \leq{}& \frac{|\lambda_1|}{|\theta|}\int_{a_0}^{\zeta}\sum_{m=2}^{p}|R_1(\zeta, \mathrm{y}; m)|e(m)\left(\int_{a_0}^{a_1} H_2(\mathrm{y}, \tau)\lim_{\ell\to\infty}\left|\mathrm{g}_\ell(\tau) - u(\tau)\right|\mathrm{d}\tau\right)\mathrm{dy}\\
&+ \frac{|\lambda_2|}{|\theta|}\int_{a_0}^{a_1}\sum_{m=2}^{p}|R_2(\zeta, \mathrm{y}; m)|e(m)\left(\int_{a_0}^{a_1} H_2(\mathrm{y}, \tau)\lim_{\ell\to\infty}\left|\mathrm{g}_\ell(\tau) - \mathrm{g}(\tau)\right|\mathrm{d}\tau\right)\mathrm{dy} = 0,\end{aligned}$$

where $e(m)$ depends on m and it is a finite positive constant. Thus, $\mathscr{W}$ is a continuous mapping on Γ_r. Then we notice that the sequence $\mathscr{W}(\mathrm{g}_\ell)$ is uniformly bounded on $\Im$ since

$$|\mathscr{W}\mathrm{g}(x)| \leq \frac{1}{|\theta|}\left(|\lambda_1|\sum_{m=2}^{p}(h_1 r)^m C_1^*(m) + |\lambda_2|\sum_{m=2}^{p} r^m C_2^*(m)\right).$$

Moreover, the sequence $\mathscr{W}(\mathrm{g}_\ell)$ is equicontinuous since $|\mathscr{W}(\mathrm{g}_\ell)(\zeta_2) - \mathscr{W}(\mathrm{g}_\ell)(\zeta_1)| < \varepsilon$, as $|\zeta_2 - \zeta_1| < \delta$, for each $\ell \in \mathbb{N}$. Then, by applying the Arzela-Ascoli theorem it follows that $\mathscr{W}(\mathrm{g}_\ell)$ contains a subsequence $\mathscr{W}(\mathrm{g}_{\ell k})$, which is uniformly convergent. Thus, the set $\mathscr{W}\Gamma_r$ is compact and the operator $\mathscr{W}$ is completely continuous. Consequently, all of the conditions of Krasnosel'skii theorem are fulfilled. Hence, there is at least one fixed point in Γ_r for $\mathscr{T} + \mathscr{W}$, which can be a solution of NVFIE (2.13). Thus, the proof is done. □

In the following theorem, we prove the uniqueness of the continuous solution in the previous theorem.

Theorem 2.7. *If the conditions (1)–(3) and (5) hold, then the* NVFIE (2.13) *has a unique continuous solution.*

Proof. Since $\mathscr{T}$ and $\mathscr{W}$ are two operators, $\mathscr{T}+\mathscr{W}$ is also an operator on Γ_r. Rewrite (2.18) as follows

$$\left\|\mathscr{T}(\mathrm{x})-\mathscr{T}(\mathrm{x}^*)\right\|_\infty \leq \frac{1}{|\theta|}\left(\alpha+h_1|\lambda_1|C_1^*(1)+|\lambda_2|C_2^*(1)\right)\|\mathrm{x}-\mathrm{x}^*\|_\infty, \tag{2.19}$$

$$\|\mathscr{W}(\mathrm{x})-\mathscr{W}(\mathrm{x}^*)\|_\infty \leq \frac{1}{|\theta|}\left(|\lambda_1|\sum_{m=2}^{p}\frac{h_1^m e(m)C_1^*(m)}{(a_1-a_0)^{3m-3}}+|\lambda_2|\sum_{m=2}^{p}\frac{e(m)C_2^*(m)}{(a_1-a_0)^{3m-3}}\right)\|\mathrm{x}-\mathrm{x}^*\|_\infty, \tag{2.20}$$

for $\mathrm{x},\mathrm{x}^*\in\Gamma_r$. Then by using (2.19) with $e(1)=1$ and (2.20), it follows that

$$\begin{aligned}\left\|(\mathscr{T}+\mathscr{W})(\mathrm{x})-(\mathscr{T}+\mathscr{W})(\mathrm{x}^*)\right\|_\infty &\leq \left\|\mathscr{T}(\mathrm{x})-\mathscr{T}(\mathrm{x}^*)\right\|_\infty+\left\|\mathscr{W}(\mathrm{x})-\mathscr{W}(\mathrm{x}^*)\right\|_\infty\\ &\leq \frac{1}{|\theta|}\left(\alpha+h_1|\lambda_1|C_1^*(1)+|\lambda_2|C_2^*(1)\right)\|\mathrm{x}-\mathrm{x}^*\|_\infty\\ &+\frac{1}{|\theta|}\left(|\lambda_1|\sum_{m=2}^{p}\frac{e(m)h_1^m C_1^*(m)}{(a_1-a_0)^{3m-3}}+|\lambda_2|\sum_{m=2}^{p}\frac{e(m)C_2^*(m)}{(a_1-a_0)^{3m-3}}\right)\|\mathrm{x}-\mathrm{x}^*\|_\infty\\ &\leq \frac{1}{|\theta|}(\alpha+(|\lambda_1|\Lambda_1+|\lambda_2|\Lambda_2)\|\mathrm{x}-\mathrm{x}^*\|_\infty\\ &\leq \|\mathrm{x}-\mathrm{x}^*\|_\infty. \end{aligned} \tag{2.21}$$

Consequently, by applying the Banach fixed point theorem and condition (6), we can deduce the operatoris contraction on Γ_r. Hence, the NVFIE (2.13) posses a unique continuous solution in Γ_r. Thus, the proof is completed. □

3. Methods of solutions

Our main section is divided into two subsections which concern the method of solutions to the proposed nonlinear equation.

3.1. The modified Adomain decomposition method solution

Recall the MADM (see [13]), given by $\mathrm{g}(\zeta):=\sum_{\ell=0}^{\infty}\mathrm{g}_\ell(\zeta)$, which approximates the NVFIE (2.13) such that the conditions of Theorem 2.7 are satisfied. With $\mathscr{F}(\zeta):=\mathscr{F}_1(\zeta)+\mathscr{F}_2(\zeta)$, we see that

$$\mathrm{g}_0(\zeta)=\frac{1}{\theta}\mathscr{F}_1(\zeta), \tag{3.1}$$

$$\mathrm{g}_1(\zeta)=\frac{1}{\theta}\mathscr{F}_2(\zeta)-\frac{1}{\theta}\left[\int_{a_0}^{a_1}\left[\mathscr{W}(\zeta,\tau)-\lambda_1\int_{a_0}^{\zeta}R_1(\zeta,\mathrm{y};1)H_2(\mathrm{y},\tau)d\mathrm{y}-\lambda_2\int_{a_0}^{a_1}R_2(\zeta,\mathrm{y};1)H_2(\mathrm{y},\tau)d\mathrm{y}\right]\mathrm{g}_0(\tau)d\tau\right]$$

$$+\frac{\lambda_1}{\theta}\int_{a_0}^{\zeta}\sum_{m=2}^{p}R_1(\zeta,\mathrm{y};m)\mathcal{A}_0(\mathrm{y},\tau)\mathrm{dy}+\frac{\lambda_2}{\theta}\int_{a_0}^{a_1}\sum_{m=2}^{p}R_2(\zeta,\mathrm{y};m)\mathcal{A}_0(\mathrm{y},\tau)\mathrm{dy}, \tag{3.2}$$

and for $\ell \geq 2$,

$$\mathrm{g}_\ell(x)=-\frac{1}{\theta}\Big[\int_{a_0}^{a_1}\Big[\mathscr{W}(\zeta,\tau)-\lambda_1\int_{a_0}^{\zeta}R_1(\zeta,\mathrm{y};1)H_2(\mathrm{y},\tau)\mathrm{dy}-\lambda_2\int_{a_0}^{a_1}R_2(\zeta,\mathrm{y};1)H_2(\mathrm{y},\tau)\mathrm{dy}\Big]\mathrm{g}_{\ell-1}(\tau)\mathrm{d}\tau\Big]$$
$$+\frac{\lambda_1}{\theta}\int_{a_0}^{\zeta}\sum_{m=2}^{p}R_1(\zeta,\mathrm{y};m)\mathcal{A}_{\ell-1}(\mathrm{y},\tau)\mathrm{dy}+\frac{\lambda_2}{\theta}\int_{a_0}^{a_1}\sum_{m=2}^{p}R_2(\zeta,\mathrm{y};m)\mathcal{A}_{\ell-1}(\mathrm{y},\tau)\mathrm{dy}, \tag{3.3}$$

where $\mathcal{A}_\ell$ given by

$$\mathcal{A}_\ell(\mathrm{g}_0(\zeta),\mathrm{g}_1(\zeta),\ldots,\mathrm{g}_\ell(\zeta),\mathrm{y};m)=\frac{1}{\ell!}\left(\frac{d^\ell}{d\rho^\ell}\left[\int_{a_0}^{a_1}H_2(\mathrm{y},\tau)\sum_{i=0}^{\infty}\rho^i\mathrm{g}_i(\tau)\mathrm{d}\tau\right]^m\right)\Bigg|_{\rho=0}, \tag{3.4}$$

is the Adomain's polynomial for $\ell \geq 0$.

If Theorem 2.7 is met, then the following consequence follows:

Theorem 3.1. *The solution* $\mathrm{g}(\zeta)$ *for the NVFIE (2.13) obtained from (3.1)–(3.3) converges to the exact solution as the number of iterations* ℓ *increases* (*i.e.*, $\lim_{\ell\to\infty}\beta_\ell(\zeta)=\mathrm{g}(\zeta)$).

Proof. Assume that $\{\beta_k(\zeta)\}$ is a sequence of partial sums with

$$\beta_k(\zeta)=\sum_{i=0}^{\imath}\mathrm{g}_i(\zeta),$$

and let $\ell, \jmath \in \mathbb{Z}^+$ with $\ell > \jmath \geq 1$. Then we have

$$\|\beta_\ell(\zeta)-\beta_\jmath(\zeta)\|_\infty=\left|\sum_{i=\jmath+1}^{\ell}\mathrm{g}_i(\zeta)\right|$$
$$\leq\frac{1}{|\theta|}\int_{a_0}^{a_1}\Big|\mathscr{W}(\zeta,\tau)\sum_{i=\jmath}^{\ell-1}\mathrm{g}_i(\tau)\Big|\mathrm{d}\tau+\frac{|\lambda_1|}{|\theta|}\int_{a_0}^{a_1}\int_{a_0}^{\zeta}\Big|R_1(\zeta,\mathrm{y};1)H_2(\mathrm{y},\tau)\sum_{i=\jmath}^{\ell-1}\mathrm{g}_i(\tau)\Big|\mathrm{dy}\mathrm{d}\tau$$
$$+\frac{|\lambda_2|}{|\theta|}\int_{a_0}^{a_1}\int_{a_0}^{a_1}\Big|R_2(\zeta,\mathrm{y};1)H_2(\mathrm{y},\tau)\sum_{i=\jmath}^{\ell-1}\mathrm{g}_i(\tau)|\mathrm{dy}\mathrm{d}\tau+\frac{|\lambda_1|}{|\theta|}\int_{a_0}^{\zeta}\sum_{m=2}^{p}\Big|R_1(\zeta,\mathrm{y};m)\sum_{i=\jmath}^{\ell-1}\mathcal{A}_i(\mathrm{y},\tau)\Big|\mathrm{dy}\mathrm{d}\tau$$
$$+\frac{|\lambda_2|}{|\theta|}\int_{a_0}^{a_1}\sum_{m=2}^{p}\Big|R_2(\zeta,\mathrm{y};m)\sum_{i=\jmath}^{\ell-1}\mathcal{A}_i(\mathrm{y},\tau)\Big|\mathrm{dy}\mathrm{d}\tau$$
$$\leq\frac{\alpha}{\theta}\|\beta_{\ell-1}-\beta_{\jmath-1}\|_\infty+\frac{h_1|\lambda_1|}{|\theta|}(a_1-a_0)^3\int_{a_0}^{a_1}\Big|R_1(\zeta,\mathrm{y};1)\sum_{i=\jmath}^{\ell-1}\mathrm{g}_i(\tau)\Big|\mathrm{d}\tau$$

$$+\frac{|\lambda_2|}{|\theta|}(a_1-a_0)^3\int_{a_0}^{a_1}\left|R_2(\zeta,\mathrm{y};1)\sum_{i=\jmath}^{\ell-1}\mathrm{g}_i(\tau)\right|\mathrm{d}\tau$$

$$+\frac{|\lambda_1|}{|\theta|}\int_{a_0}^{\zeta}\sum_{m=2}^{p}\left|R_1(\zeta,\mathrm{y};m)\left(\int_{a_0}^{a_1}\sum_{i=\jmath}^{\ell-1}\mathrm{g}_i(\tau)\mathrm{d}\tau\right)^m\right|\mathrm{dy}+\frac{|\lambda_2|}{|\theta|}\int_{a_0}^{a_1}\sum_{m=2}^{p}\left|R_2(\zeta,\mathrm{y};m)\left(\int_{a_0}^{a_1}\sum_{i=\jmath}^{\ell-1}\mathrm{g}_i(\tau)\mathrm{d}\tau\right)^m\right|\mathrm{dy}$$

$$\leq\frac{1}{|\theta|}(\alpha+(h_1|\lambda_1|C_1^*(1)+|\lambda_2|C_2^*(1)))\|\beta_{\ell-1}-\beta_{\jmath-1}\|_\infty$$

$$+\frac{|\lambda_1|}{|\theta|}\int_{a_0}^{\zeta}\sum_{m=2}^{p}(a_1-a_0)^3e(m)|R_1(\zeta,\mathrm{y};m)|\mathrm{dy}+\frac{|\lambda_2|}{|\theta|}\int_{a_0}^{a_1}\sum_{m=2}^{p}(a_1-a_0)^3e(m)|R_2(\zeta,\mathrm{y};m)|\mathrm{dy}$$

$$\leq\frac{1}{\theta}\left(\alpha+h_1|\lambda_1|C_1^*(1)+|\lambda_2|C_2^*(1)+|\lambda_1|\sum_{m=2}^{p}\frac{e(m)h^mC_1^*(m)}{(a_1-a_0)^{3m-3}}+|\lambda_2|\sum_{m=2}^{p}\frac{e(m)C_2^*(m)}{(a_1-a_0)^{3m-3}}\right)\|\beta_{\ell-1}-\beta_{\jmath-1}\|_\infty.$$

For $h_1=1$, it follows that

$$\begin{aligned}\|\beta_\ell(\zeta)-\beta_\jmath(\zeta)\|_\infty&\leq\frac{1}{|\theta|}\left(\alpha+|\lambda_1|\sum_{m=1}^{p}\frac{e(m)h^mC_1^*(m)}{(a_1-a_0)^{3m-3}}+|\lambda_2|\sum_{m=1}^{p}\frac{e(m)C_2^*(m)}{(a_1-a_0)^{3m-3}}\right)\|\beta_{\ell-1}-\beta_{\jmath-1}\|_\infty\\&=\frac{1}{|\theta|}\left(\alpha+|\lambda_1|\Lambda_1+|\lambda_2|\Lambda_2\right)\|\beta_{\ell-1}-\beta_{\jmath-1}\|_\infty\\&=\vartheta\|\beta_{\ell-1}(x)-\beta_{\jmath-1}(\zeta)\|_\infty,\end{aligned}\tag{3.5}$$

where $\vartheta:=\dfrac{(\alpha+|\lambda_1|\Lambda_1+|\lambda_2|\Lambda_2)}{|\theta|}$ with $\vartheta<1$. For $\ell=\jmath+1$, it follows that

$$\|\beta_{\jmath+1}-\beta_\jmath\|_\infty\leq\vartheta\|\beta_\jmath(\zeta)-\beta_{\jmath-1}(\zeta)\|_\infty\leq\vartheta^2\|\beta_{\jmath-1}(\zeta)-\beta_{\jmath-2}(\zeta)\|_\infty\leq\cdots\leq\vartheta^\jmath\|\beta_1(\zeta)-\beta_0(\zeta)\|_\infty=\vartheta^\jmath\|\mathrm{g}_1\|_\infty.\tag{3.6}$$

By substituting (3.6) in (3.5) and applying the triangle inequality with $\ell>\jmath>N\in\mathbb{N}$, we can deduce

$$\|\beta_\ell-\beta_\jmath\|_\infty\leq\frac{\vartheta^\ell}{1-\vartheta}\|\mathrm{g}_1\|_\infty=\varepsilon,$$

where $\lim_{\ell\to\infty}\vartheta^\ell=0$. Therefore,

$$\|\beta_\ell-\beta_\jmath\|_\infty<\varepsilon,\quad\text{for each }\ell,\jmath\in\mathbb{N}.$$

Thus, the sequence $\beta_\ell(\zeta)$ is a Cauchy sequence in $C(\Im,\mathbb{R})$, and hence $\lim_{\ell\longrightarrow\infty}\beta_\ell(\zeta)=\mathrm{g}(\zeta)$, as desired. □

3.2. The Homotopy analysis method solution

In this section, we establish an analysis for the NVFIE (2.13) under the conditions of Theorem 2.7 by applying the HAM (see [20]) to (1.2) as follows:

$$\mathrm{g}(\zeta)-\frac{1}{\theta}\left(\int_{a_0}^{a_1}\left[\mathscr{W}(\zeta,\tau)-\lambda_1\int_{a_0}^{\zeta}R_1(\zeta,\mathrm{y};1)H_2(\mathrm{y},\tau)\mathrm{dy}-\lambda_2\int_{a_0}^{a_1}R_2(\zeta,\mathrm{y};1)H_2(\mathrm{y},\tau)\mathrm{dy}\right]\mathrm{g}(\tau)\mathrm{d}\tau\right)+\frac{1}{\theta}\mathscr{F}(\zeta)$$
$$+\frac{\lambda_1}{\theta}\int_{a_0}^{\zeta}\sum_{m=2}^{p}R_1(\zeta,\mathrm{y};m)\left(\int_{a_0}^{a_1}H_2(\mathrm{y},\tau)\mathrm{g}(\tau)\mathrm{d}\tau\right)^m\mathrm{dy}+\frac{\lambda_2}{\theta}\int_{a_0}^{a_1}\sum_{m=2}^{p}R_2(\zeta,\mathrm{y};m)\left(\int_{a_0}^{a_1}H_2(\mathrm{y},\tau)\mathrm{g}(\tau)\mathrm{d}\tau\right)^m\mathrm{dy}=0. \tag{3.7}$$

We define the nonlinear operator $\mathcal{N}$ as follows:

$$\mathcal{N}[\mathrm{g}(\zeta)]=\mathrm{g}(\zeta)+\frac{1}{\theta}\left(\int_{a_0}^{a_1}\left[\mathscr{W}(\zeta,\tau)-\lambda_1\int_{a_0}^{\zeta}R_1(\zeta,\mathrm{y};1)H_2(\mathrm{y},\tau)\mathrm{dy}-\lambda_2\int_{a_0}^{a_1}R_2(\zeta,\mathrm{y};1)H_2(\mathrm{y},\tau)\mathrm{dy}\right]\mathrm{g}(\tau)\mathrm{d}\tau\right)$$
$$-\frac{1}{\theta}\mathscr{F}(\zeta)-\frac{\lambda_1}{\theta}\int_{a_0}^{\zeta}\sum_{m=2}^{p}R_1(\zeta,\mathrm{y};m)\left(\int_{a_0}^{a_1}H_2(\mathrm{y},\tau)\mathrm{g}(\tau)\mathrm{d}\tau\right)^m\mathrm{dy}-\frac{\lambda_2}{\theta}\int_{a_0}^{a_1}\sum_{m=2}^{p}R_2(\zeta,\mathrm{y};m)\left(\int_{a_0}^{a_1}H_2(\mathrm{y},\tau)\mathrm{g}(\tau)\mathrm{d}\tau\right)^m\mathrm{dy}. \tag{3.8}$$

Considering (3.7) and (3.8), we have

$$\mathcal{N}[\mathrm{g}(\zeta)]=0,\quad \text{for } \zeta\in\Im.$$

On the other hand, if we define the homotopy of $\mathrm{g}(\zeta)$ as follows:

$$\sigma^*[\kappa(\zeta;\hbar,\varrho)]=(1-\varrho)\mathcal{L}(\kappa(\zeta;\hbar,\varrho)-\mathrm{g}_0(\zeta))-\varrho\hbar\mathcal{N}[\kappa(\zeta;\hbar,\varrho)], \tag{3.9}$$

then we notice that

(i) the function $\mathrm{g}_0(\zeta)$ is the initial approximation solution of $\mathrm{g}(\zeta)$;
(ii) non-zero real parameter $\hbar$ is used to manage the convergence of suggested models;
(iii) the homotopy parameter $\varrho\in[0,1]$ is embedded in (3.9);
(iv) the auxiliary linear operator $\mathcal{L}$ can satisfy the property $\mathcal{L}[\varrho(\zeta)]=0$, where $\varrho(\zeta)=0$;
(v) the operator $\mathcal{N}$ can be represented in (3.8); that is,

$$\mathcal{N}\left[\kappa(\zeta;\hbar,\varrho)\right]$$
$$=\kappa(\zeta;\hbar,\varrho)+\frac{1}{\theta}\left(\int_{a_0}^{a_1}\left[\mathscr{W}(\zeta,\tau)-\lambda_1\int_{a_0}^{\zeta}R_1(\zeta,\mathrm{y};1)H_2(\mathrm{y},\tau)\mathrm{dy}-\lambda_2\int_{a_0}^{a_1}R_2(\zeta,\mathrm{y};1)H_2(\mathrm{y},\tau)\mathrm{dy}\right]\kappa(\tau;\hbar,\varrho)\mathrm{d}\tau\right)-\frac{1}{\theta}\mathscr{F}(\zeta)$$
$$-\frac{\lambda_1}{\theta}\int_{a_0}^{\zeta}\sum_{m=2}^{p}R_1(\zeta,\mathrm{y};m)\left(\int_{a_0}^{a_1}H_2(\mathrm{y},\tau)\kappa(\tau;\hbar,\varrho)\mathrm{d}\tau\right)^m\mathrm{dy}-\frac{\lambda_2}{\theta}\int_{a_0}^{a_1}\sum_{m=2}^{p}R_2(\zeta,\mathrm{y};m)\left(\int_{a_0}^{a_1}H_2(\mathrm{y},\tau)\kappa(\tau;\hbar,\varrho)\mathrm{d}\tau\right)^m\mathrm{dy},$$

$$\sigma^*[\kappa(\zeta;\hbar,\varrho)]=0.$$

Solving Eq (3.9) to get

$$(1-\varrho)\mathcal{L}\left[\kappa(\zeta;\hbar,\varrho)-\mathrm{g}_0(\zeta)\right]=\varrho\hbar\mathcal{N}\left[\kappa(\zeta;\hbar,\varrho)\right],$$

$$\mathrm{g}(\zeta)=\mathrm{g}_0(\zeta)+\sum_{\iota=1}^{\infty}\mathrm{g}_\iota(\zeta)=\sum_{\iota=0}^{\infty}\mathrm{g}_\iota(\zeta),$$

where

$$\begin{aligned}
&\mathrm{g}_\iota(\zeta)=\frac{1}{\iota!}\frac{\partial^\iota\kappa(\zeta;\hbar,\varrho)}{\partial\varrho^\iota}\bigg|_{\varrho=0},\\
&\mathrm{g}_1(\zeta)=\hbar\mathbf{R}_1\left[\mathrm{g}_0(\zeta)\right],\\
&\mathrm{g}_\ell(\zeta)=\mathrm{g}_{(\ell-1)}(\zeta)+\hbar\mathbf{R}_\ell\left[\mathrm{g}_{(\ell-1)}(\zeta)\right],\quad \text{for } \ell\geq 2,
\end{aligned}$$

where

$$\mathrm{g}_{(\ell-1)}(\zeta)=\left(\mathrm{g}_0(\zeta),\mathrm{g}_1(\zeta),\ldots,\mathrm{g}_{\ell-1})\right),$$

and

$$\mathbf{R}_\ell\left[\mathrm{g}_{\ell-1}(\zeta)\right]=\frac{1}{(\ell-1)!}\left[\frac{\partial^{\ell-1}}{\partial\varrho^{\ell-1}}\mathcal{N}\left(\sum_{i=0}^{\infty}\mathrm{g}_i(\zeta)\varrho^i\right)\bigg|_{\varrho=0}\right].$$

4. Numerical results

As an application of the construction of the above algorithms in Theorems 2.6 and 2.7, we can now present some numerical examples. Data calculations and graphs are implemented by MATLAB 2022a.

Example 4.1. *Consider the boundary value problem*

$$\begin{aligned}
\theta\psi''(\zeta)+\cos(\zeta)\psi'(\zeta)+\sin(\zeta)\psi(\zeta)=f(\zeta)+\lambda_1\int_0^\zeta\exp(\zeta-\tau)\psi^2(\tau)\mathrm{d}\tau+\lambda_2\int_0^1\exp(\zeta+\tau)\psi^2(\tau)\mathrm{d}\tau,\\
\psi(0)=1,\quad \psi(1)=\exp(1),
\end{aligned}\tag{4.1}$$

where $f(\zeta)=(\theta+\cos(\zeta)+\sin(\zeta))e^\zeta-\lambda_1(e^{2\zeta}-e^\zeta)-\lambda_2(\frac{e^{\zeta+3}-e^\zeta}{3})$, $\theta=1.6\times10^2$, $\lambda_1=\frac{1}{600}$, *and* $\lambda_2=\frac{1}{200}$. *Note that the exact solution for this problem is* $\psi(x)=\exp(x)$, *for* $t\in[0,1]$.

Repeating the above process as in Section 2 by setting $\mathrm{g}(\zeta):=\psi''(\zeta)$, *we can deduce a nonlinear Volterra-Fredholm integral equation in the form of (2.1). Moreover, (4.1) can satisfy the condition postulate* (5). *Thus, Theorem 2.7 confirms the uniqueness of solution of this problem. Finally, we tabulate the numerical results in Table 1 with* $\hbar=-0.3332987$ *for the proposed methods and their absolute errors between them with the exact value. Moreover, we have drawn it graphically in Figure 1 for the same value of* $\hbar$.

Table 1. Numerical solutions for Example 4.1 solved by the MADM (g_{MADM}) and HAM (g_{HAM}).

ζ	g_{exact}	g_{MADM}	g_{HAM}	$\|g_{exact} - g_{MADM}\|$	$\|g_{exact} - g_{HAM}\|$
0	1.000000000000000	0.999984104835638	0.999581887186063	0.000015895164362	0.000418112813937
0.200000000000000	1.221402758160170	1.221388089863198	1.221049812943067	0.000014668296972	0.000352945217103
0.400000000000000	1.491824697641270	1.491814748366577	1.491543753272788	0.000009949274693	0.000280944368483
0.600000000000000	1.822118800390509	1.822119095098361	1.821920429564758	0.000000294707852	0.000198370825751
0.800000000000000	2.225540928492468	2.225559875518119	2.225437477606285	0.000018947025652	0.000103450886183
1.000000000000000	2.718281828459045	2.718331874761669	2.718283517373330	0.000050046302623	0.000001688914284

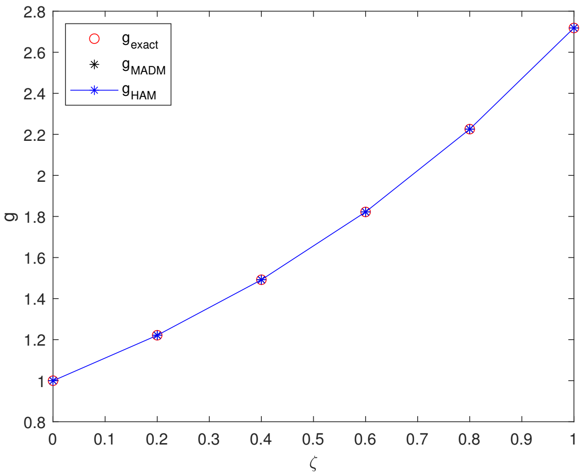

Figure 1. Plot of the proposed methods compared with the exact solution of Example 4.1.

Example 4.2. *In our second example, we consider the boundary value problem*

$$\theta\psi''(\zeta) - 2\psi'(\zeta) + \exp(\zeta)\psi(\zeta) = f(\zeta) + \lambda_1 \int_0^{\zeta} \sin(\zeta-\tau)\psi^2(\tau)\mathrm{d}\tau + \lambda_2 \int_0^{\frac{\pi}{2}} \cos(\zeta-\tau)\psi^2(\tau)\mathrm{d}\tau,$$
$$\psi(0) = 1, \quad \psi(\frac{\pi}{2}) = 0, \tag{4.2}$$

where, $f(\zeta) = -\theta\cos(\zeta) + 2\sin(\zeta) + e^{\zeta}\cos(\zeta) - \lambda_1\left(\frac{\sin^2(\zeta)-\cos(\zeta)+1}{3}\right) - \lambda_2\left(\frac{2\cos(\zeta)+\sin(\zeta)}{3}\right)$, $\theta = 2\times 10^3$, $\lambda_1 = -1\times 10^{-4}$, *and* $\lambda_2 = 2\times 10^{-4}$. *It is worth mentioning that the exact solution for this problem is* $\psi(\zeta) = \cos(\zeta)$, *for* $\zeta \in \left[0, \frac{\pi}{2}\right]$.

Again, by repeating the above process as in Section 2 with $g(\zeta) := \psi''(\zeta)$, *we can deduce a nonlinear Volterra-Fredholm integral equation in the form of (2.1). Further, (4.2) satisfies the condition postulate* (5). *Therefore, Theorem 2.7 confirms the uniqueness of solution of this problem. Finally, we tabulate the numerical results in Table 2 with* $\hbar = -0.3335010$ *for the proposed methods and their absolute errors between them with the exact value. In addition, we have shown graphically the proposed method together with the exact solution in Figure 2 for the same value of* $\hbar$.

Table 2. Numerical solutions for Example 4.2 solved by the MADM (g_MADM) and HAM (g_{HAM}).

ζ	g_{exact}	g_{MADM}	g_{HAM}	$\|\|g_{exact} - g_{MADM}\|\|$	$\|\|g_{exact} - g_{HAM}\|\|$
0	-1.000000000000000	-0.999999736951218	-1.000927604126402	0.000000263048782	0.000927604126402
0.314159265358979	-0.951056516295154	-0.951056324364335	-0.951684585338327	0.000000191930819	0.000628069043174
0.628318530717959	-0.809016994374947	-0.809016933053865	-0.809326057611100	0.000000061321082	0.000309063236153
0.942477796076938	-0.587785252292473	-0.587785406837625	-0.587805387326148	0.000000154545152	0.000020135033675
1.256637061435917	-0.309016994374947	-0.309017459304380	-0.308835063094412	0.000000464929432	0.000181931280536
1.570796326794897	-0.000000000000000	-0.000000754517346	0.000242283168664	0.000000754517346	0.000242283168664

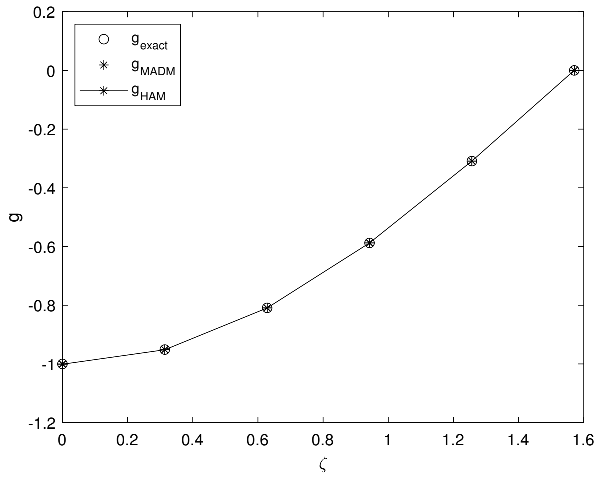

Figure 2. Plot of the proposed methods compared with the exact solution of Example 4.2.

Example 4.3. *Consider the following problem*

$$\theta\psi''(\zeta) + 2\psi'(\zeta) = f(\zeta) + \lambda_1 \int_0^{\zeta} \zeta(3\tau^2 - 2)\psi^3(\tau)d\tau + \lambda_2 \int_0^{1} \zeta^2(3\tau^2 - 2)\psi^3(\tau)d\tau,$$
$$\psi(0) = 1, \quad \psi(1) = 0, \tag{4.3}$$

where,

$$f(\zeta) = 6\zeta\theta + 6\zeta^2 - 4 - \lambda_1 \frac{\zeta^2 \cdot \left(\zeta^{11} - 8\zeta^9 + 4\zeta^8 + 24\zeta^7 - 24\zeta^6 - 26\zeta^5 + 48\zeta^4 - 8\zeta^3 - 28\zeta^2 + 24\zeta - 8\right)}{4} + \lambda_2 \frac{\zeta^2}{4},$$

$\psi(\zeta) = \zeta^3 - 2\zeta + 1$, $\theta = 600$, $\lambda_1 = \frac{1}{200}$, *and* $\lambda_2 = \frac{1}{400}$. *Note that the exact solution for this problem is* $\psi(\zeta) = \zeta^3 - 2\zeta + 1$, *for* $\zeta \in [0, 1]$.

By repeating the procedure in Section 2 by setting $g(\zeta) := \psi''(\zeta)$, *a nonlinear Volterra-Fredholm integral equation of the form (2.1) can be deduced. Moreover, (4.3) can satisfy the condition postulate* (5), *and this implies that the problem* (4.3) *has a unique solution by Theorem 2.7. Finally, the numerical results are tabulated in Table 3 with* $\hbar = -0.333335660493482$ *for the proposed methods and their absolute errors between them with the exact value. Furthermore, it has been drawn graphically in Figure 3 for the same value of* $\hbar$.

Table 3. Numerical solutions for Example 4.3 solved by the MADM (g_{MADM}) and HAM (g_{HAM}).

ζ	g_{exact}	g_{MADM}	g_{HAM}	$\|\|g_{exact} - g_{MADM}\|\|$	$\|\|g_{exact} - g_{HAM}\|\|$
0	0	0.000000000902758	-0.002221604293149	0.000000000902758	0.002221604293149
0.200000000000000	1.200000000000000	1.200000557975028	1.197157654809369	0.000000557975028	0.002842345190631
0.400000000000000	2.400000000000000	2.400001016543842	2.397069458186190	0.000001016543842	0.002930541813811
0.600000000000000	3.600000000000000	3.600001360727028	3.597513742673720	0.000001360727029	0.002486257326280
0.800000000000000	4.800000000000001	4.800001758680223	4.798490618826795	0.000001758680223	0.001509381173205
1.000000000000000	6.000000000000000	6.000002075477893	5.999999960429999	0.000002075477893	0.000000039570001

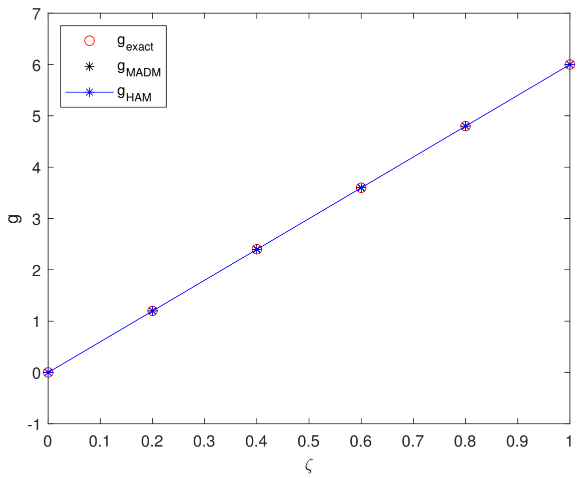

Figure 3. Plot of the proposed methods compared with the exact solution of Example 4.3.

Example 4.4. *Finally, we consider the following problem*

$$\theta\psi''(\zeta) + \sin(x)\psi'(\zeta) + \cos(x)\psi(\zeta) = f(\zeta) + \lambda_1 \int_0^{\zeta} \sinh(\zeta - \tau)\psi^2(\tau)d\tau + \lambda_2 \int_0^{\log(2)} \cosh(\zeta - \tau)psi^2(\tau)d\tau,$$
$$\psi(0) = 1, \quad \psi(\log(2)) = 2, \tag{4.4}$$

where, $f(\zeta) = (\theta + \cos(\zeta) + \sin(\zeta))\, e^{\zeta} - \lambda_1 \frac{\left(3(e^{2\zeta} - e^{\zeta}) + e^{\zeta} - e^{2\zeta}\right)}{6} + \lambda_2 \frac{\left(6e^{\zeta} + 15e^{-\zeta}\right)}{8}$, $\theta = 2 \times 10^3$, $\lambda_1 = \frac{1}{1200}$, *and* $\lambda_2 = \frac{1}{2400}$.

Numerical results for this example are tabulated and shown in Table 4 and Figure 4, respectively.

Table 4. Numerical solutions for Example 4.4 solved by the MADM (g_{MADM}) and HAM (g_{HAM}).

ζ	g_{h1HAM}	g_{h2HAM}	g_{h3HAM}	g_{MADM}
0	1.000000002395460	1.000000000895461	0.999999999395461	0.999999591561626
0.138629436111989	1.148660177561804	1.148660175838817	1.148660174115830	1.148698031776195
0.277258872223978	1.319463455596505	1.319463453617322	1.319463451638140	1.319507684383564
0.415888308335967	1.515702866711945	1.515702864438419	1.515702862164893	1.515716463317634
0.554517744447956	1.741158734029968	1.741158731418281	1.741158728806595	1.741101190901671
0.693147180559945	2.000170959481061	2.000170956480888	2.000170953480715	2.000000298539512

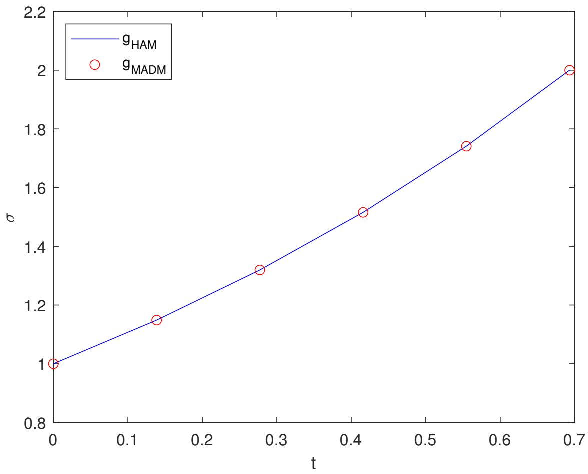

Figure 4. Plot comparison of the proposed methods in Example 4.4.

5. Conclusion and future directions

We have studied a nonlinear boundary value problem for a Volterra-Fredholm integro equation-type subjected to certain boundary conditions. For the auxiliary problem (1.1) with the simplified right-hand side, we have explicitly constructed its existence and uniqueness by applying Arzela-Ascoli Krasnoselskii fixed point theorems. In addition, based on the theory of the Banach contraction principle index, we prove existence of at most one continuous solution to the original problem as pointed out in Theorem 2.6. For a better understanding on the resulting boundary models, we have provided some numerical discussions and clear graphical demonstrations for Volterra-Fredholm integro problems for some eigenvalues and homotopy parameters. Many solutions have been obtained and represented in Figures 1–4.

As a consequence, (HAM and MADM) have the best approximate solutions to solve nonlinear integral equations in both circumstances, but we found that the HAM has much lower running durations than the MADM.

The fractional differential problems of Volterra-Fredholm integro type are great prospect as a kind of highly integrated boundary value problem in integrated fractional operators, although there is still room for improvement in transmission efficiency and numerical solutions, which is also the future direction of our work.

Conflict of interest

The authors declare that they have no competing interests.

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